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Seiberg–Witten invariants and surface singularities

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Abstract

We formulate a very general conjecture relating the analytical invariants of a normal surface singularity to the Seiberg–Witten invariants of its link provided that the link is a rational homology sphere. As supporting evidence, we establish its validity for a large class of singularities: some rational and minimally elliptic (including the cyclic quotient and "polygonal") singularities, and Brieskorn–Hamm complete intersections. Some of the verifications are based on a result which describes (in terms of the plumbing graph) the Reidemeister–Turaev sign refined torsion (or, equivalently, the Seiberg–Witten invariant) of a rational homology 3–manifold M, provided that M is given by a negative definite plumbing.

These results extend previous work of Artin, Laufer and S S-T Yau, respectively of Fintushel–Stern and Neumann–Wahl.

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1 Introduction

The main goal of the present paper is to formulate a very general conjecture which relates the topological and the analytical invariants of a complex normal surface singularity whose link is a rational homology sphere.

The motivation for such a result comes from several directions. Before we present some of them, we fix some notations.

Let $(X,0)$ be a normal two-dimensional analytic singularity. It is well-known that from a topological point of view, it is completely characterized by its link M, which is an oriented 3–manifold. Moreover, by a result of Neumann [33], any decorated resolution graph of $(X,0)$ carries the same information as M. A property of $(X,0)$ will be called *topological* if it can be determined from M, or equivalently, from any resolution graph of $(X,0)$.

It is interesting to investigate, in which cases (some of) the analytical invariants (determined, say, from the local algebra of $(X,0)$) are topological. In this article we are mainly interested in the geometric genus p_g of $(X,0)$ (for details, see section 4).

Moreover, if $(X,0)$ has a smoothing with Milnor fiber F, then one can ask the same question about the signature $\sigma(F)$ and the topological Euler characteristic $\chi_{top}(F)$ of F as well. It is known (via some results of Laufer, Durfee, Wahl and Steenbrink) that for Gorenstein singularities, any of p_g, $\sigma(F)$ and $\chi_{top}(F)$ determines the remaining two modulo a certain invariant $K^2+\#\mathcal{V}$ of the link M. Here K is the canonical divisor, and $\#\mathcal{V}$ is the number of irreducible components of the exceptional divisor of the resolution. We want to point out that this invariant coincides with an invariant introduced by Gompf in [14] (see Remark 4.8).

The above program has a long history. M Artin proved in [3, 4] that the rational singularities (ie $p_g=0$) can be characterized completely from the graph (and he computed even the multiplicity and embedding dimension of these singularities from the graph). In [21], H Laufer extended these results to minimally elliptic singularities. Additionally, he noticed that the program breaks for more complicated singularities (for details, see also section 4). On the other hand, the first author noticed in [28] that Laufer's counterexamples do not signal the end of the program. He conjectured that if we restrict ourselves to the case of those Gorenstein singularities whose links are rational homology spheres then some numerical analytical invariants (including p_g) are topological. This was carried out explicitly for elliptic singularities in [28].

On the other hand, in the literature there is no "good" topological candidate for p_g in the very general case. In fact, we are searching for a *"good" topological upper bound* in the following sense. We want a topological upper bound for p_g for *any* normal surface singularity, which, additionally, is optimal in the sense that for Gorenstein singularities it yields exactly p_g. Eg, such a "good" topological upper bound for *elliptic* singularities is the length of the elliptic sequence, introduced and studied by S S-T Yau (see, eg [53]) and Laufer.

In fact, there are some other particular cases too, when a possible candidate is present in the literature. Fintushel and Stern proved in [10] that for a hypersurface Brieskorn singularity whose link is an *integral* homology sphere, the Casson invariant $\lambda(M)$ of the link M equals $\sigma(F)/8$ (hence, by the mentioned correspondence, it determines p_g as well). This fact was generalized by Neumann and Wahl in [35]. They proved the same statement for all Brieskorn–Hamm complete intersections and suspensions of plane curve singularities (with the same assumption, that the link is an integral homology sphere). Moreover, they conjectured the validity of the formula for any isolated complete intersection singularity (with the same restriction about the link). For some other relevant conjectures the reader can also consult [36].

The result of Neumann–Wahl [35] was reproved and reinterpreted by Collin and Saveliev (see [7] and [8]) using equivariant Casson invariant and cyclic covering techniques. But still, a possible generalization for rational homology sphere links remained open. It is important to notice that the "obvious" generalization of the above identity for rational homology spheres, namely to expect that $\sigma(F)/8$ equals the Casson–Walker invariant of the link, completely fails.

In fact, our next conjecture states that one has to replace the Casson invariant $\lambda(M)$ by a certain Seiberg–Witten invariant of the link, ie, by the difference of a certain Reidemeister–Turaev sign-refined torsion invariant and the Casson–Walker invariant (the sign-change is motivated by some sign-conventions already used in the literature).

We recall (for details, see section 2 and 3) that the Seiberg–Witten invariants associates to any $spin^c$ structure σ of M a rational number $\mathbf{sw}^0_M(\sigma)$. In order to formulate our conjecture, we need to fix a "canonical" $spin^c$ structure σ_{can} of M. This can be done as follows. The (almost) complex structure on $X\setminus\{0\}$ induces a natural $spin^c$ structure on $X\setminus\{0\}$. Its restriction to M is, by definition, σ_{can}. The point is that this structure depends only on the topology of M alone.

In fact, the $spin^c$ structures correspond in a natural way to quadratic functions associated with the linking form of M; by this correspondence σ_{can} corresponds to the quadratic function $-q_{LW}$ constructed by Looijenga and Wahl in [25].

We are now ready to state our conjecture.

Main Conjecture *Assume that $(X,0)$ is a normal surface singularity whose link M is a rational homology sphere. Let σ_{can} be the canonical spin^c structure on M. Then, conjecturally, the following facts hold.*

(1) *For any $(X,0)$, there is a topological upper bound for p_g given by:*

$$\mathbf{sw}^0_M(\sigma_{can}) - \frac{K^2 + \#\mathcal{V}}{8} \geq p_g.$$

(2) *If $(X,0)$ is $\mathbb{Q}$–Gorenstein, then in (1) one has equality.*

(3) *In particular, if $(\mathcal{X},0)$ is a smoothing of a Gorenstein singularity $(X,0)$ with Milnor fiber F, then*

$$-\mathbf{sw}^0_M(\sigma_{can}) = \frac{\sigma(F)}{8}.$$

If $(X,0)$ is numerically Gorenstein and M is a $\mathbb{Z}_2$–homology sphere then σ_{can} is the unique spin structure of M; if M is an integral homology sphere then in the above formulae $-\mathbf{sw}^0_M(\sigma_{can}) = \lambda(M)$, the Casson invariant of M.

In the above Conjecture, we have automatically built in the following statements as well.

(a) For any normal singularity $(X,0)$ the topological invariant

$$\mathbf{sw}^0_M(\sigma_{can}) - \frac{K^2 + \#\mathcal{V}}{8}$$

is non-negative. Moreover, this topological invariant is zero if and only if $(X,0)$ is rational. This provides a new topological characterization of the rational singularities.

(b) Assume that $(X,0)$ (equivalently, the link) is numerically Gorenstein. Then the above topological invariant is 1 if and only if $(X,0)$ is minimally elliptic (in the sense of Laufer). Again, this is a new topological characterization of minimally elliptic singularities.

In this paper we will present evidence in support of the conjecture in the form of explicit verifications. The computations are rather arithmetical, involving non-trivial identities about generalized Fourier–Dedekind sums. For the reader's convenience, we have included a list of basic properties of the Dedekind sums in Appendix B.

In general it is not easy to compute the Seiberg–Witten invariant. In our examples we use two different approaches. First, the (modified) Seiberg–Witten invariant is the sum of the Kreck–Stolz invariant and the number of certain monopoles [6, 24, 26]. On the other hand, by a result of the second author, it can also be computed as the difference of the Reidemeister–Turaev torsion and the Casson–Walker invariant [41] (for more details, see section 3). Both methods have their advantages and difficulties. The first method is rather explicit when M is a Seifert manifold (thanks to the results of the second author in [38], cf also with [27]), but frequently the corresponding Morse function will be degenerate. Using the second method, the computation of the Reidemeister–Turaev torsion leads very often to complicated Fourier–Dedekind sums.

In section 5 we present some formulae for the needed invariant in terms of the plumbing graph. The formula for the Casson–Walker invariant was proved by Ratiu in his thesis [45], and can be deduced from Lescop's surgery formulae as well [23]. Moreover, we also provide a similar formula for the invariant $K^2+\#\mathcal{V}$ (which generalizes the corresponding formula already known for cyclic quotient singularities by Hirzebruch, see also [18, 25]). The most important result of this section describes the Reidemeister–Turaev torsion (associated with any *spin*^{c} structure) in terms of the plumbing graph. The proof is partially based on Turaev's surgery formulae [49] and the structure result [48, Theorem 4.2.1]. We have deferred it to Appendix A.

In our examples we did not try to force the verification of the conjecture in the largest generality possible, but we tried to supply a rich and convincing variety of examples which cover different aspects and cases.

In order to eliminate any confusion about different notations and conventions in the literature, in most of the cases we provide our working definitions.

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2 The link and its canonical *spin*^{c} structure

2.1 Definitions Let $(X,0)$ be a normal surface singularity embedded in $(\mathbb{C}^N,0)$. Then for ϵ sufficiently small the intersection $M := X\cap S^{2N-1}_\epsilon$ of a representative X of the germ with the sphere S^{2N-1}_ϵ (of radius ϵ) is a compact oriented 3–manifold, whose oriented C^∞ type does not depend on the choice of the embedding and ϵ. It is called the link of $(X,0)$.

In this article we will assume that M is a rational homology sphere, and we write $H := H_1(M,\mathbb{Z})$. By Poincaré duality H can be identified with $H^2(M,\mathbb{Z})$.

It is well-known that M carries a symmetric non-singular bilinear form

$$b_M\colon\ H\times H\to \mathbb{Q}/\mathbb{Z}$$

called the linking form of M. If $[v_1]$ and $[v_2]\in H$ are represented by the 1–cycles v_1 and v_2, and for some integer n one has $nv_1=\partial w$, then $b_M([v_1],[v_2]) = (w\cdot v_2)/n \pmod{\mathbb{Z}}$.

2.2 The linking form as discriminant form We briefly recall the definition of the discriminant form. Assume that L is a finitely generated free Abelian group with a symmetric bilinear form $(\,,\,)\colon L\times L\to\mathbb{Z}$. Set $L' := \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})$. Then there is a natural homomorphism $i_L\colon L\to L'$ given by $x\mapsto (x,\cdot)$ and a natural extension of the form $(\,,\,)$ to a rational bilinear form $(\,,\,)_{\mathbb{Q}}\colon L'\times L'\to\mathbb{Q}$. If $d_1,d_2\in L'$ and $nd_j=i_L(e_j)$ $(j=1,2)$ for some integer n, then $(d_1,d_2)_{\mathbb{Q}}=d_2(e_1)/n=(e_1,e_2)/n^2$.

If L is non-degenerate (ie, i_L is a monomorphism) then one defines the discriminant space $D(L)$ by $\mathrm{coker}(i_L)$. In this case there is a discriminant bilinear form

$$b_{D(L)}\colon\ D(L)\times D(L)\to\mathbb{Q}/\mathbb{Z}$$

defined by $b_{D(L)}([d_1],[d_2])=(d_1,d_2)_{\mathbb{Q}} \pmod{\mathbb{Z}}$.

Assume that M is the boundary of a oriented 4–manifold N with $H_1(N,\mathbb{Z})=0$ and $H_2(N,\mathbb{Z})$ torsion-free. Let L be the intersection lattice $\big(H_2(N,\mathbb{Z}),(\,,\,)\big)$. Then L' can be identified with $H_2(N,\partial N,\mathbb{Z})$ and one has the exact sequence $L\to L'\to H\to 0$. The fact that M is a rational homology sphere implies that L is non-degenerate. Moreover $(H,b_M)=(D(L),-b_{D(L)})$. Sometimes it is also convenient to regard L' as $H^2(N,\mathbb{Z})$ (identification by Poincaré duality).

2.3 Quadratic functions and forms associated with b_M A map $q\colon H\to\mathbb{Q}/\mathbb{Z}$ is called quadratic *function* if $b(x,y)=q(x+y)-q(x)-q(y)$ is a bilinear form on $H\times H$. If in addition $q(nx)=n^2q(x)$ for any $x\in H$ and $n\in\mathbb{Z}$ then q is called quadratic *form.* In this case we say that the quadratic function, respectively form, is associated with b. Quadratic *forms* are also called *quadratic refinements* of the bilinear form b.

In the case of the link M, we denote by $Q^c(M)$ (respectively by $Q(M)$) the set of quadratic functions (resp. forms) associated with b_M. Obviously, there is a natural inclusion $Q(M)\subset Q^c(M)$.

The set $Q(M)$ is non-empty. It is a $G := H^1(M,\mathbb{Z}_2)$ torsor, ie, G acts freely and transitively on $Q(M)$. The action can be easily described if we identify G with $\mathrm{Hom}(H,\mathbb{Z}_2)$ and we regard $\mathbb{Z}_2$ as $(\frac{1}{2}\mathbb{Z})/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$. Then, the difference of any two quadratic refinements of b_M is an element of G, which provides a natural action $G \times Q(M) \to Q(M)$ given by $(\chi, q) \mapsto \chi + q$.

Similarly, the set $Q^c(M)$ is non-empty and it is a $\hat{H} = \mathrm{Hom}(H,\mathbb{Q}/\mathbb{Z})$ torsor. The free and transitive action $\hat{H} \times Q^c(M) \to Q^c(M)$ is given by the same formula $(\chi, q) \mapsto \chi + q$. In particular, the inclusion $Q(M) \subset Q^c(M)$ is G–equivariant via the natural monomorphism $G \hookrightarrow \hat{H}$. We prefer to replace the $\hat{H}$ action on $Q^c(M)$ by an action of H. This action $H \times Q^c(M) \to Q^c(M)$ is defined by $(h,q) \mapsto q + b_M(h,\cdot)$. Then the natural monomorphism $G \hookrightarrow \hat{H}$ is replaced by the Bockstein-homomorphism $G = H^1(M,\mathbb{Z}_2) \to H^2(M,\mathbb{Z}) = H$. In the sequel we consider $Q^c(M)$ with this H–action.

Quadratic functions appear in a natural way. In order to see this, let N be as in 2.2. Pick a characteristic element, that is an element $k \in L'$, so that $(x,x) + k(x) \in 2\mathbb{Z}$ for any $x \in L$. Then for any $d \in L'$ with class $[d] \in H$, define

$$q_{D(L),k}([d]) := \frac{1}{2}\,(d+k,d)_{\mathbb{Q}} \ (\mathrm{mod}\ \mathbb{Z}).$$

Then $-q_{D(L),k}$ is a quadratic function associated with $b_M = -b_{D(L)}$. If in addition $k \in \mathrm{Im}(i_L)$, then $-q_{D(L),k}$ is a *quadratic refinement* of b_M.

There are two important examples to consider.

First assume that N is an *almost-complex* manifold, ie, its tangent bundle TN carries an almost complex structure. By Wu formula, $k = -c_1(TN) \in L'$ is a characteristic element. Hence $-q_{D(L),k}$ is a quadratic function associated with b_M. If $c_1(TN) \in \mathrm{Im}(i_L)$ then we obtain a quadratic refinement.

Next, assume that N carries a *spin structure.* Then $w_2(N)$ vanishes, hence by Wu formula $(\ ,\ )$ is an even form. Then one can take $k = 0$, and $-q_{D(L),0}$ is a quadratic refinement of b_M.

2.4 The *spin* structures of M The 3–manifold M is always spinnable. The set $Spin(M)$ of the possible spin structures of M is a G–torsor. In fact, there is a natural (equivariant) identification of $\mathfrak{q} : Spin(M) \to Q(M)$.

In order to see this, fix a spin structure $\epsilon \in Spin(M)$. Then there exists a simple connected oriented spin 4–manifold N with $\partial N = M$ whose induced spin structure on M is exactly ϵ (see, eg [15, 5.7.14]). Set $L = (H_2(N,\mathbb{Z}), (\ ,\ ))$. Then the quadratic refinement $-q_{D(L),0}$ (cf 2.3) of b_M depends only on the spin structure ϵ and not on the particular choice of N. The correspondence $\epsilon \mapsto -q_{D(L),0}$ determines the identification $\mathfrak{q}$ mentioned above.

2.5 The *spin^c* structures on M We denote by $Spin^c(M)$ the space of isomorphism classes of *spin^c* structures on M. $Spin^c(M)$ is in a natural way a $H = H^2(M,\mathbb{Z})$–torsor. We denote this action $H \times Spin^c(M) \to Spin^c(M)$ by $(h,\sigma) \mapsto h\cdot\sigma$. For every $\sigma \in Spin^c(M)$ we denote by $\mathbb{S}_\sigma$ the associated bundle of complex spinors, and by $\det\sigma$ the associated line bundle, $\det\sigma := \det\mathbb{S}_\sigma$. We set $c(\sigma) := c_1(\mathbb{S}_\sigma) \in H$. Note that $c(h\cdot\sigma) = 2h + c(\sigma)$.

$Spin^c(M)$ is equipped with a natural involution $\sigma \longleftrightarrow \bar\sigma$ such that

$$c(\bar\sigma) = -c(\sigma) \quad \text{and} \quad \overline{h\cdot\sigma} = (-h)\cdot\bar\sigma.$$

There is a natural injection $\epsilon \mapsto \sigma(\epsilon)$ of $Spin(M)$ into $Spin^c(M)$. The image of $Spin(M)$ in $Spin^c(M)$ is

$$\{\sigma \in Spin^c(M);\ \ c(\sigma) = 0\} = \{\sigma \in Spin^c(M);\ \ \sigma = \bar\sigma\}.$$

Consider now a 4–manifold N with lattices L and L' as in 2.2. We prefer to write $L' = H^2(N,\mathbb{Z})$, and denote by $d \mapsto [d]$ the restriction map $L' = H^2(N,\mathbb{Z}) \to H^2(M,\mathbb{Z}) = H$.

Then N is automatically a *spin^c* manifold. In fact, the set of *spin^c* structures on N is parametrized by the set of characteristic elements

$$\mathcal{C}_N := \{k \in L' : k(x) + (x,x) \in 2\mathbb{Z} \text{ for all } \ x \in L\}$$

via $\tilde\sigma \mapsto c(\tilde\sigma) \in \mathcal{C}_N$ (see. eg [15, 2.4.16]). The set $Spin^c(N)$ is an L' torsor with action $(d,\tilde\sigma) \mapsto d\cdot\tilde\sigma$. Let $r\colon Spin^c(N) \to Spin^c(M)$ be the restriction. Then $r(d\cdot\tilde\sigma) = [d]\cdot r(\tilde\sigma)$ and $c(r(\tilde\sigma)) = [c(\tilde\sigma)]$.

Moreover, notice that $r(\tilde\sigma) = r(d\cdot\tilde\sigma)$ if and only if $[d] = 0$, ie $d \in L$. If this is happening then $c(\tilde\sigma) - c(d\cdot\tilde\sigma) \in 2L$.

2.6 Lemma *There is a canonical H–equivariant identification*

$$\mathfrak{q}^c\colon Spin^c(M) \to Q^c(M).$$

Moreover, this identification is compatible with the G–equivariant identification $\mathfrak{q}\colon Spin(M) \to Q(M)$ via the inclusions $Spin(M) \subset Spin^c(M)$ and $Q(M) \subset Q^c(M)$.

Proof Let N be as above. We first show that r is onto. Indeed, take any $\tilde{\sigma} \in Spin^c(N)$ with restriction $\sigma \in Spin^c(M)$. Then all the elements in the H–orbit of σ are induced structures. But this orbit is the whole set. Next, define for any $\tilde{\sigma}$ corresponding to $k = c(\tilde{\sigma})$ the quadratic function $q_{D(L),k}$. Then $r(\tilde{\sigma}) = r(d \cdot \tilde{\sigma})$ if and only if $d \in L$. This means that $c(\tilde{\sigma}) - c(d \cdot \tilde{\sigma}) \in 2L$ hence $c(\tilde{\sigma})$ and $c(d \cdot \tilde{\sigma})$ induce the same quadratic function. Hence $\mathfrak{q}^c(r(\tilde{\sigma})) := -q_{D(L),c(\sigma)}$ is well-defined. Finally, notice that $\mathfrak{q}^c$ does not depend on the choice of N, fact which shows its compatibility with $\mathfrak{q}$ as well (by taking convenient spaces N). □

2.7 M **as a plumbing manifold** Fix a sufficiently small (Stein) representative X of $(X, 0)$ and let $\pi\colon \tilde{X} \to X$ be a resolution of the singular point $0 \in X$. In particular, $\tilde{X}$ is smooth, and π is a biholomorphic isomorphism above $X \setminus \{0\}$. We will assume that the exceptional divisor $E := \pi^{-1}(0)$ is a normal crossing divisor with irreducible components $\{E_v\}_{v \in \mathcal{V}}$. Let $\Gamma(\pi)$ be the dual resolution graph associated with π decorated with the self intersection numbers $\{(E_v, E_v)\}_v$. Since M is a rational homology sphere, all the irreducible components E_v of E are rational, and $\Gamma(\pi)$ is a tree.

