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On the Cut Number of a 3–manifold

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Abstract

The question was raised as to whether the cut number of a 3–manifold X is bounded from below by $\frac{1}{3}\beta_1(X)$. We show that the answer to this question is "no." For each $m \geq 1$, we construct explicit examples of closed 3–manifolds X with $\beta_1(X) = m$ and cut number 1. That is, $\pi_1(X)$ cannot map onto any non-abelian free group. Moreover, we show that these examples can be assumed to be hyperbolic.

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1 Introduction

Let X be a closed, orientable n–manifold. The *cut number of* X, $c(X)$, is defined to be the maximal number of components of a closed, 2–sided, orientable hypersurface $F \subset X$ such that $X - F$ is connected. Hence, for any $n \leq c(X)$, we can construct a map $f\colon X \to \bigvee_{i=1}^{n} S^1$ such that the induced map on π_1 is surjective. That is, there exists a surjective map $f_*\colon \pi_1(X) \twoheadrightarrow F(c)$, where $F(c)$ is the free group with $c = c(X)$ generators. Conversely, if we have any epimorphism $\phi\colon \pi_1(X) \twoheadrightarrow F(n)$, then we can find a map $f\colon X \to \bigvee_{i=1}^{n} S^1$ such that $f_* = \phi$. After making the f transverse to a non-wedge point x_i on each S^1, $f^{-1}(X)$ will give n disjoint surfaces $F = \cup F_i$ with $X - F$ connected. Hence one has the following elementary group-theoretic characterization of $c(X)$.

Proposition 1.1 *$c(X)$ is the maximal n such that there is an epimorphism $\phi\colon \pi_1(X) \twoheadrightarrow F(n)$ onto the free group with n generators.*

Example 1.2 Let $X = S^1 \times S^1 \times S^1$ be the 3–torus. Since $\pi_1(X) = \mathbb{Z}^3$ is abelian, $c(X) = 1$.

Using Proposition 1.1, we show that the cut number is additive under connected sum.

Proposition 1.3 *If $X = X_1 \# X_2$ is the connected sum of X_1 and X_2 then*

$$c(X) = c(X_1) + c(X_2).$$

Proof Let $G_i = \pi_1(X_i)$ for $i = 1, 2$ and $G = \pi_1(X) \cong G_1 * G_2$. It is clear that G maps surjectively onto $F(c(X_1)) * F(c(X_2)) \cong F(c(X_1 + X_2))$. Therefore $c(X) \geq c(X_1) + c(X_2)$.

Now suppose that there exists a map $\phi\colon G \twoheadrightarrow F(n)$. Let $\phi_i\colon G_i \to F(n)$ be the composition $G_i \to G_1 * G_2 \xrightarrow{\cong} G \overset{\phi}{\twoheadrightarrow} F(n)$. Since ϕ is surjective and $G \cong G_1 * G_2$, $\mathrm{Im}(\phi_1)$ and $\mathrm{Im}(\phi_2)$ generate $F(n)$. Morever, $\mathrm{Im}(\phi_i)$ is a subgroup of a free group, hence is free of rank less than or equal to $c(X_i)$. It follows that $n \leq c(X_1) + c(X_2)$. In particular, when n is maximal we have $c(X) = n \leq c(X_1) + c(X_2)$. □

In this paper, we will only consider 3–manifolds with $\beta_1(X) \geq 1$. Consider the surjective map $\pi_1(X) \twoheadrightarrow H_1(X) / \{\mathbb{Z}\text{–torsion}\} \cong \mathbb{Z}^{\beta_1(X)}$. Since $\beta_1(X) \geq 1$, we can find a surjective map from $\mathbb{Z}^{\beta_1(X)}$ onto $\mathbb{Z}$. It follows from Proposition 1.1 that $c(X) \geq 1$. Moreover, every map $\phi\colon \pi_1(X) \twoheadrightarrow F(n)$ gives rise to an epimorphism $\overline{\phi}\colon H_1(X) \twoheadrightarrow H_1\left(\bigvee_{i=1}^{n} S^1\right) \cong \mathbb{Z}^n$ It follows that $\beta_1(X) \geq n$ which gives us the well known result:

$$1 \leq c(X) \leq \beta_1(X). \tag{1}$$

It has recently been asked whether a (non-trivial) lower bound exists for the cut number. We make the following observations.

Remark 1.4 If S is a closed, orientable surface then $c(S) = \frac{1}{2}\beta_1(S)$.

Remark 1.5 If X has solvable fundamental group then $c(X) = 1$ and $\beta_1(X) \leq 3$.

Remark 1.6 Both c and β_1 are additive under connected sum (Proposition 1.3).

Therefore it is natural to ask the following question first asked by A Sikora and T Kerler. This question was motivated by certain results and conjectures on the divisibility of quantum 3–manifold invariants by P Gilmer–T Kerler [2] and T Cochran–P Melvin [1].

Question 1.7 Is $c(X) \geq \frac{1}{3}\beta_1(X)$ for all closed, orientable 3–manifolds X?

We show that the answer to this question is "as far from yes as possible." In fact, we show that for each $m \geq 1$ there exists a closed, *hyperbolic* 3–manifold with $\beta_1(X) = m$ and $c(X) = 1$. We actually prove a stronger statement.

Theorem 3.1 *For each $m \geq 1$ there exist closed 3–manifolds X with $\beta_1(X) = m$ such that for any infinite cyclic cover $X_\phi \to X$, $\operatorname{rank}_{\mathbb{Z}[t^{\pm 1}]} H_1(X_\phi) = 0$.*

We note the condition stated in the Theorem 3.1 is especially interesting because of the following theorem of J Howie [3]. Recall that a group G is *large* if some subgroup of finite index has a non-abelian free homomorphic image. Howie shows that if G has an infinite cyclic cover whose rank is at least 1 then G is large.

Theorem 1.8 (Howie [3]) *Suppose that $\widetilde{K}$ is a connected regular covering complex of a finite 2–complex K, with nontrivial free abelian covering transformation group A. Suppose also that $H_1\left(\widetilde{K};\mathbb{Q}\right)$ has a free $\mathbb{Q}[A]$–submodule of rank at least 1. Then $G = \pi_1(K)$ is large.*

Using the proof of Theorem 3.1 we show that the fundamental group of the aforementioned 3–manifolds cannot map onto F/F_4 where F is the free group with 2 generators and F_4 is the 4^{th} term of the lower central series of F.

