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Seiberg–Witten–Floer stable homotopy type of three-manifolds with $b_1 = 0$

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Abstract

Using Furuta's idea of finite dimensional approximation in Seiberg–Witten theory, we refine Seiberg–Witten Floer homology to obtain an invariant of homology 3–spheres which lives in the S^1–equivariant graded suspension category. In particular, this gives a construction of Seiberg–Witten Floer homology that avoids the delicate transversality problems in the standard approach. We also define a relative invariant of four-manifolds with boundary which generalizes the Bauer–Furuta stable homotopy invariant of closed four-manifolds.

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1 Introduction

Given a metric and a *spin*^{c} structure $\mathfrak{c}$ on a closed, oriented three-manifold Y with $b_1(Y)=0$, it is part of the mathematical folklore that the Seiberg–Witten equations on $\mathbb{R}\times Y$ should produce a version of Floer homology. Unfortunately, a large amount of work is necessary to take care of all the technical obstacles and to this day there are few accounts of this construction available in the literature. One difficulty is to find appropriate perturbations in order to guarantee Morse–Smale transversality. Another obstacle is the existence of a reducible solution. There are two ways of taking care of the latter problem: one could either ignore the reducible and obtain a metric dependent Floer homology, or one could do a more involved construction, taking into account the S^1–equivariance of the equations, and get a metric independent equivariant Floer homology (see [18], [20]).

In this paper we construct a pointed S^1–space $\mathrm{SWF}(Y,\mathfrak{c})$ well-defined up to stable S^1–homotopy equivalence whose reduced equivariant homology agrees with the equivariant Seiberg–Witten–Floer homology. For example, $\mathrm{SWF}(S^3,\mathfrak{c})\cong S^0$. This provides a construction of a "Floer homotopy type" (as imagined by Cohen, Jones, and Segal in [6]) in the context of Seiberg–Witten theory. It turns out that this new invariant is metric independent and its definition does not require taking particular care of the reducible solution. Moreover, many of the other complications associated with defining Floer homology, such as finding appropriate generic perturbations, are avoided.

To be more precise, $\mathrm{SWF}(Y,\mathfrak{c})$ will be an object of a category $\mathfrak{C}$, the S^1–equivariant analogue of the Spanier–Whitehead graded suspension category. We denote an object of $\mathfrak{C}$ by (X,m,n), where X is a pointed topological space with an S^1–action, $m\in\mathbb{Z}$ and $n\in\mathbb{Q}$. The interpretation is that X has index (m,n) in terms of suspensions by the representations $\mathbb{R}$ and $\mathbb{C}$ of S^1. For example, $(X,m,n),(\mathbb{R}^+\wedge X,m+1,n)$, and $(\mathbb{C}^+\wedge X,m,n+1)$ are all isomorphic in $\mathfrak{C}$. We extend the notation (X,m,n) to denote the shift of any $X\in\mathrm{Ob}\,\mathfrak{C}$. We need to allow n to be a rational number rather than an integer because the natural choice of n in the definition of our invariant will not always turn out to be an integer. This small twist causes no problems in the theory. We also use the notation $\Sigma^{-E}X$ to denote the formal desuspension of X by a vector space E with semifree S^1 action.

The main ingredient in the construction is the idea of finite dimensional approximation, as developed by M Furuta and S Bauer in [13], [4], [5]. The Seiberg–Witten map can be written as a sum $l+c: V\to V$, where $V=$

$i\ker d^* \oplus \Gamma(W_0) \subset i\Omega^1(Y) \oplus \Gamma(W_0)$, $l = *d \oplus \not\partial$ is a linear Fredholm, self-adjoint operator, and c is compact as a map between suitable Sobolev completions of V. Here V is an infinite-dimensional space, but we can restrict to V^μ_λ, the span of all eigenspaces of l with eigenvalues in the interval $(\lambda, \mu]$. Note that λ is usually taken to be negative and μ positive. If p^μ_λ denotes the projection to the finite dimensional space V^μ_λ, the map $l + p^\mu_\lambda c$ generates an S^1–equivariant flow on V^μ_λ, with trajectories

$$x : \mathbb{R} \to V, \ \ \frac{\partial}{\partial t} x(t) = -(l + p^\mu_\lambda c) x(t).$$

If we restrict to a sufficiently large ball, we can use a well-known invariant associated with such flows, the Conley index I^μ_λ. In our case this is an element in S^1–equivariant pointed homotopy type, but we will often identify it with the S^1–space that is used to define it.

In section 6 we will introduce an invariant $n(Y, \mathfrak{c}, g) \in \mathbb{Q}$ which encodes the spectral flow of the Dirac operator. For now it suffices to know that $n(Y, \mathfrak{c}, g)$ depends on the Riemannian metric g on Y, but not on λ and μ. Our main result is the following:

Theorem 1 *For $-\lambda$ and μ sufficiently large, the object $(\Sigma^{-V^0_\lambda} I^\mu_\lambda, 0, n(Y, \mathfrak{c}, g))$ depends only on Y and $\mathfrak{c}$, up to canonical isomorphism in the category $\mathfrak{C}$.*

We call the isomorphism class of $\mathrm{SWF}(Y, \mathfrak{c}) = (\Sigma^{-V^0_\lambda} I^\mu_\lambda, 0, n(Y, \mathfrak{c}, g))$ the *equivariant Seiberg–Witten–Floer stable homotopy type* of $(Y, \mathfrak{c})$.

It will follow from the construction that the equivariant homology of SWF equals the Morse–Bott homology computed from the (suitably perturbed) gradient flow of the Chern–Simons–Dirac functional on a ball in V^μ_λ. We call this the *Seiberg–Witten Floer homology* of $(Y, \mathfrak{c})$.

Note that one can think of this finite dimensional flow as a perturbation of the Seiberg–Witten flow on V. In [20], Marcolli and Wang used more standard perturbations to define equivariant Seiberg–Witten Floer homology of rational homology 3–spheres. A similar construction for all 3–manifolds is the object of forthcoming work of Kronheimer and Mrowka [18]. It might be possible to prove that our definition is equivalent to these by using a homotopy argument as $-\lambda, \mu \to \infty$. However, such an argument would have to deal with both types of perturbations at the same time. In particular, it would have to involve the whole technical machinery of [20] or [18] in order to achieve a version of Morse–Smale transversality, and this is not the goal of the present paper. We prefer to work with SWF as it is defined here.

In section 9 we construct a relative Seiberg–Witten invariant of four-manifolds with boundary. Suppose that the boundary Y of a compact, oriented four-manifold X is a (possibly empty) disjoint union of rational homology 3–spheres, and that X has a *spin*^{c} structure $\hat{\mathfrak{c}}$ which restricts to $\mathfrak{c}$ on Y. For any version of Floer homology, one expects that the solutions of the Seiberg–Witten (or instanton) equations on X induce by restriction to the boundary an element in the Floer homology of Y. In our case, let Ind be the virtual index bundle over the Picard torus $H^1(X;\mathbb{R})/H^1(X;\mathbb{Z})$ corresponding to the Dirac operators on X. If we write Ind as the difference between a vector bundle E with Thom space $T(E)$ and a trivial bundle $\mathbb{R}^\alpha \oplus \mathbb{C}^\beta$, then let us denote $T(Ind) = (T(E), \alpha, \beta + n(Y, \mathfrak{c}, g)) \in \mathrm{Ob}\,\mathfrak{C}$. The correction term $n(Y, \mathfrak{c}, g)$ is included to make $T(Ind)$ metric independent. We will prove the following:

Theorem 2 *Finite dimensional approximation of the Seiberg–Witten equations on X gives an equivariant stable homotopy class of maps:*

$$\Psi(X, \hat{\mathfrak{c}}) \in \{(T(Ind), b_2^+(X), 0), SWF(Y, \mathfrak{c})\}_{S^1}.$$

The invariant Ψ depends only on X and $\hat{\mathfrak{c}}$, up to canonical isomorphism.

In particular, when X is closed we recover the Bauer–Furuta invariant Ψ from [4]. Also, in the general case by composing Ψ with the Hurewicz map we obtain a relative invariant of X with values in the Seiberg–Witten–Floer homology of Y.

When X is a cobordism between two 3–manifolds Y_1 and Y_2 with $b_1 = 0$, we will see that the invariant Ψ can be interpreted as a morphism $\mathcal{D}_X$ between $\mathrm{SWF}(Y_1)$ and $\mathrm{SWF}(Y_2)$, with a possible shift in degree. (We omit the *spin*^{c} structures from notation for simplicity.)

We expect the following gluing result to be true:

Conjecture 1 *If X_1 is a cobordism between Y_1 and Y_2 and X_2 is a cobordism between Y_2 and Y_3, then*

$$\mathcal{D}_{X_1 \cup X_2} \cong \mathcal{D}_{X_2} \circ \mathcal{D}_{X_1}.$$

A particular case of this conjecture (for connected sums of closed four-manifolds) was proved in [5]. Note that if Conjecture 1 were true, this would give a construction of a "spectrum-valued topological quantum field theory" in 3+1 dimensions, at least for manifolds with boundary rational homology 3–spheres.

In section 10 we present an application of Theorem 2. We specialize to the case of four-manifolds with boundary that have negative definite intersection form.

For every integer $r \geq 0$ we construct an element $\gamma_r \in H^{2r+1}(\mathtt{swf}^{\mathrm{irr}}_{>0}(Y, \mathfrak{c}, g, \nu); \mathbb{Z})$, where $\mathtt{swf}^{\mathrm{irr}}_{>0}$ is a metric dependent invariant to be defined in section 8 (roughly, it equals half of the irreducible part of SWF.) We show the following bound, which parallels the one obtained by Frøyshov in [12]:

Theorem 3 *Let X be a smooth, compact, oriented 4–manifold such that $b_2^+(X) = 0$ and $\partial X = Y$ has $b_1(Y) = 0$. Then every characteristic element $c \in H_2(X, \partial X)/$ Torsion satisfies:*

$$\frac{b_2(X) + c^2}{8} \leq \max_{\mathfrak{c}} \inf_{g,\nu} \bigl(-n(Y, \mathfrak{c}, g) + \min\{r | \gamma_r = 0\}\bigr).$$

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2 Seiberg–Witten trajectories

We start by reviewing a few basic facts about the Seiberg–Witten equations on three-manifolds and cylinders. Part of our exposition is inspired from [16], [17], and [18].

Let Y be an oriented 3–manifold endowed with a metric g and a *spin*^{c} structure $\mathfrak{c}$ with spinor bundle W_0. Our orientation convention for the Clifford multiplication $\rho : TY \to \mathrm{End}\ (W_0)$ is that $\rho(e_1)\rho(e_2)\rho(e_3) = 1$ for an oriented frame e_i. Let $L = \det(W_0)$, and assume that $b_1(Y) = 0$. The fact that $b_1(Y) = 0$ implies the existence of a flat *spin*^{c} connection A_0. This allows us to identify the affine space of *spin*^{c} connections A on W_0 with $i\Omega^1(Y)$ by the correspondence which sends $a \in i\Omega^1(Y)$ to $A_0 + a$.