It is clear that $H_1(\tilde{X}, \mathbb{Z}) = 0$ and $H_2(\tilde{X}, \mathbb{Z})$ is freely generated by the fundamental classes $\{[E_v]\}_v$. Let I be the intersection matrix $\{(E_v, E_w)\}_{v,w}$. Since π identifies $\partial\tilde{X}$ with M, the results from 2.2 can be applied. In particular, $H = \operatorname{coker}(I)$ and $b_M = -b_{D(I)}$. The matrix I is negative definite.

The graph $\Gamma(\pi)$ can be identified with a plumbing graph, and M can be considered as an S^1–plumbing manifold whose plumbing graph is $\Gamma(\pi)$. In particular, any resolution graph $\Gamma(\pi)$ determines the oriented 3–manifold M completely.

We say that two plumbing graphs (with negative definite intersection forms) are equivalent if one of them can be obtained from the other by a finite sequence of blowups and/or blowdowns along rational (-1)–curves. Obviously, for a given $(X, 0)$, the resolution π, hence the graph $\Gamma(\pi)$ too, is not unique. But different resolutions provide equivalent graphs. By a result of W. Neumann [33], the oriented diffeomorphism type of M determines completely the equivalence class of $\Gamma(\pi)$. In particular, any invariant defined from the resolution graph $\Gamma(\pi)$ (which is constant in its equivalence class) is, in fact, an invariant of the oriented C^∞ 3–manifold M. This fact will be crucial in the next discussions.

Now, we fix a resolution π as above and identify $M = \partial\tilde{X}$. Let K be the canonical class (in $Pic(\tilde{X})$) of $\tilde{X}$. By the adjunction formula,

$$-K \cdot E_v = E_v \cdot E_v + 2$$

for any $v \in \mathcal{V}$. In fact, K at homological level provides an element $k_{\tilde{X}} \in L'$ which has the obvious property

$$-k_{\tilde{X}}([E_v]) = ([E_v],[E_v]) + 2 \quad \text{for any } v \in \mathcal{V}.$$

Since the matrix I is non-degenerate, this defines $k_{\tilde{X}}$ uniquely.

$-k_{\tilde{X}}$ is known in the literature as the *canonical (rational) cycle* of $(X,0)$ associated with the resolution π. More precisely, let $Z_K = \sum_{v\in\mathcal{V}} r_v E_v$, $r_v \in \mathbb{Q}$, be a rational cycle supported by the exceptional divisor E, defined by

$$Z_K \cdot E_v = -K \cdot E_v = E_v \cdot E_v + 2 \quad \text{for any } v \in \mathcal{V}. \tag{$*$}$$

Then the above linear system has a unique solution, and $\sum_v r_v[E_v] \in L \otimes \mathbb{Q}$ can be identified with $(i_L \otimes \mathbb{Q})^{-1}(-k_{\tilde{X}})$.

It is clear that $-k_{\tilde{X}} \in \mathrm{Im}(i_L)$ if and only if all the coefficients $\{r_v\}_v$ of Z_K are integers. In this case the singularity $(X,0)$ is called *numerically Gorenstein* (and we will also say that "M is numerically Gorenstein").

In particular, for any normal singularity $(X,0)$, the resolution π provides a quadratic function $-q_{D(I),k_{\tilde{X}}}$ associated with b_M, which is a quadratic form if and only if $(X,0)$ is numerically Gorenstein.

2.8 The universal property of $q_{D(I),k_{\tilde{X}}}$ In [25], Looijenga and Wahl define a quadratic function q_{LW} (denoted by q in [25]) associated with b_M from the almost complex structure of the bundle $TM \oplus \mathbb{R}_M$ (where TM is the tangent bundle and $\mathbb{R}_M$ is the trivial bundle of M). By the main universal property of q_{LW} (see [loc. cit.], Theorem 3.7) (and from the fact that any resolution π induces the same almost complex structure on $TM \oplus \mathbb{R}_M$) one gets that for any π as in 2.7, the identity $q_{LW} = -q_{D(I),k_{\tilde{X}}}$ is valid. This shows that $q_{D(I),k_{\tilde{X}}}$ does not depend on the choice of the resolution π.

This fact can be verified by elementary computation as well: one can prove that $q_{D(I),k_{\tilde{X}}}$ is stable with respect to a blow up (of points of E).

2.9 The "canonical" $spin^c$ structure of a singularity link Assume that M is the link of $(X,0)$. Fix a resolution $\pi\colon \tilde{X} \to X$ as in 2.7. Then π determines a "canonical" quadratic function $q_{can} := -q_{D(I),k_{\tilde{X}}}$ associated with b_M which does not depend on the choice of π (cf 2.8). Then the natural identification $\mathfrak{q}^c\colon Spin^c(M) \to Q^c(M)$ (cf 2.3) provides a well-defined $spin^c$ structure $(\mathfrak{q}^c)^{-1}(q_{can})$. Then the "canonical" $spin^c$ structure σ_{can} on M is $(\mathfrak{q}^c)^{-1}(q_{can})$ modified by the natural involution of $Spin^c(M)$. In particular,

$c(\sigma_{can}) = -[k_{\tilde{X}}] \in H$. (Equivalently, σ_{can} is the restriction to M of the *spin*^{c} structure given by the characteristic element $-k_{\tilde{X}} \in \mathcal{C}_{\tilde{X}}$.) If $(X,0)$ is numerically Gorenstein then σ_{can} is a spin structure. In this case we will use the notation $\epsilon_{can} = \sigma_{can}$ as well.

We want to emphasize (again) that σ_{can} depends only on the oriented C^∞ type of M (cf also with 2.11). Indeed, one can construct q_{can} as follows. Fix an arbitrary plumbing graph Γ of M with negative definite intersection form (lattice) L. Then determine Z_K by 2.7($*$), and take

$$q_{can}([d]) := -\frac{1}{2}(d - [Z_K], d)_{\mathbb{Q}} \ (\mathrm{mod}\ \mathbb{Z}).$$

Then q_{can} does not depend on the choice of Γ.

It is remarkable that this construction provides an "origin" of the torsor space $Spin^c(M)$.

2.10 Compatibility with the (almost) complex structure As we have already mentioned in 2.8, the result of Looijenga and Wahl [25] implies the following: the almost complex structure on $X \setminus \{0\}$ determines a *spin*^{c} structure, whose restriction to M is σ_{can}. Similarly, if $\pi\colon \tilde{X} \to X$ is a resolution, then the almost complex structure on $\tilde{X}$ gives a *spin*^{c} structure $\sigma_{\tilde{X}}$ on $\tilde{X}$, whose restriction to M is σ_{can}. Here we would like to add the following discussion. Assume that the intersection form $(\,,\,)_{\tilde{X}}$ is even, hence $\tilde{X}$ has a unique spin structure $\epsilon_{\tilde{X}}$. The point is that, in general, $\sigma_{\tilde{X}} \neq \epsilon_{\tilde{X}}$, and their restrictions can be different as well, even if the restriction of $\sigma_{\tilde{X}}$ is spin.

More precisely: $(\,,\,)_{\tilde{X}}$ is even if and only if $k_{\tilde{X}} \in 2L'$; $r(\sigma_{\tilde{X}}) \in Spin(M)$ if and only if $k_{\tilde{X}} \in L$; and finally, $r(\sigma_{\tilde{X}}) = r(\epsilon_{\tilde{X}})$ if and only if $k_{\tilde{X}} \in 2L$.

2.11 Remarks

(1) In fact, by the classification theorem of plumbing graphs given by Neumann [33], if M is a rational homology sphere which is not a lens space, then already $\pi_1(M)$ (ie, the homotopy type of M) determines its orientation class and its canonical *spin*^{c} structure. Indeed, if one wants to recover the oriented C^∞ type of M from its fundamental group, then by Neumann's result the only ambiguity appears for cusp singularities (which are not rational homology spheres) and for cyclic quotient singularities. The links of cyclic quotient singularities are exactly the lens spaces. In fact, if we assume the numerically Gorenstein assumption, even the lens spaces are classified by their fundamental groups (since they are exactly the du Val A_p–singularities).

(2) If M is a numerically Gorenstein $\mathbb{Z}_2$–homology sphere, the definition of ϵ_{can} is obviously simpler: it is the unique spin structure of M. If M is an *integral* homology sphere then it is automatically numerically Gorenstein, hence the above statement applies.

(3) Assume that $(X,0)$ has a smoothing with Milnor fiber F whose homology group $H_1(F,\mathbb{Z})$ has no torsion. Then the (almost) complex structure of F provides a *spin*^{c} structure on F whose restriction to M is exactly σ_{can}. This follows (again) by the universal property of q_{LM} ([25, Theorem 3.7], ; cf also with 2.8).

Moreover, F has a spin structure if and only if the intersection form $(\, ,\,)$ of F is even (see eg [15, 5.7.6]); and in this case, the spin structure is unique. If F is spin, then its spin structure ϵ_F coincides with the *spin*^{c} structure induced by the complex structure (since the canonical bundle of F is trivial). In particular, if F is spin, σ_{can} is the restriction of ϵ_F, hence it is spin. This also proves that if $(X,0)$ has a smoothing with even intersection form and without torsion in $H_1(F,\mathbb{Z})$, then it is necessarily numerically Gorenstein.

Here is worth noticing that the Milnor fiber of a smoothing of a Gorenstein singularity has even intersection form [46].

(4) Clearly, q_{can} depends only on $Z_K \pmod{2\mathbb{Z}}$.

2.12 The invariant $K^2+\#\mathcal{V}$ Fix a resolution $\pi\colon \tilde{X}\to X$ of $(X,0)$ as in 2.7, and consider Z_K or $k_{\tilde{X}}$. The rational number $Z_K\cdot Z_K=(k_{\tilde{X}},k_{\tilde{X}})_{\mathbb{Q}}$ will be denoted by K^2. Let $\#\mathcal{V}$ denote the number of irreducible components of $E=\pi^{-1}(0)$. Then $K^2+\#\mathcal{V}$ does not depend on the choice of the resolution π. In fact, the discussion in 2.7 and 2.9 shows that it is an invariant of M. Obviously, if $(X,0)$ is numerically Gorenstein, then $K^2+\#\mathcal{V}\in\mathbb{Z}$.

2.13 Notation Let $\tilde{X}$ as above. Let $\{E_v\}_{v\in\mathcal{V}}$ be the set of irreducible exceptional divisors and D_v a small transversal disc to E_v. Then $\{[E_v]\}_v$ (resp. $\{[D_v]\}_v$) are the free generators of $L=H_2(\tilde{X},\mathbb{Z})$ (resp. $L'=H_2(\tilde{X},M,\mathbb{Z})$) with $[D_v]\cdot[E_w]=1$ if $v=w$ and $=0$ otherwise. Moreover, $g_v:=[\partial D_v]$ ($v\in\mathcal{V}$) is a generator set of $L'/L=H$. In fact ∂D_v is a generic fiber of the S^1–bundle over E_v used in the plumbing construction of M. If I is the intersection matrix defined by the resolution (plumbing) graph, then i_L written in the bases $\{[E_v]\}_v$ and $\{[D_v]\}_v$ is exactly I.

Using this notation, $k_{\tilde{X}} \in L'$ can be expressed as $\sum_v(-e_v - 2)[D_v]$, where $e_v = E_v \cdot E_v$. For the degree of v (ie, for $\#\{w : E_w \cdot E_v = 1\}$) we will use the notation δ_v. Obviously

$$\sum_v \delta_v = -2 \times \text{Euler characteristic of the plumbing graph} + 2\#\mathcal{V} = 2\#\mathcal{V} - 2.$$

Most of the examples considered later are star-shaped graphs. In these cases it is convenient to express the corresponding invariants of the Seifert 3–manifold M in terms of their Seifert invariants. In order to eliminate any confusion about the different notations and conventions in the literature, we list briefly the definitions and some of the needed properties.

2.14 The unnormalized Seifert invariants Consider a Seifert fibration $\pi\colon M \to \Sigma$. In our situation M is a rational homology sphere and the base space Σ is an S^2 with genus 0 (and we will not emphasize this fact anymore).

Consider a set of points $\{x_i\}_{i=1}^{\nu}$ in such a way that the set of fibers $\{\pi^{-1}(x_i)\}_i$ contains the set of singular fibers. Set $O_i := \pi^{-1}(x_i)$. Let D_i be a small disc in X containing x_i, $\Sigma' := \Sigma \setminus \cup_i D_i$ and $M' := \pi^{-1}(\Sigma')$. Now, $\pi\colon M' \to \Sigma'$ admits sections, let $s\colon \Sigma' \to M'$ be one of them. Let $Q_i := s(\partial D_i)$ and let H_i be a circle fiber in $\pi^{-1}(\partial D_i)$. Then in $H_1(\pi^{-1}(D_i), \mathbb{Z})$ one has $H_i \sim \alpha_i O_i$ and $Q_i \sim -\beta_i O_i$ for some integers $\alpha_i > 0$ and β_i with $(\alpha_i, \beta_i) = 1$. The set $((\alpha_i, \beta_i)_{i=1}^{\nu})$ constitute the set of *(unnormalized) Seifert invariants.* The number

$$e := -\sum_i (\beta_i/\alpha_i)$$

is called the *(orbifold) Euler number* of M. If M is a link of singularity then $e < 0$.

Replacing the section by another one, a different choice changes each β_i within its residue class modulo α_i in such a way that the sum $e = -\sum_i(\beta_i/\alpha_i)$ is constant.

The elements $q_i = [Q_i]$ $(1 \le i \le \nu)$ and the class h of the generic fiber H generate the group $H = H_1(M, \mathbb{Z})$. By the above construction is clear that:

$$H = \mathrm{ab}\langle q_1, \dots q_\nu, h \mid q_1 \dots q_\nu = 1,\ q_i^{\alpha_i} h^{\beta_i} = 1,\ \text{for all } i \rangle.$$

Let $\alpha := \mathrm{lcm}(\alpha_1, \dots, \alpha_\nu)$. The order of the group H and of the subgroup $\langle h \rangle$ can be determined by (cf [32]):

$$|H| = \alpha_1 \cdots \alpha_\nu |e|, \qquad |\langle h \rangle| = \alpha |e|.$$

2.15 The normalized Seifert invariants and plumbing graph We write

$$e = b + \sum \omega_i/\alpha_i$$

for some integer b, and $0 \leq \omega_i < \alpha_i$ with $\omega_i \equiv -\beta_i \pmod{\alpha_i}$. Clearly, these properties define $\{\omega_i\}_i$ uniquely. Notice that $b \leq e < 0$. For the uniformity of the notations, in the sequel we assume $\nu \geq 3$.

For each i, consider the continued fraction $\alpha_i/\omega_i = b_{i1} - 1/(b_{i2} - 1/(\cdots - 1/b_{i\nu_i})\cdots)$. Then (a possible) plumbing graph of M is a star-shaped graph with ν arms. The central vertex has decoration b and the arm corresponding to the index i has ν_i vertices, and they are decorated by $b_{i1}, \ldots, b_{i\nu_i}$ (the vertex decorated by b_{i1} is connected by the central vertex).

We will distinguish those vertices $v \in \mathcal{V}$ of the graph which have $\delta_v \neq 2$. We will denote by $\bar{v}_0$ the central vertex (with $\delta = \nu$), and by $\bar{v}_i$ the end-vertex of the i^{th} arm (with $\delta = 1$) for all $1 \leq i \leq \nu$. In this notation, $g_{\bar{v}_0} = h$, the class of the generic fiber. Moreover, using the plumbing representation of the group H, we have another presentation for H, namely:

$$H = \mathrm{ab}\langle g_{\bar{v}_1}, \ldots g_{\bar{v}_\nu}, h \mid h^{-b} = \prod_{i=1}^{\nu} g_{\bar{v}_i}^{\omega_i}, \quad h = g_{\bar{v}_i}^{\alpha_i} \text{ for all } i\rangle.$$

3 Seiberg–Witten invariants of $\mathbb{Q}$–homology spheres

In this section we consider an oriented rational homology 3–sphere M. We set $H := H^2(M, \mathbb{Z})$. When working with the group algebra $\mathbb{Q}[H]$ of H it is more convenient to use the multiplicative notation for the group operation of H.

3.1 The Seiberg–Witten invariants of M To describe the Seiberg–Witten invariants we need to fix some additional geometric data belonging to the space of parameters

$$\mathcal{P} = \{(g, \eta); \quad g = \text{Riemann metric}, \quad \eta = \text{closed two-form}\}.$$

For each $spin^c$ structure σ on M (cf 2.5), we have the space of configurations $\mathcal{C}_\sigma$ (associated with σ) consisting of pairs $\mathsf{C} = (\psi, A)$, where ψ is a section of $\mathbb{S}_\sigma$ and A is a Hermitian connection on $\det\sigma$. The gauge group $\mathcal{G} := \mathrm{Map}\,(M, S^1)$ acts on $\mathcal{C}_\sigma$. Moreover, it acts freely on the irreducible part

$$\mathcal{C}_\sigma^{irr} = \{(\psi, A) \in \mathcal{C}_\sigma; \quad \psi \not\equiv 0\},$$

and the quotient $\mathcal{B}_\sigma^{irr} := \mathcal{C}_\sigma^{irr}/\mathcal{G}$ can be equipped with a structure of Hilbert manifold. Every parameter $u = (g, \eta) \in \mathcal{P}$ defines a $\mathcal{G}$–invariant function $\mathcal{F}_{\sigma,u}\colon \mathcal{C}_\sigma \to \mathbb{R}$ whose critical points are called the (σ, g, η)*–Seiberg–Witten monopoles*. In particular, $\mathcal{F}_{\sigma,u}$ descends to a smooth function $[\mathcal{F}_{\sigma,u}]\colon \mathcal{B}_\sigma^{irr} \to \mathbb{R}$. We denote by $\mathfrak{M}_{\sigma,u}^{irr}$ its critical set.

The first Chern class $c(\sigma)$ of $\mathbb{S}_\sigma$ is a torsion element of $H^2(M,\mathbb{Z})$, and thus the curvature of any connection on $\det\sigma$ is an *exact* 2–form. In particular we can find an unique $\mathcal{G}$–equivalence class of connections A on $\det\sigma$ with the property that

$$F_A = \mathbf{i}\eta. \qquad (\dagger)$$

Using the metric g on M (which is part of our parameter u) and a connection A_u satisfying $(\dagger)$, we obtain a *spin*^{c}–Dirac operator $\mathfrak{D}_{A_u}$. To define the Seiberg–Witten invariants we need to work with *good* parameters, ie, parameters u such that the following two things happen.

- The Dirac operator $\mathfrak{D}_{A_u}$ is invertible.
- The function $[\mathcal{F}_{\sigma,u}]$ is Morse, and $\mathfrak{M}_{\sigma,u}$ consists of finitely many points.

The space of good parameters is *generic*. Fix such a good parameter u. Then each critical point has a well defined $\mathbb{Z}_2$–valued Morse index

$$\boldsymbol{m}\colon \mathfrak{M}_{\sigma,u}^{irr} \to \{\pm 1\}$$

and we set

$$\mathbf{sw}_M(\sigma, u) = \sum_{x\in\mathfrak{M}_{\sigma,u}^{irr}} \boldsymbol{m}(x) \in \mathbb{Z}.$$

This integer depends on the choice of the parameter u and thus it is not a topological invariant. To obtain an invariant we need to alter this monopole count.

The eta invariant of $\mathfrak{D}_{A_u}$ depends only on the gauge equivalence class of A_u, and we will denote it by $\eta_{dir}(\sigma, u)$. The metric g defines an odd signature operator on M whose eta invariant we denote by $\eta_{sign}(u)$. Now define the *Kreck–Stolz* invariant associated with the data (σ, u) by

$$KS_M(\sigma, u) := 4\eta_{dir}(\sigma, u) + \eta_{sign}(u) \in \mathbb{Q}.$$

We have the following result.

3.2 Theorem [6, 24, 26] *The rational number*

$$\frac{1}{8}KS_M(\sigma,u)+\mathbf{sw}_M(\sigma,u)$$

is independent of u *and thus it is a topological invariant of the pair* (M,σ). *We denote this number by* $\mathbf{sw}^0_M(\sigma)$. *Moreover*

$$\mathbf{sw}^0_M(\sigma)=\mathbf{sw}^0_M(\bar{\sigma}). \qquad (*)$$

It is convenient to rewrite the collection $\{\mathbf{sw}^0_M(\sigma)\}_\sigma$ as a function $H\to\mathbb{Q}$ (see eg the Fourier calculus below). For every *spin*^{c} structure σ on M we consider

$$\boldsymbol{SW}^0_{M,\sigma}:=\sum_{h\in H}\mathbf{sw}^0_M(h^{-1}\cdot\sigma)h\in\mathbb{Q}[H].$$

Equivalently, $\boldsymbol{SW}^0_{M,\sigma}$, as a function $H\to\mathbb{Q}$, is defined by $\boldsymbol{SW}^0_{M,\sigma}(h)=\mathbf{sw}^0_M(h^{-1}\cdot\sigma)$. The symmetry condition 3.2($*$) implies

$$\boldsymbol{SW}^0_{M,\sigma}(h)=\boldsymbol{SW}^0_{M,\bar{\sigma}}(h^{-1})\quad\text{for all } h\in H.$$

This description is very difficult to use in concrete computations unless we have very specific information about the geometry of M. This is the case of the Seifert 3–manifolds, see [37, 38] for the complete presentation. In the next subsection we recall some facts needed in our computations. The interested reader is invited to consult [loc. cit.] for more details.

