ISSN 1364-0380 (on line) 1465-3060 (printed)


*Geometry & Topology*
Volume 7 (2003) 789–797
Published: 28 November 2003

# Hyperbolic cone–manifolds with large cone–angles

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**Abstract**

We prove that every closed oriented 3–manifold admits a hyperbolic cone–manifold structure with cone–angle arbitrarily close to $2\pi$.

**AMS Classification numbers** Primary: 57M50

Secondary: 30F40, 57M60

**Keywords:** Hyperbolic cone–manifold, Kleinian groups

Proposed: Jean-Pierre Otal
Seconded: David Gabai, Benson Farb

Received: 3 June 2003
Accepted: 13 November 2003

# 1 Introduction

Consider the hyperbolic 3–space in the upper half–space model $\mathbb{H}^3 \simeq \mathbb{C} \times \mathbb{R}_+$ and for $\alpha \in (0, 2\pi)$ set $S_\alpha = \{(e^{r+i\theta}, t) \mid \theta \in [0, \alpha], r \in [0, \infty), t \in \mathbb{R}_+\}$. The boundary of $S_\alpha$ is a union of two hyperbolic half–planes. Denote by $\mathbb{H}^3(\alpha)$ the space obtained from $S_\alpha$ by identifying both half–planes by a rotation around the vertical line $\{0\} \times \mathbb{R}_+$.

A distance on a 3–manifold $M$ determines a hyperbolic cone–manifold structure with singular locus a link $L \subset M$ and cone–angle $\alpha \in (0, 2\pi)$, if every point $x \in M$ has a neighborhood which can be isometrically embedded either in $\mathbb{H}^3$ or in $\mathbb{H}^3(\alpha)$ depending on $x \in M \setminus L$ or $x \in L$.

Jean–Pierre Otal showed that the connected sum $\#^k(\mathbb{S}^2 \times \mathbb{S}^1)$ of $k$ copies of $\mathbb{S}^2 \times \mathbb{S}^1$ admits a hyperbolic cone–manifold structure with cone–angle $2\pi - \epsilon$ for all $\epsilon > 0$ as follows: The manifold $\#^k(\mathbb{S}^2 \times \mathbb{S}^1)$ is the double of the genus $k$ handlebody $H$. There is a convex–cocompact hyperbolic metric on the interior of $H$ such that the boundary of the convex–core is bent along a simple closed curve $\gamma$ with dihedral angle $\pi - \frac{1}{2}\epsilon$ [2]; the convex–core is homeomorphic to $H$ and hence the double of the convex–core is homeomorphic to $\#^k(\mathbb{S}^2 \times \mathbb{S}^1)$. The induced distance determines a hyperbolic cone–manifold structure on $\#^k(\mathbb{S}^1 \times \mathbb{S}^2)$ with singular locus $\gamma$ and cone–angle $2\pi - \epsilon$. The same argument applies for every manifold which is the double of a compact manifold whose interior admits a convex–cocompact hyperbolic metric. Michel Boileau asked whether every 3–manifold has this property. Our goal is to give a positive answer to this question. We prove:

**Theorem 1** *Let $M$ be a closed and orientable 3–manifold. For every $\epsilon$ there is a distance $d_\epsilon$ which determines a hyperbolic cone–manifold structure on $M$ with cone–angle $2\pi - \epsilon$.*

Before going further, we remark that we do not claim that the singular locus is independent of $\epsilon$.

We now sketch the proof of Theorem 1. First, we construct a compact manifold $M^0$, whose boundary consists of tori, and such that there is a sequence $(M^0_n)$ of 3–manifolds obtained from $M^0$ by Dehn filling such that $M^0_n$ is homeomorphic to $M$ for all $n$. The especial structure of $M^0$ permits us to show that the interior $\operatorname{Int} M^0$ of the manifold $M^0$ admits, for every $\epsilon > 0$, a complete hyperbolic cone–manifold structure with cone–angle $2\pi - \epsilon$. Thus, it follows from the work of Hodgson and Kerckhoff [5] that for $n_\epsilon$ sufficiently large there is a distance $d_{n_\epsilon} = d_\epsilon$ on the manifold $M^0_{n_\epsilon} = M$ which determines a hyperbolic cone–manifold structure with cone–angle $2\pi - \epsilon$.

Let $(\epsilon_i)$ be a non-increasing sequence of positive numbers tending to $0$. If the corresponding sequence $(n_{\epsilon_i})$ grows fast enough, then the pointed Gromov–Hausdorff limit of the sequence $(M, d_{\epsilon_i})$ of metric spaces is a complete, smooth, hyperbolic manifold $X$ with finite volume. Moreover, the volume of the $(M, d_{\epsilon_i})$ converges to the volume of $X$ when $i$ tends to $\infty$; in particular the volume of $(M, d_{\epsilon_i})$ is uniformly bounded.