Let us denote by $\partial\!\!\!/_a = \rho(a) + \partial\!\!\!/ : \Gamma(W_0) \to \Gamma(W_0)$ the Dirac operator associated to the connection $A_0 + a$. In particular, $\partial\!\!\!/ = \partial\!\!\!/_0$ corresponds to the flat connection.

The gauge group $\mathcal{G} = \mathrm{Map}(Y, S^1)$ acts on the space $i\Omega^1(Y) \oplus \Gamma(W_0)$ by

$$u(a, \phi) = (a - u^{-1}du, u\phi).$$

It is convenient to work with the completions of $i\Omega^1(Y) \oplus \Gamma(W_0)$ and $\mathcal{G}$ in the L^2_{k+1} and L^2_{k+2} norms, respectively, where $k \geq 4$ is a fixed integer. In general, we denote the L^2_k completion of a space E by $L^2_k(E)$.

The Chern–Simons–Dirac functional is defined on $L^2_{k+1}(i\Omega^1(Y) \oplus \Gamma(W_0))$ by

$$CSD(a, \phi) = \frac{1}{2}\Big(-\int_Y a \wedge da + \int_Y \langle \phi, \not{\partial}_a \phi \rangle d\mathrm{vol}\Big).$$

We have $CSD(u\cdot(a, \phi)) - CSD(a, \phi) = \frac{1}{2}\int_Y u^{-1}du \wedge da = 0$ because $H^1(Y;\mathbb{Z}) = 0$, so the CSD functional is gauge invariant. A simple computation shows that its gradient (for the L^2 metric) is the vector field

$$\nabla CSD(a, \phi) = (*da + \tau(\phi, \phi), \not{\partial}_a \phi),$$

where τ is the bilinear form defined by $\tau(\phi, \psi) = \rho^{-1}(\phi\psi^*)_0$ and the subscript 0 denotes the trace-free part.

The *Seiberg–Witten equations* on Y are given by

$$*da + \tau(\phi, \phi) = 0, \quad \not{\partial}_a \phi = 0,$$

so their solutions are the critical points of the Chern–Simons–Dirac functional. A solution is called *reducible* if $\theta = 0$ and irreducible otherwise.

The following result is well-known (see [17] for the analogue in four dimensions, or see [16]):

Lemma 1 *Let (a, ϕ) be a C^2 solution to the Seiberg–Witten equations on Y. Then there exists a gauge transformation u such that $u(a, \phi)$ is smooth. Moreover, there are upper bounds on all the C^m norms of $u(a, \phi)$ which depend only on the metric on Y.*

Let us look at trajectories of the downward gradient flow of the CSD functional:

$$x = (a, \phi) : \mathbb{R} \to L^2_{k+1}(i\Omega^1(Y) \oplus \Gamma(W_0)), \ \frac{\partial}{\partial t}x(t) = -\nabla CSD(x(t)). \quad (1)$$

Seiberg–Witten trajectories $x(t)$ as above can be interpreted in a standard way as solutions of the four-dimensional monopole equations on the cylinder $\mathbb{R} \times Y$. A *spin*^{c} structure on Y induces one on $\mathbb{R} \times Y$ with spinor bundles $W^{\pm}$, and a path of spinors $\phi(t)$ on Y can be viewed as a positive spinor $\Phi \in \Gamma(W^+)$. Similarly, a path of connections $A_0 + a(t)$ on Y produces a *spin*^{c} connection

A on $\mathbb{R} \times Y$ by adding a d/dt component to the covariant derivative. There is a corresponding Dirac operator $D_A^+ : \Gamma(W^+) \to \Gamma(W^-)$. Let us denote by F_A^+ the self-dual part of half the curvature of the connection induced by A on $\det(W^\pm)$ and let us extend Clifford multiplication $\hat{\rho}$ to 2–forms in the usual way. Set $\sigma(\Phi, \Phi) = \hat{\rho}^{-1}(\Phi\Phi^*)_0$. The fact that $x(t) = (a(t), \phi(t))$ satisfies (1) can be written as

$$D_A^+\Phi = 0, \quad F_A^+ = \sigma(\Phi, \Phi).$$

These are exactly the four-dimensional Seiberg–Witten equations.

Definition 1 A Seiberg–Witten trajectory $x(t)$ is said to be *of finite type* if both $CSD(x(t))$ and $\|\phi(t)\|_{C^0}$ are bounded functions of t.

Before proving a compactness result for trajectories of finite type analogous to Lemma 1, we need to define a useful concept. If (A, Φ) are a *spin*^{c} connection and a positive spinor on a comapct 4–manifold X, we say that the *energy* of (A, Φ) is the quantity:

$$E(A, \Phi) = \frac{1}{2}\int_X \Big(|F_A|^2 + |\nabla_A \Phi|^2 + \frac{1}{4}|\Phi|^4 + \frac{s}{4}|\Phi|^2\Big),$$

where s denotes the scalar curvature. It is easy to see that E is gauge invariant.

In the case when $X = [\alpha, \beta] \times Y$ and (A, Φ) is a Seiberg–Witten trajectory $x(t) = (a(t), \phi(t)), t \in [\alpha, \beta]$, the energy can be written as the change in the CSD functional. Indeed,

$$\begin{aligned} CSD(x(\alpha)) - CSD(x(\beta)) &= \int_\alpha^\beta \Big(\|(\partial/\partial t)a(t)\|_{L^2}^2 + \|(\partial/\partial t)\phi(t)\|_{L^2}^2\Big)dt \qquad (2) \\ &= \int_X \big(|{*da} + \tau(\phi, \phi)|^2 + |\partial\!\!\!/_a \phi|^2\big) \\ &= \int_X \big(|{*da}|^2 + |\nabla_a \phi|^2 + \frac{1}{4}|\phi|^4 + \frac{s}{4}|\phi|^2\big). \end{aligned}$$

It is now easy to see that the last expression equals $E(A, \Phi)$. In the last step of the derivation we have used the Weitzenböck formula.

We have the following important result for finite type trajectories:

Proposition 1 *There exist $C_m > 0$ such that for any $(a, \phi) \in L_{k+1}^2(i\Omega^1(Y) \oplus \Gamma(W_0))$ which is equal to $x(t_0)$ for some $t_0 \in \mathbb{R}$ and some Seiberg–Witten trajectory of finite type $x : \mathbb{R} \to L_{k+1}^2(i\Omega^1(Y) \oplus \Gamma(W_0))$, there exists (a', ϕ') smooth and gauge equivalent to (a, ϕ) such that $\|(a', \phi')\|_{C^m} \leq C_m$ for all $m > 0$.*

First we must prove:

Lemma 2 *Let X be a four-dimensional Riemannian manifold with boundary such that $H^1(X;\mathbb{R}) = 0$. Denote by ν the unit normal vector to ∂X. Then there is a constant $K > 0$ such that for any $A \in \Omega^1(X)$ continuously differentiable, with $A(\nu) = 0$ on ∂X, we have:*

$$\int_X |A|^2 < K \int_X (|dA|^2 + |d^*A|^2).$$

Proof Assume there is no such K. Then we can find a sequence of normalized $A_n \in \Omega^1$ with

$$\int_X |A_n|^2 = 1, \int_X (|dA_n|^2 + |d^*A_n|^2) \to 0.$$

The additional condition $A_n(\nu) = 0$ allows us to integrate by parts in the Weitzenböck formula to obtain:

$$\int_X \big(|\nabla A_n|^2 + \langle Ric(A_n), A_n\rangle\big) = \int_X \big(|dA_n|^2 + |d^*A_n|^2\big).$$

Since Ric is a bounded tensor of A_n we obtain a uniform bound on $\|\nabla A_n\|_{L^2}$. By replacing A_n with a subsequence we can assume that A_n converge weakly in L^2_1 norm to some A such that $dA = d^*A = 0$. Furthermore, since the restriction map from $L^2_1(X)$ to $L^2(\partial X)$ is compact, we can also assume that $A_n|_{\partial X} \to A|_{\partial X}$ in $L^2(\partial X)$. Hence $A(\nu) = 0$ on ∂X (Neumann boundary value condition) and A is harmonic on X, so $A = 0$. This contradicts the strong L^2 convergence $A_n \to A$ and the fact that $\|A_n\|_{L^2} = 1$. □

Proof of Proposition 1 We start by deriving a pointwise bound on the spinorial part. Consider a trajectory of finite type $x = (a, \phi) : \mathbb{R} \to L^2_{k+1}(i\Omega^1(Y) \oplus \Gamma(W_0))$. Let S be the supremum of the pointwise norm of $\phi(t)$ over $\mathbb{R} \times Y$. If $|\phi(t)(y)| = S$ for some $(y, t) \in \mathbb{R} \times Y$, since $\phi(t) \in L^2_5 \subset C^2$, we have $\Delta|\phi|^2 \geq 0$ at that point. Here Δ is the four-dimensional Laplacian on $\mathbb{R} \times Y$. By the standard compactness argument for the Seiberg–Witten equations [17], we obtain an upper bound for $|\phi|$ which depends only on the metric on Y.

If the supremum is not attained, we can find a sequence $(y_n, t_n) \in \mathbb{R} \times Y$ with $|\phi(t_n)(y_n)| \to S$. Without loss of generality, by passing to a subsequence we can assume that $y_n \to y \in Y$ and $t_{n+1} > t_n + 2$ (hence $t_n \to \infty$). Via a reparametrization, the restriction of x to each interval $[t_n - 1, t_n + 1]$ can be interpreted as a solution (A_n, Φ_n) of the Seiberg–Witten equations on $X = [-1, 1] \times Y$. The finite type hypothesis and formula (2) give uniform bounds on

$|\Phi_n|$ and $\|dA_n\|_{L^2}$. Here we identify connections with 1–forms by comparing them to the standard product flat connection.

We can modify (A_n, Φ_n) by a gauge transformation on X so that we obtain $d^*A_n = 0$ on X and $A_n(\partial/\partial t) = 0$ on ∂X. Using Lemma 2 we get a uniform bound on $\|A_n\|_{L^2}$. After this point the Seiberg–Witten equations

$$D^+_{A_n}\Phi_n = 0, \quad d^+A_n = \sigma(\Phi_n, \Phi_n)$$

provide bounds on all the Sobolev norms of $A_n|_{X'}$ and $\Phi_n|_{X'}$ by elliptic bootstrapping. Here X' could be any compact subset in the interior of X, for example $[-1/2, 1/2] \times Y$.