3.3 The Seiberg–Witten invariants of Seifert manifolds We will use the notations of 2.14 and 2.15; nevertheless, in [11, 27, 38] (and in general, in the gauge theoretic literature) some other notations became generally accepted too. They will be mentioned accordingly.

In [27, 38] a Seifert manifold is regarded as the unit circle sub-bundle of an (orbifold) V–line bundle over a 2–dimensional V–manifold (orbifold) Σ. The 2–dimensional orbifold in our case is $\mathbb{P}^1$ (with ν conical singularities each with angle $\frac{\pi}{\alpha_i}$, $i=1,\dots,\nu$).

The space of isomorphisms classes of topological V–line bundles over Σ is an Abelian group $\mathrm{Pic}^V_{top}(\Sigma)$. Its is a subgroup of $\mathbb{Q}\times\prod_{i=1}^{\nu}\mathbb{Z}_{\alpha_i}$, and correspondingly we denote its elements by $(\nu+1)$–uples

$$L(c;\frac{\tau_1}{\alpha_1},\cdots,\frac{\tau_\nu}{\alpha_\nu}),$$

where $0 \le \tau_i < \alpha_i$, $i = 1, \cdots, \nu$. The number c is called the *rational degree*, while the fractions $\frac{\tau_i}{\alpha_i}$ are called the *singularity data*. They are subject to a single compatibility condition

$$c - \sum_{i=1}^{\nu} \frac{\tau_i}{\alpha_i} \in \mathbb{Z}.$$

To any V–line bundle $L(c; \frac{\tau_i}{\alpha_i},\ 1 \le i \le \nu)$ we canonically associate a *smooth* line bundle $|L| \to \Sigma = \mathbb{P}^1$ uniquely determined by the condition

$$\deg |L| = c - \sum_{i=1}^{\nu} \frac{\tau_i}{\alpha_i}.$$

The *canonical* V–line bundle K_Σ has singularity data $(\alpha_i - 1)/\alpha_i$ for $1 \le i \le \nu$, and $\deg |K_\Sigma| = -2$, hence rational degree

$$\kappa := \deg^V K_\Sigma = -2 + \sum_{i=1}^{\nu} \Big(1 - \frac{1}{\alpha_i}\Big).$$

The Seifert manifold M with non-normalized Seifert invariants $((\alpha_i, \beta_i)_{i=1}^{\nu})$ (or, equivalently, with normalized Seifert invariants $(b; (\alpha_i, \omega_i)_{i=1}^{\nu})$, cf 2.15), is the unit circle bundle of the V–line bundle $\mathbb{L}_0$ with rational degree $\ell = e$, and singularity data

$$\frac{\omega_i}{\alpha_i},\ \ 1 \le i \le \nu,\ \ (\omega_i \equiv -\beta_i \ (\mathrm{mod}\ \alpha_i)).$$

Denote by $\langle \mathbb{L}_0 \rangle \subset \mathrm{Pic}^V_{top}(\Sigma)$ the cyclic group generated by $\mathbb{L}_0$. Then one has the following exact sequence:

$$0 \to \langle \mathbb{L}_0 \rangle \to \mathrm{Pic}^V_{top}(\Sigma) \stackrel{\pi^*}{\to} \mathrm{Pic}_{top}(M) \to 0,$$

where π^* is the pullback map induced by the natural projection $\pi\colon M \to \Sigma$. Therefore, the above exact sequence identifies for every $L \in \mathrm{Pic}^V_{top}(\Sigma)$ the pullback $\pi^*(L)$ with the class $[L] \in \mathrm{Pic}^V_{top}(\Sigma)/\langle \mathbb{L}_0 \rangle$.

For every $L \in \mathrm{Pic}^V_{top}(\Sigma)$, $c := \deg^V L$ we set

$$\rho(L) := \frac{\deg^V K_\Sigma - 2c}{2\ell} = \frac{\kappa}{2\ell} - \frac{c}{\ell} \in \mathbb{Q}.$$

For every class $u \in \mathrm{Pic}_{top}(M)$ we can find an unique $E_u \in \mathrm{Pic}^V_{top}(\Sigma)$ such that $u = [E_u]$ and $\rho(E_u) \in [0, 1)$. We say that E_u is the *canonical representative* of u. As explained in [27, 38] there is a natural bijection

$$\mathrm{Pic}_{top}(M) \ni u \mapsto \sigma(u) \in \mathit{Spin}^c(M)$$

with the property that $\det\sigma(u)=2u-[K_\Sigma]\in \mathrm{Pic}_{top}(M)$. The canonical *spin*^{c} structure $\sigma_{can}\in Spin^c(M)$ corresponds to $u=0$. In fact, $\mathrm{Pic}_{top}(M)$ can be identified in a natural way to H via the Chern class. Then $\sigma(u)$, in terms of the H–action described in 2.5, is given by $u\cdot\sigma_{can}$. In this case, if one writes $\rho_0:=\rho(E_0)$ one has

$$\rho_0=\Big\{\frac{\kappa}{2\ell}\Big\}\ \text{ and }\ E_0:=n_0\mathbb{L}_0,\ \text{ with }\ n_0:=\Big\lfloor\frac{\kappa}{2\ell}\Big\rfloor.$$

We denote the orbifold invariants of E_0 by $\frac{\gamma_i}{\alpha_i}$. Observe that

$$\frac{\gamma_i}{\alpha_i}=\Big\{\frac{n_0\omega_i}{\alpha_i}\Big\}.$$

The Seifert manifold M admits a natural metric, the so called *Thurston metric* which we denote by g_0. The $(\sigma_{can},g_0,0)$–monopoles were explicitly described in [27, 38].

The space $\mathfrak{M}_0^*$ of irreducible $(\sigma_{can},g_0,0)$ monopoles on M consists of several components parametrized by a subset of

$$S_0=\big\{E=E_0+n\mathbb{L}_0\in\pi;\ \ 0<|\nu(E)|\le\frac12\deg^V K_\Sigma\big\},$$

where

$$\nu(E):=\deg^V(E_0+n\mathbb{L}_0)-\frac12\deg^V K_\Sigma.$$

More precisely, consider the sets

$$S_0^+:=\Big\{E\in S_0;\ \ \nu(E)<0,\ \ \deg|E|\ge 0\Big\},$$

$$S_0^-=\Big\{E\in S_0;\ \ \nu(E)>0,\ \ \deg|K_\Sigma-E|\ge 0\big\}.$$

To every $E\in S_0^+$ there corresponds a component $\mathfrak{M}_E^+$ of $\mathfrak{M}_0^*$ of dimension $2\deg|E|$, and to every $E\in S_0^-$ there corresponds a component $\mathfrak{M}_E^-$ of $\mathfrak{M}_0^*$ of dimension $2\deg|K_\Sigma-E|$.

The Kreck–Stolz invariant $KS(\sigma_{can},g_0,0)$ is given by (see [38])

$$KS_M(\sigma_{can},g_0,0)=\ell+1-4\ell\rho_0(1-\rho_0)$$

$$+4\nu\rho_0-4\sum_{i=1}^{\nu}\boldsymbol{s}(\omega_i,\alpha_i)-8\sum_{i=1}^{\nu}\boldsymbol{s}(\omega_i,\alpha_i;\frac{\gamma_i+\rho_0\omega_i}{\alpha_i},-\rho_0)$$

$$+4\begin{cases}-\sum_{i=1}^n\Big(\Big(\frac{r_i\gamma_i}{\alpha_i}\Big)\Big) & \text{if}\quad \rho_0=0\\[2ex] \frac{2+\kappa}{2}(1-2\rho_0)-\sum_{i=1}^{\nu}\Big\{\frac{r_i\gamma_i+\rho_0}{\alpha_i}\Big\} & \text{if}\quad \rho_0\neq 0,\end{cases}$$

where

$$r_i\omega_i \equiv 1 \ (\mathrm{mod}\ \alpha_i), \ \ i = 1, \dots, \nu.$$

Above, $\boldsymbol{s}(h,k;x,y)$ is the Dedekind–Rademacher sum defined in Appendix B, where we list some of its basic properties as well.

The following result is a consequence of the analysis carried out in [37, 38].

3.4 Proposition (a) *If $\rho_0 \neq 0$ and $\mathfrak{M}_0^*$ has only zero dimensional components then $(g_0, 0)$ is a good parameter and*

$$\mathbf{sw}_M^0(\sigma_{can}) = \frac{1}{8}KS_M(\sigma_{can}, g_0, 0) + |S_0^+| + |S_0^-|.$$

(b) *If g_0 has positive scalar curvature then $(g_0, 0)$ is a good parameter, $S_0^+ = S_0^- = \emptyset$ and*

$$\mathbf{sw}_M^0(\sigma_{can}) = \frac{1}{8}KS(\sigma_{can}, g_0, 0).$$

Notice that part (b) can be applied for the links of quotient singularities.

One of the main obstructions is, that in many cases, the above theorem cannot be applied (ie, the natural parameter provided by the natural Seifert metric is not "good", cf 3.1).

Fortunately, the Seiberg–Witten invariant has an alternate combinatorial description as well. To formulate it we need to review a few basic topological facts.

3.5 The Reidemeister–Turaev torsion According to Turaev [48] a choice of a *spin^c* structure on M is equivalent to a choice of an Euler structure. For every *spin^c* structure σ on M, we denote by

$$\mathcal{T}_{M,\sigma} = \sum_{h \in H} \mathcal{T}_{M,\sigma}(h)\, h \in \mathbb{Q}[H],$$

the sign refined *Reidemeister–Turaev torsion* determined by the Euler structure associated to σ. (For its detailed description, see [48].) Again, it is convenient to think of $\mathcal{T}_{M,\sigma}$ as a function $H \to \mathbb{Q}$ given by $h \mapsto \mathcal{T}_{M,\sigma}(h)$. The Poincaré duality implies that $\mathcal{T}_{M,\sigma}$ satisfies the symmetry condition

$$\mathcal{T}_{M,\sigma}(h) = \mathcal{T}_{M,\bar\sigma}(h^{-1}) \quad \text{for all } h \in H. \tag{$*$}$$

Recall that the *augmentation map* $\mathfrak{aug}\colon \mathbb{Q}[H] \to \mathbb{Q}$ is defined by

$$\sum a_h\, h \mapsto \sum a_h.$$

It is known that $\mathfrak{aug}(\mathcal{T}_{M,\sigma}) = 0$.

3.6 The Casson–Walker invariant and the modified Reidemeister–Turaev torsion Denote by $\lambda(M)$ the Casson–Walker invariant of M normalized as in Lescop's book (cf [23, Section 4.7]), and denote by $H \ni h \mapsto \mathcal{T}^0_{M,\sigma}(h)$ the *modified Reidemeister–Turaev torsion*

$$\mathcal{T}^0_{M,\sigma}(h) := \mathcal{T}_{M,\sigma}(h) - \lambda(M)/|H|.$$

We have the following result.

3.7 Theorem [41] *$\boldsymbol{SW}^0_{M,\sigma}(h) = \mathcal{T}^0_{M,\sigma}(h)$ for all $\sigma \in Spin^c(M)$ and $h \in H$.*

3.8 The Fourier transform Later we will need a dual description of these invariants in terms of Fourier transform. Denote by $\hat{H}$ the Pontryagin dual of H, namely $\hat{H} := \mathrm{Hom}(H, U(1))$. The Fourier transform of any function $f\colon H \to \mathbb{C}$ is the function

$$\hat{f}\colon \hat{H} \to \mathbb{C}, \quad \hat{f}(\chi) = \sum_{h\in H} f(h)\bar{\chi}(h).$$

The function f can be recovered from its Fourier transform via the *Fourier inversion formula*

$$f(h) = \frac{1}{|H|}\sum_{\chi\in\hat{H}} \hat{f}(\chi)\chi(h).$$

Notice that $\mathfrak{aug}(f) = \hat{f}(1)$, in particular $\hat{\mathcal{T}}_{M,\sigma}(1) = \mathfrak{aug}(\mathcal{T}_{M,\sigma}) = 0$. By the above identity,

$$\begin{aligned}\mathbf{sw}^0_M(\sigma) = \boldsymbol{SW}^0_{M,\sigma}(1) &= -\frac{1}{|H|}\lambda(M) + \frac{1}{|H|}\sum_{\chi\in\hat{H}} \hat{\mathcal{T}}_{M,\sigma}(\chi)\\ &= -\frac{1}{|H|}\lambda(M) + \mathcal{T}_{M,\sigma}(1). \qquad (1)\end{aligned}$$

The symmetry condition 3.5$(*)$ transforms into

$$\hat{\mathcal{T}}_{M,\sigma}(\chi) = \hat{\mathcal{T}}_{M,\bar{\sigma}}(\bar{\chi}). \qquad (2)$$

It is convenient to use the notation $\sum'_\chi$ for a summation where χ runs over all the *non-trivial* characters of $\hat{H}$.

3.9 The identification $Spin^c(M) \to Q^c(M)$ via the Seiberg–Witten invariant Sometimes it is important to have an efficient way to recover the *spin*^{c} structure σ (or, equivalently, the quadratic function $\mathfrak{q}^c(\sigma)$, cf 2.6) from the Seiberg–Witten invariant $\boldsymbol{SW}^0_{M,\sigma}$, or from $\mathcal{T}_{M,\sigma}$. In order to do this, we first recall that Turaev in [48, Theorem 4.3.1] proves the following identity for *any* σ and $g, h \in H$.

$$\mathcal{T}_{M,\sigma}(1) - \mathcal{T}_{M,\sigma}(h) - \mathcal{T}_{M,\sigma}(g) + \mathcal{T}_{M,\sigma}(gh) = -b_M(g,h) \pmod{\mathbb{Z}}.$$

Clearly there is a similar identity for $\mathcal{T}^0_{M,\sigma}$ instead of $\mathcal{T}_{M,\sigma}$. By Fourier inversion, this reads

$$\frac{1}{|H|}\sum_{\chi}{}' \hat{\mathcal{T}}_{M,\sigma}(\chi)(\chi(h)-1)(\chi(g)-1) = -b_M(g,h) \pmod{\mathbb{Z}}. \tag{1}$$

This identity has a "refinement" in the following sense (see [40, 3.3]): for any *spin*^{c} structure σ, the map $H \ni h \mapsto q'_\sigma(h)$, defined by,

$$\begin{aligned} q'_\sigma(h) &:= \mathcal{T}_{M,\sigma}(1) - \mathcal{T}_{M,\sigma}(h) = \mathcal{T}^0_{M,\sigma}(1) - \mathcal{T}^0_{M,\sigma}(h) \\ &= \boldsymbol{SW}^0_{M,\sigma}(1) - \boldsymbol{SW}^0_{M,\sigma}(h) \pmod{\mathbb{Z}}, \end{aligned}$$

is a quadratic function associated with b_M. Moreover, the correspondence $\mathfrak{q}^c_{sw} : Spin^c(M) \to Q^c(M)$ given by $\sigma \mapsto q'_\sigma$ is a bijection.

3.10 Proposition *$\mathfrak{q}^c_{sw} = \mathfrak{q}^c$, ie, the above bijection is exactly the canonical identification $\mathfrak{q}^c$ considered in 2.6.*

Proof Let us denote $\mathfrak{q}^c(\sigma)$ by q_σ. Since both maps $\mathfrak{q}^c$ and $\mathfrak{q}^c_{sw}$ are H–equivariant, it suffices to show that $q_\epsilon = q'_\epsilon$ for some *spin* structure ϵ on M. Fix $\epsilon \in Spin(M)$, and as in 2.4, pick a simple connected oriented 4–manifold N with $\partial N = M$ together with a $\tilde\epsilon \in Spin(N)$ such that the restriction of $\tilde\epsilon$ to M is ϵ. In particular, $q_\epsilon = -q_{D(L),0}$, cf 2.4. For every $h \in H$ set $\sigma_h := h \cdot \sigma(\epsilon)$. Pick $\tilde h \in H^2(N,\mathbb{Z}) = L'$ such that $[\tilde h] = h$. For $h = 0$ we choose $\tilde h = 0$. Set $\tilde\sigma_h := \tilde h \cdot \sigma(\tilde\epsilon) \in Spin^c(N)$. Then $c_1(\det \tilde\sigma_h) = 2\tilde h$. Observe that

$$\boldsymbol{SW}^0_{M,\epsilon}(h) = \frac{1}{8}KS(\sigma_h) \pmod{\mathbb{Z}}.$$

But the Atiyah–Patodi–Singer index theorem implies (see eg [24, page 197]):

$$\frac{1}{8}KS(\sigma_h) = \frac{1}{8}(c_1(\det\tilde\sigma_h), c_1(\det\tilde\sigma_h))_{\mathbb{Q}} - \frac{1}{8}\text{signature}(N) \pmod{\mathbb{Z}},$$

for any $h \in H$. Thus $q'_\epsilon(h) = -\frac{1}{2}(\tilde h, \tilde h)_{\mathbb{Q}} = q_\epsilon(h) \pmod{\mathbb{Z}}$. □

Via the Fourier transform, the above identity is equivalent to the following one, valid for any $h \in H$:

$$\frac{1}{|H|}{\sum_{\chi}}' \hat{\mathfrak{T}}_{M,\sigma}(\chi)(\chi(h)-1) = -\mathfrak{q}^c(\sigma)(h) \pmod{\mathbb{Z}}. \tag{2}$$

3.11 Remark The above discussion can be compared with the following identity. Let us keep the notations of 2.3. Let $\sigma(L)$ denote the signature of L, and $k \in L'$ a characteristic element. Then the $\pmod 8$–residue class of $\sigma(L) - (k,k)_{\mathbb{Q}} \in \mathbb{Q}/8\mathbb{Z}$ depends only on the quadratic function $q = q_{D(L),k}$. In fact one has the following formula of van der Blij [51] for the Gauss sum:

$$\gamma(q) := |H|^{-1/2} \sum_{x \in H} e^{2\pi i q(x)} = e^{\frac{\pi i}{4}(\sigma(L) - (k,k)_{\mathbb{Q}})}.$$

If $k \in \mathrm{Im}(i_L)$ then $\sigma(L) - (k,k)_{\mathbb{Q}} = \sigma(L) - (k,k) \in \mathbb{Z}/8\mathbb{Z}$.

4 Analytic invariants and the main conjecture

4.1 Definitions Let $(X,0)$ be a normal surface singularity. Consider the holomorphic line bundle $\Omega^2_{X\setminus\{0\}}$ of holomorphic 2–forms on $X \setminus \{0\}$. If this line bundle is holomorphically trivial then we say that $(X,0)$ is *Gorenstein*. If some power of this line bundle is holomorphically trivial then we say that $(X,0)$ is *$\mathbb{Q}$–Gorenstein*. If $\Omega^2_{X\setminus\{0\}}$ is *topologically* trivial we say that $(X,0)$ is *numerically Gorenstein*. The first two conditions are analytic, the third depends only on the link M (cf 2.7).

4.2 The geometric genus Fix a resolution $\pi\colon \tilde{X} \to X$ over a sufficiently small Stein representative X of the germ $(X,0)$. Then $p_g := \dim H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is finite and independent of the choice of π. It is called the *geometric genus* of $(X,0)$. If $p_g(X,0) = 0$ then the singularity $(X,0)$ is called *rational*.

4.3 Smoothing invariants Let $(X,0)$ be as above. By a *smoothing* of $(X,0)$ we mean a proper flat analytic germ $f\colon (\mathcal{X},0) \to (\mathbb{C},0)$ with an isomorphism $(f^{-1}(0),0) \to (X,0)$. Moreover, we assume that 0 is an isolated singular point of the germ $(\mathcal{X},0)$.

If $\mathcal{X}$ is a sufficiently small contractible Stein representative of $(\mathcal{X},0)$, then for sufficiently small η $(0 < |\eta| \ll 1)$ the fiber $F := f^{-1}(\eta) \cap \mathcal{X}$ is smooth, and its diffeomorphism type is independent of the choices. It is a connected oriented real 4–manifold with boundary ∂F which can be identified with the link M of $(X, 0)$.

We will use the following notations: $\mu(F) = \operatorname{rank} H_2(F, \mathbb{Z})$ (called the *Milnor number*); $(\ ,\ )_F=$ the intersection form of F on $H_2(F, \mathbb{Z})$; (μ_0, μ_+, μ_-) the Sylvester invariant of $(\ ,\ )_F$; $\sigma(F) := \mu_+ - \mu_-$ the signature of F. Notice that the Milnor fiber F, hence its invariants too, in general depend on the choice of the (irreducible component) of the smoothing.

If $M = \partial F$ is a rational homology sphere then $\mu_0 = 0$, hence $\mu(F) = \mu_+ + \mu_-$. It is known that for a smoothing of a Gorenstein singularity $\operatorname{rank} H_1(F, \mathbb{Z}) = 0$ [13]. Therefore, in this case $\mu(F) + 1$ is the topological Euler characteristic $\chi_{top}(F)$ of F.

The following relations connect the invariants p_g, $\mu(F)$ and $\sigma(F)$. The next statement is formulated for rational homology sphere links, for the general statements the reader can consult the original sources [9, 22, 47] (cf also with [25]).

4.4 Theorem *Assume that the link M is a rational homology sphere. Then the following identities hold.*

(1) [Wahl, Durfee, Steenbrink] $4p_g = \mu(F) + \sigma(F)$.