I would like to thank Michel Boileau for many useful suggestions and remarks which have clearly improved the paper.

The author has been supported by the Sonderforschungsbereich 611.

## 2 Preliminaries

### 2.1 Dehn filling

Let $N$ be a compact manifold whose boundary consists of tori $T_1, \dots, T_k$ and let $U_1, \dots, U_k$ be solid tori. For any collection $\{\phi_i\}_{i=1,\dots,k}$ of homeomorphisms $\phi_i : \partial U_i \to T_i$ let $N_{\phi_1,\dots,\phi_k}$ be the manifolds obtained from $N$ by attaching the solid torus $U_i$ via $\phi_i$ to $T_i$ for $i = 1, \dots, k$.

Suppose that for all $i$ we have a basis $(m_i, l_i)$ of $H_1(T_i; \mathbb{Z})$ and let $\mu_i$ be the meridian of the solid torus $U_i$. There are coprime integers $a_i, b_i$ with $\phi_i(\mu_i) = a_i m_i + b_i l_i$ in $H_1(T_i; \mathbb{Z})$ for all $i = 1, \dots, k$. It is well known that the manifold $N_{\phi_1,\dots,\phi_k}$ depends only on the set $\{a_1 m_1 + b_1 l_1, \dots, a_k m_k + b_k l_k\}$ of homology classes. We denote this manifold by $N_{(a_1 m_1 + b_1 l_1),\dots,(a_k m_k + b_k l_k)}$ and say that it has been obtained from $N$ filling the curves $a_i m_i + b_i l_i$.

The following theorem, due to Hodgson and Kerckhoff [5] (see also [3]), generalizes Thurston's Dehn filling theorem:

**Generalized Dehn filling theorem** *Let $N$ be a compact manifold whose boundary consists of tori $T_1, \dots, T_k$ and let $(m_i, l_i)$ be a basis of $H_1(T_i; \mathbb{Z})$ for $i = 1, \dots, k$. Assume that the interior* $\operatorname{Int} N$ *of $N$ admits a complete finite volume hyperbolic cone–manifold structure with cone–angle $\alpha \leq 2\pi$. Then there exists $C > 0$ with the following property:*

*The manifold $N_{(a_1 m_1 + b_1 l_1),\dots,(a_k m_k + b_k l_k)}$ admits a hyperbolic cone–manifold structure with cone–angle $\alpha$ if $|a_i| + |b_i| \geq C$ for all $i = 1, \dots, k$.*

### 2.2 Geometrically finite manifolds

The *convex–core* of a complete hyperbolic manifold $N$ with finitely generated fundamental group is the smallest closed convex set $CC(N)$ such that the inclusion $CC(N) \hookrightarrow N$ is a homotopy equivalence. The convex–core $CC(N)$ has empty interior if an only if $N$ is Fuchsian; since we will not be interested in this case we assume from now on that the interior of the convex–core is not empty. We will only work with geometrically finite manifolds, i.e. the convex–core has finite volume. If $N$ is geometrically finite then it is homeomorphic to the interior of a compact manifold $\mathcal{N}$ and the convex–core $CC(N)$ is homeomorphic to $\mathcal{N} \setminus \mathcal{P}$ where $\mathcal{P} \subset \partial\mathcal{N}$ is the union of all toroidal components of $\partial\mathcal{N}$ and of a collection of disjoint, non–parallel, essential simple closed curves. The pair $(\mathcal{N}, \mathcal{P})$ is said to be the pared manifold associated to $N$ and $\mathcal{P}$ is its parabolic locus ([6]).

A theorem of Thurston [13] states that the induced distance on the boundary $\partial CC(N)$ of the convex–core $CC(N)$ is a complete smooth hyperbolic metric with finite volume. The boundary components are in general not smoothly embedded, they are pleated surfaces bent along the so–called bending lamination. We will only consider geometrically finite manifolds for which the bending lamination is a weighted curve $\epsilon \cdot \gamma$. Here $\gamma$ is the simple closed geodesic of $N$ along which $\partial CC(N)$ is bent and $\pi - \epsilon$ is the dihedral angle.

The following theorem, due to Bonahon and Otal, is an especial case of [2, Théorème 1].