Thus, after to passing to a subsequence we can assume that $(A_n, \Phi_n)|_{X'}$ converges in C^∞ to some (A, Φ), up to some gauge transformations. Note that the energies on X'

$$E'(A_n, \Phi_n) = \big(CSD(x(t_n - \tfrac{1}{2})) - CSD(x(t_n + \tfrac{1}{2}))\big) = \int_{t_n - 1/2}^{t_n + 1/2} \|(\partial/\partial t)x(t)\|^2_{L^2} dt$$

are positive, while the series $\sum_n E'(A_n, \Phi_n)$ is convergent because CSD is bounded. It follows that $E'(A_n, \Phi_n) \to 0$ as $n \to \infty$, so $E'(A, \Phi) = 0$. In temporal gauge on X', (A, Φ) must be of the form $(a(t), \phi(t))$, where $a(t)$ and $\phi(t)$ are constant in t, giving a solution of the Seiberg–Witten equations on Y. By Lemma 1, there is an upper bound for $|\phi(0)(y)|$ which depends only on Y. Now $\phi(t_n)(y_n)$ converges to $\phi(0)(y)$ up to some gauge transformation, hence the upper bound also applies to $\lim_n |\phi(t_n)(y_n)| = S$.

Therefore, in all cases we have a uniform bound $\|\phi(t)\|_{C^0} \leq C$ for all t and for all trajectories.

The next step is to deduce a similar bound for the absolute value of $CSD(x(t))$. Observe that $CSD(x(t)) > CSD(x(n))$ for all n sufficiently large. As before, we interpret the restriction of x to each interval $[n-1, n+1]$ as a solution of the Seiberg–Witten equations on $[-1, 1] \times Y$. Then we find that a subsequence of these solutions restricted to X' converges to some (A, Φ) in C^∞. Also, (A, Φ) must be constant in temporal gauge. We deduce that a subsequence of $CSD(x(n))$ converges to $CSD(a, \phi)$, where (a, ϕ) is a solution of the Seiberg–Witten equations on Y. Using Lemma 1, we get a lower bound for $CSD(x(t))$. An upper bound can be obtained similarly.

Now let us concentrate on a specific $x(t_0)$. By a linear reparametrization, we can assume $t_0 = 0$. Let $X = [-1, 1] \times Y$. Then $(A, \Phi) = (a(t), \phi(t))$ satisfies the 4–dimensional Seiberg–Witten equations. The formula (2) and the bounds on $|\phi|$ and $|CSD|$ imply a uniform bound on $\|dA\|_{L^2}$. Via a gauge transformation on X we can assume that $d^*A = 0$ on X and $A_n(\nu) = 0$ on ∂X. By Lemma 2 we obtain a bound on $\|A\|_{L^2}$ and then, by elliptic bootstrapping, on all Sobolev norms of A and Φ. The desired C^m bounds follow. □

The same proof works in the setting of a half-trajectory of finite type glued to a four-manifold with boundary. We state here the relevant result, which will prove useful to us in section 9.

Proposition 2 *Let X be a Riemannian four-manifold with a cylindrical end U isometric to $(0,\infty) \times \mathbb{R}$, and such that $X \setminus U$ is compact. Let $t > 0$ and $X_t = X \setminus ([t,\infty) \times \mathbb{R})$. Then there exist $C_{m,t} > 0$ such that any monopole on X which is gauge equivalent to a half-trajectory of finite type over U is in fact gauge equivalent over X_t to a smooth monopole (A,Φ) such that $\|(A,\Phi)\|_{C^m} \leq C_{m,t}$ for all $m > 0$.*

3 Projection to the Coulomb gauge slice

Let $\mathcal{G}_0$ be the group of "normalized" gauge transformations, ie, $u : Y \to S^1, u = e^{i\xi}$ with $\int_{Y_j} \xi = 0$ for any connected component Y_j of Y. It will be helpful to work on the space

$$V = i \ker d^* \oplus \Gamma(W_0).$$

For $(a,\phi) \in i\Omega^1(Y) \oplus \Gamma(W_0)$, there is a unique element of V which is equivalent to (a,ϕ) by a transformation in $\mathcal{G}_0$. We call this element the *Coulomb projection* of (a,ϕ).

Denote by π the orthogonal projection from $\Omega^1(Y)$ to $\ker d^*$. The space V inherits a metric $\tilde{g}$ from the L^2 inner product on $i\Omega^1(Y) \oplus \Gamma(W_0)$ in the following way: given (b,ψ) a tangent vector at $(a,\phi) \in V$, we set

$$\|(b,\psi)\|_{\tilde{g}} = \|(b,\psi) + (-id\xi, i\xi\phi)\|_{L^2}$$

where $\xi \in \mathcal{G}_0$ is such that $(b - id\xi, \psi + i\xi\phi)$ is in Coulomb gauge, ie,

$$d^*(b - id\xi) + 2i\mathrm{Re}\langle i\phi, \psi + i\xi\phi\rangle = 0.$$

The trajectories of the CSD functional restricted to V in this metric are the same as the Coulomb projections of the trajectories of the CSD functional on $i\Omega^1(Y) \oplus \Gamma(W_0)$.

For $\phi \in \Gamma(W_0)$, note that $(1-\pi) \circ \tau(\phi,\phi) \in (\ker d^*)^{\perp} = \mathrm{Im}\, d$. Define $\xi(\phi) : Y \to \mathbb{R}$ by $d\xi(\phi) = i(1-\pi) \circ \tau(\phi,\phi)$ and $\int_{Y_j} \xi(\phi) = 0$ for all connected components $Y_j \subset Y$.

Then the gradient of $CSD|_V$ in the $\tilde{g}$ metric can be written as $l+c$, where $l, c : V \to V$ are given by

$$l(a,\phi) = (*da, \partial\!\!\!/\phi)$$
$$c(a,\phi) = \big(\pi \circ \tau(\phi,\phi), \rho(a)\phi - i\xi(\phi)\phi\big).$$

Thus from now on we can concentrate on trajectories $x : \mathbb{R} \to V, (\partial/\partial t)x(t) = -(l+c)x(t)$. More generally, we can look at such trajectories with values in the L^2_{k+1} completion of V. Note that $l+c : L^2_{k+1}(V) \to L^2_k(V)$ is a continuous map. We construct all Sobolev norms on V using l as the differentiation operator:

$$\|v\|^2_{L^2_m(V)} = \sum_{j=0}^{m} \int_Y |l^j(v)|^2 d\text{vol}.$$

Consider such trajectories $x : \mathbb{R} \to L^2_{k+1}(V), k \geq 4$. Assuming they are of finite type, from Proposition 1 we know that they are locally the projections of smooth trajectories living in the ball of of radius C_m in the C^m norm, for each m. We deduce that $x(t) \in V$ for all t, x is smooth in t and there is a uniform bound on $\|x(t)\|_{C^m}$ for each m.

4 Finite dimensional approximation

In this section we use Furuta's technique to prove an essential compactness result for *approximate* Seiberg–Witten trajectories.

Note that the operator l defined in the previous section is self-adjoint, so has only real eigenvalues. In the standard L^2 metric, let $\tilde{p}^\mu_\lambda$ be the orthogonal projection from V to the finite dimensional subspace V^μ_λ spanned by the eigenvectors of l with eigenvalues in the interval $(\lambda, \mu]$.

It is useful to consider a modification of the projections so that we have a continuous family of maps, as in [14]. Thus let $\beta : \mathbb{R} \to [0, \infty)$ be a smooth function so that $\beta(x) > 0 \iff x \in (0,1)$ and the integral of β is 1. For each $-\lambda, \mu > 1$, set

$$p^\mu_\lambda = \int_0^1 \beta(\tau)\tilde{p}^{\mu-\tau}_{\lambda+\tau} d\tau.$$

Now $p^\mu_\lambda : V \to V$ varies continuously in λ and μ. Also $V^\mu_\lambda = \text{Im}(p^\mu_\lambda)$, except when μ is an eigenvalue. Let us modify the definition of V^μ_λ slightly so that it is always the image of p^μ_λ). (However, we only do that for $\mu > 1$; later on, when we talk about $V^\lambda_{\lambda'}$ for $\lambda' < \lambda \leq 0$, for technical reasons we still want it to be the span of eigenspaces with eigenvalues in $(\lambda', \lambda]$.)

Let $k \geq 4$. Then $c : L^2_{k+1}(V) \to L^2_k(V)$ is a compact map. This follows from the following facts: ξ maps L^2_{k+1} to L^2_{k+1}; the Sobolev multiplication $L^2_{k+1} \times L^2_{k+1} \to L^2_{k+1}$ is continuous; and the inclusion $L^2_{k+1} \to L^2_k$ is compact.

A useful consequence of the compactness of c is that we have

$$\|(1 - p^{\mu}_{\lambda})c(x)\|_{L^2_k} \to 0$$

when $-\lambda, \mu \to \infty$, uniformly in x when x is bounded in $L^2_{k+1}(V)$.

Let us now denote by $B(R)$ the open ball of radius R in $L^2_{k+1}(V)$. We know that there exists $R > 0$ such that all the finite type trajectories of $l + c$ are inside $B(R)$.

Proposition 3 *For any $-\lambda$ and μ sufficiently large, if a trajectory $x : \mathbb{R} \to L^2_{k+1}(V^{\mu}_{\lambda})$,*

$$(l + p^{\mu}_{\lambda}c)(x(t)) = -\frac{\partial}{\partial t}x(t)$$

satisfies $x(t) \in \overline{B(2R)}$ for all t, then in fact $x(t) \in B(R)$ for all t.

We organize the proof in three steps.

Step 1 Assume that the conclusion is false, so there exist sequences $-\lambda_n, \mu_n \to \infty$ and corresponding trajectories $x_n : \mathbb{R} \to \overline{B(2R)}$ satisfying

$$(l + p^{\mu_n}_{\lambda_n}c)(x_n(t)) = -\frac{\partial}{\partial t}x_n(t),$$

and (after a linear reparametrization) $x_n(0) \notin B(R)$. Let us denote for simplicity $\pi_n = p^{\mu_n}_{\lambda_n}$ and $\pi^n = 1 - \pi_n$. Since l and c are bounded maps from $L^2_{k+1}(V)$ to $L^2_k(V)$, there is a uniform bound

$$\begin{aligned}
\|\frac{\partial}{\partial t}x_n(t)\|_{L^2_k} &\leq \|l(x_n(t))\|_{L^2_k} + \|\pi_n c(x_n(t))\|_{L^2_k} \\
&\leq \|l(x_n(t))\|_{L^2_k} + \|c(x_n(t))\|_{L^2_k} \\
&\leq C\|x_n(t)\|_{L^2_{k+1}} \leq 2CR
\end{aligned}$$

for some constant C, independent of n and t. Therefore x_n are equicontinuous in L^2_k norm. They also sit inside a compact subset B' of $L^2_k(V)$, the closure of $B(2R)$ in this norm. After extracting a subsequence we can assume by the Arzela–Ascoli theorem that x_n converge to some $x : \mathbb{R} \to B'$, uniformly in L^2_k norm over compact sets of $t \in \mathbb{R}$. Letting n go to infinity we obtain

$$-\frac{\partial}{\partial t}x_n(t) = (l + c)x_n(t) - \pi^n c(x_n(t)) \to (l + c)x(t)$$

in $L^2_{k-1}(V)$, uniformly on compact sets of t. From here we get that

$$x_n(t) = \int_0^t \frac{\partial}{\partial t} x_n(s) ds \to -\int_0^t (l+c)(x(s))ds.$$

On the other hand, we also know that $x_n(t)$ converges to $x(t)$, so

$$(l+c)x(t) = -\frac{\partial}{\partial t} x(t).$$

The Chern–Simons–Dirac functional and the pointwise norm of the spinorial part are bounded on the compact set B'. We conclude that $x(t)$ is the Coulomb projection of a finite type trajectory for the usual Seiberg–Witten equations on Y. In particular, $x(t)$ is smooth, both on Y and in the t direction. Also $x(0) \in B(R)$. Thus

$$\|x(0)\|_{L^2_{k+1}} < R. \tag{3}$$

We seek to obtain a contradiction between (3) and the fact that $x_n(0) \notin B(R)$ for any n.