In addition, if $(X, 0)$ is Gorenstein, then

(2) [Laufer, Steenbrink] $\mu(F) = 12p_g + K^2 + \#\mathcal{V}$, *where $K^2 + \#\mathcal{V}$ is the topological invariant of M introduced in 2.12.*

In particular, for Gorenstein singularities, (1) and (2) give $\sigma(F) + 8p_g + K^2 + \#\mathcal{V} = 0$.

This shows that modulo the link-invariant $K^2 + \#\mathcal{V}$ there are two (independent) relations connecting p_g, $\mu(F)$ and $\sigma(F)$, provided that $(X, 0)$ is Gorenstein. So, if by some other argument one can recover one of them from the topology of M, then all of them can be determined from M.

In general, these invariants cannot be computed from M. Here one has to emphasize two facts. First, if M is not a rational homology sphere, then one can construct easily (even hypersurface) singularities with the same link but different (μ, σ, p_g). On the other hand, even if we restrict ourselves to rational homology links, if we consider all the possible analytic structures of $(X, 0)$, then again p_g can vary. For example, in the case of "weakly" elliptic singularities, there is a topological upper bound of p_g (namely, the length of the elliptic sequence, found by Laufer and S S-T Yau) which equals p_g for Gorenstein singularities; but p_g drops to 1 for a generic analytic structure (fact proved by Laufer). For more details and examples, see the series of articles of S S-T Yau (eg [53]), or [28]. On the other hand, the first author in [28] conjectured that for Gorenstein singularities with rational homology sphere links the invariants (μ, σ, p_g) can be determined from the topology of $(X, 0)$ (ie, from the link M); (cf also with the list of conjectures in [36]). The conjecture is true for rational singularities [3, 4], minimally elliptic singularities [21], "weakly" elliptic singularities [28], and some special hypersurface singularities [10, 35], and special complete intersections [35]; in all cases with explicit formulae for p_g. But in general, even a conjectural topological candidate (computed from M) for p_g was completely open. The next conjecture provides exactly this topological candidate (which is also a "good" topological upper bound, cf introduction).

4.5 The Main Conjecture *Assume that $(X, 0)$ is a normal surface singularity whose link M is a rational homology sphere. Let σ_{can} be the canonical spin^c structure on M. Then, conjecturally, the following facts hold:*

(1) *For any $(X, 0)$, there is a topological upper bound for p_g given by:*

$$\mathbf{sw}^0_M(\sigma_{can}) - \frac{K^2 + \#\mathcal{V}}{8} \geq p_g.$$

(2) *If $(X, 0)$ is $\mathbb{Q}$–Gorenstein, then in (1) one has equality.*

(3) *In particular, if $(\mathcal{X}, 0)$ is a smoothing of a Gorenstein singularity $(X, 0)$ with Milnor fiber F, then*

$$-\mathbf{sw}^0_M(\epsilon_{can}) = \frac{\sigma(F)}{8}.$$

If $(X, 0)$ is numerically Gorenstein and M is a $\mathbb{Z}_2$–homology sphere then $\epsilon_{can} = \sigma_{can}$ is the unique spin structure of M; if M is an integral homology sphere then in the above formulae $-\mathbf{sw}^0_M(\epsilon_{can}) = \lambda(M)$, the Casson invariant of M.

4.6 Remarks

(1) Assume that $(X, 0)$ is a hypersurface Brieskorn singularity whose link is an *integral* homology sphere. Then $\lambda(M) = \sigma(F)/8$ by a result of Fintushel and

Stern [10]. This fact was generalized for Brieskorn–Hamm complete intersections and for suspension hypersurface singularities ($(X,0)=\{g(x,y)+z^n=0\}$) with $H_1(M,\mathbb{Z})=0$ by Neumann and Wahl [35]. In fact, for a normal complete intersection surface singularity with $H=H_1(M,\mathbb{Z})=0$, Neumann and Wahl conjectured $\lambda(M)=\sigma(F)/8$. This conjecture was one of the starting points of our investigation.

The result of Neumann–Wahl [35] was re-proved and reinterpreted by Collin and Saveliev (see [7] and [8]) using equivariant Casson invariant and cyclic covering techniques.

(2) The family of $\mathbb{Q}$–Gorenstein singularities is rather large: it contains eg the rational singularities [3, 4], the singularities with good $\mathbb{C}^*$–actions and with rational homology sphere links [32], the minimally elliptic singularities [21], and all the isolated complete intersection singularities. Neumann–Wahl have conjectured in [36] that all the singularities in 4.5 (2) are finite abelian quotients of complete intersection singularities.

(3) If one wants to test the Conjecture for rational or elliptic singularities (or in any example where p_g is known), one should compute the corresponding Seiberg–Witten invariant. But, in some cases, even if all the terms in the main conjecture can (in principle) be computed, the identification of these contributions in the main formula can create difficulties (eg involving complicated identities of Dedekind sums and lattice point counts).

4.7 Remark Notice, that in the above Conjecture, we have automatically built in the following statements as well.

(1) For any normal singularity $(X,0)$ the topological invariant

$$\mathbf{sw}^0_M(\sigma_{can})-\frac{K^2+\#\mathcal{V}}{8}$$

is non-negative. Moreover, this topological invariant is zero if and only if $(X,0)$ is rational. This provides a new topological characterization of the rational singularities.

(2) Assume that $(X,0)$ (equivalently, the link) is numerically Gorenstein. Then the above topological invariant is 1 if and only if $(X,0)$ is minimally elliptic (in the sense of Laufer). Again, this is a new topological characterization of minimally elliptic singularities.

4.8 Remark The invariant $K^2 + \#\mathcal{V}$ appears not only in the type of results listed in 4.4, but also in other topological contexts. For example, it can be identified with the Gompf invariant $\theta(\xi)$ defined in [14, 4.2] (see also [15, 11.3.3]). This appears as an "index defect" (similarly to the signature defect of Hirzebruch) (cf also with [9] and [25]).

More precisely, the almost complex structure on $TM \oplus \mathbb{R}_M$ (cf 2.8) determines a *contact structure* ξ_{can} on M (see eg [15, page 420]), with $c_1(\xi_{can})$ torsion element. Then the Gompf invariant $\theta(\xi_{can})$, computed via $\tilde{X}$, is $K^2 - 2\chi_{top}(\tilde{X}) - 3\sigma(\tilde{X}) = K^2 + \#\mathcal{V} - 2$.

In fact, in our situation, by 2.12, $\theta(\xi_{can})$ can be recovered from the oriented C^∞ type of M completely. In the Gorenstein case, in the presence of a smoothing, $\theta(\xi_{can})$ computed from the Milnor fiber F, equals $-2 - 2\mu(F) - 3\sigma(F)$. The identity $K^2 + \#\mathcal{V} + 2\mu(F) + 3\sigma(F) = 0$ can be deduced from 4.4 as well.

The goal of the remaining part of the present paper is to describe the needed topological invariants in terms of the plumbing graphs of the link, and finally, to provide a list of examples supporting the Main Conjecture.

5 Invariants computed from the plumbing graph

5.1 Notation The goal of this section is to list some formulae for the invariants $K^2{+}\#\mathcal{V}$, $\lambda(M)$ and $\mathcal{T}_{M,\sigma}$ from the resolution graph of M (or, equivalently, from any negative definite plumbing). The formulae are made explicit for star-shaped graphs in terms of their Seifert invariants. For notations, see 2.13, 2.14 and 2.15.

Let I^{-1} be the inverse of the intersection matrix I. For any $v, w \in \mathcal{V}$, I^{-1}_{vw} denotes the (v,w)–entry of I^{-1}. Since I is negative definite, and the graph is connected, $I^{-1}_{vw} < 0$ for each entry v, w. Since I is described by a tree, these entries have the following interpretation as well. For any two vertex $v, w \in \mathcal{V}$, let p_{vw} be the unique minimal path in the graph connecting v and w, and let $I_{(vw)}$ be the matrix obtained from I by deleting all the lines and columns corresponding to the vertices on the path p_{vw} (ie, $I_{(vw)}$ is the intersection matrix of the complement graph of the path). Then $I^{-1}_{vw} = -|\det(I_{(vw)})/\det(I)|$.

For simplicity we will write E_v, D_v, ... instead of $[E_v]$, $[D_v]$....

5.2 The invariant $K^2 + \#\mathcal{V}$ from the plumbing graph Let $Z_K = \sum_v r_v E_v$. Since $Z_K = (i_L \otimes \mathbb{Q})^{-1}(\sum (e_v + 2)D_v)$ (cf 2.13), one has clearly $r_v = \sum_w (e_w + 2) I^{-1}_{vw}$.

Then a "naive formula" of $K^2 = Z_K^2$ is

$$Z_K^2 = \sum_v r_v Z_K \cdot E_v = \sum_v r_v(e_v + 2) = \sum_{v,w} (e_v + 2)(e_w + 2) I^{-1}_{vw}.$$

However, we prefer a different form for r_v and Z_K^2 which involves only a small part of the entries of I^{-1}. Indeed, let us consider the class

$$D = \sum_v i_L(E_v) + \sum_v (2 - \delta_v) D_v \in L'.$$

Then clearly $D \cdot E_v = e_v + \delta_v + 2 - \delta_v = e_v + 2$, hence $D = (i_L \otimes \mathbb{Q}) Z_K$. Therefore,

$$Z_K = \sum_v E_v + \sum_v (2 - \delta_v)(i_L \otimes \mathbb{Q})^{-1}(D_v),$$

hence

$$r_v = 1 + \sum_w (2 - \delta_w) I^{-1}_{vw}.$$

Moreover,

$$Z_K^2 = Z_K \cdot D = \sum_v Z_K \cdot E_v + \sum_v (2 - \delta_v) Z_K \cdot D_v = \sum_v (e_v + 2) + \sum_v (2 - \delta_v) r_v,$$

hence by the second formula for r_v we deduce

$$K^2 + \#\mathcal{V} = \sum_v e_v + 3\#\mathcal{V} + 2 + \sum_{v,w} (2 - \delta_v)(2 - \delta_w) I^{-1}_{vw}.$$

In particular, this number depends only of those entries of I^{-1} whose index set runs over the rupture points ($\delta_v \geq 3$) and the end-vertices ($\delta_v = 1$) of the graph.

For cyclic quotient singularities, the above formula for K^2 goes back to the work of Hirzebruch. In fact, the right hand side can also be expressed in terms of Dedekind sums, see eg [25, 5.7] and [18] (or 7.1 here).

5.3 The Casson–Walker invariant from plumbing We recall a formula for the Casson–Walker invariant for plumbing 3–manifolds proved by A Ratiu in his dissertation [45]. In fact, the formula can also be recovered from the surgery formulae of Lescop [23] (since any plumbing graph can be transformed into a precise surgery data, see eg A1). The first author thanks Christine Lescop for providing him all the details and information about it. We have

$$-\frac{24}{|H|}\,\lambda(M)=\sum_v e_v+3\#\mathcal{V}+\sum_v(2-\delta_v)\,I^{-1}_{vv}.$$

If $M=L(p,q)$, then this can be transformed into $\lambda(L(p,q))=p\cdot \boldsymbol{s}(q,p)/2$. (Here we emphasize that by our notations, $L(p,q)$ is obtained by $-p/q$–surgery on the unknot in S^3, as in [15, page 158], and *not* by p/q–surgery as in [52, page 108].)

5.4 The Casson–Walker invariant for Seifert manifolds Assume that M is a Seifert manifold as in 2.14 and 2.15. Using [23], Proposition 6.1.1, one has the following expression:

$$-\frac{24}{|H|}\lambda(M)=\frac{1}{e}\Big(2-\nu+\sum_{i=1}^{\nu}\frac{1}{\alpha_i^2}\Big)+e+3+12\sum_{i=1}^{\nu}\boldsymbol{s}(\beta_i,\alpha_i).$$

(Warning: our notations for the Seifert invariants differ slightly from those used in [23]; and also, our e and b have opposite signs.)

5.5 $K^2+\#\mathcal{V}$ for Seifert manifolds Using 5.3 we deduce

$$-\frac{24}{|H|}\lambda(M)=\sum_v e_v+3\#\mathcal{V}+\sum_{i=1}^{\nu}I^{-1}_{\bar v_i\bar v_i}+(2-\nu)I^{-1}_{\bar v_0\bar v_0}.$$

For Seifert manifolds, 5.2 can be rewritten as

$$K^2+\#\mathcal{V}=$$
$$2+\sum_v e_v+3\#\mathcal{V}+\sum_{i=1}^{\nu}I^{-1}_{\bar v_i\bar v_i}+(2-\nu)^2I^{-1}_{\bar v_0\bar v_0}+2\sum_{i=1}^{\nu}(2-\nu)I^{-1}_{\bar v_0\bar v_i}+\sum_{i\neq j}I^{-1}_{\bar v_i\bar v_j}.$$

Using the interpretation of the entries of I^{-1} given in 5.1, one gets easily that

$$I^{-1}_{\bar v_0\bar v_0}=\frac{1}{e};\quad I^{-1}_{\bar v_i\bar v_0}=\frac{1}{e\alpha_i};\quad I^{-1}_{\bar v_i\bar v_j}=\frac{1}{e\alpha_i\alpha_j}\ (i\neq j;\ 1\leq i,j\leq\nu). \tag{1}$$

Therefore, these identities and 5.4 give:

$$K^2+\#\mathcal{V}=\frac{1}{e}\Big(2-\nu+\sum_{i=1}^{\nu}\frac{1}{\alpha_i}\Big)^2+e+5+12\sum_{i=1}^{\nu}\boldsymbol{s}(\beta_i,\alpha_i).$$

It is instructive to compare this expression with 5.4 and also with the coefficient r_0 of Z_K, namely with $r_0=1+(2-\nu+\sum_i 1/\alpha_i)/e$.

5.6 The Reidemeister–Turaev torsion In the remaining part of this section we provide a formula for the torsion $\mathcal{T}_M$ of M using the plumbing representation of M. We decided not to distract the reader's attention from the main message of the paper and we deferred its proof to Appendix A.

5.7 Theorem *Let M be an oriented rational homology 3–manifold represented by a negative definite plumbing graph Γ. (Eg, let M be the link of a normal surface singularity $(X,0)$, and $\Gamma = \Gamma(\pi)$ be one of its resolution graphs.) In the sequel, we keep the notations used above (cf 2.13 and 5.1). For any spin^c structure $\sigma \in Spin^c(M)$, consider the unique element $h_\sigma \in H$ such that $h_\sigma \cdot \sigma_{can} = \sigma$. Then for any $\chi \in \hat{H}$, $\chi \neq 1$, the following identity holds:*

$$\hat{\mathcal{T}}_{M,\sigma}(\bar{\chi}) = \bar{\chi}(h_\sigma) \cdot \prod_{v \in \mathcal{V}} \Big(\chi(g_v) - 1\Big)^{\delta_v - 2}.$$

The right hand side should be understood as follows. If $\chi(g_v) \neq 1$ for all v (with $\delta_v \neq 2$), then the expression is well-defined. Otherwise, the right hand side is computed via a limit (regularization procedure). More precisely, fix a vertex $v_* \in \mathcal{V}$ so that $\chi(g_{v_*}) \neq 1$, and let $\vec{b}^*$ be the column vector with entries =1 on the place v_* and zero otherwise. Then find $\vec{w}^* \in \mathbb{Z}^n$ with entries $\{w_v^*\}_v$ in such a way that $I\vec{w}^* = -m^* \cdot \vec{b}^*$ for some integer $m^* > 0$. Then

$$\hat{\mathcal{T}}_{M,\sigma}(\bar{\chi}) = \bar{\chi}(h_\sigma) \cdot \lim_{t \to 1} \prod_{v \in \mathcal{V}} \Big(t^{w_v^*}\chi(g_v) - 1\Big)^{\delta_v - 2}.$$

The above limit always exists. Moreover, once v_* is fixed, the vector $\vec{w}^*$ is unique modulo a positive multiplicative factor (which does nor alter the limit). In fact, the above limit is independent even of the choice of v_* (as long as $\chi(g_{v_*}) \neq 1$). This follows also from the general theory (cf also A.3 and A.4), but it also has an elementary combinatorial proof given in Lemma A.7. (This can be read independently from the other parts of the proof.) In fact, by Lemma A.7, the set of vertices v_*, providing a suitable $\vec{w}^*$ in the limit expression, is even larger than the set identified in the theorem: one can take any v_* which satisfy *either* $\chi(g_{v_*}) \neq 1$ *or* it has an adjacent vertex u with $\chi(g_u) \neq 1$.

5.8 The torsion of Seifert manifolds In this paragraph we use the notations of 2.14 and 2.15. Recall that we introduced $\nu + 1$ distinguished vertices $\bar{v}_i$, $0 \leq i \leq \nu$, whose degree is $\neq 2$ (and $g_{\bar{v}_0}$ is the central vertex). Fix χ and first assume that $\chi(g_{\bar{v}_i}) \neq 1$ for all $0 \leq i \leq \nu$. Then, the above theorem reads as

$$\hat{\mathfrak{T}}_{M,\sigma_{can}}(\bar{\chi}) = \frac{\big(\chi(g_{\bar{v}_0})-1\big)^{\nu-2}}{\prod_{i=1}^{\nu}\big(\chi(g_{\bar{v}_i})-1\big)}.$$

If there is one index $1 \leq i \leq \nu$ with $\chi(g_{\bar{v}_i}) = 1$, then necessarily $\chi(g_{\bar{v}_0}) = 1$ as well. If $\chi(g_{\bar{v}_0}) = 1$, then either $\mathfrak{T}_{M,\sigma_{can}}(\bar{\chi}) = 0$, or for exactly $\nu - 2$ indices i ($1 \leq i \leq \nu$) one has $\chi(g_{\bar{v}_i}) = 1$. In this later case the limit is non-zero. Let us analyze this case more closely. Assume that $\chi(g_{\bar{v}_i}) \neq 1$ for $i = 1, 2$. Using the last statement of the previous subsection, it is not difficult to verify that $\vec{w}^*$ computed from any vertex on these two arms provide the same limit. In fact, by the same argument (cf A.7), one gets that even the central vertex $\bar{v}_0$ provides a suitable set of weight $\vec{w}^*$ (for any χ). The relevant weights can be computed via 5.5(1), and with the notation $\alpha := \mathrm{lcm}(\alpha_1, \dots, \alpha_\nu)$ one has:

$$\hat{\mathfrak{T}}_{M,\sigma_{can}}(\bar{\chi}) = \lim_{t\to 1} \frac{\big(t^{\alpha}\chi(g_{\bar{v}_0})-1\big)^{\nu-2}}{\prod_{i=1}^{\nu}\big(t^{\alpha/\alpha_i}\chi(g_{\bar{v}_i})-1\big)} \quad \text{for any } \ \chi \in \hat{H} \setminus \{1\}.$$

Notice the mysterious similarity of this expression with the Poincaré series of the graded affine ring associated with the universal abelian cover of $(X, 0)$, provided that $(X, 0)$ admits a good $\mathbb{C}^*$–action, cf [32].

6 Brieskorn–Hamm rational homology spheres

6.1 Notation Fix $n \geq 3$ positive integers $a_i \geq 2$ ($i = 1, \dots, n$). For any set of complex numbers $C = \{c_{j,i}\}, i = 1, \dots, n;\ j = 1, \dots, n-2$, one can consider the affine variety

$$X_C(a_1, \dots, a_n) := \{z \in \mathbb{C}^n \ : \ c_{j,1}z_1^{a_1} + \cdots + c_{j,n}z_n^{a_n} = 0 \text{ for all } 1 \leq j \leq n-2\}.$$

It is well-known (see [16]) that for generic C, the intersection of $X_C(a_1, \dots, a_n)$ with the unit sphere $S^{2n-1} \subset \mathbb{C}^n$ is an oriented smooth 3–manifold whose diffeomorphism type is independent of the choice of the coefficients C. It is denoted by $M = \Sigma(a_1, \dots, a_n)$.

6.2 In fact (cf [19, 34]), M is an oriented Seifert 3–manifold with Seifert invariants

$$\Big(g; \underbrace{(\alpha_1, \beta_1), \cdots, (\alpha_1, \beta_1)}_{s_1}; \cdots; \underbrace{(\alpha_n, \beta_n), \cdots, (\alpha_n, \beta_n)}_{s_n}\Big)$$

where g denotes the genus of the base of the Seifert fibration, and the pairs of coprime positive integers (α_i, β_i) (each considered s_i times) are the orbit invariants (cf 2.14 and 2.15 for notations). Recall that the rational degree of this Seifert fibration is

$$e = -\sum_{i=1}^{n} s_i \cdot \frac{\beta_i}{\alpha_i} < 0. \tag{e}$$

Set

$$a := \operatorname{lcm}(a_i;\ \ 1 \le i \le n),\ \ q_i := \frac{a}{a_i},\ \ 1 \le i \le n;\ \ \ A := \prod_{i=1}^{n} a_i.$$

The Seifert invariants are as follows (see [19, Section 7] or [34]):

$$\alpha_i := \frac{a}{\operatorname{lcm}(a_j;\ \ j \ne i)},\qquad s_i := \frac{\prod_{j\ne i} a_j}{\operatorname{lcm}(a_j;\ \ j \ne i)} = \frac{A\alpha_i}{a a_i},\qquad (1 \le i \le n);$$

$$g := \frac{1}{2}\Big(2 + (n-2)\frac{A}{a} - \sum_i s_i\Big). \tag{g}$$

By these notations $-e = A/(a^2)$. Notice that the integers $\{\alpha_j\}_{j=1}^{n}$ are pairwise coprime. Therefore the integers β_j are determined from (e). In fact $\sum_j\ q_j\beta_j = 1$, hence $\beta_i q_j \equiv 1 \pmod{\alpha_j}$ for any j. Similarly as above, we set $\alpha := \operatorname{lcm}(\alpha_1, \dots, \alpha_n)$.