Proposition 3.3 *Let X be as in Theorem 3.1, $G = \pi_1(X)$ and F be the free group on 2 generators. There is no epimorphism from G onto F/F_4.*

Independently, A Sikora has recently shown that the cut number of a "generic" 3–manifold is at most 2 [8]. Also, C Leininger and A Reid have constructed specific examples of genus 2 surface bundles X satisfying (i) $\beta_1(X) = 5$ and $c(X) = 1$ and (ii) $\beta_1(X) = 7$ and $c(X) = 2$ [6].

Acknowledgements I became interested in the question as to whether the cut number of a 3–manifold was bounded below by one-third the first betti number after hearing it asked by A Sikora at a problem session of the 2001 Georgia Topology Conference. The question was also posed in a talk by T Kerler at the 2001 Lehigh Geometry and Topology Conference. The author was supported by NSF DMS-0104275 as well as by the Bob E and Lore Merten Watt Fellowship.

2 Relative Cut Number

Let ϕ be a primitive class in $H^1(X;\mathbb{Z})$. Since $H^1(X;\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(X), \mathbb{Z})$, we can assume ϕ is a surjective homomorphism, $\phi\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}$. Since X is an orientable 3–manifold, every element in $H_2(X;\mathbb{Z})$ can be represented by an embedded, oriented, 2–sided surface [10, Lemma 1]. Therefore, if $\phi \in H^1(X;\mathbb{Z}) \cong H_2(X;\mathbb{Z})$ there exists a surface (not unique) dual to ϕ. The *cut number of X relative to ϕ*, $c(X,\phi)$, is defined as the maximal number of components of a closed, 2–sided, oriented surface $F \subset X$ such that $X - F$ is connected and one of the components of F is dual to ϕ. In the above definition, we could have required that "any number" of components of F be dual to ϕ as opposed to just "one." We remark that since $X - F$ is connected, these two conditions are equivalent. Similar to $c(X)$, we can describe $c(X,\phi)$ group theoretically.

Proposition 2.1 *$c(X,\phi)$ is the maximal n such that there is an epimorphism $\psi\colon \pi_1(X) \twoheadrightarrow F(n)$ onto the free group with n generators that factors through ϕ (see diagram on next page).*

$$\begin{tikzcd} \pi_1(X) \arrow[r, "\phi"] \arrow[d, "\psi"'] & \mathbb{Z} \\ F(n) \arrow[ur] & \end{tikzcd}$$

It follows immediately from the definitions that $c(X,\phi) \leq c(X)$ for all primitive ϕ. Now let F be any surface with $c(X)$ components and let ϕ be dual to one of the components, then $c(X,\phi) = c(X)$. Hence

$$c(X) = \max\left\{c(X,\phi) \mid \phi \text{ is a primitive element of } H^1(X;\mathbb{Z})\right\}. \tag{2}$$

In particular, if $c(X,\phi) = 1$ for all ϕ then $c(X) = 1$.

We wish to find sufficient conditions for $c(X,\phi) = 1$. In [5, page 44], T Kerler develops a skein theoretic algorithm to compute the one-variable Alexander polynomial $\Delta_{X,\phi}$ from a surgery presentation of X. As a result, he shows that if $c(X,\phi) \geq 2$ then the Frohman–Nicas TQFT evaluated on the cut cobordism is zero, implying that $\Delta_{X,\phi} = 0$. Using the fact that $\mathbb{Q}\left[t^{\pm 1}\right]$ is a principal ideal domain one can prove that $\Delta_{X,\phi} = 0$ is equivalent to $\operatorname{rank}_{\mathbb{Z}[t^{\pm 1}]} H_1(X_\phi) \geq 1$. We give an elementary proof of the equivalent statement of Kerler's.

Proposition 2.2 *If* $c(X,\phi) \geq 2$ *then* $\operatorname{rank}_{\mathbb{Z}[t^{\pm 1}]} H_1(X_\phi) \geq 1$.

Proof Suppose $c(X,\phi) \geq 2$ then there is a surjective map $\psi\colon \pi_1(X) \twoheadrightarrow F(n)$ that factors through ϕ with $n \geq 2$. Let $\overline{\phi}\colon F(n) \twoheadrightarrow \mathbb{Z}$ be the homomorphism such that $\phi = \overline{\phi} \circ \psi$. ϕ surjective implies that $\psi_{|\ker\phi}\colon \ker\phi \twoheadrightarrow \ker\overline{\phi}$ is surjective. Writing $\mathbb{Z}$ as the multiplicative group generated by t, we can consider $\frac{\ker\phi}{[\ker\phi,\ker\phi]}$ and $\frac{\ker\overline{\phi}}{[\ker\overline{\phi},\ker\overline{\phi}]}$ as modules over $\mathbb{Z}\left[t^{\pm 1}\right]$. Here, the t acts by conjugating by an element that maps to t by ϕ or $\overline{\phi}$. Moreover, $\psi_{|\ker\phi}\colon \frac{\ker\phi}{[\ker\phi,\ker\phi]} \twoheadrightarrow \frac{\ker\overline{\phi}}{[\ker\overline{\phi},\ker\overline{\phi}]}$ is surjective hence

$$\operatorname{rank}_{\mathbb{Z}[t^{\pm 1}]}\left(\frac{\ker\phi}{[\ker\phi,\ker\phi]}\right) \geq \operatorname{rank}_{\mathbb{Z}[t^{\pm 1}]}\left(\frac{\ker\overline{\phi}}{[\ker\overline{\phi},\ker\overline{\phi}]}\right) = n-1.$$

Since $n \geq 2$, $\operatorname{rank}_{\mathbb{Z}[t^{\pm 1}]} H_1(X_\phi) = \operatorname{rank}_{\mathbb{Z}[t^{\pm 1}]}\left(\frac{\ker\phi}{[\ker\phi,\ker\phi]}\right) \geq 1$. $\square$

Corollary 2.3 *If* $\pi_1(X) \twoheadrightarrow F/F''$ *where* F *is a free group of rank 2 then there exists a* $\phi\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}$ *such that* $\operatorname{rank}_{\mathbb{Z}[t^{\pm 1}]} H_1(X_\phi) \geq 1$.

