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Splitting the concordance group of algebraically slice knots

Charles Livingston

Department of Mathematics, Indiana University
Bloomington, IN 47405, USA

Email: `livingst@indiana.edu`

Abstract

As a corollary of work of Ozsváth and Szabó [8], it is shown that the classical concordance group of algebraically slice knots has an infinite cyclic summand and in particular is not a divisible group.

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Let $\mathcal{A}$ denote the concordance group of algebraically slice knots, the kernel of Levine's homomorphism $\phi\colon \mathcal{C} \to \mathcal{G}$, where $\mathcal{C}$ is the classical knot concordance group and $\mathcal{G}$ is Levine's algebraic concordance group [6]. Little is known about the algebraic structure of $\mathcal{A}$: it is countable and abelian, Casson and Gordon [2] proved that $\mathcal{A}$ is nontrivial, Jiang [5] showed it contains a subgroup isomorphic to $\mathbf{Z}^\infty$, and the author [7] proved that it contains a subgroup isomorphic to $\mathbf{Z}_2^\infty$. We add the following theorem, a quick corollary of recent work of Ozsváth and Szabó [8].

Theorem 1 *The group $\mathcal{A}$ contains a summand isomorphic to $\mathbf{Z}$ and in particular $\mathcal{A}$ is not divisible.*

Proof In [8] a homomorphism $\tau\colon \mathcal{C} \to \mathbf{Z}$ is constructed. We prove that τ is nontrivial on $\mathcal{A}$. The theorem follows since, because $\mathrm{Im}(\tau)$ is free, there is the induced splitting, $\mathcal{A} \cong \mathrm{Im}(\tau) \oplus \mathrm{Ker}(\tau)$. No element representing a generator of $\mathrm{Im}(\tau)$ is divisible.

According to [8], $|\tau(K)| \leq g_4(K)$, where g_4 is the 4–ball genus of a knot, and there is the example of the $(4,5)$–torus knot T for which $\tau(T) = 6$. We will show that there is a knot T^* algebraically concordant to T with $g_4(T^*) < 6$. Hence, $T\# -T^*$ is an algebraically slice knot with nontrivial τ, as desired.

Recall that T is a fibered knot with fiber F of genus $(4-1)(5-1)/2 = 6$. Let V be the 12×12 Seifert matrix for T with respect to some basis for $H_1(F)$. The quadratic form $q(x) = xVx^t$ on $\mathbf{Z}^{12}$ is equal to the form given by $(V+V^t)/2$. Using [3] the signature of this symmetric bilinear form can be computed to be 8, so q is indefinite, and thus by Meyer's theorem [4] there is a nontrivial primitive element z with $q(z) = 0$. Since z is primitive, it is a member of a symplectic basis for $H_1(F)$. Let V^* be the Seifert matrix for T with respect to that basis. The canonical construction of a Seifert surface with Seifert matrix V^* ([9], or see [1]) yields a surface F^* such that z is represented by a simple closed curve on F^* that is unknotted in S^3. Hence, F^* can be surgered in the 4–ball to show that its boundary T^* satisfies $g_4(T^*) < 6$. Since T^* and T have the same Seifert form, they are algebraically concordant. □

Addendum An alternative proof of Theorem 1 follows from the construction of knots with trivial Alexander polynomial for which τ is nontrivial, to appear in a forthcoming paper.

References

[1] **G Burde**, **H Zieschang**, *Knots*, de Gruyter Studies in Mathematics, 5, Walter de Gruyter & Co., Berlin (1985)

[2] **A Casson**, **C McA Gordon**, *Cobordism of classical knots*, from: “A la recherche de la Topologie perdue”, (Guillou and Marin, editors), Progress in Mathematics, Volume 62 (1986), originally published as an Orsay Preprint (1975)

[3] **C McA Gordon**, **R Litherland**, **K Murasugi**, *Signatures of covering links*, Canad. J. Math. 33 (1981) 381–394

[4] **D Husemoller**, **J Milnor**, *Symmetric Bilinear Forms.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73, Springer-Verlag, New York-Heidelberg (1973)

[5] **B Jiang**, *A simple proof that the concordance group of algebraically slice knots is infinitely generated*, Proc. Amer. Math. Soc. 83 (1981) 189–192

[6] **J Levine** *Knot cobordism groups in codimension two*, Comment. Math. Helv. 44 (1969) 229–244

[7] **C Livingston**, *Order 2 algebraically slice knots*, from Proceedings of the Kirbyfest (Berkeley, CA, 1998) 335–342, Geom. Topol. Monogr., 2, Geom. Topol. Publ., Coventry (1999)

[8] **P Ozsváth**, **Z Szabó**, *Knot Floer homology and the four-ball genus*, Geometry and Topology 7 (2003) 615-639, `arXiv:math.GT/0301149`

[9] **H Seifert**, *Über das Geschlecht von Knoten*, Math. Ann. 110 (1934) 571–592