Step 2 Let W be the vector space of trajectories $x : [-1,1] \to V, x(t) = (a(t), \phi(t))$. We can introduce Sobolev norms L^2_m on this space by looking at $a(t), \phi(t)$ as sections of bundles over $[-1,1] \times Y$.

We will prove that $x_n(t) \to x(t)$ in $L^2_k(W)$.

To do this, it suffices to prove that for every $j (0 \le j \le k)$ we have

$$\Big(\frac{\partial}{\partial t}\Big)^j x_n(t) \to \Big(\frac{\partial}{\partial t}\Big)^j x(t)$$

in $L^2_{k-j}(V)$, uniformly in t, for $t \in [-1,1]$. We already know this statement to be true for $j = 0$, so we proceed by induction on j.

Assume that

$$\Big(\frac{\partial}{\partial t}\Big)^s x_n(t) \to \Big(\frac{\partial}{\partial t}\Big)^s x(t) \text{ in } L^2_{k-s}(V),$$

uniformly in t, for all $s \le j$. Then

$$-\Big(\frac{\partial}{\partial t}\Big)^{j+1}(x_n(t) - x(t)) = \Big(\frac{\partial}{\partial t}\Big)^j \big((l+\pi_n c)(x_n(t)) - (l+c)(x(t))\big)$$

$$= l\Big(\Big(\frac{\partial}{\partial t}\Big)^j (x_n(t) - x(t))\Big) + \pi_n\Big(\Big(\frac{\partial}{\partial t}\Big)^j (c(x_n(t)) - c(x(t)))\Big) - \pi^n\Big(\Big(\frac{\partial}{\partial t}\Big)^j c(x(t))\Big).$$

Here we have used the linearity of l, π_n and π^n. We discuss each of the three terms in the sum above separately and prove that each of them converges to 0 in L^2_{k-j-1} uniformly in t. For the first term this is clear, because l is a bounded linear map from L^2_{k-j} to L^2_{k-j-1}.

For the third term, $y(t) = (\partial/\partial t)^j c(x(t))$ is smooth over $[-1,1] \times Y$ by what we showed in Step 1, and $\|\pi^n y(t)\| \to 0$ for each $t \in [-1,1]$ by the spectral theorem. Here the norm can be any Sobolev norm, in particular L^2_{k-j-1}. The convergence is uniform in t because of smoothness in the t direction. Indeed, assume that we can find $\epsilon > 0$ and $t_n \in [-1,1]$ so that $\|\pi^n y(t_n)\| \geq \epsilon$ for all n. By going to a subsequence we can assume $t_n \to t \in [-1,1]$. Then

$$\epsilon \leq \|\pi^n y(t_n)\| \leq \|\pi^n(y(t_n) - y(t))\| + \|\pi^n y(t)\|$$

This is a contradiction. The last expression converges to zero because the first term is less or equal to $\|y(t_n) - y(t)\|$.

All that remains is to deal with the second term. Since $\|\pi_n y\| \leq \|y\|$ for every Sobolev norm, it suffices to show that

$$\left(\frac{\partial}{\partial t}\right)^j c(x_n(t)) \to \left(\frac{\partial}{\partial t}\right)^j c(x(t)) \text{ in } L^2_{k-j-1}(V)$$

uniformly in t. In fact we will prove a stronger L^2_{k-j} convergence. Note that $c(x_n)$ is quadratic in $x_n = (a_n, \phi_n)$ except for the term $-i\xi(\phi_n)\phi_n$. Expanding $(\partial/\partial t)^j c(x_n(t))$ by the Leibniz rule, we get expressions of the form

$$\left(\frac{\partial}{\partial t}\right)^s z_n(t) \cdot \left(\frac{\partial}{\partial t}\right)^{j-s} w_n(t),$$

where z_n, w_n are either $\xi(\phi_n)$ or local coordinates of x_n. Assume they are both local coordinates of x_n. By the inductive hypothesis, we have $(\partial/\partial t)^s z_n(t) \to (\partial/\partial t)^s z(t)$ in L^2_{k-s} and $(\partial/\partial t)^{j-s} w_n(t) \to (\partial/\partial t)^{j-s} w(t)$ in L^2_{k-j+s}, both uniformly in t. Note that $\max\,(k-s, k-j+s) \geq (k-s+k-j+s)/2 = k-(j/2) \geq k/2 \geq 2$. Therefore there is a Sobolev multiplication

$$L^2_{k-s} \times L^2_{k-j+s} \to L^2_{\min(k-s,k-j+s)}$$

and the last space is contained in L^2_{k-j}. It follows that

$$\left(\frac{\partial}{\partial t}\right)^s z_n(t) \cdot \left(\frac{\partial}{\partial t}\right)^{j-s} w_n(t) \to \left(\frac{\partial}{\partial t}\right)^s z(t) \cdot \left(\frac{\partial}{\partial t}\right)^{j-s} w(t)$$

in L^2_{k-j}, uniformly in t.

The same is true when one or both of z_n, w_n are $\xi(\phi_n)$. Clearly it is enough to show that $(\partial/\partial t)^s \xi(\phi_n(t)) \to (\partial/\partial t)^s \xi(\phi(t))$ in L^2_{k-s}, uniformly in t, for $s \leq j$. But from the discussion above we know that this is true if instead of ξ we had

$d\xi$, because this is quadratic in ϕ_n. Hence the convergence is also true for ξ. This concludes the inductive step.

Step 3 The argument in this part is based on elliptic bootstrapping for the equations on $X = [-1, 1] \times Y$. Namely, the operator $D = -\partial/\partial t - l$ acting on W is Fredholm (being the restriction of an elliptic operator). We know that

$$Dx_n(t) = \pi_n c(x_n(t)),$$

where $x_n(t) \to x(t)$ in $L^2_k(W)$. We prove by induction on $m \geq k$ that $x_n(t) \to x(t)$ in $L^2_m(W_m)$, where W_m is the restriction of W to $X_m = I_m \times Y$ and $I_m = [-1/2 - 1/m, 1/2 + 1/m]$. Assume this is true for m and we prove it for $m + 1$. The elliptic estimate gives

$$\begin{aligned}\|x_n(t) - x(t)\|_{L^2_{m+1}(W_{m+1})} &\leq C\big(\|D(x_n(t) - x(t))\|_{L^2_m(W_m)} + \|x_n(t) - x(t)\|_{L^2_m(W_m)}\big) \\ &\leq C\big(\|\pi_n c(x_n(t)) - \pi_n c(x(t))\| + \|\pi^n c(x(t))\| + \|x_n(t) - x(t)\|\big). \qquad (4)\end{aligned}$$

In the last expression all norms are taken in the $L^2_m(W_m)$ norm. We prove that each of the three terms converges to zero when $n \to \infty$. This is clear for the third term from the inductive hypothesis.

For the first term, note that $\pi_n c$ is quadratic in $x_n(t)$, apart from the term involving $\xi(\phi_n(t))$. Looking at $x_n(t)$ as L^2_m sections of a bundle over X_m, the Sobolev multiplication $L^2_m \times L^2_m \to L^2_m$ tells us that the quadratic terms are continuous maps from $L^2_m(W_m)$ to itself. From here we also deduce that $d\xi(\phi_n(t))$, which is quadratic in its argument, converges to $d\xi(\phi(t))$. By integrating over I_m we get:

$$\int_{I_m} \|\xi(\phi_n(t)) - \xi(\phi(t))\|_{L^2_{m+1}(V)} \leq \int_{I_m} C \cdot \|d\xi(\phi_n(t)) - d\xi(\phi(t))\|_{L^2_m(V)}$$

The right hand side of this inequality converges to zero as $n \to \infty$, hence so does the left hand side. Furthermore, the same is true if we replace ξ by $(\partial/\partial t)^s \xi$ and m by $m - s$. Therefore, $\xi_n(\phi_n(t)) \to \xi(\phi(t))$ in $L^2_m(W_m)$, so by the Sobolev multiplication the first term in (4) converges to zero.

Finally, for the second term in (4), recall from Step 1 that $c(x(t))$ is smooth. Hence $\pi^n(\partial/\partial t)^s c(x(t))$ converges to zero in $L^2_m(V)$, for each t and for all $s \geq 0$. The convergence is uniform in t because of smoothness in the t direction, by an argument similar to the one in Step 2. We deduce that

$$\|\pi^n(\partial/\partial t)^s c(x(t))\|_{L^2_m(W_m)} \to 0$$

as well.

Now we can conclude that the inductive step works, so $x_n(t) \to x(t)$ in $L^2_m(W')$ for all m if we take W' to be the restriction of W to $[-1/2, 1/2] \times Y$. Convergence in all Sobolev norms means C^∞ convergence, so in particular $x_n(0) \in C^\infty(V)$ and $x_n(0) \to x(0)$ in C^∞. Hence

$$\|x_n(0)\|_{L^2_{k+1}(V)} \to \|x(0)\|_{L^2_{k+1}(V)}.$$

We obtain a contradiction with the fact that $x_n(0) \notin B(R)$, so $\|x_n(0)\|_{L^2_{k+1}(V)} \geq R$, while $\|x(0)\|_{L^2_{k+1}(V)} < R$. $\square$

5 The Conley index

The Conley index is a well-known invariant in dynamics, developed by C. Conley in the 70's. Here we summarize its construction and basic properties, as presented in [7] and [24].

Let M be a finite dimensional manifold and φ a flow on M, ie, a continuous map $\varphi : M \times \mathbb{R} \to M, (x,t) \to \varphi_t(x)$, satisfying $\varphi_0 = \text{id}$ and $\varphi_s \circ \varphi_t = \varphi_{s+t}$. For a subset $A \subset M$ we define

$$\begin{aligned} A^+ &= \{x \in A : \forall t > 0, \varphi_t(x) \in A\}; \\ A^- &= \{x \in A : \forall t < 0, \varphi_t(x) \in A\}; \\ \text{Inv } A &= A^+ \cap A^-. \end{aligned}$$

It is easy to see that all of these are compact subsets of A, provided that A itself is compact.

