ISSN 1364-0380 (on line) 1465-3060 (printed)

Geometry & Topology
Volume 6 (2002) 69–89
Published: 1 March 2002

Bounded cohomology of subgroups of mapping class groups

Mladen Bestvina
Koji Fujiwara

Mathematics Department, University of Utah
155 South 1400 East, JWB 233
Salt Lake City, UT 84112, USA
and
Mathematics Institute, Tohoku University
Sendai, 980-8578, Japan

Email: `bestvina@math.utah.edu` and `fujiwara@math.tohoku.ac.jp`

Abstract

We show that every subgroup of the mapping class group $MCG(S)$ of a compact surface S is either virtually abelian or it has infinite dimensional second bounded cohomology. As an application, we give another proof of the Farb–Kaimanovich–Masur rigidity theorem that states that $MCG(S)$ does not contain a higher rank lattice as a subgroup.

AMS Classification numbers Primary: 57M07, 57N05

Secondary: 57M99

Keywords: Bounded cohomology, mapping class groups, hyperbolic groups

Proposed: Joan Birman — Received: 15 December 2000
Seconded: Dieter Kotschick, Steven Ferry — Revised: 28 February 2002

1 Introduction

When G is a discrete group, a *quasi-homomorphism* on G is a function $h\colon G \to \mathbb{R}$ such that

$$\Delta(h) := \sup_{\gamma_1,\gamma_2\in G} |h(\gamma_1\gamma_2) - h(\gamma_1) - h(\gamma_2)| < \infty.$$

The number $\Delta(h)$ is the *defect* of h. Let $\mathcal{V}(G)$ be the vector space of all quasi-homomorphisms $G \to \mathbb{R}$. By $BDD(G)$ and respectively $HOM(G) = H^1(G;\mathbb{R})$ denote the subspaces of $\mathcal{V}(G)$ consisting of bounded functions and respectively homomorphisms. Note that $BDD(G) \cap HOM(G) = 0$. We will be concerned with the quotient spaces

$$QH(G) = \mathcal{V}(G)/BDD(G) \qquad \text{and}$$

$$\widetilde{QH}(G) = \mathcal{V}(G)/(BDD(G) + HOM(G)) \cong QH(G)/H^1(G;\mathbb{R}).$$

There is an exact sequence

$$0 \to H^1(G;\mathbb{R}) \to QH(G) \to H^2_b(G;\mathbb{R}) \to H^2(G;\mathbb{R})$$

where $H^2_b(G;\mathbb{R})$ denotes the second bounded cohomology of G (for the background on bounded cohomology the reader is referred to [14] and [24]). Since $\widetilde{QH}(G)$ is the quotient $QH(G)/H^1(G;\mathbb{R})$ we see that $\widetilde{QH}(G)$ can be also identified with the kernel of $H^2_b(G;\mathbb{R}) \to H^2(G;\mathbb{R})$. If $G \to G'$ is an epimorphism then the induced maps $QH(G') \to QH(G)$ and $\widetilde{QH}(G') \to \widetilde{QH}(G)$ are injective.

Calculations of $\widetilde{QH}(G)$ have been made for many groups G. In all such cases $\widetilde{QH}(G)$ is either 0 or infinite dimensional. $\widetilde{QH}(G)$ vanishes when G is amenable (see [14]) and also notably when G is a cocompact irreducible lattice in a semisimple Lie group of real rank > 1 [5].

In the sequence of papers [8, 11, 12] the second author has established a method for showing that $\widetilde{QH}(G)$ is infinite dimensional for groups G acting on hyperbolic spaces and satisfying certain additional conditions. This represents a generalization of the argument of Brooks [2] that $\dim \widetilde{QH}(G) = \infty$ when G is a nonabelian free group. Theorem 1 can be viewed as a refinement of that method.

Not every group G acting on a hyperbolic space has $\dim \widetilde{QH}(G) = \infty$. A nontrivial example is provided by an irreducible cocompact lattice in $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ that acts (discretely) on the product $\mathbb{H}^2 \times \mathbb{H}^2$ of two hyperbolic planes.

Notice that the action given by projecting to a single factor is highly non-discrete. Our contribution in this paper is to identify what we believe to be the "right" condition on the action that guarantees $\dim \widetilde{QH}(G) = \infty$. The condition is termed *WPD* ("weak proper discontinuity").

The main application is to the action of mapping class groups on curve complexes. These were shown to be hyperbolic by Masur–Minsky [22]. The action is far from discrete — indeed, the vertex stabilizers are infinite. However, we will show that *WPD* holds for this action. As a consequence we will deduce the rigidity theorem of Farb–Kaimanovich–Masur that mapping class groups don't contain higher rank lattices as subgroups. More generally, if Γ is not virtually abelian and $\dim \widetilde{QH}(\Gamma) < \infty$ then Γ does not occur as a subgroup of a mapping class group. In particular, if the mapping class group $MCG(S)$ is not virtually abelian, then $\dim \widetilde{QH}(G) = \infty$. This settles Morita's Conjectures 6.19 and 6.21 [25] in the affirmative.

We now proceed with a review of hyperbolic spaces and we introduce some terminology needed in the paper.

When X is a connected graph, we consider the path metric $d = d_X$ on X by declaring that each edge has length 1. A *geodesic arc* is a path whose length is equal to the distance between its endpoints. A *bi-infinite geodesic* is a line in X such that every finite segment is geodesic. Recall [13] that X is said to be *δ–hyperbolic* if for any three geodesic arcs α, β, γ in X that form a triangle we have that α is contained in the δ–neighborhood of $\beta \cup \gamma$.

A map $\phi\colon Y \to X$ from a metric space Y is a *(K,L)–quasi-isometric (qi) embedding* if

$$\frac{1}{K} d_Y(y,y') - L \le d_X(\phi(y),\phi(y')) \le K d_Y(y,y') + L$$

for all $y,y' \in Y$. A *(K,L)–quasi-geodesic* (or just quasi-geodesic when (K,L) are understood) is a (K,L)–qi embedding of an interval (finite or infinite). A fundamental property of δ–hyperbolic spaces is that there is $B = B(K,L,\delta)$ such that any two finite (K,L)–quasi-geodesics with common endpoints are within B of each other, and also any two bi-infinite quasi-geodesics that are finite distance from each other are within B of each other. A qi embedding $Y \to X$ is a *quasi-isometry* if the distance between points of X and the image of the map is uniformly bounded.