Proof This follows immediately from the proof of Proposition 2.2 after noticing that $F'' \subset \left[\ker\left(\overline{\phi}\right), \ker\left(\overline{\phi}\right)\right]$ and $\mathrm{Hom}\left(F/F'', \mathbb{Z}\right) \cong \mathrm{Hom}\left(F, \mathbb{Z}\right)$. $\square$

3 The Examples

We construct closed 3–manifolds all of whose infinite cyclic covers have first homology that is $\mathbb{Z}\left[t^{\pm 1}\right]$–torsion. The 3–manifolds we consider are 0–surgery on an m–component link that is obtained from the trivial link by tying a Whitehead link interaction between each two components.

Theorem 3.1 *For each $m \geq 1$ there exist closed 3–manifolds X with $\beta_1(X) = m$ such that for any infinite cyclic cover $X_\phi \to X$, $\mathrm{rank}_{\mathbb{Z}[t^{\pm 1}]} H_1(X_\phi) = 0$.*

It follows from Propostion 2.2 that the cut number of the manifolds in Theorem 3.1 is 1. In fact, Corollary 2.3 implies that $\pi_1(X)$ does not map onto F/F'' where F is a free group of rank 2. Moreover, the proof of this theorem shows that $\pi_1(X)$ does not even map onto F/F_4 where F_n is the n^{th} term of the lower central series of F (see Proposition 3.3).

By a theorem of Ruberman [7], we can assume that the manifolds with cut number 1 are hyperbolic.

Corollary 3.2 *For each $m \geq 1$ there exist closed, orientable, hyperbolic 3–manifolds Y with $\beta_1(Y) = m$ such that for any infinite cyclic cover $Y_\phi \to Y$, $\mathrm{rank}_{\mathbb{Z}[t^{\pm 1}]} H_1(Y_\phi) = 0$.*

Proof Let X be one of the 3–manifolds in Theorem 3.1. By [7, Theorem 2.6], there exists a degree one map $f\colon Y \to X$ where Y is hyperbolic and f_* is an isomorphism on H_*. Denote by $G = \pi_1(X)$ and $P = \pi_1(Y)$. It is then well-known that f is surjective on π_1. It follows from Stalling's theorem [9, page 170] that the kernel of f_* is $P_\omega \equiv \cap P_n$. Now, suppose $\phi\colon P \overset{f_*}{\twoheadrightarrow} G \overset{\overline{\phi}}{\twoheadrightarrow} \mathbb{Z}$ defines an infinite cyclic cover of Y. Then $H_1(Y_\phi) \twoheadrightarrow H_1\left(X_{\overline{\phi}}\right)$ has kernel $P_\omega/\left[\ker\phi, \ker\phi\right]$. To show that $\mathrm{rank}_{\mathbb{Z}[t^{\pm 1}]} H_1(Y_\phi) = 0$ it suffices to show that P_ω vanishes under the map $H_1(Y_\phi) \to H_1(Y_\phi) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Q}\left[t^{\pm 1}\right] \to H_1(Y_\phi) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Q}(t)$ since then $\mathrm{rank}_{\mathbb{Z}[t^{\pm 1}]} H_1(Y_\phi) = \mathrm{rank}_{\mathbb{Z}[t^{\pm 1}]} H_1\left(X_{\overline{\phi}}\right) = 0$.

Note that $H_1(Y_\phi) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Q}\left[t^{\pm 1}\right] \cong \bigoplus_{i=1}^{n} \mathbb{Q}\left[t^{\pm 1}\right] \oplus T$ where T is a $\mathbb{Q}\left[t^{\pm 1}\right]$ torsion module. Moreover, P_n is generated by elements of the form $\gamma =$

$[p_1 [p_2 [p_3, \ldots [p_{n-2}, \alpha]]]]$ where $\alpha \in P_2 \subseteq \ker \phi$. Therefore

$$[\gamma] = (\phi(p_i) - 1) \cdots (\phi(p_{n-2}) - 1) [\alpha]$$

in $H_1(Y_\phi)$ which implies that $P_n \subseteq J^{n-2}(H_1(Y_\phi))$ for $n \geq 2$ where J is the augmentation ideal of $\mathbb{Z}[t^{\pm 1}]$. It follows that any element of P_ω considered as an element of $H_1(Y_\phi) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Q}[t^{\pm 1}]$ is infinitely divisible by $t-1$ and hence is torsion. □

Proof of Theorem 3.1 Let $L = \sqcup L_i$ be the oriented trivial link with m components in S^3 and $\sqcup D_i$ be oriented disjoint disks with $\partial D_i = L_i$. The fundamental group of $S^3 - L$ is freely generated by $\{x_i\}$ where x_i is a meridian curve of L_i which intersects D_i exactly once and $D_i \cdot x_i = 1$. For all i, j with $1 \leq i < j \leq m$ let $\alpha_{ij}\colon I \to S^3$ be oriented disjointly embedded arcs such that $\alpha_{ij}(0) \in L_i$ and $\alpha_{ij}(1) \in L_j$ and $\alpha_{ij}(I)$ does not intersect $\sqcup D_i$. For each arc α_{ij}, let γ_{ij} be the curve embedded in a small neighborhood of α_{ij} representing the class $[x_i, x_j]$ as in Figure 1. Let X be the 3–manifold obtained performing

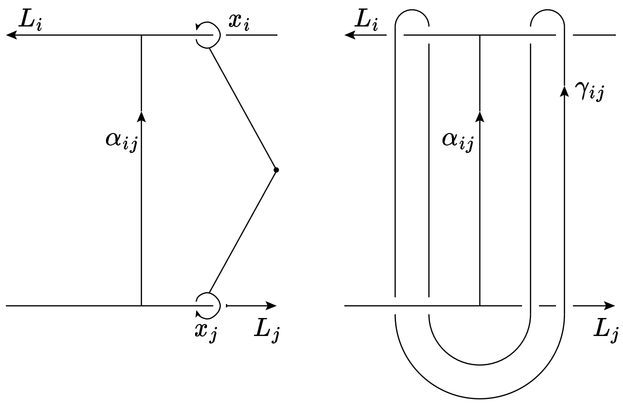

Figure 1

0–framed Dehn surgery on L and -1–framed Dehn surgery on each $\gamma = \sqcup \gamma_{ij}$. See Figure 2 for an example of X when $m = 5$.