A compact subset $S \subset M$ is called an *isolated invariant set* if there exists a compact neighborhood A such that $S = \text{Inv } A \subset \text{int}(A)$. Such an A is called an *isolating neighborhood* of S. It follows from here that $\text{Inv } S = S$.

A pair (N, L) of compact subsets $L \subset N \subset M$ is said to be an *index pair* for S if the following conditions are satisfied:

(1) $\text{Inv } (N \setminus L) = S \subset \text{int}(N \setminus L)$;

(2) L is an exit set for N, ie, for any $x \in N$ and $t > 0$ such that $\varphi_t(x) \notin N$, there exists $\tau \in [0, t)$ with $\varphi_\tau(x) \in L$.

(3) L is positively invariant in N, ie, if for $x \in L$ and $t > 0$ we have $\varphi_{[0,t]}(x) \subset N$, then in fact $\varphi_{[0,t]}(x) \subset L$.

Consider an isolated invariant set $S \subset M$ with an isolating neighborhood A. The fundamental result in Conley index theory is that there exists an index pair (N, L) for S such that $N \subset A$. We prove this theorem in a slightly stronger form which will be useful to us in section 9; the proof is relegated to Appendix A:

Theorem 4 *Let $S \subset M$ be an isolated invariant set with a compact isolating neighborhood A, and let $K_1, K_2 \subset A$ be compact sets which satisfy the following conditions:*

(i) *If $x \in K_1 \cap A^+$, then $\varphi_t(x) \notin \partial A$ for any $t \geq 0$;*

(ii) $K_2 \cap A^+ = \emptyset$.

Then there exists an index pair (N, L) for S such that $K_1 \subset N \subset A$ and $K_2 \subset L$.

Given an isolated invariant set S with index pair (N, L), one defines the *Conley index* of S to be the pointed homotopy type

$$I(\varphi, S) = (N/L, [L]).$$

The Conley index has the following properties:

(1) It depends only on S. In fact, there are natural pointed homotopy equivalences between the spaces N/L for different choices of the index pair.

(2) If φ_i is a flow on $M_i, i = 1, 2$, then $I(\varphi_1 \times \varphi_2, S_1 \times S_2) \cong I(\varphi_1, S_1) \wedge I(\varphi_2, S_2)$.

(3) If A is an isolating neighborhood for $S_t = \text{Inv } A$ for a continuous family of flows $\varphi_t, t \in [0, 1]$, then $I(\varphi_0, S_0) \cong I(\varphi_1, S_1)$. Again, there are canonical homotopy equivalences between the respective spaces.

By abuse of notation, we will often use I to denote the pointed space N/L, and say that N/L "is" the Conley index.

To give a few examples of Conley indices, for any flow $I(\varphi, \emptyset)$ is the homotopy type of a point. If p is a nondegenerate critical point of a gradient flow φ on M, then $I(\varphi, \{p\}) \cong S^k$, where k is the Morse index of p. More generally, when φ is a gradient flow and S is an isolated invariant set composed of critical points and trajectories between them satisfying the Morse-Smale condition, then one can compute a Morse homology in the usual way (as in [25]), and it turns out that it equals $\tilde{H}_*(I(\varphi, S))$.

Another useful property of the Conley index is its behavior in the presence of attractor-repeller pairs. Given a subset $A \subset M$, we define its α–limit set and its ω–limit set as:

$$\alpha(A) = \bigcap_{t<0} \overline{\varphi_{(-\infty,t]}(A)}; \quad \omega(A) = \bigcap_{t>0} \overline{\varphi_{[t,\infty)}(A)}.$$

If S is an isolated invariant set, a subset $T \subset S$ is called a *repeller* (resp. *attractor*) relative to S is there exists a neighborhood U of T in S such that $\alpha(U) = T$ (resp. $\omega(U) = T$). If $T \subset S$ is an attractor, then the set T^* of $x \in S$ such that $\omega(x) \cap T = \emptyset$ is a repeller in S, and (T, T^*) is called an *attractor-repeller pair* in S. To give an example, let S be a set of critical points and the trajectories between them in a gradient flow φ generated by a Morse function f on M. Then, for some $a \in \mathbb{R}$, we could let $T \subset S$ be the set of critical points x for which $f(x) \leq a$, together with the trajectories connecting them. In this case T^* is the set of critical points $x \in S$ for which $f(x) > a$, together with the trajectories between them.

In general, for an attractor-repeller pair (T, T^*) in S, we have the following:

Proposition 4 *Let A be an isolating neighborhood for S. Then there exist compact sets $N_3 \subset N_2 \subset N_1 \subset A$ such that $(N_1, N_2), (N_1, N_3)$, and (N_2, N_3) are index pairs for T^*, S, and T, respectively. Hence there is a coexact sequence:*

$$I(\varphi, T) \to I(\varphi, S) \to I(\varphi, T^*) \to \Sigma I(\varphi, T) \to \Sigma I(\varphi, S) \to \dots$$

Finally, we must note that an equivariant version of the Conley index was constructed by A. Floer in [11] and extended by A. M. Pruszko in [23]. Let G be a compact Lie group; in this paper we will be concerned only with $G = S^1$. If the flow φ preserves a G–symmetry on M and S is an isolated invariant set which is also invariant under the action of G, then one can generalize Theorem 4 to prove the existence of an G–invariant index pair with the required properties. The resulting Conley index $I_G(\varphi, S)$ is an element of G–equivariant pointed homotopy type. It has the same three basic properties described above, as well as a similar behavior in the presence of attractor-repeller pairs.

6 Construction of the invariant

Let us start by defining the equivariant graded suspension category $\mathfrak{C}$. Our construction is inspired from [1], [9], and [19]. However, for the sake of simplicity we do not work with a universe, but we follow a more classical approach. There are several potential dangers in doing this in an equivariant setting (see [1], [19]). However, in our case the Burnside ring $A(S^1) = \mathbb{Z}$ is particularly simple, and it turns out that our construction does not involve additional complications compared to its non-equivariant analogue in [27] and [21].

We are only interested in suspensions by the representations $\mathbb{R}$ and $\mathbb{C}$ of S^1. Thus, the objects of $\mathfrak{C}$ are triplets (X, m, n), where X is a pointed topological space with an S^1–action, $m \in \mathbb{Z}$ and $n \in \mathbb{Q}$. We require that X has the S^1–homotopy type of a S^1–CW complex (this is always true for Conley indices on manifolds). The set $\{(X, m, n), (X', m', n')\}_{S^1}$ of morphisms between two objects is nonempty only for $n - n' \in \mathbb{Z}$ and in this case it equals

$$\{(X, m, n), (X', m', n')\}_{S^1} = \operatorname{colim} [(\mathbb{R}^k \oplus \mathbb{C}^l)^+ \wedge X, (\mathbb{R}^{k+m-m'} \oplus \mathbb{C}^{l+n-n'})^+ \wedge X']_{S^1}$$

The colimit is taken over $k, l \in \mathbb{Z}$. The maps that define the colimit are given by suspensions, ie, smashing on the left at each step with either $\mathrm{id}_{(\mathbb{R}^+ \wedge * \wedge *)}$ or $\mathrm{id}_{(* \wedge \mathbb{C}^+ \wedge *)}$.

Inside of $\mathfrak{C}$ we have a subcategory $\mathfrak{C}_0$ consisting of the objects $(X, 0, 0)$. We usually denote such an object by X. Also, in general, if $X' = (X, m, n)$ is any object of $\mathfrak{C}$, we write (X', m', n') for $(X, m + m', n + n')$.

Given a finite dimensional vector space E with trivial S^1 action, we can define the desuspension of $X \in \mathrm{Ob}\, \mathfrak{C}_0$ by E to be $\Sigma^{-E} X = (E^+ \wedge X, 2 \dim E, 0)$. Alternatively, we can set $\Sigma^{-E} X = \Omega^E X$, the set of pointed maps from E^+ to X. It is easy to check that these two definitions give the same object in $\mathfrak{C}$, up to canonical isomorphism. Similarly, when E has free S^1 action apart from the origin, one can define $\Sigma^{-E} X = \Omega^E X$. This is naturally isomorphic to $(X, 0, \dim_{\mathbb{C}} E)$, because E has a canonical orientation coming from its complex structure.

Now recall the notations from section 4. We would like to consider the downward gradient flow of the Chern–Simons–Dirac functional on V_λ^μ in the metric $\tilde{g}$. However, there could be trajectories that go to infinity in finite time, so this is not well-defined. We need to take a compactly supported, smooth cut-off function u_λ^μ on V_λ^μ which is identically 1 on $B(3R)$, where R is the constant from Proposition 3. For consistency purposes we require $u_\lambda^\mu = u_{\lambda'}^{\mu'}|_{V_\lambda^\mu}$ for $\lambda' \leq \lambda$ and $\mu' \geq \mu$. Now for each λ and μ the vector field $u_\lambda^\mu \cdot (l + p_\lambda^\mu c)$ is compactly supported, so it generates a well-defined flow φ_λ^μ on V_λ^μ.

From Proposition 3 we know that there exist $-\lambda^*, \mu^* > 1$ such that for all $\lambda \leq \lambda^*, \mu \geq \mu^*$, all trajectories of φ_λ^μ inside $\overline{B(2R)}$ are in fact contained in $B(R)$. It follows that $\mathrm{Inv}\, \left(V_\lambda^\mu \cap \overline{B(2R)}\right) = S_\lambda^\mu$, the compact union of all such trajectories, and S_λ^μ is an isolated invariant set.

There is an S^1 symmetry in our case as a result of the division by $\mathcal{G}_0$ rather than the full gauge group. We have the following S^1 action on V : $e^{i\theta} \in S^1$ sends (a, ϕ) to $(a, e^{i\theta}\phi)$. The maps l and c are equivariant, and there is an induced S^1–action on the spaces V_λ^μ. Since both φ_λ^μ and S_λ^μ are invariant under the S^1 action, using the notion of equivariant Conley index from the previous section we can set

$$I_\lambda^\mu = I_{S^1}(\varphi_\lambda^\mu, S_\lambda^\mu).$$

It is now the time to explain why we desuspended by V_λ^0 in the definition of $\mathrm{SWF}(Y, \mathfrak{c})$ from the introduction. We also have to figure out what the value of $n(Y, \mathfrak{c}, g)$ should be.