An isometry g of X is *axial* if there is a bi-infinite geodesic (called an *axis* of g) on which g acts as a nontrivial translation. Any axis of g is contained in the 2δ–neighborhood of any other axis of g. More generally, an isometry g of

X is *hyperbolic* if it admits an invariant quasi-geodesic (we will refer to it as a *quasi-axis* or a (K,L)–quasi-axis if we want to emphasize K and L). We will often blur the distinction between a quasi-axis and its image. There are easy examples of hyperbolic isometries that are not axial, but whose squares are axial (eg, take the "infinite ladder" consisting of two parallel lines and rungs joining corresponding integer points, and the isometry that interchanges the lines and moves rungs one unit up). When the graph is allowed to be locally infinite, there are similar examples of hyperbolic isometries none of whose powers are axial. In our main application, the action of the mapping class group on the curve complex, it is unknown whether powers of hyperbolic elements are axial. We are thankful to Howie Masur and Yair Minsky for bringing up this point. Note that any two (K,L)–quasi-axes of g are within $B(K,L,\delta)$ of each other.

Every quasi-axis of g is oriented by the requirement that g acts as a positive translation. We call this orientation the *g–orientation* of the quasi-axis. Of course, the g^{-1}–orientation is the opposite of the g–orientation. More generally, any sufficiently long (K,L)–quasi-geodesic arc J inside the $B(K,L,\delta)$–neighborhood of a (K,L)–quasi-axis ℓ of g has a natural orientation given by g: a point of ℓ within $B(K,L,\delta)$ of the terminal endpoint of J is ahead (with respect to the g–orientation of ℓ) of a point of ℓ within $B(K,L,\delta)$ of the initial endpoint of J. We call this orientation of J the *g–orientation*. We say that two quasi-geodesic arcs are *C–close* if each is contained in the C–neighborhood of the other, and we say that two oriented quasi-geodesic arcs are *oriented C–close* if they are C–close and the distance between their initial and also their terminal endpoints is $\le C$.

Definition When g_1 and g_2 are hyperbolic elements of G we will write

$$g_1 \sim g_2$$

if for an arbitrarily long segment J in a (K,L)–quasi-axis for g_1 there is $g \in G$ such that $g(J)$ is within $B(K,L,\delta)$ of a (K,L)–quasi-axis of g_2 and $g\colon J \to g(J)$ is orientation-preserving with respect to the g_1–orientation on J and the g_2–orientation on $g(J)$.

Replacing the constant $B(K,L,\delta)$ by a larger constant would not change the concept since for every $C>0$ there is $C'>0$ such that for any (K,L)–quasi-geodesic arc J contained in the C–neighborhood of a (K,L)–quasi-geodesic ℓ it follows that the arc obtained by removing the C'–neighborhood of each vertex is contained in the $B(K,L,\delta)$–neighborhood of ℓ. Similarly, the concept does not depend on the choice of (K,L). In particular:

- $\sim$ is an equivalence relation.
- $g_1 \sim g_2$ if and only if $g_1^k \sim g_2^l$ for any k, l with $kl > 0$.
- If g_1 and g_2 have positive powers which are conjugate in G then $g_1 \sim g_2$.

Under our condition *WPD* (see Section 3) the converse of the third bullet also holds.

Definition We say that the action of G on X is *nonelementary* if there exist at least two hyperbolic elements whose (K, L)–quasi-axes do not contain rays within finite distance of each other (this distance can be taken to be $B(K, L, \delta)$). The two hyperbolic elements are then called *independent*.

Theorem 1 *Suppose a group G acts on a δ–hyperbolic graph X by isometries. Suppose also that the action is nonelementary and that there exist independent hyperbolic elements $g_1, g_2 \in G$ such that $g_1 \not\sim g_2$.*

Then $\widetilde{QH}(G)$ is infinite dimensional.

Remark Special cases of this theorem are discussed in the earlier papers of the second author:

- [8] G is a word-hyperbolic group acting on its Cayley graph,
- [11] the action of G on X is properly discontinuous,
- [12] G is a graph of groups acting on the associated Bass–Serre tree.

Proposition 2 *Under the hypotheses of Theorem 1 there is a sequence $f_1, f_2, \cdots \in G$ of hyperbolic elements such that*

- $f_i \not\sim f_i^{-1}$ *for* $i = 1, 2, \cdots,$ *and*
- $f_i \not\sim f_j^{\pm 1}$ *for* $j < i$.

Proof Since g_1 and g_2 are independent, we may replace g_1, g_2 by high positive powers of conjugates to ensure that the subgroup F of G generated by g_1, g_2 is free with basis $\{g_1, g_2\}$, each nontrivial element of F is hyperbolic, and F is quasi-convex with respect to the action on X (see [13, Section 5.3]). We will call such free subgroups *Schottky groups*. Let T be the Cayley graph of F with respect to the generating set $\{g_1, g_2\}$. Then T is a tree and each oriented edge has a label $g_i^{\pm 1}$. Choose a basepoint $x_0 \in X$ and construct an F–equivariant map $\Phi\colon T \to X$ that sends 1 to x_0 and sends each edge to a geodesic arc. Quasi-convexity implies that Φ is a (K, L)–quasi-isometric embedding for some

(K,L) and in particular for every $1 \neq f \in F$ the Φ–image of the axis of f in T is a (K,L)–quasi-axis of f. By ℓ_i denote the axis of g_i in T, $i=1,2$.

Choose positive constants

$$0 \ll n_1 \ll m_1 \ll k_1 \ll l_1 \ll n_2 \ll m_2 \ll \cdots$$

and define

$$f_i = g_1^{n_i} g_2^{m_i} g_1^{k_i} g_2^{-l_i}$$

for $i = 1,2,3,\cdots$.

Claim 1 $f_1 \not\sim f_2$.

The key to the proof is the following observation. If K', L', C are fixed and the exponents $n_1, m_1, \cdots, k_2, l_2$ are chosen suitably large, then for any sufficiently long f_2–oriented segment S in the axis $\ell_2 \subset T$ of f_2 and any orientation preserving (K',L')–qi embedding $\phi\colon S \to \ell_1$ (with respect to the f_1–orientation of ℓ_1) there is a subsegment S' of S containing a string of $\geq C$ edges labeled g_2 whose image (pulled tight) is a segment in ℓ_1 consisting only of edges labeled g_1 (this is true because $m_2 \gg n_1 + m_1 + k_1 + l_1$ so the image will contain a whole fundamental domain for the action of f_1 on ℓ_1). Figure 1 illustrates the situation.