Denote by X_0, the manifold obtained by performing 0–framed Dehn surgery on L. Let W be the 4–manifold obtained by adding a 2–handle to $X_0 \times I$ along each curve $\gamma_{ij} \times \{1\}$ with framing coefficient -1. The boundary of W is $\partial W = X_0 \sqcup -X$. We note that

$$\pi_1(W) = \langle x_1, \ldots, x_m | [x_i, x_j] = 1 \text{ for all } 1 \leq i < j \leq m \rangle \cong \mathbb{Z}^m.$$

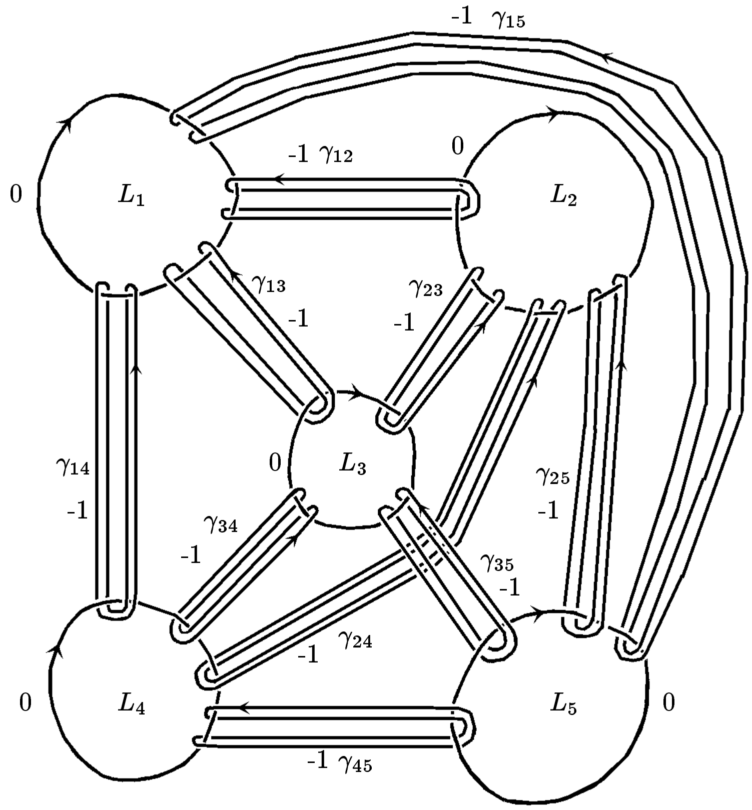

Figure 2: The surgered manifold X when $m = 5$

Let $\{x_{ik}, \mu_{ijl}\}$ be the generators of $\pi_1\left(S^3 - (L \sqcup \gamma)\right)$ that are obtained from a Wirtinger presentation where x_{ik} are meridians of the i^{th} component of L and μ_{ijl} are meridians of the $(i,j)^{th}$ component of γ. Note that $\{x_{ik}, \mu_{ijl}\}$ generate $G \equiv \pi_1(X)$. For each $1 \leq i \leq m$ let $\overline{x}_i = x_{i1}$ and $\overline{\mu}_{ij}$ be the specific μ_{ijl} that is denoted in Figure 3. We will use the convention that

$$[a,b] = aba^{-1}b^{-1}$$

and

$$a^b = bab^{-1}.$$

We can choose a projection of the trivial link so that the arcs α_{ij} do not pass under a component of L. Since $\overline{\mu}_{ij}$ is equal to a longitude of the curve γ_{ij} in X, we have $\overline{\mu}_{ij} = \left[x_{in_{ij}}, \lambda x_{jn_{ji}} \lambda^{-1}\right]$ for some n_{ij} and n_{ji} and λ where λ is a product of conjugates of meridian curves $\overline{\mu}_{lk}$ and $\overline{\mu}_{lk}^{-1}$. Moreover, we can find

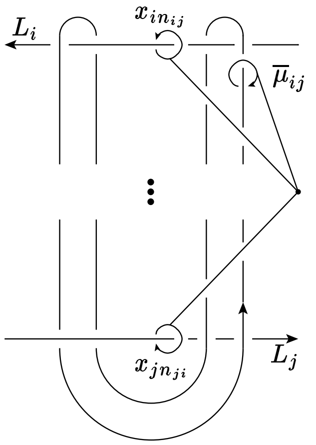

Figure 3

a projection of $L \sqcup \gamma$ so that the individual components of L do not pass under or over one another. Hence $x_{ij} = \omega \overline{x}_i \omega^{-1}$ where ω is a product of conjugates of the meridian curves $\overline{\mu}_{lk}$ and $\overline{\mu}_{lk}^{-1}$. As a result, we have

$$\begin{aligned} \overline{\mu}_{ij} &= \left[x_{in_{ij}}, \lambda x_{jn_{ji}} \lambda^{-1}\right] \\ &= \left[\omega_1 \overline{x}_i \omega_1^{-1}, \lambda \omega_2 \overline{x}_j \omega_2^{-1} \lambda^{-1}\right] \\ &= \left[\overline{x}_i, \omega_1^{-1} \lambda \omega_2 \overline{x}_j \omega_2^{-1} \lambda^{-1} \omega_1^{-1}\right]^{\omega_1} \end{aligned} \tag{3}$$

for some λ, ω_1, and ω_2.

We note that $\overline{\mu}_{ij} = \left[x_{in_{ij}}, \lambda x_{jn_{ji}} \lambda^{-1}\right]$ hence $\overline{\mu}_{ij} \in G'$ for all $i < j$. Setting $v = \omega_1^{-1} \lambda \omega_2$ and using the equality

$$[a, bc] = [a, b]\,[a, c]^b \tag{4}$$

we see that

$$\begin{aligned} \overline{\mu}_{ij} &= \left[\overline{x}_i, v \overline{x}_j v^{-1}\right]^{\omega_1} \\ &= \left[\overline{x}_i, v \overline{x}_j v^{-1}\right] \mod G'' \\ &= \left[\overline{x}_i, [v, \overline{x}_j]\,\overline{x}_j\right] \\ &= \left[\overline{x}_i, [v, \overline{x}_j]\right] [\overline{x}_i, \overline{x}_j]^{[v, \overline{x}_j]} \\ &= \left[\overline{x}_i, [v, \overline{x}_j]\right] [\overline{x}_i, \overline{x}_j] \mod G'' \end{aligned} \tag{5}$$

since $\omega_1, v \in G'$.

