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Area preserving group actions on surfaces

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Abstract

Suppose $\mathcal{G}$ is an almost simple group containing a subgroup isomorphic to the three-dimensional integer Heisenberg group. For example any finite index subgroup of $SL(3,\mathbb{Z})$ is such a group. The main result of this paper is that every action of $\mathcal{G}$ on a closed oriented surface by area preserving diffeomorphisms factors through a finite group.

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1 Introduction and notation

This article is motivated by the program of classifying actions of higher rank lattices in simple Lie groups on closed manifolds. More specifically, we are concerned here with actions of $SL(n,\mathbb{Z})$ with $n \geq 3$, on closed oriented surfaces. A standard example of such an action is given by projectivizing the usual action of $SL(3,\mathbb{Z})$ on $\mathbb{R}^3$. Our objective here is to show that there are essentially no such actions which are symplectic or, what amounts to the same thing in this case, area preserving.

R. Zimmer conjectured (see Conjecture 2 of [30]) that any C^∞ volume preserving action of a finite index subgroup of $SL(n,\mathbb{Z})$ on a closed manifold with dimension less than n, factors through an action of a finite group. We prove this in the case that the dimension of the manifold is 2. L Polterovich [24] previously provided a proof of this result for surfaces of genus at least one. Our aim was to extend this to the case of the sphere. As it happens our techniques, which are quite different from his, are equally applicable to any genus so we present the argument in full generality.

Definition 1.1 A group $\mathcal{G}$ is called *almost simple* if every normal subgroup is either finite or has finite index.

The Margulis normal subgroups theorem (see Theorem IX.5.4 of [21] or 8.1.2 of [29]) asserts that an irreducible lattice in a semi-simple Lie group with $\mathbb{R}-$rank ≥ 2 is almost simple. In particular, any finite index subgroup of $SL(n,\mathbb{Z})$ with $n \geq 3$ is almost simple.

Suppose S is a closed oriented surface and ω is a smooth volume form. We will generally assume a fixed choice of ω and refer to a diffeomorphism $F\colon S \to S$ which preserves ω as an *area preserving diffeomorphism.* We denote the group of diffeomorphisms preserving ω by $\mathrm{Diff}_\omega(S)$ and its identity component by $\mathrm{Diff}_\omega(S)_0$. Equivalently $\mathrm{Diff}_\omega(S)_0$ is the group of diffeomorphisms which preserve ω and are isotopic to id.

A key ingredient (and perhaps limitation) of our approach to this problem is the fact that a finite index subgroup of $SL(n,\mathbb{Z})$ with $n \geq 3$ always contains a subgroup isomorphic to the three-dimensional integer Heisenberg group. Recall that this is the group of upper triangular 3×3 integer matrices with all diagonal entries equal to 1. Our main result is:

Theorem 1.2 *Suppose $\mathcal{G}$ is an almost simple group containing a subgroup isomorphic to the three-dimensional integer Heisenberg group, e.g. any finite* *index subgroup of $SL(n,\mathbb{Z})$ with $n \geq 3$. If S is a closed oriented surface then every homomorphism $\phi\colon \mathcal{G} \to \mathrm{Diff}_\omega(S)$ factors through a finite group.*

Given $\mathcal{G}, S$ and ϕ as in Theorem 1.2, let $\phi'\colon \mathcal{G} \to MCG(S)$ be the homomorphism to the mapping class group $MCG(S)$ induced by ϕ. Since nilpotent subgroups of $MCG(S)$ are virtually abelian (see [2]), the kernel $\mathcal{G}_0$ of ϕ' has an infinite order element in the integer Heisenberg subgroup. Hence by almost simplicity $\mathcal{G}_0$ has finite index in $\mathcal{G}$. Moreover, $\mathcal{G}_0$ contains a finite index subgroup of the integer Heisenberg subgroup and hence contains a subgroup isomorphic to the three-dimensional integer Heisenberg group. Thus in proving Theorem 1.2 there is no loss in replacing $\mathcal{G}$ with $\mathcal{G}_0$. In other words, we may assume that the image of ϕ is contained in $\mathrm{Diff}_\omega(S)_0$.

If $n \geq 4$, then any analytic action by $SL(n,\mathbb{Z})$ on a closed oriented surface S factors through a finite group. This was proved by Farb and Shalen [7] for $S \neq T^2$ and by Rebelo [25] for $S = T^2$. (Farb and Shalen proved this for $S = T^2$ under the assumption that the action is area preserving.) Polterovich (see Corollary 1.1.D of [24]) proved that if $n \geq 3$, then any action by $SL(n,\mathbb{Z})$ on a closed surface S other than S^2 and T^2 by area preserving diffeomorphisms factors through a finite group. All of these results can be stated in greater generality than we give here. There is also an analogous result of D. Witte ([28]), which asserts that a homomorphism $\phi\colon SL(n,\mathbb{Z}) \to Homeo(S^1)$ must factor through a finite group if $n \geq 3$.