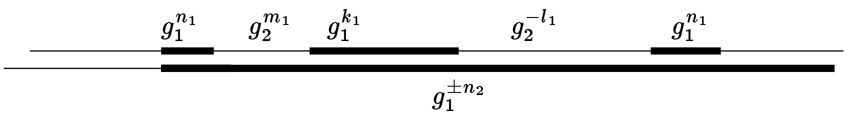

Figure 1: Thick (thin) lines represent strings of edges labeled g_1 (g_2).

Now assuming that $f_1 \sim f_2$ let $I_2 \subset \ell_2$ be a long arc, let $J = \Phi(I_2)$, and let $g \in G$ be such that $g(J)$ is $B(K,L,\delta)$–close to the (K,L)–quasi-axis $f_1(\ell_1)$ of f_1, with matching orientations. Choose an arc $I_1 \subset \ell_1$ so that $\Phi(I_1)$ is $B(K,L,\delta)$–close to $g(J)$. Then there is a (K',L')–quasi-isometry $I_2 \to I_1$ obtained by composing $I_2 \to \Phi(I_2) = J \to g(J) \to \Phi(I_1) \to I_1$ and (K',L') does not depend on the choices of $n_1, m_1, \cdots, k_2, l_2$, only on δ, T, and Φ. Combining this with the observation above, we conclude that g takes a long segment in an axis of a conjugate of g_2 uniformly close to an axis of a conjugate of g_1 with matching orientation, contradicting the assumption that $g_1 \not\sim g_2$.

Similarly, $f_i \not\sim f_j$ for $i \neq j$.

Claim 2 $f_1 \not\sim f_2^{-1}$.

The proof is similar to the proof of Claim 1, only now one uses $l_2 \gg n_1 + m_1 + k_1 + l_1$.

Similarly, $f_i \not\sim f_j^{-1}$ for $i \neq j$.

Claim 3 If in addition $g_1 \not\sim g_2^{-1}$ then $f_1 \not\sim f_1^{-1}$.

If $f_1 \sim f_1^{-1}$, we obtain the situation pictured in Figure 2 where a long string of g_1's is close to a long string of g_2's with either the same or opposite orientation. Note that it is possible that all such pairs of strings have opposite orientation so the assumption $g_1 \not\sim g_2^{-1}$ is necessary.

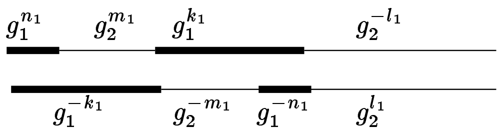

Figure 2: Whenever thick and thin lines are close, they are anti-parallel.

Similarly, if $g_1 \not\sim g_2^{-1}$ then $f_i \not\sim f_i^{-1}$ for all i.

We now finish the proof. If $g_1 \not\sim g_2^{-1}$ then the above claims conclude the argument. Otherwise, note that by Claims 1 and 2 we have $f_1 \not\sim f_2^{\pm 1}$. Now replace (g_1, g_2) by (f_1, f_2) and repeat the construction. $\square$

2 Proof of Theorem 1

We will only give a sketch of the proof since it is a minor generalization of results in [8, 11, 12] and the proof uses the same techniques.

We start by recalling the basic construction of quasi-homomorphisms in this setting. The model case of the free group is due to Brooks [2].

Let w be a finite (oriented) path in X. By $|w|$ denote the length of w. For $g \in G$ the composition $g \circ w$ is a *copy* of w. Obviously $|g \circ w| = |w|$.

Let α be a finite path. We define

$$|\alpha|_w = \{\text{the maximal number of non-overlapping copies of } w \text{ in } \alpha\}.$$

Suppose that $x, y \in X$ are two vertices and that W is an integer with $0 < W < |w|$. We define the integer

$$c_{w,W}(x, y) = d(x, y) - \inf_{\alpha}(|\alpha| - W|\alpha|_w),$$

where α ranges over all paths from x to y. Note that if α is such a path that contains a subpath whose length is large compared to the distance between its endpoints, then replacing this subpath by a geodesic arc between the endpoints produces a new path with smaller $|\alpha| - W|\alpha|_w$. This observation leads to the following lemma.

Lemma 3 [11, Lemma 3.3] *Suppose a path β realizes the infimum above. Then β is a $(\frac{|w|}{|w|-W}, \frac{2W|w|}{|w|-W})$–quasi-geodesic.* □

Replace g_1, g_2 by large positive powers if necessary, let F be the subgroup of G generated by g_1, g_2, and let $\Phi\colon T \to X$ be an F–equivariant map with $\Phi(1) = x_0$ as in the proof of Proposition 2. If $w \in F$ is cyclically reduced as a word in g_1, g_2 (equivalently, if its axis passes through $1 \in T$) then by the quasi-convexity of F in G we have (see Figure 3)

$$d(x_0, w^n(x_0)) \geq n(d(x_0, w(x_0)) - 2B)$$

where $B = B(K, L, \delta) > 0$ is independent of w and n.

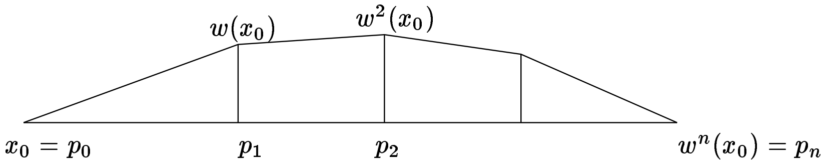

Figure 3: $nd(x_0, w(x_0)) \leq d(x_0, w^n(x_0)) + d(p_1, w(x_0)) + d(p_2, w^2(x_0)) + \cdots \leq d(x_0, w^n(x_0)) + 2nB$

Following [11] we fix an integer $W \geq 3B$ and will only consider w with $|w| > W$. Thus, paths β as in Lemma 3 will be quasi-geodesics with constants independent of w and the endpoints, and β is contained in a uniform neighborhood, say D–neighborhood, of any geodesic joining the endpoints of β. We will also omit W from notation and write c_w. For every $f \in F$ choose a geodesic γ_f from x_0 to $f(x_0)$. We find it convenient to denote the concatenation

$$\gamma_f f(\gamma_f) f^2(\gamma_f) \cdots f^{a-1}(\gamma_f)$$

by f^a.