It is clear that M is a $\mathbb{Q}$–homology sphere if and only if $g = 0$. In order to be able to compute the Reidemeister–Turaev torsion, we need a good characterization of $g = 0$ in terms of the integers $\{a_i\}_i$. For hypersurface singularities this is given in [5]. This characterization was partially extended for complete intersections in [16]. The next proposition provides a complete characterization (for the case when the link is 3–dimensional).

6.3 Proposition *Assume that $X_C(a_1, \dots, a_n)$ is a Brieskorn–Hamm isolated complete intersection singularity as in 6.1 such that its link M is a 3–dimensional rational homology sphere. Then $(a_1, \dots, a_n)$ (after a possible permutation) has (exactly) one of the following forms:*

(i) $(a_1, \dots, a_n) = (db_1, db_2, b_3, \dots, b_n)$, *where the integers $\{b_j\}_{j=1}^{n}$ are pairwise coprime, and $\gcd(d, b_j) = 1$ for any $j \ge 3$;*

(ii) $(a_1, \dots, a_n) = (2^c b_1, 2b_2, 2b_3, b_4 \dots, b_n)$, *where the integers $\{b_j\}_{j=1}^{n}$ are odd and pairwise coprime, and $c \ge 1$.*

Proof The proof will be carried out in several steps.

Step 1 Fix any four distinct indices i, j, k, l. Then $d_{ijkl} := \gcd(a_i, a_j, a_k, a_l) = 1$. Indeed, if a prime p divides d_{ijkl}, then $p^2|(A/a)$ and $p^2|s_i$ for all i. Hence by 6.2(g) one has $p^2|2$.

Step 2 Fix any three distinct indices i, j, k and set $d_{ijk} := \gcd(a_i, a_j, a_k)$. If a prime p divides d_{ijk} then $p = 2$. Indeed, if $p|d_{ijk}$ then $p|(A/a)$ and $p|s_j$ for all j, hence $p|2$ by 6.2(g).

Step 3 There is at most one triple $i < j < k$ with $d_{ijk} \neq 1$. This follows from steps 1 and 2.

Step 4 Assume that $d_{ijk} = 1$ for all triples i, j, k. For any $i \neq k$ set $d_{ik} := \gcd(a_i, a_k)$. Then $A/a = \prod_{i<k} d_{ik}$, and there are similar identities for each s_j. Then 6.2(g) reads as

$$2 + (n-2) \prod_{i<k} d_{ik} - \sum_{j=1}^{n} \prod_{i<k; i,k \neq j} d_{ik} = 0. \qquad \text{(eq1)}$$

Step 5 Assume that $d_{123} \neq 1$, hence

$$(a_1, \dots, a_n) = (2^c b_1, 2^u b_2, 2^v b_3, b_4 \dots, b_n),$$

with $c \geq u \geq v$, and all b_i odd numbers (after a permutation of the indices). Then $u = v = 1$. For this use similar argument as above with $4|(A/a)$, $4|s_i$ for $i \geq 4$, s_1 and s_2 (resp. s_3) is divisible exactly by the v^{th} (resp. u^{th}) power of 2.

Using this fact, write for each par $i \neq k$, $d_{ik} := \gcd(b_i, b_k)$. We deduce as above that $A/a = 4 \prod_{i<k} d_{ik}$, and there are similar identities for each s_j. Then 6.2(b) transforms into

$$\frac{1}{2} + (n-2) \prod_{i<k} d_{ik} - \sum_{j=1}^{n} \varepsilon_j \prod_{i<k; i,k \neq j} d_{ik} = 0, \qquad \text{(eq2)}$$

where $\varepsilon_j = 1/2$ for $j \leq 3$ and $= 1$ for $j \geq 4$.

Step 6 The equation (eq2) has only one solution with all d_{ik} strict positive integer, namely $d_{ik} = 1$ for all i, k. Similarly, any set of solutions of (eq1) has at most one d_{ik} strict greater than 1, all the others being equal to 1.

This can be proved eg by induction. For example, in the case of (eq2), if one replaces the set of integers d_{ik} in the left hand side of the equation with the same set but in which one of them is increased by one unit, then the new expression is strictly greater than the old one. A similar argument works for (eq1) as well. The details are left to the reader. □

6.4 Verification of the conjecture in the case 6.3(i) We start to list the properties of Brieskorn–Hamm complete intersections of the form (i).

- $\alpha_j = b_j$ for $j = 1, \dots, n$;
- $s_1 = 1$, $s_2 = 1$, and $s_j = d$ for $j \geq 3$; in particular, the number of "arms" is $\nu = 2 + (n-2)d$;
- $\alpha = B := \prod_{j=1}^n b_j$, and $-e \cdot B = 1$, hence by 2.14 the generic orbit h is homologically trivial.
- Using the group representation 2.15, and the fact that h is trivial, one has

$$H = \text{ab}\langle\, g_{ij},\ \ 1 \leq j \leq n,\ \ 1 \leq i \leq s_j \mid g_{ij}^{\alpha_j} = 1 \text{ for all } \ i,j;\ \prod_{i,j} g_{ij}^{\omega_j} = 1\,\rangle.$$

Since the integers α_j are pairwise coprime, taking the α/α_j power of the last relation, and using that $\gcd(\alpha_j, \omega_j) = 1$, one obtains that

$$H = \bigoplus_{j \geq 3} \text{ab}\langle\, g_{ij},\ \ 1 \leq i \leq d \mid g_{ij}^{b_j} = 1 \text{ for all } \ i;\ \prod_i g_{ij} = 1\,\rangle \ \cong \bigoplus_{j\geq 3} (\mathbb{Z}_{b_j})^{d-1}.$$

In particular, $|H| = \prod_{j\geq 3} b_j^{d-1}$.

- **The Reidemeister–Turaev torsion of M** By 5.8

$$\hat{\mathcal{T}}_{M,\sigma_{can}}(\chi) = \lim_{t\to 1} \frac{(t^\alpha - 1)^{d(n-2)}}{(t^{\alpha/\alpha_1} - 1)(t^{\alpha/\alpha_2} - 1)\prod_{j\geq 3}\prod_{i=1}^d (t^{\alpha/\alpha_j}\bar{\chi}(g_{ij}) - 1)}.$$

As explained in 5.8, for a fixed χ, the expression $\hat{\mathcal{T}}_{M,\sigma_{can}}(\chi)$ is nonzero if and only if for exactly two pairs (i,j) (where $j \geq 3$ and $1 \leq i \leq b$) $\chi(g_{ij}) \neq 1$. Analyzing the group structure, one gets easily that these two pairs must have the same j. For a fixed j, there are $d(d-1)/2$ choices for the set of indices $\{i_1, i_2\}$. For fixed j, the set of nontrivial characters of the group

$$\text{ab}\langle\, g_{ij},\ \ 1 \leq i \leq d \mid g_{ij}^{b_j} = 1 \text{ for all } \ i;\ \prod_i g_{ij} = 1\,\rangle,$$

satisfying $\chi(g_{ij}) = 1$ for all $i \neq i_1, i_2$ is clearly $\hat{\mathbb{Z}}_{b_j} \setminus \{1\}$, and in this case $\chi(g_{i_1 j})\chi(g_{i_2 j}) = 1$ as well. Therefore

$$\mathcal{T}_{M,\sigma_{can}}(1) = \frac{1}{|H|} \cdot \sum_{j\geq 3} \frac{\alpha^{d(n-2)}}{\frac{\alpha}{\alpha_1}\frac{\alpha}{\alpha_2}(\frac{\alpha}{\alpha_3})^d \cdots (\frac{\alpha}{\alpha_j})^{d-2} \cdots (\frac{\alpha}{\alpha_n})^d}$$

$$\times \frac{d(d-1)}{2} \cdot {\sum_{\mathbb{Z}_{b_j}}}' \frac{1}{(\zeta - 1)(\bar{\zeta} - 1)}.$$

Recall that $|H| = \prod_{j\geq 3} b_j^{d-1}$. Hence, by (B.9) of Appendix B and an easy computation:

$$\mathcal{T}_{M,\sigma_{can}}(1) = \frac{B\cdot d(d-1)}{24}\sum_{j\geq 3}\big(1-\frac{1}{b_j^2}\big).$$

- **The Casson–Walker invariant** From 5.4 we get

$$-\frac{\lambda(M)}{|H|} = -\frac{B}{24}\Big(-d(n-2)+\sum_{j=1}^{n}\frac{s_j}{b_j^2}\Big)-\frac{1}{24B}+\frac{1}{8}+\frac{1}{2}\sum_{j=1}^{n}s_j\boldsymbol{s}(\beta_j,b_j).$$

- **The signature of the Milnor fiber** Since X_C is an isolated complete intersection singularity, its singular point is Gorenstein. Hence, by 4.4, it is enough to verify only part (3) of the main conjecture, part (2) will follow automatically.

The signature $\sigma(F) = \sigma(a_1,\dots,a_n)$ of the Milnor fiber F of a Brieskorn–Hamm singularity is computed by Hirzebruch [17] in terms of cotangent sums. Nevertheless, we will use the version proved in [35, 1.12]. This, in the case (i), (via B.10) reads as

$$\sigma(F) = -1+\frac{1}{3B}\Big(1-(n-2)d^2B^2+B^2\sum_{j=1}^{n}\frac{s_j^2}{b_j^2}\Big)-4\sum_{j=1}^{n}s_j\boldsymbol{s}(q_j,b_j),$$

where $q_j = s_jB/b_j$ for all $1\leq j\leq n$. Since $\beta_jq_j\equiv 1 \pmod{b_j}$ (cf 6.2), one has $\boldsymbol{s}(q_j,b_j) = \boldsymbol{s}(\beta_j,b_j)$ for all j. Now, by a simple computation one can verify the conjecture.

6.5 Verification of the conjecture in the case 6.3(ii) The discussion is rather similar to the previous case, the only difference (which is not absolutely negligible) is that now h is not trivial. This creates some extra work in the torsion computation. In the sequel we write $B := \prod_j b_j$.

- $a = 2^cB$ and $A = 2^{c+2}B$. Moreover, $\alpha_1 = 2^{c-1}b_1$ and $\alpha_j = b_j$ for $j\geq 2$.
- $s_j = 2$ for $j\leq 3$ and $s_j = 4$ for $j\geq 4$. The number of "arms" is $\nu = 4n-6$.
- $\alpha = 2^{c-1}B$, hence $-e^{-1} = 2^{c-2}B$. Therefore $|\langle h\rangle| = 2$ and $|H| = 2^cB^3/(b_1b_2b_3)^2$.
- The self intersection number b of the central exceptional divisor is even. Indeed, equation (e) implies that $1\equiv\beta_1b_2\cdots b_n \pmod{\alpha_1}$. Since $\omega_1\equiv -\beta_1 \pmod{\alpha_1}$, one has $2^{c-1}|1+\omega_1b_2\cdots b_n$. This, and the first formula of 2.15 implies that b is even.

• Using 2.15, and the fact that $h^{-b} = 1$ is automatically satisfied, we obtain the following presentation for H:

$$\mathrm{ab}\Big\langle g_{ij},\ 1 \le j \le n,\ 1 \le i \le s_j;\ h \Big|\ g_{ij}^{\alpha_j} = h \text{ for all } i,j;\ \prod_{i,j} g_{ij}^{\omega_j} = 1,\ h^2 = 1\Big\rangle.$$

Clearly $\langle h\rangle \approx \mathbb{Z}_2$, and there is an exact sequence $0 \to \langle h\rangle \to H \to Q \to 0$ with

$$Q = \bigoplus_{j\ge1} \mathrm{ab}\langle\ g_{ij},\ 1 \le i \le s_j \mid g_{ij}^{\alpha_j} = 1 \text{ for all } i;\ \prod_i g_{ij} = 1\ \rangle\ \cong \bigoplus_{j\ge1} (\mathbb{Z}_{\alpha_j})^{s_j-1}.$$

• **The Reidemeister–Turaev torsion of** M We have to distinguish two types of characters $\chi \in \hat H \setminus \{1\}$ since $\chi(h)$ is either $+1$ or -1. The sum over characters χ with $\chi(h) = 1$ (ie, over $\hat Q \setminus \{1\}$) can be computed similarly as in case (i), namely it is

$$\frac{1}{|H|}\cdot\sum_{j\ge1} \frac{\alpha^{\nu-2}}{(\frac{\alpha}{\alpha_1})^{s_1}\cdots(\frac{\alpha}{\alpha_{j-1}})^{s_{j-1}}(\frac{\alpha}{\alpha_j})^{s_j-2}(\frac{\alpha}{\alpha_{j+1}})^{s_{j+1}}\cdots(\frac{\alpha}{\alpha_n})^{s_n}}\cdot{\sum_{\mathbb{Z}_{\alpha_j}}}' \frac{s_j(s_j-1)}{2(\zeta-1)(\bar\zeta-1)}$$

$$= \frac{2^{c-1}B}{24}\sum_{j\ge1}\frac{s_j(s_j-1)}{2}\Big(1-\frac{1}{\alpha_j^2}\Big).$$

The sum over characters χ with $\chi(h) = -1$ requires no “limit regularization”, hence it is $(-2)^{\nu-2}/(\prod_j P_j)$, where for any fixed j the expression P_j has the form $\prod_*(\zeta_i - 1)$, where the product runs over $1 \le i \le s_j$, $\zeta_i^{\alpha_j} = -1$ with restriction $\prod_i \zeta_i = 1$.

Using the identity $-2 = \zeta_i^{\alpha_j} - 1$ one gets

$$\frac{1}{P_j} = \frac{1}{(-2)^{s_j}}\prod\nolimits_*(1 + \zeta_i + \cdots + \zeta_i^{\alpha_j-1}).$$

By an elementary argument, this is exactly $(\alpha_j/2)^{s_j}$. Therefore, this second contribution is $2^{c-1}B/8$, hence

$$\mathcal{T}_{M,\sigma_{can}}(1) = \frac{2^{c-1}B}{8} + \frac{2^{c-1}B}{24}\sum_{j\ge1}\frac{s_j(s_j-1)}{2}\Big(1-\frac{1}{\alpha_j^2}\Big).$$

• **The Casson–Walker invariant** From 5.4 and $\nu = 4n-6$ one gets

$$-\frac{\lambda(M)}{|H|} = -\frac{2^{c-2}B}{24}\Big(-4(n-2) + \sum_{j=1}^n \frac{s_j}{\alpha_j^2}\Big) - \frac{1}{3\cdot 2^{c+1}B} + \frac{1}{8} + \frac{1}{2}\sum_{j=1}^n s_j\boldsymbol{s}(\beta_j,\alpha_j).$$

• **The signature of the Milnor fiber** Using the above identities about the Seifert invariants (and $a_j = 4\alpha_j/s_j$ too), [35, 1.12] reads as

$$\sigma(F) = -1 + \frac{1}{3\cdot 2^{c-2}B}\Big(1-(n-2)2^{2c}B^2 + 2^{2c-4}B^2\sum_{j=1}^{n}\frac{s_j^2}{\alpha_j^2}\Big) - 4\sum_{j=1}^{n} s_j\,\mathbf{s}(\beta_j,\alpha_j).$$

Now, the verification of the statement of the conjecture is elementary.

7 Some rational singularities

7.1 Cyclic quotient singularities The link of a cyclic quotient singularity $(X_{p,q},0)$ $(0<q<p,\ (p,q)=1)$ is the lens space $L(p,q)$. $X_{p,q}$ is numerically Gorenstein if and only if $q=p-1$, case which will be considered in 7.2. In all other cases σ_{can} is not spin. In all the cases $p_g=0$. Moreover (see eg [25, 5.9], or [18], or 5.2):

$$K^2 + \#\mathcal{V} = \frac{2(p-1)}{p} - 12\cdot\mathbf{s}(q,p).$$

On the other hand, the Seiberg–Witten invariants of $L(p,q)$ are computed in [39] (where a careful reading will identify $\mathbf{sw}^0_M(\sigma_{can})$ as well). In fact, cf ([39, 3.16]):

$$\hat{\mathcal{T}}_{M,\sigma_{can}}(\chi) = \frac{1}{(\bar\chi-1)(\bar\chi^q-1)},$$

fact which follows also from 5.7. Therefore, using [44, 18a] (or B.8), one gets that

$$\mathcal{T}_{M,\sigma_{can}}(1) = \frac{p-1}{4p} - \mathbf{s}(q,p).$$

The Casson–Walker contribution is $\lambda(L(p,q))/p = \mathbf{s}(q,p)/2$ (cf 5.3). Hence one has equality in part (1) of the Conjecture.

7.2 Particular case: the A_{p-1}–singularities Assume that $(X,0)=(\{x^2+y^2+z^p=0\},0)$. Then by 7.1 (or [39, Section 2.2.A]) one has $\mathbf{sw}^0_M(\epsilon_{can}) = (p-1)/8$ (in [39] this spin structure is denoted by σ_{spin}). On the other hand, $(\,,\,)_F$ is negative definite of rank $p-1$, hence $\sigma(F)=-(p-1)$. (Obviously, the A_{p-1}–case can also be deduced from section 6.)

7.3 The D_n–singularities For each $n \geq 4$, one denotes by D_n the singularity at the origin of the weighted homogeneous complex hypersurface $x^2y + y^{n-1} + z^2 = 0$. It is convenient to write $p := n - 2$. We invite the reader to recall the notations of 3.3 about orbifold invariants.

The normalized Seifert invariants are

$$(b, (\omega_1, \alpha_1), (\omega_2, \alpha_2), (\omega_3, \alpha_3)) = (-2, (p, p-1), (2, 1), (2, 1)).$$

Its rational degree is $\ell = -1/p$. Observe that $\kappa = \ell$.

The link M is the unit circle bundle of the V–line bundle $\mathbb{L}_0$ with rational degree ℓ and singularity data $((p-1)/p, 1/2, 1/2)$. Therefore, $\mathbb{L}_0 = K_\Sigma$. Hence, $\rho(\sigma_{can}) = \{\kappa/2\ell\} = 0$. The canonical representative of σ_{can} is then the trivial line bundle E_0. It has rational degree 0 and singularity data $\vec{\gamma} = (0, 0, 0)$. The Kreck–Stolz invariant is then

$$KS_M(\sigma_{can}, g_0, 0) = 7 - 4\sum_{i=1}^{3} \boldsymbol{s}(\omega_i, \alpha_i) - 8\sum_{i=1}^{3} \boldsymbol{s}(\omega_i, \alpha_i; \frac{\omega_i}{2\alpha_i}, -1/2) - 4\sum_{i=1}^{3}\left\{\frac{1}{2\alpha_i}\right\}.$$

Using the fact that $\boldsymbol{s}(1, 2; 1/4, -1/2) = 0$, this expression equals:

$$6 + \frac{p}{3} + \frac{2}{3p} + 8\boldsymbol{s}(1, p, \frac{1}{2p}, 1/2).$$

Now using the reciprocity formula for the generalized Dedekind sum, one has

$$8\boldsymbol{s}(1, p, \frac{1}{2p}, 1/2) = \frac{4/3 - 2p + 2p^2/3}{p} = \frac{4}{3p} - 2 + \frac{2p}{3}.$$

Thus

$$8\mathbf{sw}^0_M(\sigma_{can}) = KS_M(\sigma_{can}, g_0) = 4 + \frac{p}{3} - \frac{4}{3p} + \frac{4}{3p} - 2 + \frac{2p}{3} = 2 + p.$$

On the other hand, the signature of the Milnor fiber is $-n = -(p+2)$, confirming again the Main Conjecture.

7.4 The E_6 and E_8 singularities Both E_6 (ie, $x^4 + y^3 + z^2 = 0$) and E_8 (ie, $x^5 + y^3 + z^2 = 0$) are Brieskorn (hypersurface) singularities, hence the result of section 6 can be applied. The link of E_8 is an *integral* homology sphere, hence the validity of the conjecture in this case was proved in [10]. The interested reader can verify the conjecture using the machinery of 3.3 and 3.4 as well.

7.5 The E_7 singularity It is given by the complex hypersurface $x^3 + xy^3 + z^2 = 0$. The group H is $\mathbb{Z}_2$. The normalized Seifert invariants are $(-2,(2,1),(3,2),(4,3))$, the rational degree is $-1/12$. We deduce as above that $\mathbb{L}_0 = K_\Sigma$, with $\rho_0 = \rho(\sigma_{can}) = 1/2$. The canonical representative is again the trivial line bundle E_0. Its singularity data are trivial. The Seiberg–Witten invariant of σ_{can} is determined by the Kreck–Stolz invariant alone. A direct computation shows that $KS_M(\sigma_{can}, g_0) = 7$. But the signature of the Milnor fiber is $\sigma(E_7) = -7$ as well, hence the statement of the conjecture is true.

7.6 Another family of rational singularities Consider a singularity $(X,0)$ whose link M is described by the negative definite plumbing given in Figure 1. (It is clear that in this case M it is *not* numerically Gorenstein.)