**Realization theorem** *Let $\mathcal{N}$ be a compact 3–manifold with incompressible boundary whose interior* $\operatorname{Int}\mathcal{N}$ *admits a complete hyperbolic metric with parabolic locus $\mathcal{P}$. If $\gamma \subset \partial\mathcal{N} \setminus \mathcal{P}$ is an essential simple closed curve such that $\partial\mathcal{N} \setminus (\gamma \cup \mathcal{P})$ is acylindrical then for every $\epsilon > 0$ there is a unique geometrically finite hyperbolic metric on* $\operatorname{Int}\mathcal{N}$ *with parabolic locus $\mathcal{P}$ and bending lamination $\epsilon \cdot \gamma$.*

We refer to [4] and to [6] for more about the geometry of the convex–core of geometrically finite manifolds.

## 3 Proof of Theorem 1

Let $S \subset M$ be a closed embedded surface which determines a Heegaard splitting $M = H_1 \cup_\phi H_2$ of $M$. Here $H_1$ and $H_2$ are handlebodies and $\phi : \partial H_1 \to \partial H_2$

is the attaching homeomorphism. Without loss of generality we may assume that $S$ has genus $g \geq 2$.

**Lemma 2** *There is a pant decomposition $P$ of $\partial H_1$ such that both pared manifolds $(H_1, P)$ and $(H_2, \phi(P))$ have incompressible and acylindrical boundary.*

**Proof** The Masur domain $\mathcal{O}(H_i)$ of the handlebody $H_i$ is an open subset of $\mathcal{PML}(\partial H_i)$, the space of projective measured laminations on $\partial H_i$. If $\gamma$ is a weighted multicurve in the Masur domain then the pared manifold $(H_i, \mathrm{supp}(\gamma))$ has incompressible and acylindrical boundary, where $\mathrm{supp}(\gamma)$ is the support of $\gamma$ (see [9, 10] for the properties of the Masur domain). Kerckhoff [7] proved that the Masur domain has full measure with respect to the measure class induced by the PL–structure of $\mathcal{PML}(\partial H_i)$. The map $\phi : \partial H_1 \to \partial H_2$ induces a homeomorphism $\phi_* : \mathcal{PML}(\partial H_1) \to \mathcal{PML}(\partial H_2)$ which preserves the canonical measure class. In particular, the intersection of $\mathcal{O}(H_1)$ and $\phi_*^{-1}(\mathcal{O}(H_2))$ is not empty and open in $\mathcal{PML}(\partial H_1)$. Since weighted multicurves are dense in $\mathcal{PML}(\partial H_1)$ the result follows. □

Now, choose a pant decomposition $P = \{p_1, \ldots, p_{3g-3}\}$ of $\partial H_1$ as in Lemma 2 and identify it with a pant decomposition $P$ of $S$. Let $S \times [-2, 2]$ be a regular neighborhood of $S$ in $M$ and $\mathcal{U}$ a regular neighborhood of $P \times \{-1, 1\}$ in $S \times [-2, 2]$; $\mathcal{U}$ is a union of disjoint open solid tori $U_1^+, \ldots, U_{3g-3}^+, U_1^-, \ldots, U_{3g-3}^-$ with $p_j^{\pm} = p_j \times \{\pm 1\} \subset U_j^{\pm}$ for all $j$. The boundary of the manifold $M^0 = M \backslash \mathcal{U}$ is a collection of tori

$$\partial M^0 = T_1^+ \cup \cdots \cup T_{3g-3}^+ \cup T_1^- \cup \cdots \cup T_{3g-3}^-$$

where $T_j^{\pm}$ bounds $U_j^{\pm}$. We choose a basis $(l_j^{\pm}, m_j^{\pm})$ of $H_1(T_j^{\pm}; \mathbb{Z})$ for $j = 1, \ldots, 3g-3$ as follows:

$l_j^{\pm}$: For all $j$ there is a properly embedded annulus

$$\mathcal{A}_j : (\mathbb{S}^1 \times [-1, 1], \mathbb{S}^1 \times \{\pm 1\}) \to (M^0 \cap S \times [-2, 2], T_j^{\pm});$$

set $l_j^{\pm} = \mathcal{A}_j|_{\mathbb{S}^1 \times \{\pm 1\}}$.

$m_j^{\pm}$: The curve $m_j^{\pm}$ is the meridian of the solid torus $U_j^{\pm}$ with the orientation chosen such that the algebraic intersection number of $m_j^{\pm}$ and $l_j^{\pm}$ is equal to 1.

For $n \in \mathbb{Z}$ let $M_n^0$ be the manifold

$$M_n^0 \stackrel{\text{def}}{=} M^0_{(nl_1^+ + m_1^+),\dots,(nl_{3g-3}^+ + m_{3g-3}^+),(-nl_1^- + m_1^-),\dots,(-nl_{3g-3}^- + m_{3g-3}^-)}$$

obtained by filling the curve $\pm nl_j^\pm + m_j^\pm$ for all $j$.