Define $h_w\colon G \to \mathbb{R}$ by

$$h_w(g) = c_w(x_0, g(x_0)) - c_{w^{-1}}(x_0, g(x_0)).$$

Proposition 4 [11, Proposition 3.10] *The map $h_w\colon G \to \mathbb{R}$ is a quasi-homomorphism. Moreover, the defect $\Delta(h_w)$ is uniformly bounded independently of w.* □

Proposition 5 *Suppose $1 \neq f \in F$ is cyclically reduced and $f \not\sim f^{-1}$. Then there is $a > 0$ such that h_{f^a} is unbounded on $< f >$. Moreover, if $f^{\pm 1} \not\sim f' \in F$ then h_{f^a} is 0 on $< f' >$ for sufficiently large $a > 0$.*

Proof It is clear that c_{f^a} is unbounded on $< f >$ for any $a > 0$: If we use $\alpha = (f^a)^n$ as a competitor path we have

$$c_{f^a}(f^{an}) \geq d(x, f^{an}(x)) - (n|f^a| - 3Bn) \geq Bn.$$

If $a > 0$ is large, then there are no copies of f^{-a} in the D–neighborhood of an axis of f, which implies that $c_{f^{-a}}$ is zero on $< f >$.

The proof of the other claim is similar. □

Proof of Theorem 1 Let $f_1, f_2, \cdots$ be the sequence from Proposition 2. We assume in addition (without loss of generality) that each f_i is cyclically reduced. Define $h_i\colon G \to \mathbb{R}$ as $h_i = h_{f^{a_i}}$ where $a_i > 0$ is chosen as in Proposition 5 so that h_i is unbounded on $< f_i >$ and so that it is 0 on $< f_j >$ for $j < i$ (a high power of f_i cannot be translated into a B–neighborhood of an axis of f_j). It follows that $[h_i] \in QH(G)$ is not a linear combination of $[h_1], \cdots, [h_{i-1}]$, ie, the sequence $[h_i]$ consists of linearly independent elements. We can easily arrange that F is contained in the commutator subgroup of G (this is automatic if g_1, g_2 are in the commutator subgroup; otherwise, replace g_1, g_2 by $g_1' = g_1^N g_2^M g_1^{-M} g_2^{-N}, g_2' = g_1^K g_2^L g_1^{-K} g_2^{-L}$ with $0 \ll N \ll M \ll K \ll L$ — as in Proposition 2 it follows that $g_1' \not\sim g_2'$). In that case any homomorphism $G \to \mathbb{R}$ vanishes on F and it follows that the sequence $[h_i]$ in $\widetilde{QH}(G)$ consists of linearly independent elements. □

Remark The argument shows that there is an embedding of ℓ^1 into $QH(G)$. If $t = (t_1, t_2, \cdots) \in \ell^1$ then

$$h_t = \sum t_i h_i\colon G \to \mathbb{R}$$

is a well-defined function ($|f_i| \to \infty$ implies that for any $g \in G$ only finitely many $h_i(g)$ are nonzero) and it is a quasi-homomorphism (because the defect of h_i is uniformly bounded independently of i), and $t \neq 0$ implies h_t is unbounded (on $< f_i >$ where i is the smallest index with $t_i \neq 0$). Similarly, one can argue that $\widetilde{QH}(G)$ contains ℓ^1.

Remark Instead over $\mathbb{R}$ one can work over $\mathbb{Z}$ and consider $H_b^2(G;\mathbb{Z})$ and quasi-homomorphisms $G \to \mathbb{Z}$. The quasi-homomorphisms h_w constructed above are integer-valued, and therefore it follows that there are infinitely many linearly independent elements in the kernel of $H_b^2(G;\mathbb{R}) \to H^2(G;\mathbb{R})$ which are in the image of $H_b^2(G;\mathbb{Z})$.

3 Weak Proper Discontinuity

Definition We say that the action of G on X satisfies *WPD* if

- G is not virtually cyclic,
- G contains at least one element that acts on X as a hyperbolic isometry, and
- for every hyperbolic element $g \in G$, every $x \in X$, and every $C > 0$ there exists $N > 0$ such that the set
$$\{\gamma \in G | d(x, \gamma(x)) \leq C, d(g^N(x), \gamma g^N(x)) \leq C\}$$
is finite.

Proposition 6 *Suppose that G and X satisfy WPD. Then*

(1) *for every hyperbolic $g \in G$ the centralizer $C(g)$ is virtually cyclic,*

(2) *for every hyperbolic $g \in G$ and every (K,L)–quasi-axis ℓ for g there is a constant $M = M(g,K,L)$ such that if two translates ℓ_1, ℓ_2 of ℓ contain (oriented) segments of length $> M$ that are oriented $B(K,L,\delta)$–close then ℓ_1 and ℓ_2 are oriented $B(K,L,\delta)$–close and moreover the corresponding conjugates g_1, g_2 of g have positive powers which are equal,*

(3) *the action of G on X is nonelementary,*

(4) *$f_1 \sim f_2$ if and only if some positive powers of f_1 and f_2 are conjugate,*

(5) *there exist hyperbolic g_1, g_2 such that $g_1 \not\sim g_2$.*

Proof (1) We will show that $< g >$ has finite index in $C(g)$. Let $f_1, f_2, \cdots$ be an infinite sequence of elements of $C(g)$. Choose a (K,L)–quasi-axis ℓ for g and let $x \in \ell$. Since g and f_i commute, $f_i(\ell)$ is also a (K,L)–quasi-axis for g and the distance between ℓ and $f_i(\ell)$ is uniformly bounded by $B(K,L,\delta)$. Let $k_i \in \mathbb{Z}$ be such that $d(f_i(x), g^{k_i}(x))$ minimizes the distance between $f_i(x)$ and the g–orbit of x. Thus $d(f_i(x), g^{k_i}(x))$ is uniformly bounded (by $B(K,L,\delta)$ plus the diameter of the fundamental domain for the action of g on ℓ); call such a bound C. Let N be from the definition of *WPD*. We note that $f_i g^{-k_i}$ move both x and $g^N(x)$ by $\leq C$. Therefore, the set of such elements is finite. From $f_i g^{-k_i} = f_j g^{-k_j}$ we conclude that f_i and f_j represent the same $< g >$–coset and the claim is proved.