Consider the dual relative handlebody decomposition (W, X). W can be obtained from X by adding a 0–framed 2–handle to $X \times I$ along each of the meridian curves $\overline{\mu}_{ij} \times \{1\}$. (3) implies that $\overline{\mu}_{ij}$ is trivial in $H_1(X)$ hence the inclusion map $j\colon X \to W$ induces an isomorphism $j_*\colon H_1(X) \overset{\cong}{\to} H_1(W)$. Therefore if $\phi\colon G \twoheadrightarrow \Lambda$ where Λ is abelian then there exists a $\psi\colon \pi_1(W) \twoheadrightarrow \Lambda$ such that $\psi \circ j_* = \phi$.

Suppose $\phi\colon G \twoheadrightarrow \langle t\rangle \cong \mathbb{Z}$ and $\psi\colon \pi_1(W) \twoheadrightarrow \langle t\rangle$ is an extension of ϕ to $\pi_1(W)$. Let X_ϕ and W_ψ be the infinite cyclic covers of W and X corresponding to ψ and ϕ respectively. Consider the long exact sequence of pairs,

$$\to H_2(W_\psi, X_\phi) \overset{\partial_*}{\to} H_1(X_\phi) \to H_1(W_\psi) \to \tag{6}$$

Since $\pi_1(W) \cong \mathbb{Z}^m$, $H_1(W_\psi) \cong \mathbb{Z}^{m-1}$ where t acts trivially so that $H_1(W_\psi)$ has rank 0 as a $\mathbb{Z}\left[t^{\pm1}\right]$–module. $H_2(W_\psi, X_\phi) \cong \left(\mathbb{Z}\left[t^{\pm1}\right]\right)^{\binom{m}{2}}$ generated by the core of each 2–handle (extended by $\overline{\mu}_{ij} \times I$) attached to X. Therefore, $\mathrm{Im}\partial_*$ is generated by a lift of $\overline{\mu}_{ij}$ in $H_1(X_\phi)$ for all $1 \le i < j \le m$. To show that $H_1(X_\phi)$ has rank 0 it suffices to show that each of the $\overline{\mu}_{ij}$ are $\mathbb{Z}\left[t^{\pm1}\right]$–torsion in $H_1(X_\phi)$.

Let $F = \langle \overline{x}_1, \dots, \overline{x}_m\rangle$ be the free group of rank m and $f\colon F \to G$ be defined by $f(\overline{x}_i) = \overline{x}_i$. We have the following $\binom{m}{3}$ Jacobi relations in F/F'' [4, Proposition 7.3.6]. For all $1 \le i < j < k \le m$,

$$[\overline{x}_i, [\overline{x}_j, \overline{x}_k]]\,[\overline{x}_j, [\overline{x}_k, \overline{x}_i]]\,[\overline{x}_k, [\overline{x}_i, \overline{x}_j]] = 1 \bmod F''.$$

Using f, we see that these relations hold in G/G'' as well. From (5), we can write

$$[\overline{x}_i, \overline{x}_j] = [[v_{ij}, \overline{x}_j], \overline{x}_i]\,\overline{\mu}_{ij} \bmod G''.$$

Hence for each $1 \le i < j < k \le m$ we have the Jacobi relation $J(i,j,k)$ in G/G'',

$$\begin{aligned}
1 &= [\overline{x}_i, [\overline{x}_j, \overline{x}_k]]\left[\overline{x}_j, [\overline{x}_i, \overline{x}_k]^{-1}\right][\overline{x}_k, [\overline{x}_i, \overline{x}_j]] \bmod G'' \\
&= \left[\overline{x}_i, [[v_{jk}, \overline{x}_k], \overline{x}_j]\,\overline{\mu}_{jk}\right]\left[\overline{x}_j, \overline{\mu}_{ik}^{-1}[\overline{x}_i, [v_{ik}, \overline{x}_k]]\right] \\
&\quad \left[\overline{x}_k, [[v_{ij}, \overline{x}_j], \overline{x}_i]\,\overline{\mu}_{ij}\right] \bmod G'' \\
&= [\overline{x}_i, [[v_{jk}, \overline{x}_k], \overline{x}_j]]\left[\overline{x}_i, \overline{\mu}_{jk}\right]\left[\overline{x}_j, \overline{\mu}_{ik}^{-1}\right][\overline{x}_j, [\overline{x}_i, [v_{ik}, \overline{x}_k]]] \\
&\quad [\overline{x}_k, [[v_{ij}, \overline{x}_j], \overline{x}_i]]\left[\overline{x}_k, \overline{\mu}_{ij}\right] \bmod G'' \\
&= \left[\overline{x}_i, \overline{\mu}_{jk}\right]\left[\overline{x}_j, \overline{\mu}_{ik}^{-1}\right]\left[\overline{x}_k, \overline{\mu}_{ij}\right][\overline{x}_i, [[v_{jk}, \overline{x}_k], \overline{x}_j]]\,[\overline{x}_j, [\overline{x}_i, [v_{ik}, \overline{x}_k]]] \\
&\quad [\overline{x}_k, [[v_{ij}, \overline{x}_j], \overline{x}_i]] \bmod G''.
\end{aligned} \tag{7}$$

Moreover, for each component of the trivial link L_i the longitude, l_i, of L_i is trivial in G and is a product of commutators of $\overline{\mu}_{ij}$ with a conjugate of $\overline{x}_j$. We can write each of the longitudes (see Figure 4) as

$$\begin{aligned}
l_i &= \prod_{j<i} \alpha_j \lambda_j^{-1} \overline{\mu}_{ji}^{-1} \lambda_j \cdot \prod_{k>i} \overline{\mu}_{ik} \beta_k \bmod G'' \\
&= \prod_{j<i} \left(\lambda_j^{-1} x_{jn_{ji}}^{-1} \overline{\mu}_{ji} x_{jn_{ji}} \lambda_j\right) \lambda_j^{-1} \overline{\mu}_{ji}^{-1} \lambda_j \cdot \\
&\quad \prod_{k>i} \overline{\mu}_{ik} \left(\lambda_k x_{kn_{ki}}^{-1} \lambda_k^{-1} \overline{\mu}_{ik}^{-1} \lambda_k x_{kn_{ki}} \lambda_k^{-1}\right) \\
&= \prod_{j<i} \left[x_{jn_{ji}}^{-1}, \overline{\mu}_{ji}\right]^{\lambda_j^{-1}} \cdot \prod_{k>i} \left[\overline{\mu}_{ik}, \lambda_k x_{kn_{ki}}^{-1} \lambda_k^{-1}\right] \\
&= \prod_{j<i} \left[\overline{x}_j^{-1}, \overline{\mu}_{ji}\right] \cdot \prod_{k>i} \left[\overline{\mu}_{ik}, \overline{x}_k^{-1}\right] \bmod G''. \qquad (8)
\end{aligned}$$