We are grateful to Benson Farb for introducing us to the problem and for several very helpful converations. We are also grateful to David Fisher for suggestions that significantly streamlined and helped organize the paper.

2 Hyperbolic structures and normal form

Some of our proofs rely on mapping class group techniques that use hyperbolic geometry. In this section we establish notation and recall a result from [16]. For further details see, for example, [3]).

Let S be a closed orientable surface. We will say that a connected open subset M of S has *negative Euler characteristic* if $H_1(M,\mathbb{R})$ is infinite dimensional or if M is of finite type and the usual definition of Euler characteristic has a negative value. If M has negative Euler characteristic then M supports a complete hyperbolic structure.

We use the Poincaré disk model for the hyperbolic plane H. In this model, H is identified with the interior of the unit disk and geodesics are segments of Euclidean circles and straight lines that meet the boundary in right angles. A choice of complete hyperbolic structure on M provides an identification of the universal cover $\tilde{M}$ of M with H. Under this identification covering translations become isometries of H and geodesics in M lift to geodesics in H. The compactification of the interior of the unit disk by the unit circle induces a compactification of H by the 'circle at infinity' S_∞. Geodesics in H have unique endpoints on S_∞. Conversely, any pair of distinct points on S_∞ are the endpoints of a unique geodesic.

Suppose that $F\colon S \to S$ is an orientation preserving homeomorphism of a closed surface S and that $M \subset S$ is an open connected F-invariant set with negative Euler characteristic. Equip M with a complete hyperbolic structure and let $f = F|_M\colon M \to M$. We use the identification of H with $\tilde{M}$ and write $\tilde{f}\colon H \to H$ for lifts of $f\colon M \to M$ to the universal cover. A fundamental result of Nielsen theory is that every lift $\tilde{f}\colon H \to H$ extends uniquely to a homeomorphism (also called) $\tilde{f}\colon H \cup S_\infty \to H \cup S_\infty$. (A proof of this fact appears in Proposition 3.1 of [18]). If $f\colon M \to M$ is isotopic to the identity then there is a unique lift $\tilde{f}$, called the *identity lift,* that commutes with all covering translations and whose extension over S_∞ is the identity.

Every covering translation $T\colon H \to H$ extends to a homeomorphism (also called) $T\colon H \cup S_\infty \to H \cup S_\infty$. If $\gamma \subset M$ is a closed geodesic and $\tilde{\gamma} \subset H$ is a lift to H then there is an extended covering translation T whose only fixed points T^+ and T^- are the endpoints of $\tilde{\gamma}$. If $f(\gamma)$ is isotopic to γ then there is a lift $\tilde{f}\colon H \cup S_\infty \to H \cup S_\infty$ that fixes $T^\pm$ and commutes with T. The quotient space of $H \cup (S_\infty \setminus T^\pm)$ by the action of T is a closed annulus on which $\tilde{f}$ induces a homeomorphism denoted $\hat{f}\colon \mathbb{A} \to \mathbb{A}$.

The following result is an immediate corollary of Theorem 1.2 and Lemma 6.3 of [16]. If $\mathrm{Fix}(F)$ is finite, then it is just a special case of the Thurston classification theorem [27].

Theorem 2.1 *Suppose that $F\colon S \to S$ is a diffeomorphism of an orientable closed surface, that F is isotopic to the identity and that $\mathrm{Fix}(F) \neq \emptyset$. Then there is a finite set R of simple closed curves in $S \setminus \mathrm{Fix}(F)$ and a homeomorphism ϕ isotopic to F rel $\mathrm{Fix}(F)$ such that:*

(1) *ϕ setwise fixes disjoint open annulus neighborhoods $\mathbb{A}_j \subset (S \setminus \mathrm{Fix}(F))$ of the elements $\gamma_j \in R$. The restriction of ϕ to $cl(\mathbb{A}_i)$ is a non-trivial Dehn twist of a closed annulus.*

Let $\{S_i\}$ be the components of $S \setminus \cup \mathbb{A}_j$, let $X_i = \mathrm{Fix}(F) \cap S_i$, and if X_i is finite, let $M_i = S_i \setminus X_i$.

(2) *If X_i is infinite then $\phi|_{S_i} =$ identity.*

(3) *If X_i is finite, then M_i has negative Euler characteristic and $\phi|_{M_i}$ is either pseudo-Anosov or the identity.*

We say that ϕ is a *normal form for F* and that R is *the set of reducing curves for ϕ*. If R has minimal cardinality among all sets of reducing curves for all normal forms for F, then we say that *R is a minimal set of reducing curves.*

3 The proof of Theorem 1.2

We denote by $[G,H]$ the commutator $G^{-1}H^{-1}GH$.