(2) Denote by g_1 and g_2 the corresponding conjugates of g. For notational simplicity, we will first assume that g is axial and that ℓ is an axis of g.

Without loss of generality we assume $g_1 = g$. Choose $x \in \ell$ and let N be as in the definition of *WPD* for $g, x, C = 4\delta$. Let P be the size of the finite set from the definition of *WPD*. If $\ell_1 = \ell$ and ℓ_2 contain oriented 2δ–close arcs J_1 and J_2 of length $> (P + N + 2)\tau_g$ (τ_g is the translation length of g) then the elements $g_1^i g_2^{-i}$ move each point of the terminal subarc of J_2 of length $(N+2)\tau_g$ a distance $\leq 4\delta$ for $i = 0, 1, \cdots, P$. It follows that $g_1^i g_2^{-i} = g_1^j g_2^{-j}$ for distinct i, j so that g_1, g_2 have equal positive powers.

In general, when ℓ is only a quasi-axis, one can generalize the above paragraph by replacing 4δ etc. by larger constants that depend on (K, L) and δ. Alternatively, one can modify X to make g axial: simply attach an infinite ladder (the 1–skeleton of an infinite strip) along one of the two infinite lines to ℓ; then attach such ladders equivariantly to obtain a G–space. Finally, subdivide each rung and each edge in X into a large number Q of edges in order to arrange that the "free" lines in the attached ladders are geodesics and axes for the corresponding conjugates of g. The group G continues to act on the new space X' which is quasi-isometric to X. The statement for X' implies the statement for X.

(3) Let g be a hyperbolic element. Again, without loss of generality, we will assume that g has an axis ℓ. We aim to show that some translate of ℓ, say $h(\ell)$, has both ends distinct from ℓ, since then g and hgh^{-1} are independent hyperbolic elements.

Suppose first that there is $h \in G$ such that $h(\ell)$ is not in the 2δ–neighborhood of ℓ, but it is asymptotic in one direction, ie, a ray in $h(\ell)$ is contained in the 2δ–neighborhood of ℓ. From (2) we see that ℓ and $h(\ell)$ cannot contain segments of length $> M(g)$ that are oriented 2δ–close to each other; in particular, one of g, hgh^{-1} moves towards the common end, say ∞, and the other moves away from it. Now consider $g^N(h(\ell))$ for large N. This is a bi-infinite geodesic with one end ∞ and the other end distinct from the ends of ℓ, $h(\ell)$. The translates $h(\ell)$ and $g^N(h(\ell))$ violate (2) as they have oriented rays within 2δ of each other.

It remains to consider the case when every translate of ℓ is within 2δ of ℓ. After passing to a subgroup of G of index 2 if necessary, we may assume that

G preserves the ends of ℓ. Now proceed as in (1) to show that $< g >$ has finite index in G.

(4) This is similar to (2). We assume for simplicity that f_1, f_2 are axial. By τ_k denote the translation length of f_k, $k = 1, 2$. Let N be as in the definition of *WPD* for $g = f_1$ with respect to some x in an axis ℓ of f_1 and $C = 4\delta$. Let P be the size of the corresponding finite set. Now assume that f_2 has an axis that admits a segment of length $> (N+2)\tau_1 + P\tau_1\tau_2$ which is oriented 2δ–close to a segment in ℓ. Consider $f_2^{i\tau_1} f_1^{-i\tau_2}$, $i = 0, 1, \cdots, P$. As before, we conclude that $f_2^{i\tau_1} f_1^{-i\tau_2} = f_2^{j\tau_1} f_1^{-j\tau_2}$ for some $i \neq j$; thus f_1 and f_2 have common positive powers.

(5) Since the action of G on X is nonelementary, we can choose a Schottky subgroup $F \subset G$. Let $1 \neq f \in F$. For notational simplicity we will assume that all nontrivial elements of F are axial and in fact that there is an F–invariant totally geodesic tree $T \subset X$ (this can be arranged by modifying X as in the proof of (2) except that now one attaches the 1–skeleton of (tree)$\times I$ instead of (line)$\times I$). Let $\ell \subset T$ be an axis of f. Then *WPD* provides a segment J in ℓ such that the set of $g \in G$ that move each point of J by $\leq \tau_f + 2\delta$ is finite.

Now consider an infinite sequence $f_1, f_2, \cdots$ of elements of F with distinct (and hence non-parallel) axes $\ell_1, \ell_2, \cdots \subset T$ that overlap ℓ in (oriented) finite intervals that contain J. If $f_i \sim f$ then, according to (4), there is $g_i \in G$ such that $g_i(\ell_i)$ is 2δ–close to ℓ. Replacing g_i by $f^{a_i} g_i$ if necessary, we may assume that g_i moves each point of J by $\leq \tau_f + 2\delta$. Thus $g_i = g_j$ for some $i \neq j$, so that ℓ_i and ℓ_j are within 2δ of each other, contradicting the choice of the sequence.

Finally, note that we could have taken $f_i = gf^i$ for some $g \in F$ that does not commute with f, and that the argument shows that $f_i \not\sim f$ for all but finitely many i. $\square$

Theorem 7 *Suppose that G and X satisfy WPD. Then $\widetilde{QH}(G)$ is infinite dimensional.*

Proof This is a consequence of Theorem 1 and Proposition 6. $\square$

In order to avoid passage to finite index subgroups, we will need a slight extension of Theorem 7.

Theorem 8 *Suppose that G and X satisfy WPD. For $p \geq 1$ form the semi-direct product $\tilde{G} = G^p \rtimes S_p$ where $G^p = G \times G \times \cdots \times G$ is the p–fold cartesian* *product and S_p is the symmetric group on p letters acting on G^p by permuting the factors. Let $\tilde{H} < \tilde{G}$ be any subgroup and let $H = \tilde{H} \cap G^p$ (subgroup of $\tilde{H}$ of index $\leq p!$). Thus H has p actions on X obtained by projecting to various coordinates. If at least one of these actions satisfies WPD (equivalently, it is nonelementary) then $\widetilde{QH}(\tilde{H})$ is infinite dimensional.*

Note that Theorem 7 implies that $\widetilde{QH}(H)$ is infinite dimensional.