One solution of the Seiberg–Witten equations on Y is the reducible $\theta = (0,0)$. Let X be a simply-connected oriented Riemannian four-manifold with boundary Y. Suppose that a neighborhood of the boundary is isometric to $[0,1] \times Y$ such that $\partial X = \{1\} \times Y$. Choose a *spin*^{c} structure on X which extends the one on Y and let $\hat{L}$ be its determinant line bundle. Let $\hat{A}$ be a smooth connection on $\hat{L}$ such that on the end it equals the pullback of the flat connection A_0 on Y. Then we can define $c_1(\hat{L})^2 \in \mathbb{Q}$ in the following way. Let N be the cardinality of $H_1(Y;\mathbb{Z})$. Then the exact sequence

$$H_c^2(X) \xrightarrow{j^*} H^2(X) \longrightarrow H^2(Y) \cong H_1(Y)$$

tells us that $Nc_1(\hat{L}) = j^*(\alpha)$ for some $\alpha \in H_c^2(X)$. Using the intersection form induced by Poincaré duality

$$H_c^2(X) \times H^2(X) \to \mathbb{Z}$$

we set

$$c_1(\hat{L})^2 = (\alpha \cdot c_1(\hat{L}))/N \in \frac{1}{N}\mathbb{Z}.$$

Denote by $D_{\hat{A}}^{\pm}$ the Dirac operators on X coupled with the connection $\hat{A}$, with spectral boundary conditions as in [2]. One can look at solutions of the Seiberg–Witten equations on X which restrict to θ on the boundary. The space $\mathcal{M}(X, \theta)$ of such solutions has a "virtual dimension"

$$\text{v.dim } \mathcal{M}(X,\theta) = 2\mathrm{ind}_{\mathbb{C}}(D_{\hat{A}}^{+}) - \frac{\chi(X) + \sigma(X) + 1}{2}. \tag{5}$$

Here $\chi(X)$ and $\sigma(X)$ are the Euler characteristic and the signature of X, respectively.

In Seiberg–Witten theory, when one tries to define a version of Floer homology it is customary to assign to the reducible a real index equal to

$$\frac{c_1(\hat{L})^2 - (2\chi(X) + 3\sigma(X) + 2)}{4} - \text{v.dim } \mathcal{M}(X,\theta)$$

On the other hand, if one were to compute the homology of the Conley index I^μ_λ by a Morse homology recipe (counting moduli spaces of gradient flow lines), the Morse index of $(0,0)$ would be different. In fact we can approximate $l + p^\mu_\lambda c$ near $(0,0)$ by its linear part l. The Morse index is then the number of negative eigenvalues of l on V^μ_λ, which is the dimension of V^0_λ. (Our convention is to also count the zero eigenvalues.) To account for this discrepancy in what the index of θ should be, we need to desuspend by $V^0_\lambda \oplus \mathbb{C}^{n(Y,\mathfrak{c},g)}$ where it is natural to set

$$n(Y,\mathfrak{c},g) = \frac{1}{2}\Big(\text{v.dim } \mathcal{M}(X,\theta) - \frac{c_1(\hat{L})^2 - (2\chi(X) + 3\sigma(X) + 2)}{4}\Big).$$

We can simplify this expression using (5):

$$n(Y,\mathfrak{c},g) = \text{ind}_{\mathbb{C}}(D^+_{\hat{A}}) - \frac{c_1(\hat{L})^2 - \sigma(X)}{8}. \tag{6}$$

We have $n(Y,\mathfrak{c},g) \in \frac{1}{8N}\mathbb{Z}$, where N is the cardinality of $H_1(Y;\mathbb{Z})$.

Moreover, if Y is an integral homology sphere the intersection form on $H_2(X)$ is unimodular and this implies that $c_1(\hat{L})^2 \equiv \sigma(X) \bmod 8$ (see for example [15]). Therefore in this case $n(Y,\mathfrak{c},g)$ is an integer.

In general, we need to see that $n(Y,\mathfrak{c},g)$ does not depend on X. We follow [22] and express it in terms of two eta invariants of Y. First, the index theorem of Atiyah, Patodi and Singer for four-manifolds with boundary [2] gives

$$\text{ind}_{\mathbb{C}}(D^+_{\hat{A}}) = \frac{1}{8}\int_X \Big(-\frac{1}{3}p_1 + c_1(\hat{A})^2\Big) + \frac{\eta_{dir} - k(\not\partial)}{2}. \tag{7}$$

Here p_1 and $c_1(\hat{A}) = \frac{i}{2\pi}F_{\hat{A}}$ are the Pontryagin and Chern forms on X, while $k(\not\partial) = \dim\ker \not\partial = \dim\ker l$ as $H_1(Y;\mathbb{R}) = 0$. The eta invariant of a self-adjoint elliptic operator D on Y is defined to be the value at 0 of the analytic continuation of the function

$$\eta_D(s) = \sum_{\lambda \neq 0} \text{sign}(\lambda)|\lambda|^{-s},$$

where λ runs over the eigenvalues of D. In our case $\eta_{dir} = \eta_{\not\partial}(0)$.

Let us also introduce the odd signature operator on $\Omega^1(Y) \oplus \Omega^0(Y)$ by

$$sign = \begin{bmatrix} *d & -d \\ -d^* & 0 \end{bmatrix}.$$

Then the signature theorem for manifolds with boundary [2] gives

$$\sigma(X) = \frac{1}{3}\int_X p_1 - \eta_{sign}. \tag{8}$$

Putting (6), (7), and (8) together we obtain

$$\begin{aligned} n(Y,\mathfrak{c},g) &= \frac{1}{8}\Big(\int_X -\frac{1}{3}p_1 + c_1(\hat{A})^2\Big) + \frac{\eta_{dir} - k(\not\partial)}{2} - \frac{c_1(\hat{L})^2 - \sigma(X)}{8} \\ &= \frac{1}{2}\big(\eta_{dir} - k(\not\partial) - \frac{\eta_{sign}}{4}\big). \end{aligned}$$

Warning Our sign conventions are somewhat different from those in [2]. In our setting the manifold X has its boundary "on the right," so that $\partial/\partial t$ on $[0,1]\times Y$ is the outward normal. Atiyah, Patodi, and Singer formulated their theorem using the inward normal, so in order to be consistent we have applied their theorem with $-\not\partial$ as the operator on the 3–manifold.

7 Proof of the main theorem

It is now the time to use the tools that we have developed so far to prove Theorem 1 announced in the introduction. Recall that we are interested in comparing the spectra $(\Sigma^{-V^0_\lambda} I^\mu_\lambda, 0, n(Y,\mathfrak{c},g))$. We will denote by m_λ the dimension of the real part of V^0_λ (coming from eigenspaces of $*d$), and by n_λ the complex dimension of the spinorial part of V^0_λ.

Proof of Theorem 1 First let us keep the metric on Y fixed and prove that $(\Sigma^{-V^0_\lambda} I^\mu_\lambda,\ 0, n(Y,\mathfrak{c},g))$ are naturally isomorphic for different $\lambda \leq \lambda^*, \mu \geq \mu^*$. In fact we just need to do this for $\Sigma^{-V^0_\lambda} I^\mu_\lambda$, because $n(Y,\mathfrak{c},g)$ does not depend on λ and μ.

It is not hard to see that for any λ and μ, the finite energy trajectories of $l+p^\mu_\lambda c p^\mu_\lambda$ are contained in V^μ_λ. Let $\lambda' \leq \lambda \leq \lambda^*, \mu' \geq \mu \geq \mu^*$. Then $\overline{B(2R)} \cap V^{\mu'}_{\lambda'}$ is an isolating neighborhood for all $S^b_a, a \in [\lambda', \lambda], b \in [\mu, \mu']$. By Property 3 of the Conley index, for $\tilde{\varphi}^\mu_\lambda$ the flow of $u^{\mu'}_{\lambda'} \cdot (l + p^\mu_\lambda c)$ on $V^{\mu'}_{\lambda'}$,

$$I^{\mu'}_{\lambda'} \cong I_{S^1}(\tilde{\varphi}^\mu_\lambda, S^\mu_\lambda).$$

Let $V^{\mu'}_{\lambda'} = V^\mu_\lambda \oplus \bar{V}$ so that $\bar{V}$ is the orthogonal complement of V^μ_λ in the L^2 metric, a span of eigenspaces of l. Another isolating neighborhood of S^μ_λ is then $(\overline{B(3R/2)} \cap V^\mu_\lambda) \times D$, where D is a small closed ball in $\bar{V}$ centered at the origin. The flow $\tilde{\varphi}^\mu_\lambda$ is then homotopic to the product of φ^μ_λ and a flow ψ on $\bar{V}$ which is generated by a vector field that is identical to l on D. From the definition of the Conley index it is easy to see that $I_{S^1}(\psi, \{0\}) \cong I_{S^1}(\psi_0, \{0\})$. Here ψ_0 is the linear flow generated by l on $\bar{V}$, and the corresponding equivariant Conley index can be computed using Property 2 of the Conley index: it equals $(V^{\lambda}_{\lambda'})^+$.

By the same Property 2 we obtain

$$I^{\mu'}_{\lambda'} \cong I_{S^1}(\tilde{\varphi}^{\mu}_{\lambda}, S^{\mu}_{\lambda}) \cong (V^{\lambda}_{\lambda'})^+ \wedge I^{\mu}_{\lambda}.$$

This implies that $\Sigma^{-V^0_{\lambda}} I^{\mu}_{\lambda}$ and $\Sigma^{-V^0_{\lambda'}} I^{\mu'}_{\lambda'}$ are canonically isomorphic.

Next we study what happens when we vary the metric on Y. We start by exhibiting an isomorphism between the objects $(\Sigma^{-V^0_{\lambda}} I^{\mu}_{\lambda}, 0, n(Y, \mathfrak{c}, g))$ constructed for two metrics g_0, g_1 on Y sufficiently close to each other. Consider a smooth homotopy $(g_t)_{0 \leq t \leq 1}$ between the two metrics, which is constant near $t = 0$. We will use the subscript t to describe that the metric in which each object is constructed is g_t. Assuming all the g_t are very close to each other, we can arrange so that:

- there exist $R, -\lambda^*, \mu^*$ large enough and independent of t so that Proposition 3 is true for all metrics g_t and for all values $\lambda \leq \lambda^*, \mu \geq \mu^*$;

- there exist some $\lambda < \lambda^*$ and $\mu > \mu^*$ such that neither λ nor μ is an eigenvalue for any l_t. Hence the spaces $(V^{\mu}_{\lambda})_t$ have the same dimension for all t, so they make up a vector bundle over $[0, 1]$. Via a linear isomorphism that varies continuously in t we can identify all $(V^{\mu}_{\lambda})_t$ as being the same space V^{μ}_{λ};

- for any $t_1, t_2 \in [0, 1]$ we have $B(R)_{t_1} \subset B(2R)_{t_2}$. Here we already think of the balls as subsets of the same space V^{μ}_{λ}.

Then

$$\bigcap_{t \in [0,1]} \overline{B(2R)}_t$$

is a compact isolating neighborhood for S^{μ}_{λ} in any metric g_t with the flow $(\varphi^{\mu}_{\lambda})_t$ on V^{μ}_{λ}. Note that $(\varphi^{\mu}_{\lambda})_t$ varies continuously in t. By Property 3 of the Conley index,

$$(I^{\mu}_{\lambda})_0 \cong (I^{\mu}_{\lambda})_1.$$

The difference $n_{\lambda,0} - n_{\lambda,1}$ is the number of eigenvalue lines of $-\not{\partial}_t, t \in [0, 1]$ that cross the $-\epsilon$ line, counted with sign, ie, the spectral flow $SF(-\not{\partial}_t)$ as defined in [3]. Atiyah, Patodi and Singer prove that it equals the index of the operator $\partial/\partial t + \not{\partial}_t$ on $\hat{Y} = [0, 1] \times Y$ with the metric g_t on the slice $t \times Y$ and with the vector $\partial/\partial t$ always of unit length.