Proposition 3.1 *If $G,H\colon S \to S$ are diffeomorphisms that are isotopic to the identity and that commute with their commutator $F = [G,H]$ then F is isotopic to the identity rel* $\mathrm{Fix}(F)$.

Remark 3.2 In any group containing elements G and H, if $F = [G,H]$ commutes with G and H then it is easy to see that $[G^k,H] = [G,H]^k = [G,H^k]$ for any $k \in \mathbb{Z}$. Hence $F^{k_1k_2} = [G^{k_1},H^{k_2}]$ for all k_1,k_2.

As an immediate consequence of this and Proposition 3.1 we have

Corollary 3.3 *Suppose that $G,H\colon S \to S$ are diffeomorphisms that are isotopic to the identity and that commute with their commutator $F = [G,H]$. Then for all $n > 0$, F^n is isotopic to the identity rel* $\mathrm{Fix}(F^n)$.

Before proving Proposition 3.1 we state and prove some required lemmas. Denote the closed annulus by $\mathbb{A}$ and its boundary components by $\partial_0\mathbb{A}$ and $\partial_1\mathbb{A}$. The universal cover $\tilde{\mathbb{A}}$ is identified with $\mathbb{R} \times [0,1]$.

Lemma 3.4 *Assume that $u_i\colon \mathbb{A} \to \mathbb{A}$, $i = 0,1$, are homeomorphisms that preserve $\partial_0\mathbb{A}$ and $\partial_1\mathbb{A}$, that u_i commutes with $v = [u_1,u_2]$ and that $\tilde{u}_i\colon \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ are lifts of u_i. Then $\tilde{v} = [\tilde{u}_1,\tilde{u}_2]$ fixes at least one point in both $\partial_0\tilde{\mathbb{A}}$ and $\partial_1\tilde{\mathbb{A}}$.*

Proof Let $p_1\colon \tilde{\mathbb{A}} \to \mathbb{R}$ be projection onto the first coordinate and let $T\colon \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ be the indivisible covering translation $T(x,y) = (x+1,y)$. We claim that $|p_1(\tilde{v}(\tilde{x})) - p_1(\tilde{x})| < 8$ for all $\tilde{x} \in \partial\tilde{\mathbb{A}}$.

Since the lift $\tilde{u}_i$ commutes with T, $\tilde{v}$ is independent of the choice of $\tilde{u}_i$. We may therefore assume that $|p_1(\tilde{u}_i(\tilde{x}_i)) - p_1(\tilde{x}_i)| < 1$ for some $\tilde{x}_i \in \partial_0\tilde{\mathbb{A}}$ and hence that $|p_1(\tilde{u}_i(\tilde{x})) - p_1(\tilde{x})| < 2$ for all $\tilde{x} \in \partial_0\tilde{\mathbb{A}}$. The claim for $\tilde{x} \in \partial_0\tilde{\mathbb{A}}$ follows immediately. The analogous argument on $\partial_1\tilde{\mathbb{A}}$ completes the proof of the claim.

The preceding argument holds for any u_i that preserve the components of $\partial\mathbb{A}$ and in particular for all iterates u_i^N. We conclude (Remark 3.2) that $|p_1(\tilde{v}^{N^2}(\tilde{x})) - p_1(\tilde{x})| < 8$ for all $\tilde{x} \in \partial\tilde{\mathbb{A}}$ and all N. Since the restriction of $\tilde{v}$ to each boundary component of $\tilde{\mathbb{A}}$ is an orientation preserving homeomorphism of the line, these homeomorphisms both have points with a bounded forward orbit, and any such homeomorphisms of the line must have a fixed point. □

An isolated end E of an open set $U \subset S$ has neighborhoods of the form $N(E) = S^1 \times [0,1)$. The set $fr(E) = cl_S(N(E)) \setminus N(E)$, called the *frontier of* E, is independent of the choice of $N(E)$. We will say that an end is *trivial* if $fr(E)$ is a single point.

Lemma 3.5 *If E is a non-trivial isolated end of an open subset U of S then there is a manifold compactification of E by a circle C satisfying the following property. If F is any orientation preserving homeomorphism of $U \cup fr(E)$ and if U is F-invariant then $F|_U$ extends to a homeomorphism f of $U \cup C$. Moreover, if G is another orientation preserving homeomorphism of $U \cup fr(E)$ and if g is the extension of $G|_U$ over $U \cup C$ then:*

1) *if F is isotopic to G relative to $fr(E)$ then f is isotopic to g relative to C. In particular, if F is isotopic to the identity relative to $fr(E)$ then f is isotopic to the identity relative to C.*

2) *fg is the extension of $(FG)|_U$.*

If E is a trivial end with $fr(E) = \{x\}$ and F and G are local diffeomorphisms on a neighborhood of x, there is a manifold compactification of E by a circle C and extensions f,g to homeomorphisms of $U \cup C$ satisfying property 2).