Proof The details of this proof are similar to the proof of Theorem 7 and we only give a sketch. We will use the following principle in this proof. If F is a rank 2 free group and $\phi\colon F \to G$ a homomorphism then there is a rank 2 free subgroup $F' < F$ such that either $\phi(F')$ contains no hyperbolic elements or else ϕ is injective on F' and $\phi(F')$ is Schottky (it follows from *WPD* that either $\phi(F)$ contains two independent hyperbolic elements, in which case the latter possibility can be arranged, or $\phi(F)$ contains no hyperbolic elements, or $\phi(F)$ is virtually cyclic, and then the first possibility holds).

Say the first projection of H induces an action which is *WPD*. Therefore there is a free group $F =< x, y >\subset H$ such that the first projection of F is Schottky. Now apply the above principle with respect to each coordinate to replace F by a subgroup so that each coordinate action is either Schottky or has no hyperbolic elements. For concreteness, we assume that coordinates $1, 2, \cdots, k$ are Schottky and $k+1, \cdots, p$ have no hyperbolic elements $(1 \leq k \leq p)$. We still call x and y the basis elements of F.

We will adopt the convention in this proof that for $f \in F$ the r^{th} projection of f is denoted by ${}_rf$.

The proof of Proposition 6(5) (see the last sentence) shows that after replacing y by xy^N for some N if necessary, we may assume that ${}_rx \not\sim {}_ry$ for $r = 1, 2, \cdots, k$. Next, elements $f = x^{n_1}y^{m_1}x^{k_1}y^{-l_1}$ and $g = x^{n_2}y^{m_2}x^{k_2}y^{-l_2}$ for $0 \ll n_1 \ll m_1 \ll k_1 \ll l_1 \ll n_2 \ll m_2 \ll k_2 \ll l_2$ will have the property that ${}_rf \not\sim {}_rg^{\pm 1}$ for $r = 1, 2, \cdots, k$ (see Claims 1 and 2 in the proof of Proposition 2). We could then construct a sequence $f_1, f_2, \cdots$ as in the proof of Proposition 2 (in the same manner as in the previous sentence) so that ${}_rf_i \not\sim {}_rf_j{}^{\pm 1}$ for $i \neq j$ and ${}_rf_i \not\sim {}_rf_i{}^{-1}$.

In addition, we want to arrange that ${}_1f_j \not\sim {}_rf_j{}^{-1}$ for $j = 1, 2, \cdots$ (note that we cannot hope to arrange ${}_1f_j \not\sim {}_rf_j$ since G might have the same 1^{st} and r^{th} projections). This can be done by modifying the expression for f_j so that it reads (for example)

$$f_j = x^{-s_j}y^{-t_j}x^{n_j}y^{m_j}x^{k_j}y^{-l_j}$$

with $0 \ll s_1 \ll t_1 \ll n_1 \ll m_1 \ll k_1 \ll l_1 \ll s_2 \ll \cdots$. The idea is that ${}_1f_j \sim {}_rf_j^{-1}$ would force the situation where a long string of ${}_1y$'s is close to both a long string of ${}_rx$'s and a long string of ${}_rx^{-1}$'s, implying ${}_rx \sim {}_rx^{-1}$. Of course, it can be arranged that this is false by replacing (x, y) with (f_1, f_2) from the previous paragraph.

We now define quasi-homomorphisms $h_i\colon G^p \to \mathbb{R}$ by the formula

$$h_i(g_1, g_2, \cdots, g_p) = h_{({}_1f_i{}^{a_i})}(g_1) + \cdots + h_{({}_1f_i{}^{a_i})}(g_p)$$

for large a_i. These maps clearly extend to quasi-homomorphisms on $\tilde{G} = G^p \rtimes S_p$. The first summand in the above formula is unbounded on the cyclic subgroup $< f_i >$ and it is positive on large positive powers of f_i. The second through k^{th} summands are nonnegative on large powers of f_i thanks to the fact that ${}_1f_i \not\sim {}_rf_i^{-1}$ for $k \geq r > 1$. Finally, the other summands are bounded on $< f_i >$ since ${}_rf_i$ is not hyperbolic for $r > k$. Thus h_i is unbounded on $< f_i >$. A similar argument shows that h_i is bounded on $< f_j >$ for $j < i$, so that the elements of $QH(\tilde{H})$ induced by $h_1, h_2, \cdots$ are linearly independent. By choosing the f_i's to lie in the commutator subgroup of G as before, we obtain an infinite linearly independent set in $\widetilde{QH}(F)$ and hence in $\widetilde{QH}(\tilde{H})$. $\square$

4 Mapping class groups

Let S be a compact orientable surface of genus g and p punctures. We consider the associated mapping class group $MCG(S)$ of S. This group acts on the *curve complex* X of S defined by Harvey [17] and successfully used in the study of mapping class groups by Harer [16], [15] and by N V Ivanov [18], [19]. For our purposes, we will restrict to the 1–skeleton of (the barycentric subdivision of) Harvey's complex, so that X is a graph whose vertices are isotopy classes of essential, nonparallel, nonperipheral, pairwise disjoint simple closed curves in S (also called *curve systems*) and two distinct vertices are joined by an edge if the corresponding curve systems can be realized simultaneously by pairwise disjoint curves. In certain sporadic cases X as defined above is 0–dimensional or empty (this happens when there are no curve systems consisting of two curves, ie, when $g = 0$, $p \leq 4$ and when $g = 1$, $p \leq 1$). In the theorem below these cases are excluded (one could rectify the situation by declaring that in those cases two vertices are joined by an edge if the corresponding curves can be realized with only one intersection point). The mapping class group $MCG(S)$ acts on X by $f \cdot a = f(a)$.

H Masur and Y Minsky proved the following remarkable result.

Theorem 9 [22] *The curve complex X is δ–hyperbolic. An element of $MCG(S)$ acts hyperbolically on X if and only if it is pseudo-Anosov .* □

The following lemma is well-known (see [6, Theorem 2.7]).

Lemma 10 *Suppose that a and b are two curve systems on S that intersect minimally and such that $a \cup b$ fills S. Then the intersection $S(a, b)$ of the stabilizers of a and of b in $MCG(S)$ is finite.*

We remark that $a \cup b$ fills S if and only if $d(a, b) \geq 3$ in the curve complex.