Choose a 4–manifold X_0 as in the previous section, with a neighborhood of the boundary isometric to $\mathbb{R}_+ \times Y$. We can glue $\hat{Y}$ to the end of X_0 to obtain a manifold X_1 diffeomorphic to X_0. Then

$$\mathrm{ind}_{\mathbb{C}}(D^+_{\hat{A},1}) = \mathrm{ind}_{\mathbb{C}}(D^+_{\hat{A},0}) + SF(-\not{\partial}_t)$$

by excision. From the formula (6) and using the fact that $c_1(L), \chi$ and σ do not depend on the metric we get

$$n(Y,\mathfrak{c},g_1) - n(Y,\mathfrak{c},g_0) = SF(-\not{\partial}_t) = n_{\lambda,0} - n_{\lambda,1}.$$

It follows that $(V_\lambda^0)_0 \oplus \mathbb{C}^{n(Y,\mathfrak{c},g_0)} \cong (V_\lambda^0)_1 \oplus \mathbb{C}^{n(Y,\mathfrak{c},g_1)}$ because the $*d$ operator has no spectral flow (for any metric its kernel is zero since $H_1(Y;\mathbb{R}) = 0$). The orientation class of this isomorphism is canonical, because complex vector spaces carry canonical orientations.

Thus we have constructed an isomorphism between the objects $(\Sigma^{-V_\lambda^0} I_\lambda^\mu, 0, n(Y, \mathfrak{c}, g))$ for two different metrics close to each other. Since the space of metrics Met is path connected (in fact contractible), we can compose such isomorphisms and reach any metric from any other one.

In order to have an object in $\mathfrak{C}$ well-defined up to *canonical* isomorphism, we need to make sure that the isomorphisms obtained by going from one metric to another along different paths are identical. Because Met is contractible, this reduces to proving that when we go around a small loop in Met the construction above induces the identity morphism on $(\Sigma^{-V_\lambda^0} I_\lambda^\mu, 0, n(Y,\mathfrak{c},g))$. Such a small loop bounds a disc D in Met, and we can find μ and λ so that they are not in the spectrum of $*d \oplus \not{\partial}$ for any metric in D. Then the vector spaces V_λ^μ form a vector bundle over D, which implies that they can all be identified with one vector space, on which the Conley indices for different metrics are the same up to canonical isomorphism. The vector spaces $V_\lambda^0 \oplus \mathbb{C}^{n(Y,\mathfrak{c},g)}$ are also related to each other by canonical isomorphisms in the homotopy category. Hence going around the loop must give back the identity morphism in $\mathfrak{C}$.

A similar homotopy argument proves independence of the choice of R in Proposition 3. Thus $\mathrm{SWF}(Y,\mathfrak{c})$ must depend only on Y and on its *spin*^{c} structure, up to canonical isomorphism in the category $\mathfrak{C}$. □

8 The irreducible Seiberg–Witten–Floer invariants

In this section we construct a decomposition of the Seiberg–Witten–Floer invariant into its reducible and irreducible parts. This decomposition only exists provided that the reducible θ is an isolated critical point of the Chern–Simons–Dirac functional. To make sure that this condition is satisfied, we need to depart here from our nonperturbative approach to Seiberg–Witten theory. We introduce the *perturbed Seiberg–Witten equations* on Y :

$$*(da - d\nu) + \tau(\phi,\phi) = 0, \quad \not{\partial}_a\phi = 0, \tag{9}$$

where ν is a fixed L^2_{k+1} imaginary 1-form on Y such that $d^*\nu = 0$.

In general, the solutions to (9) are the critical points of the perturbed Chern–Simons–Dirac functional:

$$CSD_\nu(a,\phi) = CSD(a,\phi) + \frac{1}{2}\int_Y a \wedge d\nu.$$

Our compactness results (Proposition 1 and Proposition 3) are still true for the perturbed Seiberg–Witten trajectories and their approximations in the finite dimensional subspaces. The only difference consists in replacing the compact map c with $c_\nu = c - *d\nu$. There is still a unique reducible solution to the equation $(l + c_\nu)(a,\phi) = 0$, namely $\theta_\nu = (\nu, 0)$. Homotopy arguments similar to those in the proof of Theorem 1 show that the SWF invariant obtained from the perturbed Seiberg–Witten trajectories (in the same way as before) is isomorphic to $\mathrm{SWF}(Y, \mathfrak{c})$.

The advantage of working with the perturbed equations is that we can assume any nice properties which are satisfied for generic ν. The conditions that are needed for our discussion are pretty mild:

Definition 2 A perturbation $\nu \in L^2_{k+1}(i\Omega^1(Y))$ is called *good* if $\ker \not{\partial}_\nu = 0$ and there exists $\epsilon > 0$ such that there are no critical points x of CSD_ν with $CSD_\nu(x) \in (0, \epsilon)$.

Lemma 3 *There is a Baire set of perturbations ν which are good.*

Proof Proposition 3 in [12] states that there is a Baire set of forms ν for which all the critical points of CSD_ν are nondegenerate. Nondegenerate critical points are isolated. Since their moduli space is compact, we deduce that it is finite, so there exists ϵ as required in Definition 2. Furthermore, the condition $\ker \not{\partial}_\nu = 0$ is equivalent to the fact that the reducible θ_ν is nondegenerate. □

Let us choose a good perturbation ν, and let us look at the finite dimensional approximation in the space V^μ_λ. For large μ and $-\lambda$, it is easy to see that we must have $\ker(\not{\partial}_{p^\mu_\lambda \nu}) = 0$. This implies that $\theta_{p^\mu_\lambda \nu}$, the reducible solution to $l + p^\mu_\lambda c_\nu = 0$, is an isolated critical point of $CSD_\nu|_{V^\mu_\lambda}$. Note that $CSD_\nu(\theta_{p^\mu_\lambda \nu}) = 0$, and there are no critical points x of $CSD_\nu|_{V^\mu_\lambda}$ with $CSD_\nu(x) \in (0, \epsilon/2)$. Thus, in addition to $S = S^\mu_\lambda$, we can construct four other interesting isolated invariant sets for the flow φ^μ_λ :

- $S^{\mathrm{irr}}_{>0}$ =the set of critical points x of $CSD_\nu|_{V^\mu_\lambda}$ with $CSD_\nu(x) > 0$, together with all the trajectories between them; when it becomes necessary to indicate the dependence on cutoffs, we will write $(S^{\mathrm{irr}}_{>0})^\mu_\lambda$;
- $S^{\mathrm{irr}}_{\leq 0}$ = same as above, but with $CSD_\nu(x) \leq 0$ and requiring x to be irreducible;
- $S_{\leq 0}$ = same as above, with $CSD_\nu(x) \leq 0$ but allowing x to be the reducible;
- finally, $\Theta = \{\theta_{p^\mu_\lambda \nu}\}$.

Since every trajectory contained in S must end up in a critical point and CSD_ν is decreasing along trajectories, $S_{\leq 0}$ must be an attractor in S. Its dual repeller is $S^{\mathrm{irr}}_{>0}$, so by Proposition 4 we have a coexact sequence (omitting the flow φ^μ_λ and the group S^1 from the notation):

$$I(S_{\leq 0}) \to I(S) \to I(S^{\mathrm{irr}}_{>0}) \to \Sigma I(S_{\leq 0}) \to \cdots \tag{10}$$

Similarly, $(S^{\mathrm{irr}}_{\leq 0}, \Theta)$ is an attractor-repeller pair in $S_{\leq 0}$, so there is another coexact sequence:

$$I(S^{\mathrm{irr}}_{\leq 0}) \to I(S_{\leq 0}) \to I(\Theta) \to \Sigma I(S^{\mathrm{irr}}_{\leq 0}) \to \cdots \tag{11}$$

These two sequences give a decomposition of $I(S)$ into several pieces which are easier to understand. Indeed, $I(\Theta) \cong (V^0_\lambda)^+$. Also, the intersection of S with the fixed point set of V^μ_λ is simply Θ. This implies that $S^{\mathrm{irr}}_{>0}$ and $S^{\mathrm{irr}}_{\leq 0}$ have neighborhoods in which the action of S^1 is free, so $I(S^{\mathrm{irr}}_{>0})$ and $I(S^{\mathrm{irr}}_{\leq 0})$ are S^1–free as well (apart from the basepoint). Denote by $(I^{\mathrm{irr}}_{>0})^\mu_\lambda$ the quotient of $I(S^{\mathrm{irr}}_{>0}) \setminus \{*\}$ by the action of S^1.

Let us now rewrite these constructions to get something independent of the cutoffs. Just like we did in the construction of SWF, we can consider the following object of $\mathfrak{C}$:

$$\mathrm{SWF}^{\mathrm{irr}}_{>0}(Y, \mathfrak{c}, g, \nu) = \Sigma^{-V^0_\lambda} I(S^{\mathrm{irr}}_{>0})^\mu_\lambda$$

and prove that it is independent of μ and λ (but not on the metric!) up to canonical isomorphism. Similarly we get invariants $\mathrm{SWF}^{\mathrm{irr}}_{\leq 0}$ and $\mathrm{SWF}_{\leq 0}$. The coexact sequences (10) and (11) give rise to exact triangles in the category $\mathfrak{C}$ (in the terminology of [21]):

$$\mathrm{SWF}_{\leq 0} \to (\mathrm{SWF}, 0, n(Y, \mathfrak{c}, g)) \to \mathrm{SWF}^{\mathrm{irr}}_{>0} \to \Sigma(\mathrm{SWF}_{\leq 0}) \to \cdots$$
$$\mathrm{SWF}^{\mathrm{irr}}_{\leq 0} \to \mathrm{SWF}_{\leq 0} \to S^0 \to \Sigma(\mathrm{SWF}^{\mathrm{irr}}_{\leq 0}) \to \cdots$$

Furthermore, we could also consider the object

$$\mathtt{swf}^{\mathrm{irr}}_{>0}(Y, \mathfrak{c}, g, \nu) = \Sigma^{-V^0_\lambda} (I^{\mathrm{irr}}_{>0})^\mu_\lambda$$

which lives in the nonequivariant graded suspension category (see [21]). This is basically the "quotient" of $\mathrm{SWF}^{\mathrm{irr}}_{>0}$ under the S^1 action. It is independent of λ and μ, but not of the metric and perturbation. We could call it the (nonequivariant) *positive irreducible Seiberg–Witten–Floer stable homotopy type* of $(Y, \mathfrak{c}, g, \nu)$. Similarly we can define another metric-dependent invariant $\mathtt{swf}^{\mathrm{irr}}_{\leq 0}$.