Proof Assume at first that E is non-trivial. The existence of C and f is a consequence of the theory of prime ends (see [23] for a good modern exposition).

Since C is a boundary component of $U \cup C$, the extension f is the unique extension of $F|_U$ over C. Property (2) follows immediately.

If F and G are as (1), then FG^{-1} is isotopic to the identity relative to $fr(E)$. It suffices to show that fg^{-1} is isotopic to the identity relative to C since precomposing with g then gives the desired isotopy of f to g. We may therefore assume that G is the identity.

Given an isotopy H_t of F to the identity relative to $fr(E)$, extend $H_t|_U$ by the identity on C to define $h_t\colon U \cup C \to U \cup C$. It suffices to show that $h_t(x)$ varies continuously in both t and x. This is clear if $x \in U$ so suppose that $x \in C$. Choose a disk neighborhood system $\{W_i\}$ for x. It suffices to show that for all i and t there exists j so that $h_s(W_j \cap U) \subset W_i \cap U$ for all s sufficiently close to t. The frontier of W_i is an arc σ_i that intersects C exactly in its endpoints. It suffices to show that $h_s(\sigma_j)$ is disjoint from σ_i and lies in the same component of $U \setminus \sigma_i$ as does σ_j.

The system $\{W_i\}$ can be chosen with three important properties (see [23] for details.) First, the interior of σ_i, thought of as an open arc in U, is the interior of a closed arc σ_i' in $U \cup fr(E)$ with endpoints in $fr(E)$. Second, the paths σ_i's are mutually disjoint and converge to a point $z \in fr(E)$. The third property is that for all i and t, the interior of $H_t(\sigma_i')$, thought of as an open arc in U is the interior of a closed arc in $U \cup C$ with the same endpoints as σ_i; this closed arc is by definition $h_t(\sigma_i)$.

Fix i and t. Since $H_t(z) = z$, there exists $j > i$ such that $H_t(\sigma_j') \cap \sigma_i' = \emptyset$. By compactness of the closed arcs, $H_s(\sigma_j') \cap \sigma_i' = \emptyset$ for all s sufficiently close to t. Since $h_s(\sigma_j)$ and σ_j have the same endpoints, $h_s(\sigma_j)$ lies in the same component of $U \setminus \sigma_i$ as does σ_j. This completes the proof in the case that E is non-trivial.

In the case that E is trivial and F, G are local diffeomorphisms we can construct C by blowing up the point x to obtain C and the continuous extensions to $U \cup C$. The blowing up construction is functorial so property 2) is satisfied. □

Proof of Proposition 3.1 We may assume without loss that $\mathrm{Fix}(F) \neq \emptyset$. Since G and H commute with F, $\mathrm{Fix}(F)$ is G-invariant and H-invariant. Let ϕ be a canonical form for F with minimal reducing set R and let X_i, S_i and M_i be as in Theorem 2.1. It suffices to show that if X_i is finite then $\phi|_{M_i}$ is not pseudo-Anosov and that $R = \emptyset$. These properties are unchanged if F is replaced by an iterate so there is no loss in replacing G, H and F by G^n, H^n and F^{n^2} for some $n > 0$. Lemma 6.2 of [16] implies that G and H permute the elements of R up to isotopy relative to $\mathrm{Fix}(F)$. We may therefore assume that G and H fix the elements of R up to isotopy relative to $\mathrm{Fix}(F)$.

We first rule out the possibility that some $\phi|_{M_i}$ is pseudo-Anosov. For each $M_i \subset M$ there are well defined elements $< F|_{M_i} >, < G|_{M_i} >$ and $< H|_{M_i} >$ in the mapping class group of M_i defined by 'restricting' F, G and H to M_i. For concreteness we give the argument for G. Since G preserves the isotopy class of each element of R, G preserves the isotopy class of each component of ∂M_i and $G \simeq G_1$ where $G_1(M_i) = M_i$. If G_2 is another such homeomorphism then, since $G_1 \simeq G_2$, $G_1|_{M_i} \simeq G_2|_{M_i}$ are isotopic as homeomorphisms of M_i. Define $< G|_{M_i} >$ to be the isotopy class determined by $G_1|_{M_i}$.

The isotopy classes $< F|_{M_i} >, < G|_{M_i} >$ and $< H|_{M_i} >$ are determined by the actions of F, G and H on isotopy classes of simple closed curves in M_i. Thus $< F|_{M_i} >= [< G|_{M_i} >< H|_{M_i} >]$ commutes with both $< G|_{M_i} >$ and $< H|_{M_i} >$ for all i.