Let $V_j$ be a regular neighborhood of the image of $\mathcal{A}_j$ in $M^0$; we may assume that $V_i \cap V_j = \emptyset$ for all $i \neq j$. The interior of the manifold $M^0 \setminus \cup_j V_j$ is homeomorphic to $M \setminus P$ and its boundary is a collection $T_1, \dots, T_{3g-3}$ of tori. The complement of $M^0 \setminus \cup_j V_j$ in $M_n^0$ is a union of $3g-3$ solid tori whose meridians do not depend on $n$. In particular, $M_n^0$ is homeomorphic to $M_0^0$ for all $n$. Since $M_0^0$ is, by construction, homeomorphic to $M$, we obtain

**Lemma 3** *The manifold $M_n^0$ is homeomorphic to $M$ for all $n \in \mathbb{Z}$.* □

In order to complete the proof of Theorem 1 we make use of the following result which we will show later on.

**Proposition 4** *There is a link $L \subset \operatorname{Int} M^0$ such that for all $\epsilon > 0$ the manifold $\operatorname{Int} M^0$ admits a complete, finite volume hyperbolic cone–manifold structure with singular locus $L$ and cone–angle $2\pi - \epsilon$.*

We continue with the proof of Theorem 1. Since the manifold $\operatorname{Int} M^0$ admits a complete finite volume hyperbolic cone–manifold structure with cone–angle $2\pi - \epsilon$ it follows from the Generalized Dehn filling theorem that there is some $n$ such that $M_n^0$ admits a hyperbolic cone–manifold structure with cone–angle $2\pi - \epsilon$, too. This concludes the proof of Theorem 1 since $M$ and $M_n^0$ are homeomorphic by Lemma 3.

We now prove Proposition 4. The surface $S$ separates $M^0$ in two manifolds $M_-^0$ and $M_+^0$. The boundary $\partial M_\pm^0$ is the union of a copy of $S$ and the collection $\mathcal{P}_\pm = \cup_{j=1,\dots,3g-3} T_j^{\pm 1}$ of tori. It follows from the choice of $P$ that the manifold $M_\pm^0$ is irreducible, atoroidal and has incompressible boundary. In particular, Thurston's Hyperbolization theorem [11] implies that the interior of $M_\pm^0$ admits a complete hyperbolic metric with parabolic locus $\mathcal{P}_\pm$.

If $L \subset S$ is a simple closed curve such that $P \cup L$ fills $S$, then the pared manifold $(M_\pm^0, L)$ is acylindrical. Bonahon and Otal's Realization theorem implies that for all $\epsilon > 0$ there is a geometrically finite hyperbolic metric $g_\pm$ on the interior of $M_\pm^0$ with parabolic locus $\mathcal{P}_\pm$ and with bending lamination $\epsilon/2 \cdot L$. The convex–core $CC(M_\pm^0, g_\pm)$ can be identified with $M_\pm^0 \setminus \mathcal{P}_\pm$ and hence the boundary of the convex–core consists of a copy $S_\pm$ of the surface $S$; the identification of $S_\pm$ with $S$ induces a map $\psi : S_- \to S_+$ with

$$\operatorname{Int} M^0 = CC(M^0_-, g_-) \cup_\psi CC(M^0_+, g_+).$$

The hyperbolic surface $S_\pm$ is bent along $L$ with dihedral angle $\frac{1}{2}\epsilon$. The following lemma concludes the proof of Proposition 4.

**Lemma 5** *The map $\psi : S_- \to S_+$ is isotopic to an isometry.*

**Proof** The cover $(N_\pm, h_\pm)$ of $(\operatorname{Int} M^0_\pm, g_\pm)$ corresponding to the surface $S_\pm$ is geometrically finite. Since $S_\pm$ is incompressible we obtain that $N_\pm$ is homeomorphic to the interior of $S_\pm \times [-1, 1]$ and the parabolic locus of $(N_\pm, h_\pm)$ is the collection $P \times \{\pm 1\}$. The convex surface $S_\pm \subset \operatorname{Int} M^0_\pm$ lifts to one of the components of the boundary of the convex–core of$(N_\pm, h_\pm)$; the other components are spheres with three punctures, and hence totally geodesic. The map $\psi$ can be extended to the map $\tilde{\psi} : N_- = S_- \times (-1, 1) \to N_+ = S_+ \times (-1, 1)$ given by $(x, t) \mapsto (\psi(x), -t)$. The map $\tilde{\psi}$ maps, up to isotopy, $\mathcal{P}_-$ to $\mathcal{P}_+$ and $L$ to $L$. Hence, the uniqueness part of Bonahon and Otal's Realization theorem implies that $\tilde{\psi}$ is isotopic to an isometry and this gives the desired result. □