Proof Let g be in the stabilizer of both a and b. Then there is an isotopy of g so that $g(a) = a$ and $g(b) = b$. It follows that for some $N > 0$ depending only on the complexity of the graph $a \cup b$ we have that g^N is isotopic to the identity. Therefore $S(a, b)$ consists of elements of finite order and is consequently finite (every torsion subgroup of a finitely generated virtually torsion-free group is finite). □

Proposition 11 *Let S be a nonsporadic surface. The action of $MCG(S)$ on the curve complex X satisfies WPD.*

Proof The first two bullets in the definition of *WPD* are clear. Our proof of the remaining property is motivated by Feng Luo's proof (as explained in [22]) that the curve complex has infinite diameter. We recall the construction and the basic properties of Thurston's space of projective measured foliations on S (see [26] and [10]). Let $\mathcal{C}$ be the set of all curve systems in S and by

$$I\colon\ \mathcal{C} \times \mathcal{C} \to [0, \infty)$$

denote the intersection pairing, ie, $I(a, b)$ is the smallest number of intersection points between a and b after a possible isotopy. Let $\mathbb{R}_+ = (0, \infty)$ and by $\mathbb{R}_+\mathcal{C}$ denote the space of formal products ta for $t \in \mathbb{R}_+$ and $a \in \mathcal{C}$ where we identify $\mathcal{C}$ with the subset $1\mathcal{C}$. Extend I to $\mathbb{R}_+\mathcal{C} \times \mathbb{R}_+\mathcal{C}$ by

$$I(ta, sb) = tsI(a, b).$$

Consider the associated function

$$J\colon\ \mathbb{R}_+\mathcal{C} \to [0, \infty)^{\mathcal{C}}$$

defined by

$$J(ta) = (sb \mapsto I(ta, sb)).$$

Then J is injective and we let $\mathcal{MF}$ denote the closure of the image of J. An element of $\mathcal{MF}$ can be viewed as a measured foliation on S. The pairing I extends to a continuous function

$$I\colon\ \mathcal{MF}\times\mathcal{MF}\to[0,\infty).$$

There is a natural action of $\mathbb{R}_+$ on $\mathcal{MF}$ given by scaling. The orbit space $\mathcal{PMF}$ is Thurston's space of projective measured foliations and it is homeomorphic to the sphere of dimension $6g+2p-7$ (assuming this number is positive). The intersection pairing is not defined on $\mathcal{PMF}\times\mathcal{PMF}$ but note that the statement $I(\Lambda,\Lambda')=0$ makes sense for $\Lambda,\Lambda'\in\mathcal{PMF}$. The mapping class group $MCG(S)$ of S acts on $\mathcal{C}$ by $f\cdot a=f(a)$ and there is an induced action on $\mathbb{R}_+\mathcal{C}$, $\mathcal{MF}$, and $\mathcal{PMF}$.

Let $f\in MCG(S)$ be a pseudo-Anosov mapping class. Then f fixes exactly two points in $\mathcal{PMF}$ and one point Λ_+ is attracting while the other Λ_- is repelling. All other points converge to Λ_+ under forward iteration and to Λ_- under backward iteration. It is known that $I(\Lambda_+,\Lambda)=0$ implies $\Lambda=\Lambda_+$ and similarly for Λ_-. Continuity of I implies the following fact:

If U is a neighborhood of Λ_+ then there is a neighborhood V of Λ_+ such that if $\Lambda,\Lambda'\in\mathcal{PMF}$, $I(\Lambda,\Lambda')=0$ and $\Lambda\in V$ then $\Lambda'\in U$.

We will use the terminology that V is *adequate* for U if the above sentence holds. A similar fact (and terminology) holds for neighborhoods of Λ_-.

Given $C>0$, choose closed neighborhoods $U_0\supset U_1\supset U_2\supset\cdots\supset U_N$ of Λ_+ and $V_0\supset V_1\supset V_2\supset\cdots\supset V_N$ of Λ_- with $N>C$ so that

- U_{i+1} is adequate for U_i and V_{i+1} is adequate for V_i, and
- if $\Lambda\in U_0$ and $\Lambda'\in V_0$ then $I(\Lambda,\Lambda')\neq 0$.

Assume now that two curve systems a and b belong to a quasi-axis ℓ of a pseudo-Anosov mapping class f and that they are sufficiently far away from each other, so that after applying a power of f and possibly interchanging a and b we may assume that $a\in U_N$ and $b\in V_N$. Assume, by way of contradiction, that g_n is an infinite sequence in $MCG(S)$ and $d(a,g_n(a))\le N$, $d(b,g_n(b))\le N$ for all n. Note that if c is a curve system disjoint from a then $c\in U_{N-1}$, and inductively if $d(a,c)\le N$ then $c\in U_0$. We therefore conclude that $g_n(a)\in U_0$ and $g_n(b)\in V_0$. After passing to a subsequence, we may assume that the sequence $g_n(a)$ converges to $A\in U_0$ and $g_n(b)\to B\in V_0$. Note that $I(A,B)\neq 0$ by the choice of U_0 and V_0.

First suppose that the curve systems $g_n(a)$ are all different. To obtain convergence in $\mathcal{MF}$ one is required to first rescale by some $r_n>0$, ie, $\frac{1}{r_n}g_n(a)\to\tilde{A}\in$

$\mathcal{MF}$ where r_n can be taken to be the length of $g_n(a)$ in some fixed hyperbolic structure on S. Under the assumption that $g_n(a)$ are all distinct, we see that $r_n \to \infty$ and this implies that $I(\tilde{A}, \tilde{B}) = 0$ ie, that $I(A, B) = 0$, contradiction. The case when $g_n(b)$ are all distinct is similar.

Finally, if $g_n(a)$ and $g_n(b)$ take only finitely many values, we may assume by passing to a subsequence that both $g_n(a)$ and $g_n(b)$ are constant. But then $g_n^{-1} g_m \in S(a, b)$ and Lemma 10 implies that the sequence g_n is finite. □

The following is the main theorem in this note. H Endo and D Kotschick [7] have shown using 4–manifold topology and Seiberg–Witten invariants that $\widetilde{QH}(MCG(S)) \neq 0$ when S is hyperbolic. M Korkmaz [21] also proved $\widetilde{QH}(MCG(S)) \neq 0$, and in addition that $\widetilde{QH}(MCG(S))$ is infinite dimensional when S has low genus. The nontriviality, and even infinite-dimensionality, of $\widetilde{QH}(MCG(S))$ was conjectured by Morita [25, Conjecture 6.19].

Theorem 12 *Let G be a subgroup of $MCG(S)$ which is not virtually abelian. Then* $\dim \widetilde{QH}(G) = \infty$.