If some $< F|_{M_i} >=< \phi|_{M_i} >$ is pseudo-Anosov with expanding lamination Λ, then $< G|_{M_i} >$ and $< H|_{M_i} >$ are contained in the stabilizer of Λ. But this stabilizer is virtually Abelian (see, for example, Lemma 2.3 of [19]) in contradiction to the fact that $< F|_{M_i} >^{m^2} = [< G|_{M_i} >^m < H|_{M_i} >^m]$ is non-trivial for all m. This completes the proof that $\phi|_{M_i}$ is not pseudo-Anosov. Thus ϕ is the identity on the complement of the annuli $\mathbb{A}_j$.

If $R \neq \emptyset$ choose $\gamma \in R$, let U be the component of $S \setminus \mathrm{Fix}(F)$ that contains γ and let f, g and h be $F|_U, G|_U$ and $H|_U$ respectively. There are three cases to consider. The first is that U is an open annulus. By Lemma 3.5 there is a compactification of U to a closed annulus $\mathbb{A}$ and homeomorphisms $\hat{f}, \hat{g}, \hat{h}, \hat{\phi}\colon \mathbb{A} \to \mathbb{A}$ that respectively extend $f, g, h, \phi|U\colon U \to U$ and that satisfy

(1) $\hat{f}$ commutes with both $\hat{g}$ and $\hat{h}$.

(2) $\hat{f} = [\hat{g}, \hat{h}]$.

(3) $\hat{f}$ is isotopic rel $\partial\mathbb{A}$ to $\hat{\phi}$.

By hypothesis,

(4) $\hat{\phi}$ is isotopic rel $\partial\mathbb{A}$ to a non-trivial Dehn twist.

so

(5) if $\hat{\alpha}$ is an arc with endpoints on distinct components of $\partial\mathbb{A}$, then $\hat{f}(\hat{\alpha})$ is not isotopic rel endpoints to α.

Property (5) contradicts Lemma 3.4 and so completes the proof in this first case.

We may now assume that U has negative Euler characteristic and hence supports a complete hyperbolic structure. The second case is that γ is not peripheral in U. There is no loss in assuming that γ is a geodesic. Choose a lift $\tilde{\gamma} \subset H$ to the universal cover of U and let $T\colon H \cup S_\infty \to H \cup S_\infty$ be the extended indivisible covering translation that preserves $\tilde{\gamma}$. Choose lifts $\tilde{g}, \tilde{h}\colon H \cup S_\infty \to H \cup S_\infty$ of g, h that commute with T and so fix the endpoints $T^\pm$ of $\tilde{\gamma}$ in S_∞. Then $\tilde{f} = [\tilde{g}, \tilde{h}]$ is a lift of f that commutes with T and there is a lift $\tilde{\phi}$ of $\phi|_U$ that is equivariantly isotopic to $\tilde{f}$. Let $\mathbb{A}$ be the annulus obtained as the quotient space of $H \cup (S_\infty \setminus T^\pm)$ by the action of T, and let $\hat{\phi}, \hat{f}, \hat{g}, \hat{h}\colon \mathbb{A} \to \mathbb{A}$ be the homeomorphisms induced by $\tilde{\phi}, \tilde{f}, \tilde{g}$ and $\tilde{h}$. Items (2) and (3) above are immediate. Since $\tilde{f}\tilde{g}$ and $\tilde{g}\tilde{f}$ project to the same homeomorphism of U and commute with T, they differ by an iterate of T. Thus $\hat{f}\hat{g} = \hat{g}\hat{f}$. The symmetric argument shows that $\hat{f}$ commutes with $\hat{h}$ so (1) is satisfied. In this case the fixed point set of $\hat{\phi}$ intersects each component of $\partial\mathbb{A}$ in a Cantor set so we must replace (4) with a weaker property (see the proof of Proposition 7.1 of [16]) for details):

(4′) $\hat{\phi}$ is isotopic to a non-trivial Dehn twist relative to a closed set that intersects both components of $\partial\mathbb{A}$.

Property (5) follows from (4′) so the proof concludes as in the previous case.

The last case is that γ is peripheral in U and is a minor variation on the second case. By Lemma 3.5, the end of U corresponding to γ can be compactified by a circle and the homeomorphisms $f, g, h, \phi|_U$ can be extended to homeomorphisms of the resulting space U_*. There is a hyperbolic structure on U_* in which γ is isotopic to a peripheral geodesic γ_*. The universal cover $\tilde{U}_*$ is naturally identified with the interior of the convex hull in H of a Cantor set $C \subset S_\infty$ and is compactifed by a circle consisting of the union of C with the full pre-image of γ_*. (See for example page 175 of [20].) The proof now proceeds exactly as in the second case using $\tilde{U}_*$ and its circle compactification in place of H and S_∞. □