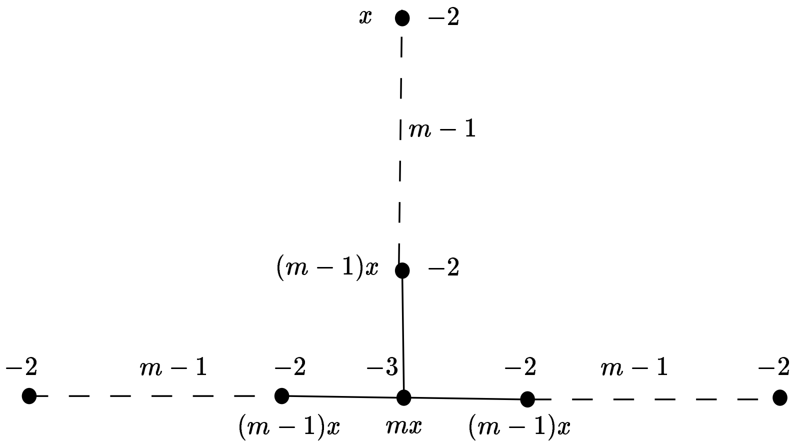

Figure 1: The resolution graph of the rational singularity $(X,0)$

The number of -2 spheres on any branch is $m-1$, where $m \geq 2$. It is easy to verify that the $(X,0)$ is a rational (with Artin cycle $\sum_v E_v$) (see, eg [29]). M is Seifert manifold with normalized Seifert invariants $(-3,(m,m-1),(m,m-1),(m,m-1))$ and rational degree $l = -3/m$.

To compute the Seiberg–Witten invariant of M associated with σ_{can} we use again 3.4.

The canonical V–line bundle of Σ has singularity data $(m-1)/m$ (three times) and rational degree $\kappa = 1 + \ell$. The link M is the unit circle bundle of the V–line bundle $\mathbb{L}_0$ with rational degree ℓ and singularity data $(m-1)/m$ (three times). Therefore

$$\rho_0 = \Big\{ -\frac{m-3}{6} \Big\}.$$

To apply 3.4 we need $\rho_0 \neq 0$, ie,

$$m \not\equiv 3 \pmod 6.$$

The canonical representative of σ_{can} is the V–line bundle E_0 with

$$E_0 = n_0\mathbb{L}_0, \quad n_0 = \Big\lfloor \frac{3-m}{6} \Big\rfloor.$$

The reader is invited to recall the definition of $S_0^{\pm}$. We start with the computation of S_0^+. Notice that

$$-\frac{1}{2}\deg^V K_\Sigma \leq \nu(E) < 0 \Longleftrightarrow 0 \leq n\ell < \frac{1}{2}\deg^V K_\Sigma.$$

Hence

$$\frac{1}{2\ell}\deg^V K_\Sigma < n \leq 0, \quad \text{ie,} \quad -\frac{m-3}{6} < n \leq 0. \tag{1}$$

The singularity data of $n\mathbb{L}_0$ are all equal to $\{-n/m\}$ (three times). We deduce

$$\deg|n\mathbb{L}_0| = \deg^V(n\mathbb{L}_0) - 3\Big\{\frac{-n}{m}\Big\} = 3\Big\lfloor\frac{-n}{m}\Big\rfloor.$$

Now observe that (1) implies

$$0 \leq \frac{-n}{m} < \frac{m-3}{6m}, \text{ hence } \Big\lfloor\frac{-n}{m}\Big\rfloor = 0$$

for every n subject to the condition (1). Since $m \not\equiv 3 \pmod 6$, we deduce

$$|S_0^+| = \Big\lfloor\frac{m-3}{6}\Big\rfloor + 1 = -n_0. \tag{2}$$

Moreover, all the connected components corresponding to the elements in S_0^+ are points. Similarly, the condition $0 < \nu(E) \leq \frac{1}{2}\deg^V K_\Sigma$ implies

$$\frac{1}{2}\deg^V K_\Sigma < n\ell \leq \deg^V K_\Sigma \Longrightarrow \frac{1}{\ell}\deg^V K_\Sigma \leq n < \frac{1}{2\ell}\deg^V K_\Sigma.$$

Hence

$$-\frac{m-3}{3} \leq n < -\frac{m-3}{6}. \tag{3}$$

The singularity data of the V–line bundle $K_\Sigma - n\mathbb{L}_0$ are all equal to $\{(n-1)/m\}$. We deduce

$$\deg|K_\Sigma - n\mathbb{L}_0| = 1 + 3\frac{n-1}{m} - 3\{(n-1)/m\} = 1 + 3\Big\lfloor\frac{n-1}{m}\Big\rfloor.$$

But this number is negative (because of (3)), hence $S_0^- = \emptyset$. These considerations show that Proposition 3.4 is applicable. Set

$$-\frac{m-3}{6} = -k + \rho_0, \quad 0 < \rho_0 < 1, \quad k \text{ non-negative integer.}$$

Then $n_0 = -k$. The canonical representative is $E_0 = -k\mathbb{L}_0$. It has degree $-k\ell$. Its singularity data are all equal to $\gamma_i/\alpha_i = k/m$. Then in the formula of KS_M one has $\omega_i = m-1$, $r_i = -1$, $\gamma_i = k$ for all i. Hence

$$KS_M(\sigma_{can}) = \ell + 1 - 4\ell\rho_0(1-\rho_0) + 12\rho_0 + 2(3+\ell)(1-2\rho_0) - 12\Big\{\frac{\rho_0 - k}{m}\Big\}$$

$$-12\boldsymbol{s}(m-1,m) - 24\boldsymbol{s}(m-1,m;\frac{k+\rho_0(m-1)}{m},-\rho_0).$$

Observe now that

$$-\boldsymbol{s}(m-1,m) = \boldsymbol{s}(1,m) = \frac{m}{12} + \frac{1}{6m} - \frac{1}{4},$$

$$-\boldsymbol{s}(m-1,m;\frac{k+\rho_0(m-1)}{m},-\rho_0)) = -\boldsymbol{s}(-1,m,\frac{k+\rho_0(m-1)}{m} - \rho_0,-\rho_0)$$

$$= \boldsymbol{s}(1,m,\frac{\rho_0-k}{m},-\rho_0).$$

Moreover, from the definition of Dedekind sum we obtain

$$\boldsymbol{s}(1,m,\frac{\rho_0-k}{m},-\rho_0) = \boldsymbol{s}(1,m,\frac{k-\rho_0}{m},0) = \boldsymbol{s}(1,m) + \frac{k(k-1)}{2m} - \frac{k-1}{2}.$$

Finally, by an elementary but tedious computation we get

$$KS_M(\sigma_{can}) = 3m - \frac{m}{3} - 2 - 8k.$$

The Seiberg–Witten invariant of the canonical *spin*^{c} structure is then

$$8\mathbf{sw}^0_M(\sigma_{can}) = KS_M(\sigma_{can}) + 8|S_0^+| = KS_M(\sigma_{can}) + 8k = 3m - \frac{m}{3} - 2.$$

The coefficients of Z_K are labelled on the graph, where the unknown x is determined from the adjunction formula applied to the central -3–sphere; namely $-3mx + 3(m-1)x = -1$, hence $x = 1/3$. Then $Z_K^2 = \sum r_v(e_v+2) = -r_0 = -m/3$. The number of vertices of this graph is $3m-2$ so

$$8p_g + K^2 + \#\mathcal{V} = 3m - 2 - \frac{m}{3}. \tag{4}$$

This confirms once again the Main Conjecture.

7.7 The case $m = 3$ In the previous example we verified the conjecture for all $m \not\equiv 3 \pmod 6$. For the other values the method given by 3.4 is not working. But this fact does not contradict the conjecture. In order to show this, we indicate briefly how one can verify the conjecture in the case $m = 3$ by the torsion computation.

In this case $|H|=27$ and h has order 3. First consider the set of characters χ with $\chi(g_{\bar{v}_0})=1$ (there are 9 altogether). They satisfy $\prod_i \chi(g_{\bar{v}_i})=1$. If $\chi(g_{\bar{v}_i})\neq 1$ for all i (2 cases), or if $\chi=1$ (1 case) then $\hat{\mathcal{T}}(\chi)=0$. If $\chi(g_{\bar{v}_i})=1$ for exactly one index i, then the contribution in $\sum_\chi \hat{\mathcal{T}}(\chi)$ is 2 for each choice of the index, hence altogether 6.

Then, we consider those characters χ for which $\chi(g_{\bar{v}_0})\neq 1$ (18 cases). Then one has to compute the sum

$$\sum \frac{1-\zeta_1^3}{(1-\zeta_1)(1-\zeta_2)(1-\zeta_1^{-1}\zeta_2^{-1})},$$

where the sum runs over $\zeta_1,\zeta_2\in\mathbb{Z}_9$, $\zeta_1^3=\zeta_2^3\neq 1$. A computation shows that this is 9. Therefore, $\mathcal{T}_M(1)/|H|=(6+9)/27=5/9$.

The Casson–Walker invariant can be computed easily from the Seifert invariants, the result is $\lambda(M)/|H|=-7/36$. Therefore, the Seiberg–Witten invariant is 5/9+7/36=3/4. But this number equals $(K^2+\#\mathcal{V})/8$ (cf 7.6(4) for $m=3$ and $p_g=0$).

8 Some minimally elliptic singularities

8.1 "Polygonal" singularities Let $(X,0)$ be a normal surface singularity with resolution graph given by Figure 2.

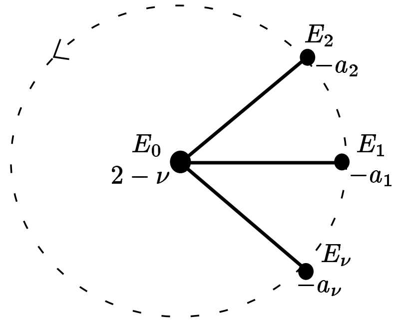

Figure 2: The resolution graph of a "polygonal" singularity

Here we assume that $\nu>2$. The negative definiteness of the intersection form implies that the integers $(a_1,\cdots,a_\nu)\in\mathbb{Z}^\nu_{>1}$ satisfy

$$\ell := 2-\nu+\sum_{i=1}^{\nu}\frac{1}{a_i}<0.$$

An elementary computation shows that $Z_K = 2E_0 + \sum_{i=1}^{\nu} E_i$. If $\nu > 3$ then this cycle is exactly the minimal cycle Z_{min} of Artin. If $\nu = 3$ then the graph is not minimal, but after blowing down the central irreducible exceptional divisor one gets the identity $Z_K = Z_{min}$. In particular, $(X, 0)$ is minimally elliptic (by Laufer's criterion, see [21]). Hence $p_g = 1$. Moreover, by a calculation $K^2 = 8 - \sum_{i=1}^{\nu} a_1$, and thus

$$8p_g + (K^2 + \nu + 1) = 17 + \nu - \sum_{i=1}^{\nu} a_i.$$

Now we will compute the Seiberg–Witten invariant via 3.4. The Seifert manifold M is the unit circle bundle of the V–line bundle $\mathbb{L}_0$ with rational degree ℓ, and singularity data $1/a_i$, $1 \le i \le \nu$.

The canonical V–line bundle K_Σ has singularity data $(a_i - 1)/a_i$, $1 \le i \le \nu$, and rational degree $\kappa := -\ell > 0$. Note that $K_0 = -\mathbb{L}_0 \in \mathrm{Pic}^V_{top}(\Sigma)$. We have $\rho_0 = 1/2$, $n_0 = \lfloor -1/2 \rfloor = -1$. The canonical representative of σ_{can} is the V–line bundle $E_0 = -\mathbb{L}_0 = K_\Sigma$.

Resolving the inequalities for $S_0^{\pm}$, one gets

$$S_0^+ = \{n\mathbb{L}_0 \in \mathbb{Z};\ \ell/2 \le (n + 1/2)\ell < 0\} = \{0 \cdot \mathbb{L}_0\},$$

$$S_0^- = \{n\mathbb{L}_0 \in \mathbb{Z};\ 0 < (n + 1/2)\ell \le -\ell/2\} = \{-1 \cdot \mathbb{L}_0\}.$$

Hence, we have only two components $\mathfrak{M}_0^+$, $\mathfrak{M}_{K_\Sigma}^-$, both of dimension 0. Thus $\mathfrak{M}_0^*$ consists of only two monopoles. Thus $(g_0, 0)$ is a good parameter. The Kreck–Stolz invariant of σ_{can} is

$$KS_M(\sigma_{can}) = 1 - 2\nu - 8\sum_{i=1}^{\nu} \boldsymbol{s}(1, a_i, \frac{a_i - 1/2}{a_i}, -1/2) - 4\sum_{i=1}^{\nu} \boldsymbol{s}(1, a_i) + 2\sum_{i=1}^{3} \frac{1}{a_i}.$$

The last identity can be further simplified using the identities from the Appendix B, namely

$$\boldsymbol{s}(1, a_i, \frac{a_i - 1/2}{a_i}, -1/2) = \boldsymbol{s}(1, a_i) = \frac{a_i}{12} + \frac{1}{6a_i} - \frac{1}{4}.$$

Therefore

$$8\mathbf{sw}_M^0(\sigma_{can}) = 17 - 2\nu - 12\Big(\sum_{i=1}^{\nu}(\frac{a_i}{12} + \frac{1}{6a_i} - \frac{1}{4})\Big) + 2\sum_{i=1}^{\nu} \frac{1}{a_i} = 17 + \nu - \sum_{i=1}^{\nu} a_i.$$

We have thus verified the conjecture in this case too.

8.2 A singularity whose graph is not star-shaped All the examples we have analyzed so far had star shaped resolution graphs. In this section we consider a different situations which will indicate that the validity of the Main Conjecture extends beyond singularities whose link is a Seifert manifold. (In this subsection we will use some standard result about hypersurface singularities. For these result and the terminology, the interested reader can consult [2].)

Consider the isolated plane curve singularity given by the local equation $g(x,y) := (x^2+y^3)(x^3+y^2) = 0$. We define the surface singularity $(X,0)$ as the 3–fold cyclic cover of f, namely $(X,0)$ is a hypersurface singularity in $(\mathbb{C}^3,0)$ given by $f(x,y,z) := g(x,y)+z^3 = 0$.

The singularity (smoothing) invariants of f can be computed in many different ways. First notice that it is not difficult to draw the embedded resolution graph of g, which gives all the numerical smoothing invariants of g. For example, by A'Campo's formula [1] one gets that the Milnor number of g is 11. Then by Thom–Sebastiani theorem (see, eg [2], page 60) $\mu(f) = 11\cdot 2 = 22$. The signature $\sigma(F)$ of the Milnor fiber of F can be computed by the method described in [30] or [31]; and it is -18. Now, by the relations 4.4 one gets $p_g(X,0) = 1$ and $K^2+\#\mathcal{V} = 10$.

In fact, by the algorithm given in [30], one can compute easily the resolution graph of $(X,0)$ as well (see Figure 3).

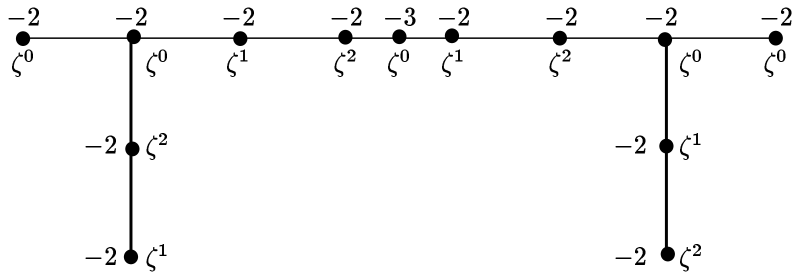

Figure 3: The resolution graph of $(x^2+y^3)(x^3+y^2)=0$

Then it is not difficult to verify that the graph satisfies Laufer's criterion for a minimally elliptic singularity, in particular this also gives that $p_g = 1$.

Using either way, finally one obtains $p_g + (K^2+\#\mathcal{V})/8 = 9/4$. Using the correspondence between the characteristic polynomial $\Delta(t)$ of the monodromy action (which can be again easily computed from the Thom–Sebastiani theorem) and the torsion of H (namely that $|\Delta(1)| = |H|$), one obtains $H = \mathbb{Z}_3$.

Using the formula for the Casson–Walker invariant from the plumbing graph one gets $\lambda(M)/|H| = -49/36$.

Finally we have to compute the torsion. There are only two non-trivial characters. One of them appears one the resolution graph (ie, $\chi(g_v) = \zeta^{n_v}$ with $\zeta^3 = 1$). The other is its conjugate. Using the general formula for plumbing graphs, one gets $\mathcal{T}_{M,\sigma_{can}}(1) = 8/9$.

Since $8/9 + 49/36 = 9/4$, the conjecture is true.

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Appendices

A The Reidemeister–Turaev torsion for plumbings

In this section we prove Theorem 5.7, which describes the torsion $\mathcal{T}_M$ of M in terms of plumbing data. The proof has two parts.

In the first part we use Turaev's surgery results [49] formulated in Fourier theoretic terms which will allow us to replace formal objects (elements in group algebras) by analytic ones (meromorphic functions of several variables). We obtain a first rough description of $\mathcal{T}_M$ in terms of surgery data which has a *spin*^{c} structure ambiguity.

In the second part, we eliminate the ambiguity about the *spin*^{c} structure using Turaev's structure theorem [48, Theorem 4.2.1], and the identities 3.9 (1) and (2) from our section 3, which completely determine the *spin*^{c} structure from the Fourier transform of a sign-refined torsion.

A.1 The surgery data We consider an *integral* surgery data: M is a rational homology 3–sphere described by the Dehn surgery on the oriented link $\mathcal{L} = L_1 \cup \cdots \cup L_n \subset S^3$ with integral surgery coefficients. We will assume that $n > 1$. We denote by E the complement of this link. The manifold M is obtained from E by attaching n solid tori $Z_1, \cdots, Z_n$. We denote by $\mu_i \in H_1(E, \mathbb{Z})$ the meridian of L_i. Similarly as in the case of plumbing, we can construct lattices $G := H_2(M, E, \mathbb{Z})$ and $G' := \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Z})$ and a presentation $P\colon G \to G'$ for $H := H_1(M, \mathbb{Z})$.

Indeed, the exact sequence $0 \to H_2(M, E, \mathbb{Z}) \to H_1(E, \mathbb{Z}) \to H_1(M, \mathbb{Z}) \to 0$ produces a short exact sequence $0 \to G \to G' \to H \to 0$. Moreover, we can identify $G = \mathbb{Z}^n$ via the canonical basis consisting of classes $[D \times *]$, one for each solid torus $Z_i = D \times S^1$; and $G' = \mathbb{Z}^n$ via the canonical basis determined by the oriented meridians $\{\mu_i\}_i$. Sometimes we regard P as a matrix written in these bases.

Recall that a plumbing graph provides a canonical surgery presentation in such a way that the 3–manifolds obtained by plumbing, respectively by the surgery, are the same. This presentation is the following: the components of the link $\mathcal{L} \in S^3$ are in one-to-one correspondence with the vertices of the graph (in particular, the index set

$I_n=\{1,\dots,n\}$ is identified with $\mathcal{V}$); all these components are trivial knots in S^3; their framings are the decorations of the corresponding vertices; two knots corresponding to two vertices connected by an edge form a Hopf link, otherwise the link is "the simplest possible". In this way we obtain an integral surgery data in such a way that the matrix P becomes exactly the intersection matrix I.

If $\pi\colon G'\to H$ is the natural projection, then the cores K_i of the attached solid tori Z_i determine the homology classes $\pi(K_i)$ in H, and $\pi(K_i)=\pi(-\mu_i)$.

For every fixed $i\in I_n$ we denote by E_i the manifold obtained by performing the surgery only along the knots K_j, $j\in I_n\setminus\{i\}$. Equivalently, E_i is the exterior of K_i in M. We set $G_i:=H_2(M,E_i,\mathbb{Z})$ and $G_i':=H_1(E_i,\mathbb{Z})$. G_i' is generated by the set $\{\mu_j\}_{j=1}^n$ subject to the relations provided by the j-th columns of P for each $j\neq i$. There is a natural projection $G'\to G_i'$ denoted by π_i. Sometimes, for simplicity, we write K_j $(j\neq i)$ for its projections as well. We write also $\tilde{G}:=\mathrm{Hom}(G',\mathbb{C}^*)$ and $\tilde{G}_i:=\mathrm{Hom}(G_i',\mathbb{C}^*)$. It is natural and convenient to introduce the following definition.

A.2 Definition A surgery presentation of a rational homology sphere is called *non-degenerate* if the homology class $\pi_i(K_j)$ has infinite order in G_i' for any $j\neq i$.

The non-degenerate surgeries can be recognized as follows: *the surgery is non-degenerate if and only if every off diagonal element of P^{-1} is nontrivial.*

Indeed, the fact that for some $j\neq i$ the class $\pi_i(\mu_j)$ has finite order in G_i' is equivalent with the existence of $n\in\mathbb{Z}^*$ and $\vec{v}\in G$ with i-th component $v_i=0$ such that $n\mu_j=P\cdot\vec{v}$. But this says $v_i=nP^{-1}_{ij}$. Notice that in our case, when the matrix P is exactly the negative definite intersection matrix I associated with a connected (resolution) graph, by a well-known result, the surgery presentation is non-degenerate.