Proof We first deal with the sporadic cases. When $g = 0$, $p \leq 3$ and when $g = 1$, $p = 0$ the mapping class group $MCG(S)$ is finite. When $g = 0$, $p = 4$ and when $g = p = 1$ then $MCG(S)$ is word hyperbolic (in fact, the quotient by the finite center is virtually free) and instead of considering the action on a curve complex we can look at the action on the Cayley graph. This action is properly discontinuous and therefore the restriction to any subgroup which is not virtually cyclic satisfies *WPD*. The statement then follows from Theorem 7.

Now we assume that S is not sporadic. By the classification of subgroups (see [23, Theorem 4.6]) there are 4 cases.

- G contains two independent pseudo-Anosov homeomorphisms. Then the action of G on the curve complex X for S satisfies the assumptions of Theorem 7 so $\widetilde{QH}(G)$ is infinite dimensional.
- G fixes a pair $\Lambda_{\pm}$ of foliations corresponding to a pseudo-Anosov homeomorphism. Then G is virtually cyclic.
- G is finite.
- There is a curve system c on S invariant under G. Choose c to be maximal possible and cut S open along c. Consider the mapping class group of the cut open surface S' where we collapse each boundary component

to a puncture. Since c is maximal and G is not virtually abelian, there is an orbit $S'_1, S'_2, \cdots, S'_p$ of components of S' and there is a subgroup of G that preserves these components and whose restriction to a component contains two independent pseudo-Anosov homeomorphisms. Pass to the quotient of G corresponding to the restriction to $S'_1 \cup \cdots \cup S'_p$. The mapping class group of $S'_1 \cup \cdots \cup S'_p$ can be identified with $MCG(S'_1)^p \rtimes S_p$ so the result in this case follows from Theorem 8. □

The following is a version of superrigidity for mapping class groups. It was conjectured by N V Ivanov and proved by Kaimanovich and Masur [20] in the case when the image group contains independent pseudo-Anosov homeomorphisms and it was extended to the general case by Farb and Masur [9] using the classification of subgroups of $MCG(S)$ as above. Our proof is different in that it does not use random walks on mapping class groups, but instead uses the work of M Burger and N Monod [5] on bounded cohomology of lattices. Note also that for this application we only need a weak version of our result, namely that $\widetilde{QH}(G) \neq 0$ when $G \subset MCG(S)$ is not virtually abelian.

Corollary 13 *Let Γ be an irreducible lattice in a connected semi-simple Lie group G with no compact factors, with finite center, and of rank > 1. Then every homomorphism $\Gamma \to MCG(S)$ has finite image.*

Proof Let $\phi\colon \Gamma \to MCG(S)$ be a homomorphism. By the Margulis–Kazhdan theorem [27, Theorem 8.1.2] either the image of ϕ is finite or the kernel of ϕ is contained in the center. When Γ is a nonuniform lattice, the proof is easier and was known to Ivanov before the work of Kaimanovich–Masur (see Ivanov's comments to Problem 2.15 on Kirby's list). Since the rank is ≥ 2 the lattice Γ then contains a solvable subgroup N which does not become abelian after quotienting out a finite normal subgroup. If the kernel is finite, then $\phi(N)$ is a solvable subgroup of $MCG(S)$ which is not virtually abelian, contradicting [1].

Now assume that Γ is a uniform lattice. If the kernel $Ker(\phi)$ is finite then there is an unbounded quasi-homomorphism $q\colon Im(\phi) \to \mathbb{R}$ by Theorem 12. But then $q\phi\colon \Gamma \to \mathbb{R}$ is an unbounded quasi-homomorphism contradicting the Burger–Monod result that every quasi-homomorphism $\Gamma \to \mathbb{R}$ is bounded. □

Remark When the center $Z(G)$ of G is infinite, one can show that every homomorphism $\phi\colon \Gamma \to MCG(S)$ has virtually abelian image, as follows. The key is that in this case the intersection $\Gamma \cap Z(G)$ has finite index in $Z(G)$ and the projection of Γ in $G/Z(G)$ is a lattice L in $G/Z(G)$, which is a Lie group of rank > 1. Choose $g \in \Gamma \cap Z(G)$ of infinite order such that the Nielsen–Thurston representative of $\phi(g)$ has no rotations (ie, the canonical invariant curve system cuts the surface into invariant subsurfaces and on each subsurface (with boundary components collapsed to punctures) $\phi(g)$ is isotopic to a pseudo-Anosov homeomorphism or to the identity). There is an induced map $\phi\colon \Gamma \to C(\phi(g))$, the centralizer of $\phi(g)$. Moreover, from the Nielsen–Thurston theory we have a homomorphism $\psi\colon C(\phi(g)) \to MCG(S_1) \times MCG(S_2) \times \cdots \times MCG(S_k)$ given by restricting to the subsurfaces on which $\phi(g)$ is identity. The kernel of ψ is virtually abelian. So it suffices to argue that the image of the composition $\psi\phi\colon \Gamma/<g> \to MCG(S_1) \times MCG(S_2) \times \cdots \times MCG(S_k)$ is virtually abelian. Now $\Gamma/<g>$ is a lattice in $G/<g>$, a Lie group of rank > 1 and the rank of the center $Z(G/<g>) = Z(G)/<g>$ is smaller than the rank of $Z(G)$. So the statement follows by induction on the rank of the center.

Remark It can also be shown that the image of $\Gamma \to MCG(S)$ must be finite, even if $Z(G)$ has infinite center. If the image is infinite, we can assume (by passing to a subgroup of Γ of finite index) that it is torsion-free and abelian of finite rank. If G and Γ satisfy Kazhdan's property (T), then the abelianization of Γ is finite and this is the case when G has no rank 1 factors. It is also true that the abelianization of Γ is finite when G is any higher rank group and this follows from a deep work of Prasad–Raghunathan and Deligne.

Remark Theorem 12 also implies that S–arithmetic groups and certain groups of automorphisms of trees do not occur as subgroups of mapping class groups. These are the groups Γ for which Burger–Monod show $\widetilde{QH}(\Gamma) = 0$ [5], [3], [4].

Acknowledgments The authors would like to thank Yair Minsky and Gopal Prasad for useful discussions. The first author gratefully acknowledges the support by the National Science Foundation. The second author appreciates the hospitality of the mathematics department of the University of Utah.

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