Lemma 3.6 *Suppose $G, H\colon S \to S$ are area preserving, orientation preserving diffeomorphisms that are isotopic to the identity and that commute with their commutator $F = [G, H]$. Then except in one case all components of $S \setminus \mathrm{Fix}(F)$ have negative Euler characteristic. The one exception is the case that $S = S^2$ and $\mathrm{Fix}(F)$ consists of exactly two points. In this case all components of $S \setminus \mathrm{Fix}(F^2)$ have negative Euler characteristic.*

Proof Let U be a component of $S \setminus \mathrm{Fix}(F)$. By [4] it is F-invariant. The Poincaré recurrence theorem and the Brouwer plane translation theorem imply that every area preserving homeomorphism of the open disk has a fixed point; thus U can not be an open disk. If $S = T^2$, then the mean rotation vector of F is zero since F is a commutator; by [5], F has fixed points and $U \neq T^2$. We are left only with the case that U is an open annulus. By Lemma 3.5 we can compactify U to a closed annulus $\mathbb{A}$ and extend F to $\mathbb{A}$ continuously. Suppose first that we are not in the exceptional case that $S = S^2$ and $\mathrm{Fix}(F)$ consists of exactly two points. Then by Lemma 3.5 the map F is the identity on at least one component of $\partial\mathbb{A}$.

Because G and H are area preserving and preserve $\mathrm{Fix}(F)$ there exist $k, l > 0$ such that $G^k(U) = U$ and $H^l(U) = U$. By Lemma 3.5 we can extend G^k, H^l to $\mathbb{A}$ and by doubling k and l if necessary we may assume G^k, H^l preserve the boundary components of $\mathbb{A}$.

Then $F^{kl} = [G^k, H^l]$ by Remark 3.2. Let $\tilde{\mathbb{A}}$ be the universal covering space of $\mathbb{A}$ and let $u, v\colon \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ be lifts of G^k and H^l respectively. Then $w = [u, v]$ is a lift of F^{kl}. Let $\tilde{F}$ be a lift of F which is the identity on one boundary component of $\tilde{\mathbb{A}}$. Then $\tilde{F}^{kl}$ is also a lift of F^{kl}. By Lemma 3.4 the map w has fixed points in both boundary components of $\tilde{\mathbb{A}}$. It follows that w and $\tilde{F}^{kl}$ have a common fixed point and hence they are equal since they are both lifts of the same map.

But the mean rotation of the lift w is zero since it is a commutator. It follows from Theorem 2.1 of [15], that it has an interior fixed point. This implies that $\tilde{F}$ has an interior periodic point and hence an interior fixed point by the Brouwer plane translation theorem. This, in turn, implies F has a fixed point in U which is a contradiction.

We are left with the single exceptional case that $S = S^2$ and $\mathrm{Fix}(F) = \{p, q\}$ so U is the open annulus $S^2 \setminus \{p, q\}$. By Lemma 3.5 we can compactify U to form an annulus $\mathbb{A}$ and extend F, G and H to orientation preserving, area preserving homeomorphisms of $\mathbb{A}$. Then G^2 and H^2 must preserve the boundary components of $\mathbb{A}$. Suppose first that in addition one of G and H (say G for definiteness) preserves the boundary components of $\mathbb{A}$. If u, v are lifts of G and H^2 respectively to $\tilde{\mathbb{A}}$ then $[u, v]$ has mean rotation zero and is a lift of $F^2 = [G, H^2]$. Again applying Theorem 2.1 of [15] we conclude that F^2 has a fixed point in the interior of U.

We want now to show this is also true in the case that both G and H switch the boundary components of $\mathbb{A}$. In that case GH and HG must preserve the boundary components of $\mathbb{A}$. Let g and h be lifts of G and H respectively to $\tilde{\mathbb{A}}$.

They switch the ends of $\tilde{\mathbb{A}}$ and hence do not have mean rotation numbers. But all elements of the subgroup of the group generated by g and h consisting of elements which preserve the ends of of $\tilde{\mathbb{A}}$ will have well defined mean rotation numbers. This subgroup consists of all elements which can be expressed as words of even length in g and h. Let $\rho_\mu(\alpha)$ denote the mean rotation number of an element α in this subgroup and recall ρ_μ is a homomorphism. Then

$$\begin{aligned}\rho_\mu([g^2,h]) &= \rho_\mu(g^{-2}h^{-1}g^2h) = \rho_\mu((g^{-2}h^{-1}g)gh)\\ &= \rho_\mu(g^{-2}h^{-1}g^{-1}g^2) + \rho_\mu(gh) = \rho_\mu(h^{-1}g^{-1}) + \rho_\mu(gh) = 0.\end{aligned}$$

It follows from Theorem 2.1 of [15] again that $[g^2,h]$ has a fixed point, but it is a lift of $F^2 = [G^2,H]$, which must also have a fixed point.