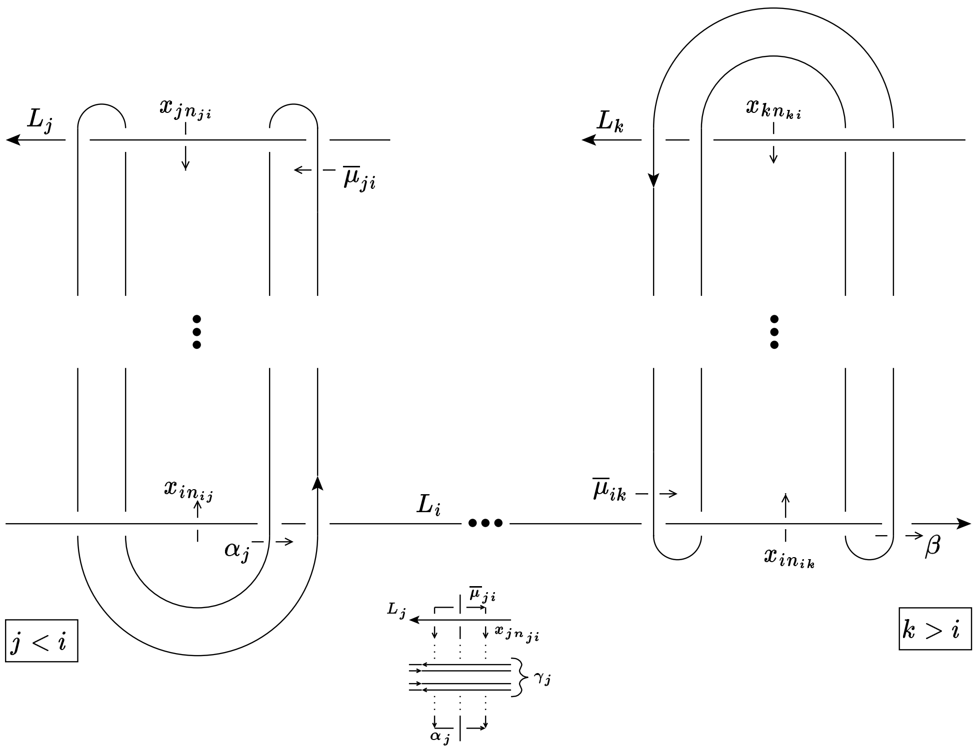

Figure 4

It follows that
$$\prod_{j<i}\left[\overline{x}_j^{-1},\overline{\mu}_{ji}\right]\cdot\prod_{k>i}\left[\overline{\mu}_{ik},\overline{x}_k^{-1}\right]=1 \bmod G''.$$
Since $G''\subset[\ker\phi,\ker\phi]$, the relations in (7) and (8) hold in $H_1(X_\phi)$ $(=\ker\phi/[\ker\phi,\ker\phi])$ as well. Suppose $\phi\colon G\twoheadrightarrow\mathbb{Z}$ is defined by sending $\overline{x}_i\longmapsto t^{n_i}$. Since ϕ is surjective, $n_N\neq 0$ for some N. We consider a subset of $\binom{m}{2}$ relations in $H_1(X_\phi)$ that we index by (i,j) for $1\leq i<j\leq m$. When $i=N$ or $j=N$ we consider the $m-1$ relations
$$\text{(i)}\qquad R_{iN}=l_i \qquad\text{and}\qquad \text{(ii)}\qquad R_{Nj}=l_j^{-1}.$$
Rewriting l_i as an element of the $\mathbb{Z}\left[t^{\pm1}\right]$ -module $H_1(X_\phi)$ generated by $\left\{\overline{\mu}_{ij}|1\leq i<j\leq m\right\}$ from (8) we have
$$\begin{aligned}R_{iN} &= \sum_{j<i}\left(t^{-n_j}-1\right)\overline{\mu}_{ji}+\sum_{k>i}\left(1-t^{-n_k}\right)\overline{\mu}_{ik}\\ &= \sum_{j<i}t^{-n_j}\left(1-t^{n_j}\right)\overline{\mu}_{ji}+\sum_{k>i}t^{-n_k}\left(t^{n_k}-1\right)\overline{\mu}_{ik}\\ &= \sum_{j<i}\left[\left(1-t^{n_j}\right)+\left(t^{-n_j}-1\right)\left(1-t^{n_j}\right)\right]\overline{\mu}_{ji}+\\ &\quad\ \sum_{k>i}\left[\left(t^{n_k}-1\right)+\left(t^{-n_k}-1\right)\left(t^{n_k}-1\right)\right]\overline{\mu}_{ik}.\end{aligned}\tag{9}$$
Similarily, we have
$$\begin{aligned}R_{Nj} &= \sum_{i<j}\left[\left(t^{n_i}-1\right)+\left(t^{-n_i}-1\right)\left(t^{n_i}-1\right)\right]\overline{\mu}_{ij}+\\ &\quad\ \sum_{k>j}\left[\left(1-t^{n_k}\right)+\left(t^{-n_k}-1\right)\left(1-t^{n_k}\right)\right]\overline{\mu}_{jk}.\end{aligned}\tag{10}$$
For the other $\binom{m-1}{3}$ relations, we use the Jacobi relations from (7). Define R_{ij} to be
$$R_{ij}=\begin{cases}J(N,i,j) & \text{for } N<i<j\\ J(i,N,j)^{-1} & \text{for } i<N<j\\ J(i,j,N) & \text{for } i<j<N\end{cases}.$$
We can write these relations as
$$R_{ij}=\begin{cases}\left(t^{n_j}-1\right)\overline{\mu}_{Ni}+\left(1-t^{n_i}\right)\overline{\mu}_{Nj}+\left(t^{n_N}-1\right)\overline{\mu}_{ij}+\\ \left(t^{n_N}-1\right)\left(t^{n_i}-1\right)\left(t^{n_j}-1\right)\left(\widetilde{v}_{ij}+\widetilde{v}_{Nj}-\widetilde{v}_{Nj}\right)\ \text{for } N<i<j\\ \left(1-t^{n_j}\right)\overline{\mu}_{iN}+\left(t^{n_N}-1\right)\overline{\mu}_{ij}+\left(1-t^{n_i}\right)\overline{\mu}_{Nj}+\\ \left(t^{n_N}-1\right)\left(t^{n_i}-1\right)\left(t^{n_j}-1\right)\left(-\widetilde{v}_{iN}-\widetilde{v}_{Nj}+\widetilde{v}_{ij}\right)\ \text{for } i<N<j\\ \left(t^{n_N}-1\right)\overline{\mu}_{ij}+\left(1-t^{n_j}\right)\overline{\mu}_{iN}+\left(t^{n_i}-1\right)\overline{\mu}_{jN}+\\ \left(t^{n_N}-1\right)\left(t^{n_i}-1\right)\left(t^{n_j}-1\right)\left(\widetilde{v}_{ij}+\widetilde{v}_{jN}-\widetilde{v}_{iN}\right)\ \text{for } i<j<N\end{cases}\tag{11}$$