We can now begin the presentation of the surgery formula for the Reidemeister–Turaev torsion.

A.3 Proposition *Suppose that the rational homology 3–sphere M is described by a non-degenerate Dehn surgery. Fix a relative spin^c structure $\tilde{\sigma}$ on E. For any j, it induces a relative spin^c structures σ_j on E_j and a spin^c structure σ on M. Let $\mathcal{T}_{E_j,\sigma}$ be the (sign-refined) Reidemeister–Turaev torsion of E_j determined by σ_j. Fix $\chi\in\hat{H}\setminus\{1\}$ and $i\in I_n$ such that $\chi(K_i)\neq 1$. Then the following hold.*

(a) *The Fourier transform $\hat{\mathcal{T}}_{E_i,\sigma}$ of the torsion of E_i extends to a holomorphic function on $\tilde{G}_i\setminus\{1\}$ uniquely determined by the equality*

$$\hat{\mathcal{T}}_{E_i,\sigma}(\chi)\cdot\prod_{j\neq i}(\chi^{-1}(K_j)-1)=\hat{\mathcal{T}}_{E,\tilde{\sigma}}(\chi),\quad \textit{for all } \chi\in\tilde{G}_i.$$

Here $\hat{\mathcal{T}}_{E,\tilde{\sigma}}$ is the holomorphic extension of Fourier transform of the Alexander–Conway polynomial $\mathcal{T}_{E,\tilde{\sigma}}$ of the link complement E, associated with the spin^c structure σ (normalized as in [49, Section 8]).

(b)

$$\hat{\mathfrak{T}}_{M,\sigma}(\chi) = \frac{\hat{\mathfrak{T}}_{E_i,\sigma}(\chi)}{\bar{\chi}(K_i) - 1}.$$

Proof $\tilde{G}$ is complex n–dimensional torus, and the Fourier transform of the torsion of E extends to a holomorphic function $\chi \mapsto \hat{\mathfrak{T}}_E(\chi)$ on $\tilde{G}$. The elements K_j also define holomorphic functions on $\tilde{G}$ by $\chi \mapsto \chi(K_j)^{-1} - 1$. Moreover, $\tilde{G}_i$ is an union of 1–dimensional complex tori and the Fourier transform of $\mathfrak{T}_{E_i}$ extends a holomorphic function $\chi \mapsto \hat{\mathfrak{T}}_{E_i}(\chi)$ on $\tilde{G}_i$. Since the elements K_j $(j \neq i)$ have infinite orders in G'_i, we deduce from [50, Lemma 17.1], [42, Section 2.5], that $\hat{\mathfrak{T}}_{E_i}$ is *the unique holomorphic extension of the meromorphic function*

$$\tilde{G}_i \setminus \{1\} \ni \chi \mapsto \frac{\hat{\mathfrak{T}}_{E,\tilde{\sigma}}(\chi)}{\prod_{j \neq i}(\chi^{-1}(K_j) - 1)}.$$

Part (b) follows from the surgery formula [49, Lemma 5.1]. □

A.4 The "limit" expression Let us now explain how we will use the above theoretical results. For each $\chi \in \hat{H} \setminus \{1\}$ pick an arbitrary i with $\chi(K_i) \neq 1$. Then χ belongs to $\tilde{G}_i$ too. The group $\tilde{G}_i$ is an union of complex tori, we denote by $\mathbb{T}_{\chi,i}$ the irreducible component containing χ. In fact, there exists $w^i \in G$ such that $\mathbb{T}_{\chi,i} = \{t *_{w^i} \chi; t \in \mathbb{C}^*\}$, where

$$t *_{w^i} \chi(v) := t^{v(w^i)}\chi(v) \ \text{ for all } t \in \mathbb{C}^* \text{ and } \ v \in G'.$$

A possible set of "weights" w^i can be determined easily. G_i is a *free* Abelian group of rank 1 which injects into G. We can choose w^i to be an arbitrary non-trivial element of G_i. Obviously, w^i depends on the index i. In general, there is no universal choice of the index i which is suitable for any character χ.

Using the matrix notation, w^i can be regarded as a vector $\vec{w}^i$ so that $I\vec{w}^i$ is an (integer) multiple of $\vec{b}^i$, where $\vec{b}^i$ is the vector whose i-th entry is 1, all the other entries are zero. Then the above proposition reads as follows:

$$\hat{\mathfrak{T}}_{M,\sigma}(\chi) = \frac{1}{\bar{\chi}(K_i) - 1} \cdot \lim_{t \to 1} \frac{\hat{\mathfrak{T}}_{E,\tilde{\sigma}}(t *_{w^i} \chi)}{\prod_{j \neq i}\big(t *_{w^i} \chi(K_j)^{-1} - 1\big)}.$$

In particular, by switching the index set I_n to $\mathcal{V}$, if $\chi(\mu_v) \neq 1$ for all v, one has:

$$\hat{\mathfrak{T}}_{M,\sigma}(\chi) = \frac{\hat{\mathfrak{T}}_{E,\tilde{\sigma}}(\chi)}{\prod_v\big(\chi(\mu_v) - 1\big)}.$$

A.5 The first part of the proof According to Turaev [49], for any $\tilde{\sigma} \in Spin^c(E, \partial E)$ the Alexander–Conway polynomial $\mathcal{T}_{E,\tilde{\sigma}}$ has the form

$$\mathcal{T}_{E,\tilde{\sigma}} = g \prod_v (\mu_v - 1)^{\delta_v - 1},$$

where $g \in G'$ depends on $\tilde{\sigma}$ by a normalization rule established by Turaev [49, Section 8] (described in terms of "charges"). The generator set $\{g_v\}_{v\in\mathcal{V}}$ of H (defined via the plumbing) and $\{[\mu_v]\}_{v\in\mathcal{V}}$ (defined via the surgery) can be identified as follows. (Here we will identify their Poincaré duals.) Consider a resolution $\tilde{X} \to X$ of $(X, 0)$ as above. The lattice inclusion $I\colon L \to L'$ (ie, $H^2(\tilde{X}, \partial\tilde{X}, \mathbb{Z}) \to H^2(\tilde{X}, \mathbb{Z})$) can be identified with the lattice inclusion $P = G \to G'$ (ie, $H^2(E, \mathbb{Z}) \to H^2(E, \partial E, \mathbb{Z})$). Indeed, let D^4 be the 4–dimensional ball with boundary S^3 with $\mathcal{L} \in S^3$. Then $\tilde{X}$ can be obtained from D^4 by attaching $n = \#\mathcal{V}$ copies of 2–handles $D^2 \times D^2$. Let the union of these handles be denoted by H. Clearly, $S^3 \setminus \mathrm{int}(E)$ is a union T of solid tori. Then the isomorphism $L' \to G'$ is given by the following sequence of isomorphism:

$$H^2(E, \partial E) \xleftarrow{(1)} H^2(S^3, T) \xleftarrow{(2)} H^2(D^4, T) \xleftarrow{(3)} H^2(\tilde{X}, H) \xrightarrow{(4)} H^2(\tilde{X}).$$

Above, (1) is an excision, (2) is given by the triple (D^4, S^3, T), (3) is excision, and (4) is a restriction isomorphism. Moreover, under this isomorphism, the basis $\{\mu_v\}_v$ correspond exactly to the basis $\{D_v\}_v$. This also shows that $g_v = [\mu_v]$ for all v. Now, fix a character $\chi \in \hat{H}$, $\chi \neq 1$. Set v_* and $\vec{w}^* \in \mathbb{Z}^n$ as in Theorem 5.7, ie, with $I\vec{w}^* = -m^*\vec{b}^*$. Then, using the notations of A.4, clearly $\vec{w}^* \in \mathbb{Z}^n = G$, $\mu_v \in G'$, and $\mu_v(\vec{w}^*)$ is exactly the v–component w_v^* of $\vec{w}^*$. Hence $t *_{\vec{w}^*} \chi(\mu_v) = t^{w_v^*}\chi(g_v)$. Then by the above notations and A.4 we conclude that for any $\sigma \in Spin^c(M)$

$$\hat{\mathcal{T}}_{M,\sigma}(\chi) = \chi(h) \cdot \lim_{t\to 1} \prod_{v\in\mathcal{V}} \left(t^{w_v^*}\, \chi(g_v) - 1\right)^{\delta_v - 2},$$

for some $h = h(\sigma) \in H$ which depends (bijectively) on σ. (Clearly, the limit is not effected by the choice of m^*.) Now, notice that if we use the identity $\hat{\mathcal{T}}_{M,\sigma}(\bar{\chi}) = \hat{\mathcal{T}}_{M,\bar{\sigma}}(\chi)$ (cf 3.8(3)), Theorem 5.7 is equivalent with the following identity

$$\hat{\mathcal{T}}_{M,\bar{\sigma}}(\chi) = \bar{\chi}(h_\sigma) \cdot \lim_{t\to 1} \prod_{v\in\mathcal{V}} \left(t^{w_v^*}\, \chi(g_v) - 1\right)^{\delta_v - 2}. \tag{t}$$

The above discussion shows clearly that this is true, modulo the ambiguity about h_σ. This ambiguity (ie, the fact that in the above expression exactly h_σ should be inserted) is verified via 3.9(2) (since there is exactly one h which satisfies 3.9(2) with a fixed $spin^c$ structure σ).

A.6 Additional discussion about the "weights" Before we start the second part, we clarify an important fact about the behavior of the weights considered above. Recall that above, for a fixed $\chi \neq 1$, we chose v_* with $\chi(g_{v_*}) \neq 1$. This can be rather unpleasant in any Fourier formula, since for different characters we have to take different vertices v_*. Therefore, we also wish to analyze the case of an arbitrary v_0 (disregarding the fact that $\chi(g_{v_0})$ is 1 or not) instead of v_*.

A.7 Lemma *Fix a character $\chi \in \hat{H} \setminus \{1\}$.*

(a) *For an arbitrary vertex v_0, consider a vector $\vec{w}^0$, with components $\{w_v^0\}_v$, satisfying $I\vec{w}^0 = -m^0\vec{b}^0$ for some positive m^0. Then the limit*

$$\lim_{t\to 1} \prod_{v\in\mathcal{V}} \left(t^{w_v^0}\chi(g_v) - 1\right)^{\delta_v - 2}$$

exists and it is finite.

(b) *Let $I_\chi := \{v : \text{either } \chi(g_v) \neq 1 \text{ or } v \text{ has an adjacent vertex } u \text{ with } \chi(g_u) \neq 1\}$. Then the above limit is the same for any $v_0 \in I_\chi$.*

Proof First we fix some notations. We say that

- a subgraph Γ' of the plumbing graph Γ satisfies the property (P) if $\sum_{\Gamma'}(\delta_v - 2) \geq 0$, where the sum $\sum_{\Gamma'}$ runs over the vertices of Γ' (and δ_v is the degree of v in Γ).
- Γ' is a "full" subgraph of Γ if any two vertices of Γ' adjacent in Γ are adjacent in Γ' as well. For any subgraph Γ', we denote by $\mathcal{V}(\Gamma')$ its set of vertices.
- a full proper subgraph Γ' of Γ has property (C) if it has a unique vertex (say v_{end}) which is connected by an edge of Γ with a vertex in $\mathcal{V}(\Gamma) \setminus \mathcal{V}(\Gamma')$. For any $\chi \in \hat{H}$, let Γ^1 be the full subgraph of Γ with set of vertices $\{v \in \mathcal{V} \,|\, \chi(g_v) = 1\}$. Next, fix a character $\chi \in \hat{H} \setminus \{1\}$ and a vertex v of Γ. Then

$$e_v g_v + \sum g_u = 0 \ \text{ in } H\text{, hence } \quad \chi(g_v)^{e_v} \cdot \prod \chi(g_u) = 1, \tag{1}$$

where the sum (resp. product) runs over the adjacent vertices u of v in Γ. Therefore, if v is in Γ^1 then

$$\#\{u : u \text{ adjacent to } v \text{ and } u \notin \mathcal{V}(\Gamma^1)\} \neq 1. \tag{2}$$

The proof of (a) We have to show that Γ^1 satisfies (P). Let $\Gamma^{1,c}$ be one of its connected components, and denote by $\delta_v^{1,c}$ the degree of v in $\Gamma^{1,c}$. Since $\Gamma^{1,c}$ is a tree, one has $\sum_{\Gamma^{1,c}}(\delta_v^{1,c} - 2) = -2$. Since $\Gamma^{1,c}$ is a proper subgraph of the connected graph Γ, there exists at least one edge of Γ which is not an edge of $\Gamma^{1,c}$, but it has one of its end-vertices in $\Gamma^{1,c}$. In fact, (2) shows that there are at least two such edges. Therefore, $\Gamma^{1,c}$ satisfies (P).

The proof of (b) First we claim the following fact.

(F) *Let Γ' be a full proper subgraph of Γ which satisfies (C). Then for any*

$$v_0 \in (\mathcal{V}(\Gamma) \setminus \mathcal{V}(\Gamma')) \cup \{v_{end}\}$$

the solution $\{w_v^0\}_v$ of $I\vec{w}^0 = -m^0\vec{b}^0$ has the following special property: the subset $\{v_v^0\}_{v\in\mathcal{V}(\Gamma')}$, modulo a multiplicative constant, is independent of the choice of v_0.

Indeed, the subset $\{v_v^0\}_{v\in\mathcal{V}(\Gamma')}$, modulo a multiplicative constant, is completely determined by the set of relations of type (1) considered for vertices $v\in\mathcal{V}(\Gamma')\setminus\{v_{end}\}$. Since the intersection form associated with Γ' is non-degenerate, this system has a maximal rank. Now, we make a partition of $\mathcal{V}(\Gamma^{1,c})$ (cf part (a) for the notation). Each set S of the partition defines a full subgraph $\Gamma^{1,c,j}$ of $\Gamma^{1,c}$ with $S=\mathcal{V}(\Gamma^{1,c,j})$. The partition is defined in such a way that each $\Gamma^{1,c,j}$ is a maximal subgraph satisfying both properties (P) and (C). One way to construct such a partition is the following.

Let us start with $\Gamma^{1,c}$. By (a), it satisfies (P). If it does not satisfies (C), then take two of its vertices, both having adjacent vertices outside of $\Gamma^{1,c}$. Eliminate next all the edges of $\Gamma^{1,c}$ situated on the path connecting these two vertices, and then, if necessary, repeat the above procedure for the connected components of the remaining graph. After a finite number of steps all the connected components will satisfy both properties (P) and (C).

Now, fact (F) can be applied for all these subgraphs $\Gamma^{1,c,j}$. In the limit we regroup the product corresponding to the subsets $\mathcal{V}(\Gamma^{1,c,j})$, and the result follows. □

A.8 The second part of the proof: preliminaries Our next goal is to show that the right hand side of (t) satisfies the formulae 3.9(1) and (2) for the *spin*^{c} structure $\bar{\sigma}$. This clearly ends our proof.

For this, let us fix a vertex $v_0\in\mathcal{V}$ and we plan to verify 3.9 (1) and (2) for $h=g_{v_0}$. In the sequel we prefer to fix m^0 in the equation of v_0, namely we let $m>0$ be the smallest positive integer so that

$$I\vec{w}^0=-m\vec{b}^0 \text{ has a solution } \vec{w}^0=\{w_v^0\}_{v\in\mathcal{V}}\in\mathbb{Z}^{\#\mathcal{V}}. \tag{3}$$

Clearly $\gcd(\{w_v^0\}_v)=1$ (and each $w_v^0>0$, fact not really important here). For a non-trivial character $\chi\in\hat{H}$ with $\chi(g_{v_0})\neq 1$, the vertex v_0 is a good candidate for v_* (or, at least, the weights $\vec{w}^*$ in (t) can be replaced by the weights $\vec{w}^0$ since they provide the same limit, cf A.7). But for characters χ with $\chi(g_{v_0})=1$ the limit in A.7 (consider for v_0) can be different from the limit needed in (t) (where one has v_*). Nevertheless, the products of these (probably different) limits with $\chi(g_{v_0})-1$ are the same (namely zero) (and in 3.9(1) and (2) we need only these type of products !). More precisely, for *any* $\chi\in\hat{H}\setminus\{1\}$:

$$(\chi(g_{v_0})-1)\cdot\lim_{t\to 1}\prod_{v\in\mathcal{V}}\Big(t^{w_v^*}\chi(g_v)-1\Big)^{\delta_v-2}=(\chi(g_{v_0})-1)\cdot\lim_{t\to 1}\prod_{v\in\mathcal{V}}\Big(t^{w_v^0}\chi(g_v)-1\Big)^{\delta_v-2}$$

Therefore, in all our verifications, we can use only one set of weights, namely $\vec{w}^0=\{w_v^0\}_v$, given exactly by the vertex v_0, and this is good for all $\chi\in\hat{H}\setminus\{1\}$. In the sequel we drop the upper index 0, and we simply write w_v instead w_v^0. Let us introduce the notation

$$\hat{R}_\chi(t):=\prod_{v\in\mathcal{V}}\Big(t^{w_v}\chi(g_v)-1\Big)^{\delta_v-2}.$$

We have to show that $\lim_{t\to 1}\bar{\chi}(h_\sigma)\hat{R}_\chi(t)$ satisfies the formulae 3.9(1) and (2) for the *spin*^{c} structure $\bar{\sigma}$. Since in these formulae we need the product of this limit with $\chi(g_{v_0})-1$, we set $\bar{\delta}_v := \delta_v$ for any $v\neq v_0$, but $\bar{\delta}_{v_0} := \delta_{v_0}+1$, and define

$$\hat{P}_\chi(t) := \hat{R}_\chi(t)\cdot\Big(t^{w_{v_0}}\,\chi(g_{v_0})-1\Big) = \prod_{v\in\mathcal{V}}\Big(t^{w_v}\,\chi(g_v)-1\Big)^{\bar{\delta}_v-2}.$$

In the case of the trivial character $\chi=1$, we define $\Delta(t)$ via the identity:

$$\frac{\Delta(t)}{t-1} := \hat{P}_1(t) = \prod_{v\in\mathcal{V}}(t^{w_v}-1)^{\bar{\delta}_v-2}. \tag{4}$$

Since $\sum_v(\bar{\delta}_v-2)=-1$, one gets that $\Delta(t)$ has no pole or zero at $t=1$, in fact:

$$\Delta(t) = \prod_{v\in\mathcal{V}}(t^{w_v-1}+\cdots+t+1)^{\bar{\delta}_v-2}. \tag{5}$$

Let L_0 be a fixed generic fiber of the S^1–bundle over E_{v_0} used in the plumbing construction of M (cf 2.13). Set $G_0 := H_1(M\setminus L_0,\mathbb{Z})$.

The reader familiar with the theorem of A'Campo about the zeta function associated with the monodromy action of a Milnor fibration, certainly realizes that $\hat{P}_1(t)$ is such a zeta function, and $\Delta(t)$ is a characteristic polynomial of a monodromy operator. The next proof will not use this possible interpretation. Nevertheless, in A.10 we will show that $\Delta(t)\in\mathbb{Z}[t]$, and $\Delta(1)$ is the order of the torsion subgroup of G_0.

Since $H_2(M,M\setminus L_0,\mathbb{Z})=\mathbb{Z}$, one has the exact sequence:

$$0\to\mathbb{Z}\xrightarrow{i}G_0\xrightarrow{p}H\to 0,$$

where $i(1_\mathbb{Z})=\tilde{g}_\infty$:=the homology class in $M\setminus L_0$ of the meridian of L_0 viewed as a knot in M. Let $\tilde{g}_v$ be the homology class in G_0 of ∂D_v, defined similarly as $g_v\in H$, cf 2.13. Obviously, $\{\tilde{g}_v\}_{v\in\mathcal{V}}$ is a generator set for G_0. Define $\varphi\colon G_0\to\mathbb{Z}$ by $\tilde{g}_v\mapsto w_v$. The equations (3) guarantee that this is well-defined. Moreover, since $\gcd(\{w_v\}_v)=1$, φ is onto. Then clearly, its kernel T is exactly the subgroup of torsion elements of G_0. Let $j\colon T\to G_0$ be the natural inclusion. Again by (3), $\varphi(\tilde{g}_\infty)=m$, hence the composition $\varphi\circ i$ in multiplication by m. These facts can be summarized in the following diagram (where r is induced by φ):

$$\begin{tikzcd}
 & T \arrow[r, leftrightarrow, "1"] \arrow[d, "j"] & T \arrow[d, "j'"] \\
\mathbb{Z} \arrow[r, hook, "i"] \arrow[d, leftrightarrow, "1"] & G_0 \arrow[r, "p"] \arrow[d, "\varphi"] & H \arrow[d, "r"] \\
\mathbb{Z} \arrow[r, hook, "\cdot m"] & \mathbb{Z} \arrow[r, "q"] & \mathbb{Z}_m
\end{tikzcd}$$

It is convenient to identify $\mathbb{Z}_m$ with a subgroup of $\mathbb{Q}/\mathbb{Z}$ via $\mathbb{Z}_m\ni\hat{a}\mapsto\frac{a}{m}\in\mathbb{Q}/\mathbb{Z}$.