Hence in all cases $\mathrm{Fix}(F^2)$ contains at least three points and we can conclude from the previous case that if $S \setminus \mathrm{Fix}(F^2)$ is non-empty, it has negative Euler characteristic. □

Lemma 3.7 *Suppose $F\colon S \to S$ is a homeomorphism, that each component M of $S \setminus \mathrm{Fix}(F)$ has negative Euler characteristic and that for every $n > 0$, F^n is isotopic to the identity rel $\mathrm{Fix}(F^n)$. Then $\mathrm{Per}(F) = \mathrm{Fix}(F)$.*

Proof Let f be the restriction of F to M. We must show $\mathrm{Per}(f) = \emptyset$. Suppose to the contrary that $x \in \mathrm{Per}(f)$. Say it has period $p > 1$. Choose an arc α that initiates at x, terminates at a point $y \in \mathrm{Fix}(F)$ and is otherwise disjoint from $\mathrm{Fix}(F)$. By hypothesis, $F^p(\alpha)$ is isotopic to α relative to $\mathrm{Fix}(F^p)$ and hence relative to $\mathrm{Fix}(F) \cup \{x\}$. Let $\tilde{f}\colon \tilde{M} \to \tilde{M}$ be the identity lift and let $\tilde{\alpha}_0 \subset H$ be a lift of $\alpha_0 = \alpha \setminus \{y\}$. The initial endpoint $\tilde{x}$ of $\tilde{\alpha}$ is a lift of x and the terminal end of $\tilde{\alpha}_0$ converges to a point in S_∞. The isotopy of $F^p(\alpha)$ to α relative to $\mathrm{Fix}(F) \cup \{x\}$ lifts to an isotopy of $\tilde{f}^p(\tilde{\alpha}_0)$ to α_0 relative to $S_\infty \cup \{\tilde{x}\}$. But this implies that $\tilde{f}^p(\tilde{x}) = \tilde{x}$ in contradiction to the Brouwer plane translation theorem and that fact that $\tilde{f}$ is fixed point free. This contradicts the assumption that $\mathrm{Per}(f) \neq \emptyset$. □

Proposition 3.8 *Suppose $G, H\colon S \to S$ are area preserving diffeomorphisms that are isotopic to the identity and that commute with $F = [G,H]$. Then $F^2 = id$. If each component M of $S \setminus \mathrm{Fix}(F)$ has negative Euler characteristic then $F = id$.*

Proof We consider the second part first, so we assume M has negative Euler characteristic. Then according to Theorem 1.1 of [16] either F has points of arbitrarily high period or is the identity. But according to Corollary 3.3 and Lemma 3.7, F has no points of period greater than one. Hence it is the identity.

For the more general case we need only show that each component of $S \setminus \mathrm{Fix}(F^2)$ has negative Euler characteristic. But this follows from Lemma 3.6. $\square$

Example 3.9 Let S^2 be the unit sphere in $\mathbb{R}^3$ and let $G\colon S^2 \to S^2$ be rotation through the angle π around the x-axis. Let $H\colon S^2 \to S^2$ be rotation through the angle π around an axis in the xy-plane which makes an angle of $\pi/4$ with the x-axis. Both G and H have order 2. One checks easily that $F = [G, H]\colon S^2 \to S^2$ is rotation around the z-axis through an angle of π. Rotations through angle π around perpendicular axes commute. Hence F commutes with G and H and $F^2 = id$ but $F \neq id$ which shows we cannot remove the exceptional case in the preceding proposition. The group generated by G and H is the dihedral group.

We are now prepared to prove our main result.

Theorem 1.2 *Suppose $\mathcal{G}$ is an almost simple group containing a subgroup isomorphic to the three-dimensional integer Heisenberg group. If S is a closed oriented surface then every homomorphism $\phi\colon \mathcal{G} \to \mathrm{Diff}_\omega(S)$ factors through a finite group.*

Proof As shown in the introduction we may assume that the action is by diffeomorphisms isotopic to the identity. Since $\mathcal{G}$ contains a subgroup isomorphic to the three-dimensional integer Heisenberg group there are elements G and H such that $F = [G, H]$ has infinite order in $\mathcal{G}$ and commutes with G and H. By Proposition 3.8 $\phi(F^2)$ is the identity. It follows that $K = ker(\phi)$ has infinite order. Since $\mathcal{G}$ is almost simple $K \subset \mathcal{G}$ has finite index. Hence ϕ factors through the finite group $\mathcal{G}/K$. $\square$

4 Nilpotent groups

Suppose $\mathcal{G}$ is a finitely generated group and inductively define $\mathcal{G}_i$ for $i > 0$ by $\mathcal{G}_0 = \mathcal{G}$, $\mathcal{G}_i = [\mathcal{G}, \mathcal{G}_{i-1}] :=$ the group generated by $\{[g, h] \mid g \in \mathcal{G},\ h \in \mathcal{G}_{i-1}\}$. The group $\mathcal{G}$ is called *nilpotent* if for some n, $\mathcal{G}_n = \{e\}$. If n is the smallest integer such that $\mathcal{G}_n = \{e\}$ then $\mathcal{G}$ said to be of *nilpotent length* n. $\mathcal{G}$ is Abelian, if and only if its nilpotent length is 1.