where $\widetilde{v}_{ij}$ is a lift of v_{ij}.

For $1 \leq i < j \leq m$ order the pairs ij by the dictionary ordering. That is, $ij < lk$ provided either $i < l$ or $j < k$ when $i = l$. The relations above give us an $\binom{m}{2} \times \binom{m}{2}$ matrix M with coefficients in $\mathbb{Z}\left[t^{\pm 1}\right]$. The $(ij, kl)^{th}$ component of M is the coefficient of $\overline{\mu}_{kl}$ in R_{ij}. We claim for now that

$$M = \left(t^{n_N} - 1\right) I + (t-1)\, S + (t-1)^2\, E \tag{12}$$

for some "error" matrix E where I is the identity matrix and S is a skew-symmetric matrix. For an example, when $m = 4$ and $N = 1$, M is the matrix

$$\begin{bmatrix} t^{n_1}-1 & 0 & 0 & 1-t^{n_3} & 1-t^{n_4} & 0 \\ 0 & t^{n_1}-1 & 0 & t^{n_2}-1 & 0 & 1-t^{n_4} \\ 0 & 0 & t^{n_1}-1 & 0 & t^{n_2}-1 & t^{n_3}-1 \\ t^{n_3}-1 & 1-t^{n_2} & 0 & t^{n_1}-1 & 0 & 0 \\ t^{n_4}-1 & 0 & 1-t^{n_2} & 0 & t^{n_1}-1 & 0 \\ 0 & t^{n_4}-1 & 1-t^{n_3} & 0 & 0 & t^{n_1}-1 \end{bmatrix} + (t-1)^2\, E.$$

The proof of (12) is left until the end.

We will show that M is non-singular as a matrix over the quotient field $\mathbb{Q}(t)$. Consider the matrix $A = \frac{1}{t-1} M$. We note that A is a matrix with entries in $\mathbb{Z}\left[t^{\pm 1}\right]$ and $A(1)$ evaluated at $t = 1$ is

$$A(1) = NI + S(1).$$

To show that M is non-singular, it suffices to show that $A(1)$ is non-singular.

Consider the quadratic form $q\colon \mathbb{Q}^{\binom{m}{2}} \to \mathbb{Q}^{\binom{m}{2}}$ defined by $q(z) \equiv z^T A(1)\, z$ where z^T is the transpose of z. Since $A(1) = NI + S(1)$ where $S(1)$ is skew-symmetric we have,

$$q(z) = N \sum z_i^2.$$

Moreover, $N \neq 0$ so $q(z) = 0$ if and only if $z = 0$. Let z be a vector satisfying $A(1)\, z = 0$. We have $q(z) = z^T A(1)\, z = z^T 0 = 0$ which implies that $z = 0$. Therefore M is a non-singular matrix. This implies that each element $\overline{\mu}_{ij}$ is $\mathbb{Z}\left[t^{\pm 1}\right]$–torsion which will complete the the proof once we have established the above claim.

We ignore entries in M that lie in J^2 where J is the augmentation ideal of $\mathbb{Z}\left[t^{\pm 1}\right]$ since they only contribute to the error matrix E. Using (9), (10), and (11) above we can explicitely write the entries in $M \pmod{J^2}$. Let $m_{ij,lk}$ denote the (ij, lk) entry of $M \pmod{J^2}$.

Case 1 $(j = N)$: From (9) we have

$$m_{iN,li} = 1 - t^{n_l},\, m_{iN,ik} = t^{n_k} - 1,$$

and $m_{iN,lk} = 0$ when neither l nor k is equal to N.

Case 2 $(i = N)$: From (10) we have

$$m_{Nj,lj} = t^{n_l} - 1,\, m_{Nj,jk} = 1 - t^{n_k},$$

and $m_{Nj,lk} = 0$ when neither l nor k is equal to N.

Case 3 $(N < i < j)$: From (11) we have

$$m_{ij,Ni} = t^{n_j} - 1,\, m_{ij,Nj} = 1 - t^{n_i},\, m_{ij,ij} = t^{n_N} - 1,$$

and $m_{ij,lk} = 0$ otherwise.

Case 4 $(i < N < j)$: From (11) we have

$$m_{ij,iN} = 1 - t^{n_j},\, m_{ij,ij} = t^{n_N} - 1,\, m_{ij,Nj} = 1 - t^{n_i},$$

and $m_{ij,lk} = 0$ otherwise.

Case 5 $(i < j < N)$: From (11) we have

$$m_{ij,ij} = t^{n_N} - 1,\, m_{ij,iN} = 1 - t^{n_j},\, m_{ij,jN} = t^{n_i} - 1,$$

and $m_{ij,lk} = 0$ otherwise.