A.9 Lemma *For any $h \in H$, $r(h) = b_M(g_{v_0}, h)$ via the above identification.*

Proof It is enough to verify the identity for each g_v, $v \in \mathcal{V}$. In that case, $r(g_v) = q(\varphi(\tilde{g}_v)) = \hat{w}_v = w_v/m \in \mathbb{Q}/\mathbb{Z}$. But by 2.2 and A.8(3), $b_M(g_{v_0}, g_v) = -(\mu_{v_0}, \mu_v)_{\mathbb{Q}} = -\mu_v(I^{-1}\mu_{v_0}) = w_v/m$ as well. □

Fix $g \in G_0$ so that $\varphi(g) = 1_{\mathbb{Z}}$. This provides automatically a splitting of the exact sequence $0 \to T \to G_0 \to \mathbb{Z} \to 0$, ie, a morphism $s\colon G_0 \to T$ with $s \circ j = id_T$ and $s(g) = 1_T$. In the sequel, we extend any morphism and character to the corresponding group-algebras over $\mathbb{Z}$ (and we denote them by the same symbol). For any character ξ of G_0 we define the representation $\xi_t\colon G_0 \to \mathbb{C}[[t, t^{-1}]]^*$ given by $\xi_t(x) = \xi(x)t^{\varphi(x)}$. (This can be identified with a family of characters. Indeed, for any fixed $t \in \mathbb{C}^*$ and character ξ, one can define the character ξ_t given by $x \mapsto \xi(x)t^{\varphi(x)}$. Eg, for $\xi = p \circ \chi$, ξ_t is just a more convenient notation for the action $t *_{\vec{w}^0} \chi$, cf A.4 and A.5.)

Now, the point is that the identity (4) has a generalization in the following sense.

A.10 Theorem *For any character $\xi \in \hat{G}_0$ define*

$$\hat{P}_\xi(t) := \prod_{v \in \mathcal{V}} \Big(\xi_t(\tilde{g}_v) - 1\Big)^{\bar{\delta}_v - 2} = \prod_{v \in \mathcal{V}} \Big(t^{w_v}\, \xi(\tilde{g}_v) - 1\Big)^{\bar{\delta}_v - 2}.$$

Then there exist an element $\delta \in \mathbb{Z}[G_0]$ such that the following hold.

(a) *For any $\xi \in \hat{G}_0$*

$$\hat{P}_\xi(t) = \frac{\xi_t(\delta)}{\xi_t(g) - 1}.$$

(b) $1_t(\delta) = \Delta(t)$; $1(\delta) = \mathfrak{aug}(\delta) = \Delta(1) = |T|$.

(c) $s(\delta) = \Sigma_T$, *where* $\Sigma_T := \sum_{x \in T} x \in \mathbb{Z}[T]$.

Proof From the first part of the proof (cf A.3.(b) and A.5(t)) follows that $\lim_{t\to 1} \hat{P}_\xi(t)$, modulo a multiplicative factor of type $\mp\xi(x)$, for some $x \in G_0$, is the Fourier transform of the Reidemeister–Turaev torsion $\mathcal{T}$ on $M \setminus L_0$ associated with some *spin^c* structure (whose identification is not needed here). By [48, 4.2.1], $\mathcal{T} - \Sigma_T/(1-g) \in \mathbb{Z}[G_0]$, identity valid in $Q(G_0)$, the ring of quotients of $\mathbb{Q}[G_0]$. By the first statement $\hat{P}_\xi(t) = \mp\xi_t(x\mathcal{T})$ for any $\xi \in \hat{G}_0$. Hence, for some $A \in \mathbb{Z}[G_0]$ one has:

$$\hat{P}_\xi(t) = \xi_t(A) \pm \xi_t\Big(\frac{x \cdot \Sigma_T}{g-1}\Big). \tag{$*$}$$

This identity multiplied by $\xi_t(g-1)$, for $\xi = 1$ and $t \to 1$, and via (4), provides $\Delta(1) = \pm|T|$. By (5), $\Delta(1)$ is positive, hence in $(*)$ $\pm 1 = +1$. Moreover, $\Delta(1) = |T|$. Now, if one defines $\delta := A(g-1) + x\Sigma_T$, then (a) and (c) follow easily, and $1_t(\delta) = \Delta(t)$ is exactly (4). □

In order to verify 3.9(1) and (2), we will apply the above theorem to special characters of the type $\xi = \chi \circ p$, where $\chi \in \hat{H}$. It is clear that for any $y \in \mathbb{Z}[G_0]$ and $h \in H$, the sum $\sum \chi(h) \cdot \chi \circ p(y) = \sum \chi(h \cdot p(y))$ over $\chi \in \hat{H}$ is an integer multiple of $|H|$. Hence:

$$\frac{1}{|H|} {\sum_{\chi \in \hat{H}}}' \chi(h) \cdot \chi \circ p(y) = -\frac{1}{|H|} 1(h) \cdot 1(y) = -\frac{1}{|H|} \mathfrak{aug}(y) \pmod{\mathbb{Z}}. \tag{6}$$

Using the splitting of G_0 into $T \times \mathbb{Z}$ given by g, one can easily verify that in $Q(G_0)$

$$\frac{y - s(y)}{g - 1} \in \mathbb{Z}[G_0] \text{ for any } y \in \mathbb{Z}[G_0]. \tag{7}$$

In the sequel, we write simply χ_t for $(\chi \circ p)_t$.

A.11 Verification of 3.9(1) Now we will verify that $\bar{\chi}(h)\hat{R}_\chi(t)$ $(\chi \in \hat{H} \setminus \{1\})$ satisfies 3.9(1) for $h = h_\sigma$ (in fact, for any $h \in H$). For this, fix a vertex v_0, and $g \in G_0$ with $\varphi(g) = 1$ as above. Take an arbitrary $x \in G_0$. Then we have to show that

$$\frac{1}{|H|} \lim_{t \to 1} {\sum_{\chi \in \hat{H}}}' \bar{\chi}(h)\hat{R}_\chi(t)(t^{w_{v_0}} \chi(g_{v_0}) - 1)(\chi_t(x) - 1) = -b_M(g_{v_0}, p(x)) \pmod{\mathbb{Z}}. \tag{8}$$

Via A.10, the left hand side of (8) is

$$\frac{1}{|H|} \lim_{t \to 1} {\sum_{\chi \in \hat{H}}}' \bar{\chi}(h) \cdot \chi_t(\delta) \cdot \frac{\chi_t(x) - 1}{\chi_t(g) - 1}. \tag{9}$$

Set $a := \varphi(x)$. Since $s(\delta x) = s(\delta)$, (9) transforms as follows (use (6), (7) and A.10):

$$\frac{1}{|H|} \lim_{t \to 1} {\sum_{\chi \in \hat{H}}}' \bar{\chi}(h) \cdot \chi_t\Big(\frac{\delta x - s(\delta x)}{g - 1} - \frac{\delta - s(\delta)}{g - 1}\Big) = -\frac{1}{|H|} \lim_{t \to 1} 1(h) \cdot 1_t\Big(\frac{\delta x - s(\delta x)}{g - 1} - \frac{\delta - s(\delta)}{g - 1}\Big)$$

$$= -\frac{1}{|H|} \lim_{t \to 1} \frac{\Delta(t)t^a - |T| - \Delta(t) + |T|}{t - 1} = -\frac{1}{|H|} \Delta(1) a = -\frac{a}{m} \pmod{\mathbb{Z}}.$$

But the right hand side of (8), via A.9, is the same $-a/m \in \mathbb{Q}/\mathbb{Z}$.

A.12 Verification of 3.9(2) Now we will verify

$$\frac{1}{|H|} \lim_{t \to 1} {\sum_{\chi \in \hat{H}}}' \bar{\chi}(h_\sigma)\hat{R}_\chi(t)(t^{w_{v_0}} \chi(g_{v_0}) - 1) = -\mathfrak{q}^c(\bar{\sigma})(g_{v_0}) \pmod{\mathbb{Z}}. \tag{10}$$

The left hand side is

$$\frac{1}{|H|} \lim_{t \to 1} {\sum_{\chi \in \hat{H}}}' \bar{\chi}(h_\sigma) \frac{\chi_t(\delta)}{\chi_t(g) - 1}.$$

The fraction in this expression can be written as (cf (7))

$$\chi_t\Big(\frac{\delta - s(\delta)}{g-1}\Big) + \frac{\chi(s(\delta))}{\chi_t(g)-1}. \tag{11}$$

This sum-decomposition provides two contributions. The first via (6), (7) and A.10 gives:

$$\frac{1}{|H|}\lim_{t\to 1}\sum_{\chi\in\hat{H}}{}' \bar{\chi}(h_\sigma)\cdot\chi_t(\frac{\delta-s(\delta)}{g-1}) = -\frac{1}{|H|}\lim_{t\to 1}\frac{\Delta(t)-\Delta(1)}{t-1} = -\frac{1}{|H|}\Delta'(1) \pmod{\mathbb{Z}},$$

where $\Delta'(t)$ denoted the derivative of Δ with respect to t. On the other hand, cf (5),

$$\frac{\Delta'(t)}{\Delta(t)} = \sum_v (\bar{\delta}_v - 2)\frac{(t^{w_v-1}+\cdots+t+1)'}{t^{w_v-1}+\cdots+t+1},$$

hence

$$\frac{\Delta'(1)}{\Delta(1)} = \frac{1}{2}\sum_v(\bar{\delta}_v-2)(w_v-1).$$

Since $\Delta(1) = |T| = |H|/m$, the first contribution is

$$\frac{1}{|H|}\lim_{t\to 1}\sum_{\chi\in\hat{H}}{}' \bar{\chi}(h_\sigma)\cdot\chi_t(\frac{\delta-s(\delta)}{g-1}) = -\frac{1}{2m}\sum_v(\bar{\delta}_v-2)(w_v-1).$$

For the second contribution, notice that $\chi(s(\delta)) = \chi(\Sigma_T)$ is zero unless χ is in the image of $\hat{r}\colon\ \hat{\mathbb{Z}}_m \to \hat{H}$; if χ is in this image then $\chi(\Sigma_T) = |T|$. For any $\chi\in\hat{\mathbb{Z}}_m$ we write $\chi(\hat{1}) = \zeta$. Assume that $r(h_\sigma) = -\hat{a}$ (or equivalently, $r(h_\sigma) = -\frac{a}{m}\in\mathbb{Q}/\mathbb{Z}$). Then

$$\frac{1}{|H|}\lim_{t\to 1}\sum_{\chi\in\hat{H}}{}' \bar{\chi}(h_\sigma)\cdot\frac{\chi(\Sigma_T)}{\chi_t(g)-1} = \frac{1}{|H|}\lim_{t\to 1}\sum_{\chi\in\hat{\mathbb{Z}}_m}{}' \bar{\chi}(h_\sigma)\cdot\frac{|T|}{\chi_t(g)-1} = \frac{1}{m}\sum_{\zeta\in\mathbb{Z}_m}{}' \zeta^a\cdot\frac{1}{\zeta-1}.$$

Since

$$\frac{1}{m}\sum_{\zeta\in\mathbb{Z}_m}\frac{\zeta^a-1}{\zeta-1} = 0 \pmod{\mathbb{Z}},$$

one gets that the second contribution is (cf B.6):

$$\frac{1}{m}\sum_{\zeta\in\mathbb{Z}_m}{}' \zeta^a\cdot\frac{1}{\zeta-1} = -\frac{a}{m} + \frac{1}{m}\sum_{\zeta\in\mathbb{Z}_m}{}'\frac{1}{\zeta-1} = -\frac{a}{m} - \frac{1}{2m}(m-1) \pmod{\mathbb{Z}}.$$

Therefore, the left hand side of (10), modulo $\mathbb{Z}$, is

$$-\frac{1}{2m}\sum_v(\bar{\delta}_v-2)(w_v-1) - \frac{a}{m} - \frac{1}{2m}(m-1).$$

Notice that $\sum_v(\bar{\delta}_v - 2) = -1$. Moreover $w_v = -mI^{-1}_{vv_0}$, and the coefficient r_{v_0} of Z_K equals $1 - \sum_v(\delta_v - 2)I^{-1}_{vv_0}$ (cf 5.2), hence the above expression can be transformed into

$$-\frac{1}{2} - \frac{1}{2m}\sum_v(\bar{\delta}_v - 2)w_v - \frac{a}{m} = -\frac{1}{2}r_{v_0} + \frac{1}{2}I^{-1}_{v_0v_0} - \frac{a}{m}.$$

Now, let us compute the right hand side of (10). Since $h_\sigma \cdot \sigma_{can} = \sigma$ one has $2h_\sigma + Z_K = c(\sigma)$. Then the characteristic element which provides $\bar{\sigma}$ is $-c(\sigma) = -2h_\sigma - Z_K$. Therefore

$$-\mathfrak{q}^c(\bar{\sigma})(g_{v_0}) = \frac{1}{2}(D_{v_0} - Z_K - 2h_\sigma, D_{v_0})_{\mathbb{Q}} = \frac{1}{2}(\sum_v I^{-1}_{vv_0}E_v - \sum_v r_vE_v, D_{v_0})_{\mathbb{Q}} - (h_\sigma, D_{v_0})_{\mathbb{Q}}$$

$$= \frac{1}{2}I^{-1}_{v_0v_0} - \frac{1}{2}r_{v_0} - (h_\sigma, D_{v_0})_{\mathbb{Q}}.$$

But, using 2.2 and A.9, $(h_\sigma, D_{v_0})_{\mathbb{Q}} = -b_M(h_\sigma, g_{v_0}) = -r(h) = \frac{a}{m}$. This proves A.12(10). At this point we invoke the following elementary fact.

Suppose q_1, q_2 are two quadratic functions on the finite abelian group H associated with the bilinear forms b_1, b_2; and $S \subset H$ is a generating set such that $q_1(s) = q_2(s)$ and $b_1(s,h) = b_2(s,h)$ for all $s \in S$ and $h \in H$. Then $q_1(h) = q_2(h)$ for all $h \in H$.

Using A.11(8), A.12(10) and the above fact we obtain 3.9(2), for any h. The identity 3.9(2) implies that $\bar{\chi}(h_\sigma)\hat{R}_\chi(t) = \hat{\mathfrak{T}}_{M,\bar{\sigma}}(\chi)$. This concludes the proof of Theorem 5.7.

(Notice that, in fact, we verified even more. First recall, cf [49], that the *sign* of the (sign-refined) torsion is decided by universal rules. In some cases its identification is rather involved. The point is that the above verification also reassures us that in (t) we have the right sign.) □

B Basic facts concerning the Dedekind–Rademacher sums

In this Appendix we collected some facts about (generalized) Dedekind sums which constitute a necessary minimum in the concrete computation of the Seiberg–Witten invariants (and in the understanding of the relationship between Dedekind sums and Fourier analysis). Let $\lfloor x \rfloor$ be the integer part of x, and $\{x\} := x - \lfloor x \rfloor$ its fractional part. In the paper [43], Rademacher introduces for every pair of coprime integers h, k and any real numbers x, y the following generalization of the classical Dedekind sum

$$\boldsymbol{s}(h,k;x,y) = \sum_{\mu=0}^{k-1}\left(\left(\frac{\mu+y}{k}\right)\right)\left(\left(\frac{h(\mu+y)}{k}+x\right)\right) = -\boldsymbol{s}(-h,k;-x,y),$$

where $((x))$ denotes the Dedekind symbol

$$((x)) = \begin{cases} \{x\} - 1/2 & \text{if } \quad x \in \mathbb{R} \setminus \mathbb{Z} \\ 0 & \text{if } \quad x \in \mathbb{Z}. \end{cases}$$

A simple computations shows that $\boldsymbol{s}(h,k;x,y)$ depends only on $x,y \pmod 1$. Additionally

$$\boldsymbol{s}(h,k;x,y) = \boldsymbol{s}(h-mk,k;x+my,y).$$

Moreover, have the following result

$$\boldsymbol{s}(1,k;0,y) = \begin{cases} \frac{k}{12} + \frac{1}{6k} - \frac{1}{4} = \frac{(k-1)(k-2)}{12k} & y \in \mathbb{Z} \\ \\ \frac{k}{12} + \frac{1}{k} B_2(\{y\}) & y \in \mathbb{R} \setminus \mathbb{Z}\,, \end{cases} \tag{B.1}$$

where $B_2(t) = t^2 - t + 1/6$ is the second Bernoulli polynomial. If $x = y = 0$ then we simply write $\boldsymbol{s}(h,k)$. Perhaps the most important property of these Dedekind–Rademacher sums is their reciprocity law which makes them computationally very friendly: their computational complexity is comparable with the complexity of the classical Euclid's algorithm. To formulate it we must distinguish two cases.

- Both x and y are integers. Then

$$\boldsymbol{s}(h,k) + \boldsymbol{s}(k,h) = -\frac{1}{4} + \frac{h^2 + k^2 + 1}{12hk}. \tag{B.2}$$

- x and/or y is not an integer. Then

$$\boldsymbol{s}(h,k;x,y) + \boldsymbol{s}(k,h;y,x)$$

$$= ((x)) \cdot ((y)) + \frac{h^2\psi_2(y) + \psi_2(hy+kx) + k^2\psi_2(x)}{2hk} \tag{B.3}$$

where $\psi_2(x) := B_2(\{x\})$. In particular, if $x, y \in \mathbb{R}$ are not both integers we deduce

$$\boldsymbol{s}(1,m;x,y) = -((x)) \cdot ((mx+y)) + ((x))((y)) + \frac{\psi_2(y) + \psi_2(y+mx) + m^2\psi_2(x)}{2m}. \tag{B.4}$$

An important ingredient behind the reciprocity law is the following identity ([43, Lemma 1])

$$\sum_{\mu=0}^{k-1} \left(\left(\frac{\mu + w}{k}\right)\right) = ((w)) \ \text{ for any } \ w \in \mathbb{R}. \tag{B.5}$$

The various Fourier–Dedekind sums we use in this paper can be expressed in terms of Dedekind–Rademacher sums. This follows from the identity ([20, page 170])

$$\frac{1}{p}\sum_{\zeta^p=1}{}' \frac{\zeta^t}{1-\zeta} = \left(\left(\frac{2t-1}{2p}\right)\right), \quad \text{for all } p,q\in\mathbb{Z},\ p>1. \tag{B.6}$$

In other words, the function

$$\{\zeta^p=1\}\subset\mathbb{C}^* \to \mathbb{C},\quad \zeta\mapsto\begin{cases} 0 & \text{if } \ \zeta=1 \\ \frac{1}{1-\zeta} & \text{if } \ \zeta\neq 1\end{cases}$$

is the Fourier transform of the function

$$\mathbb{Z}_p\to\mathbb{C},\quad \hat{t}\mapsto\left(\left(\frac{2t-1}{2p}\right)\right).$$

The identity (B.6) implies that

$$\frac{1}{p}\sum_{\zeta^p=1}{}' \frac{\zeta^{tq'}}{1-\zeta^q} = \left(\left(\frac{2tq'-1}{2p}\right)\right), \text{ for all } p,q\in\mathbb{Z},\ p>1,\ (p,q)=1,$$

where $q'=q^{-1}\pmod p$. Using the fact that Fourier transform of the convolution product of two functions $\mathbb{Z}_p\to\mathbb{C}$ is the pointwise product of the Fourier transforms of these functions we deduce after some simple manipulations the following identity.

$$\frac{1}{p}\sum_{\zeta^p=1}{}' \frac{\zeta^t}{(\zeta-1)(\zeta^q-1)} = -\boldsymbol{s}\big(q,p;\frac{q+1-2t}{2p},-\frac{1}{2}\big). \tag{B.7}$$

If $t=0$ then by (B.5) (and a computation), or by [44, 18a], one has

$$\frac{1}{p}\sum_{\zeta^p=1}{}' \frac{1}{(\zeta-1)(\zeta^q-1)} = -\boldsymbol{s}(q,p)+\frac{p-1}{4p}. \tag{B.8}$$

By setting $q=-1$ and $t=0$ in the above equality we deduce

$$\frac{1}{p}\sum_{\zeta^p=1}{}' \frac{1}{|\zeta-1|^2} = -\boldsymbol{s}(-1,p,0,-1/2)=\boldsymbol{s}(1,p,0,1/2)\overset{(B.1)}{=}\frac{p}{12}-\frac{1}{12p}. \tag{B.9}$$

The Fourier transform of the function $\eth_{p,q}:\mathbb{Z}_p\to\mathbb{C},\ \ \hat{t}\mapsto((qt/p))$ is the function (see [44, Chapter 2, Section C])

$$\{\zeta^p=1\}\to\mathbb{C},\quad \zeta\mapsto\begin{cases} \frac{1}{2}-\frac{\zeta^q}{\zeta^q-1} & \text{if } \ \zeta\neq 1 \\ 0 & \text{if } \ \zeta=1.\end{cases}$$

Then

$$\boldsymbol{s}(-q,p)=\sum_{t+s=0 \pmod p}\eth_{p,1}(t)\eth_{p,q}(s)=(\eth_{p,1}*\eth_{p,q})(0)$$

$$= \frac{1}{p} \sum_{\zeta^p = 1}{}' \Big(\frac{1}{2} - \frac{\zeta}{\zeta - 1} \Big) \Big(\frac{1}{2} - \frac{\zeta^q}{\zeta^q - 1} \Big).$$

This implies (cf also with [44])

$$\frac{1}{p} \sum_{\zeta^p = 1}{}' \frac{\zeta + 1}{\zeta - 1} \cdot \frac{\zeta^q + 1}{\zeta^q - 1} = -4\boldsymbol{s}(q, p). \tag{B.10}$$