Theorem 4.1 *Suppose that $\mathcal{G}$ is a finitely generated nilpotent subgroup of $\mathrm{Diff}_\omega(S)_0$. If $S \neq S^2$ then $\mathcal{G}$ is Abelian; if $S = S^2$ then $\mathcal{G}$ has an Abelian subgroup of index two.*

Proof Suppose first that $S \neq S^2$. If $\mathcal{G}$ has nilpotent length $n > 1$ then $\mathcal{G}_{n-1}$ is non-trivial and its elements commute with all elements of $\mathcal{G}$. Since $\mathcal{G}_{n-1}$ is generated by commutators there is a non-trivial element $F \in \mathcal{G}_{n-1}$ and elements $G, H \in \mathcal{G}$ such that $F = [G, H]$. But Lemma 3.6 and Proposition 3.8 then assert that $F = id$ contradicting the assumption that $n > 1$.

In case $S = S^2$ we again assume $\mathcal{G}$ has nilpotent length $n > 1$ and hence that there is a non-trivial element $F = [G, H] \in \mathcal{G}$ which commutes with all elements of $\mathcal{G}$. If $Fix(F)$ has at least three elements then again by Lemma 3.6 and Proposition 3.8 $F = id$ contradicting the assumption that $n > 1$. Hence we are reduced to the case $\mathrm{Fix}(F)$ consists of exactly two points, $\{p, q\}$.

Let $\mathcal{G}'$ be subgroup of $\mathcal{G}$ consisting of all elements which fix both p and q. Since every element of $\mathcal{G}$ commutes with F every element either fixes the two points p and q or switches them. Hence $\mathcal{G}'$ has index two in $\mathcal{G}$ and is nilpotent.

We will prove that $\mathcal{G}'$ is Abelian. If not then then there is a non-trivial element F_0 commuting will all of $\mathcal{G}'$ and elements $G_0, H_0 \in \mathcal{G}'$ such that $F_0 = [G_0, H_0]$. Blow up the two points p and q for the three diffeomorphisms F_0, G_0 and H_0. Since $F_0 = [G_0, H_0]$ the mean rotation number of F_0 is zero. It follows from Theorem 2.1 of [15] that F_0 has an interior fixed point. It then follows by the argument above that $F_0 = id$ which is a contradiction. □

Example 4.2 One cannot replace $\mathrm{Diff}_\omega(S)_0$ with $\mathrm{Diff}_\omega(S)$ in the preceding theorem. For example, there is an action of the three-dimensional integer Heisenberg group on the two dimensional torus T^2 by area preserving diffeomorphisms. We will define the action first on the universal cover $\mathbb{R}^2$. Choose an irrational number α and define $\tilde{G}, \tilde{H}$ by $(x, y) \mapsto (x+\alpha, y)$ and $(x, y) \mapsto (x, x+y)$ respectively. The commutator $\tilde{F}$ of $\tilde{G}$ and $\tilde{H}$ satisfies $(x, y) \mapsto (x, y+\alpha)$ and so commutes with both $\tilde{G}$ and $\tilde{H}$. The maps $\tilde{G}$ and $\tilde{H}$ descend to area preserving diffeomorphisms $G, H\colon T^2 \to T^2$ that commute with their commutator. It is easy to check that G and H generate a subgroup of $\mathrm{Diff}_\omega(S)$ that is isomorphic to the three-dimensional integer Heisenberg group. The map H is a Dehn twist and so is not contained in $\mathrm{Diff}_\omega(S)_0$.

A group $\mathcal{G}$ is *metabelian* if there is a homomorphism to an abelian group whose kernel is abelian.

Corollary 4.3 *Any finitely generated nilpotent subgroup $\mathcal{G}$ of $\mathrm{Diff}_\omega(S)$ has a finite index metabelian subgroup.*

Proof If $S = S^2$, then $\mathcal{G}$ has an index two subgroup that is contained in $\mathrm{Diff}_\omega(S)_0$ so Theorem 4.1 implies that $\mathcal{G}$ is virtually abelian. For $S \neq S^2$ let $\Phi\colon \mathrm{Diff}_\omega(S) \to MCG(S)$ be the natural map. By [2], $\Phi(\mathcal{G})$ is virtually abelian. Thus $\mathcal{G}$ has a finite index subgroup $\mathcal{G}_0$ whose image $\Phi(\mathcal{G}_0)$ is abelian. Theorem 4.1 implies that the kernel of $\Phi|_{G_0}$ is abelian. □

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