We first note that in each of the cases, the diagonal entries $m_{ij,ij}$ are all $t^{n_N} - 1$. Next, we will show that the off diagonal entries have the property that $m_{ij,lk} = -m_{lk,ij}$ for $ij < lk$. This will complete the proof of the claim since we see that each entry is divisible by $t - 1$.

We verify the skew symmetry in Cases 1 and 3. The other cases are similar and we leave the verifications to the reader.

Case 1 $(j = N)$:

$$m_{iN,li} = 1 - t^{n_l} = -m_{li,iN} \text{ (case 5)}$$

and

$$m_{iN,ik} = t^{n_k} - 1 = -m_{ik,iN} \text{ (case 4)}.$$

Case 3 $(N < i < j)$:

$$m_{ij,Ni} = t^{n_j} - 1 = -m_{Ni,ij} \text{ (case 2)}$$

and

$$m_{ij,Nj} = 1 - t^{n_i} = -m_{Nj,ij} \text{ (case 2)}. \qquad \square$$

Proposition 3.3 *Let X be as in Theorem 3.1, $G = \pi_1(X)$ and F be the free group on 2 generators. There is no epimorphism from G onto F/F_4.*

Proof Let $F = \langle x, y \rangle$ be the free group and $\phi\colon F/F_4 \twoheadrightarrow \langle t \rangle$ be defined by $x \longmapsto t$ and $y \longmapsto 1$. Suppose that there exists a surjective map $\eta\colon G \twoheadrightarrow F/F_4$. Let $N = \ker\phi$ and $H = \ker(\eta \circ \phi)$. Since η is surjective we get an epimorphism of $\mathbb{Z}\left[t^{\pm 1}\right]$–modules $\widetilde{\eta}\colon H/H' \twoheadrightarrow N/N'$. From (6) we get the short exact sequence

$$0 \to \mathrm{Im}\partial_* \xrightarrow{i} H_1(X_{\eta\circ\phi}) \to H_1(W_\psi) \to 0.$$

Let J be the augmentation ideal of $\mathbb{Z}\left[t^{\pm 1}\right]$. We compute $N/N' \cong \mathbb{Z}\left[t^{\pm 1}\right]/J^3$ so that $\widetilde{\eta}\colon H_1(X_{\eta\circ\phi}) \twoheadrightarrow \mathbb{Z}\left[t^{\pm 1}\right]/J^3$. Let $\sigma \in H_1(X_{\eta\circ\phi})$ such that $\widetilde{\eta}(\sigma) = 1$. Since every element in $H_1(W_\psi) \cong \bigoplus_{i=1}^{m-1} \frac{\mathbb{Z}[t^{\pm 1}]}{J}$ is $(t-1)$–torsion, $(t-1)\sigma \in \mathrm{Im}\partial_*$ hence $t - 1 \in \mathrm{Im}(\widetilde{\eta} \circ i)$. Recall that in the proof of the Theorem 3.1, we showed that there exists a surjective $\mathbb{Z}\left[t^{\pm 1}\right]$–module homomorphism $\rho\colon P \twoheadrightarrow \mathrm{Im}\partial_*$ where P is finitely presented as

$$0 \to \mathbb{Z}\left[t^{\pm 1}\right]^{\binom{m}{2}} \xrightarrow{(t-1)A} \mathbb{Z}\left[t^{\pm 1}\right]^{\binom{m}{2}} \xrightarrow{\pi} P \to 0.$$

Let $g\colon P \to \mathbb{Z}\left[t^{\pm 1}\right]/J^3$ defined by $g \equiv \widetilde{\eta} \circ i \circ \rho$. Since ρ is surjective, $t - 1 \in \mathrm{Im}g$. After tensoring with $\mathbb{Q}\left[t^{\pm 1}\right]$, we get a map $g\colon P \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Q}\left[t^{\pm 1}\right] \to \mathbb{Q}\left[t^{\pm 1}\right]/J^3$. It is easy to see that either g is surjective or the image of g is the submodule generated by $t-1$. Note that the submodule generated by $t-1$ is isomorphic $\mathbb{Q}\left[t^{\pm 1}\right]/J^2$. Hence, in either case, we get a surjective map $h\colon P \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Q}\left[t^{\pm 1}\right] \to \mathbb{Q}\left[t^{\pm 1}\right]/J^2$.

Consider the $\mathbb{Q}\left[t^{\pm 1}\right]$–module P' presented by A. Let $h'\colon \mathbb{Q}\left[t^{\pm 1}\right]^{\binom{m}{2}} \to \mathbb{Q}\left[t^{\pm 1}\right]/J^2$ be defined by $h' = (t-1)\, h \circ \pi$. Since

$$h'(A(\sigma)) = (t-1)\, h(\pi(A(\sigma))) = h(\pi((t-1)\, A(\sigma))) = h(0) = 0,$$

this defines a map $h'\colon P' \to \mathbb{Q}\left[t^{\pm 1}\right]/J^2$ whose image is the submodule generated by $t-1$. It follows that P' maps onto $\mathbb{Q}\left[t^{\pm 1}\right]/J$. Setting $t = 1$, the vector space over $\mathbb{Q}$ presented by $A(1)$ maps onto $\mathbb{Q}$. Therefore $det(A(1)) = 0$. However, it was previously shown that $A(1)$ was non-singular which is a contradiction. □

Corollary 3.4 *For any closed, orientable 3–manifold Y with $P/P_4 \cong G/G_4$ where $P = \pi_1(Y)$ and $G = \pi_1(X)$ is the fundamental group of the examples in Theorem 3.1, $c(Y) = 1$.*

Using Proposition 3.3, it is much easier to show that there exist *hyperbolic* 3–manifolds with cut number 1.

Corollary 3.5 *For each $m \geq 1$ there exist closed, orientable, hyperbolic 3–manifolds Y with $\beta_1(Y) = m$ such that $\pi_1(Y)$ cannot map onto F/F_4 where F is the free group on 2 generators.*

Proof Let X be one of the 3–manifolds in Theorem 3.1. By [7, Theorem 2.6], there exists a degree one map $f\colon Y \to X$ where Y is hyperbolic and f_* is an isomorphism on H_*. Denote by $G = \pi_1(X)$ and $P = \pi_1(Y)$. It follows from Stalling's theorem [9] that f induces an isomorphism $f_*\colon P/P_n \to G/G_n$. In particular this is true for $n = 4$ which completes the proof. □

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