## Concluding remarks

Recall that in Theorem 1 we do not claim that the singular locus of $d_\epsilon$ is independent of $\epsilon$. If $M$ is the double of a compact manifold with incompressible boundary whose interior admits a convex–cocompact hyperbolic metric, then, using Otal's trick, it is possible to construct a link $L$ such that $M$ admits a hyperbolic cone–manifold structure with singular locus $L$ and cone–angle $2\pi - \epsilon$ for all $\epsilon$. Proposition 4 suggests that this may be a more general phenomenon but the author does not think that it is always possible to choose the singular locus independently of $\epsilon$.

**Question 1** *Let $L$ be a link in $\mathbb{S}^2 \times \mathbb{S}^1$ which intersects an essential sphere $n$ times. Is there a hyperbolic cone–manifold structure on $\mathbb{S}^2 \times \mathbb{S}^1$ with singular locus $L$ and with cone–angle greater than $\frac{n-2}{n} 2\pi$?*

**Question 2** *Is there a link $L \subset \mathbb{S}^3$ such that for every $\epsilon > 0$ there is a hyperbolic cone–manifold structure on $\mathbb{S}^3$ with singular locus $L$ and with cone–angle $2\pi - \epsilon$?*

We suspect that both questions have a negative answer.

We define, as suggested by Michel Boileau, the *hyperbolic volume* $\mathrm{Hypvol}(M)$ of a closed 3–manifold $M$ as the infimum of the volumes of all possible hyperbolic cone–manifold structures on $M$ with cone–angle less or equal to $2\pi$. It follows from [5] and from the Schläfli formula that the hyperbolic volume of a manifold $M$ is achieved if and only if $M$ is hyperbolic. A sequence of hyperbolic cone–manifold structures realizes the hyperbolic volume if the associated volumes converge to $\mathrm{Hypvol}(M)$. From the arguments used in the proof of the Orbifold theorem [1] it is easy to deduce that the hyperbolic volume is realized by a sequence of hyperbolic cone–manifold structures whose cone–angles are all greater or equal to $\pi$.

**Question 3** *Is there a sequence of metrics realizing the hyperbolic volume and such that the associated cone–angles tend to $2\pi$?*

As remarked in the introduction, it follows from our construction that there are sequences of hyperbolic cone–manifold structures whose cone–angles tend to $2\pi$ and which have uniformly bounded volume.

Let $M$ now be a closed orientable and irreducible 3–manifold $M$. We say that $M$ is *geometrizable* if Thurston's Geometrization Conjecture holds for it. If $M$ is geometrizable then let $M_{\mathrm{hyp}}$ be the associated complete finite volume hyperbolic manifold. In [12] we proved:

**Theorem** *Let $M$ be a closed, orientable, geometrizable and prime 3–manifold. Then the minimal volume $\mathrm{Minvol}(M)$ of $M$ is equal to $\mathrm{vol}(M_{\mathrm{hyp}})$ and moreover, the manifolds $(M, g_i)$ converge in geometrically to $M_{\mathrm{hyp}}$ for every sequence $(g_i)$ of metrics realizing $\mathrm{Minvol}(M)$. In particular, the minimal volume is achieved if and only if $M$ is hyperbolic.*

Recall that the minimal volume $\mathrm{Minvol}(M)$ of $M$ is the infimum of the volumes $\mathrm{vol}(M, g)$ of all Riemannian metrics $g$ on $M$ with sectional curvature bounded in absolute value by one. A sequence of metrics $(g_i)$ realizes the minimal volume if their sectional curvatures are bounded in absolute value by one and if $\mathrm{vol}(M, g_i)$ converges to $\mathrm{Minvol}(M)$.

Under the assumption that the manifold $M$ is geometrizable and prime, it follows with the same arguments as in [12] that the hyperbolic volume can be bounded from below by the minimal volume.

**Question 4** *If $M$ is geometrizable and prime, do the hyperbolic and the minimal volume coincide?*

This question has a positive answer if the manifold $M$ is the double of a manifold which admits a convex–cocompact metric and the answer should be also positive without this restriction. If this is the case, then it should also be possible to show that the Gromov–Hausdorff limit of every sequence of hyperbolic cone–manifold structures which realizes the hyperbolic volume is isometric to $M_{\text{hyp}}$. We do not dare to ask if the assumption on $M$ to be geometrizable can be dropped.

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