Remark If the flows φ^{μ}_{λ} satisfy the Morse-Smale condition, then the homology of $\mathtt{swf}^{\mathrm{irr}}_{>0}$ (resp. $\mathtt{swf}^{\mathrm{irr}}_{\leq 0}$) coincides with the Morse homology computed from the irreducible critical points with $CSD_{\nu} > 0$ (resp. $CSD_{\nu} \leq 0$) and the trajectories between them. But we could also consider all the irreducible critical points and compute a Morse homology $SWHF(Y, \mathfrak{c}, g, \nu)$, which is the usual irreducible Seiberg–Witten–Floer homology (see [16], [20]). We expect a long exact sequence:

$$\cdots \to H_*(\mathtt{swf}^{\mathrm{irr}}_{\leq 0}) \to SWHF_* \to H_*(\mathtt{swf}^{\mathrm{irr}}_{>0}) \to H_{*-1}(\mathtt{swf}^{\mathrm{irr}}_{\leq 0}) \to \cdots \tag{12}$$

However, it is important to note that (12) does not come from an exact triangle and, in fact, there is no natural stable homotopy invariant whose homology is $SWHF$. The reason is that the interaction of the reducible with the trajectories between irreducibles can be ignored in homology (it is a substratum of higher codimension than the relevant one), but it cannot be ignored in homotopy.

9 Four-manifolds with boundary

In this section we prove Theorem 2. Let X be a compact oriented 4–manifold with boundary Y. As in section 6, we let X have a metric such that a neighborhood of its boundary is isometric to $[0,1] \times Y$, with $\partial X = \{1\} \times Y$. Assume that X has a *spin*^{c} structure $\hat{\mathfrak{c}}$ which extends $\mathfrak{c}$. Let W^+, W^- be the two spinor bundles, $W = W^+ \oplus W^-$, and $\hat{L}$ the determinant line bundle. (We shall often put a hat over the four-dimensional objects.) We also suppose that X is *homology oriented*, which means that we are given orientations on $H^1(X; \mathbb{R})$ and $H^2_+(X; \mathbb{R})$.

Our goal is to obtain a morphism Ψ between the Thom space of a bundle over the Picard torus $Pic^0(X)$ and the stable homotopy invariant $\mathrm{SWF}(Y, \mathfrak{c})$, with a possible shift in degree. We construct a representative for this morphism as the finite dimensional approximation of the Seiberg–Witten map for X.

Let $\hat{A}_0$ be a fixed *spin*^{c} connection on W. Then every other *spin*^{c} connection on W can be written $\hat{A}_0 + \hat{a}, \hat{a} \in i\Omega^1(X)$. There is a corresponding Dirac operator

$$D_{\hat{A}_0 + \hat{a}} = D_{\hat{A}_0} + \hat{\rho}(\hat{a}),$$

where $\hat{\rho}$ denotes Clifford multiplication on the four-dimensional spinors. Let $\mathcal{C}$ be the space of *spin*^{c} connections of the form $\hat{A}_0 + \ker \hat{d}$. An appropriate Coulomb gauge condition for the forms on X is $\hat{a} \in \mathrm{Im}(\hat{d}^*), \hat{a}|_{\partial X}(\nu) = 0$, where ν is the unit normal to the boundary and $\hat{d}^*$ is the four-dimensional d^* operator. Denote by $\Omega^1_g(X)$ the space of such forms. Then, for each $\mu \geq 0$ we have a *Seiberg–Witten map*

$$\begin{aligned} \widetilde{SW}^\mu : \mathcal{C} \times (i\Omega^1_g(X) \oplus \Gamma(W^+)) &\to \mathcal{C} \times (i\Omega^2_+(X) \oplus \Gamma(W^-) \oplus V^\mu) \\ (\hat{A}, \hat{a}, \hat{\phi}) &\to (\hat{A}, F^+_{\hat{A}+\hat{a}} - \sigma(\hat{\phi}, \hat{\phi}), D_{\hat{A}+\hat{a}}(\hat{\phi}), p^\mu \Pi i^*(\hat{a}, \hat{\phi})) \end{aligned}$$

Here i^* is the restriction to Y, Π denotes Coulomb projection (the nonlinear map defined in section 3), and p^μ is the orthogonal projection to $V^\mu = V^\mu_{-\infty}$. Note that $\widetilde{SW}^\mu$ is equivariant under the action of the based gauge group $\hat{\mathcal{G}}_0 = \mathrm{Map}_0(X, S^1)$; this acts on connections in the usual way, on spinors by multiplication, and on forms trivially. The quotient $\widetilde{SW}^\mu / \hat{\mathcal{G}}_0$ is an S^1–equivariant, fiber preserving map over the Picard torus

$$Pic^0(X) = H^1(X; \mathbb{R})/H^1(X; \mathbb{Z}) = \mathcal{C}/\hat{\mathcal{G}}_0.$$

Let us study the restriction of this map to a fiber (corresponding to a fixed $\hat{A} \in \mathcal{C}$):

$$SW^\mu : i\Omega^1_g(X) \oplus \Gamma(W^+) \to i\Omega^2_+(X) \oplus \Gamma(W^-) \oplus V^\mu.$$

Note that SW^μ depends on μ only through its V^μ–valued direct summand i^μ; we write $SW^\mu = sw \oplus i^\mu$. The reason for introducing the cut-off μ is that we want the linearization of the Seiberg–Witten map to be Fredholm.

Let us decompose SW^μ into its linear and nonlinear parts:

$$L^\mu = \big(d^+, D_{\hat{A}}, p^\mu(\mathrm{pr}_{\ker d^*} i^*)\big); \;\; C^\mu = SW^\mu - L^\mu.$$

Here $\mathrm{pr}_{\ker d^*}$ is a shorthand for $(\mathrm{pr}_{\ker d^*}, \mathrm{id})$ acting on the 1-forms and spinors on Y, respectively.

As in [26], we need to introduce fractionary Sobolev norms. For the following result we refer to [2] and [26]:

Proposition 5 *The linear map*

$$L^\mu : L^2_{k+3/2}\big(i\Omega^1_g(X) \oplus \Gamma(W^+)\big) \to L^2_{k+1/2}\big(i\Omega^2_+(X) \oplus \Gamma(W^-)\big) \oplus L^2_{k+1}(V^\mu)$$

is Fredholm and has index

$$2ind_{\mathbb{C}}(D^+_{\hat{A}}) - b^+_2(X) - \dim V^\mu_0.$$

Here $\operatorname{ind}_{\mathbb{C}}(D^+_{\hat{A}})$ *is the index of the operator* $D^+_{\hat{A}}$ *acting on the positive spinors* $\hat{\phi}$ *with spectral boundary condition* $p^0 i^*(\hat{\phi}) = 0$.

Equivalently, there is a uniform bound for all $\hat{x} \in i\Omega^1_g(X) \oplus \Gamma(W^+)$*:*

$$\|\hat{x}\|_{L^2_{k+3/2}} \leq C(\mu)\cdot\big(\|(d^+ \oplus D_{\hat{A}})\hat{x}\|_{L^2_{k+1/2}} + \|p^\mu \mathrm{pr}_{\ker d^*} i^*(\hat{x})\|_{L^2_{k+1}} + \|\hat{x}\|_{L^2}\big),$$

for some constant $C(\mu) > 0$.

The nonlinear part is:

$$\begin{aligned} C^\mu : L^2_{k+3/2}\big(i\Omega^1_g(X) \oplus \Gamma(W^+)\big) \to& \\ & L^2_{k+1/2}\big(i\Omega^2_+(X) \oplus \Gamma(W^-) \oplus H^1(X;\mathbb{R})\big) \oplus L^2_{k+1}(V^\mu) \\ C^\mu(\hat{a}, \hat{\phi}) = \big(F^+_{\hat{A}} - \sigma(\hat{\phi}, \hat{\phi}), \hat{\rho}(\hat{a})\hat{\phi}, &0, p^\mu(\Pi - (\mathrm{pr}_{\ker d^*}, \mathrm{id}))i^*(\hat{a}, \hat{\phi})\big). \end{aligned}$$

Just like in three dimensions, the first three terms are either constant or quadratic in the variables so they define compact maps between the respective Sobolev spaces $L^2_{k+3/2}$ and $L^2_{k+1/2}$. The last term is not compact. However, as will be seen in the proof, it does not pose problems to doing finite dimensional approximation. The use of this technique will lead us to the definition of Ψ, the invariant of 4–manifolds with boundary mentioned in the introduction.

Let U_n be any sequence of finite dimensional subspaces of $L^2_{k+1/2}(i\Omega^2_+(X) \oplus \Gamma(W^-))$ such that $\mathrm{pr}_{U_n} \to 1$ pointwise. For each $\lambda < 0$, let $U'_n = (L^\mu)^{-1}(U_n \times V^\mu_\lambda)$ and consider the map

$$\mathrm{pr}_{U_n \times V^\mu_\lambda} SW^\mu = L^\mu + \mathrm{pr}_{U_n \times V^\mu_\lambda} C^\mu : U'_n \to U_n \times V^\mu_\lambda.$$

It is easy to see that for all n sufficiently large, L^μ restricted to U'_n (with values in $U_n \times V^\mu_\lambda$) has the same index as L^μ. Indeed, the kernel is the same, while the cokernel has the same dimension provided that $U_n \times V^\mu_\lambda$ is transversal to the image of L^μ. Since $\mathrm{pr}_{U_n} \to 1$ pointwise when $n \to \infty$, it suffices to show that V^μ_λ is transversal to the image of $p^\mu(\mathrm{pr}_{\ker d^*} i^*)$ in V^μ. But it is easy to see that $p^\mu(\mathrm{pr}_{\ker d^*} i^*)$ is surjective.

We have obtained a map between finite dimensional spaces, and we seek to get from it an element in a stable homotopy group of I^μ_λ in the form of a map between $(U'_n)^+$ and $(U_n)^+ \wedge I^\mu_\lambda$.

This can be done as follows. Choose a sequence $\epsilon_n \to 0$, and denote by $B(U_n, \epsilon_n)$ and $S(U_n, \epsilon_n)$ the closed ball and the sphere of radius ϵ_n in U_n (with the $L^2_{k+1/2}$ norm), respectively. Let $\tilde{K}$ be the preimage of $B(U_n, \epsilon_n) \times V^\mu_\lambda$ under the map L^μ, and let $\tilde{K}_1$ and $\tilde{K}_2$ be the intersections of $\tilde{K}$ with $B(U'_n, R_0)$

and $S(U'_n, R_0)$, respectively. Here R_0 is a constant to be defined later, and U'_n has the $L^2_{k+3/2}$ norm. Finally, let K_1, K_2 be the images of $\tilde{K}_1$ and $\tilde{K}_2$ under the composition of SW^μ with projection to the factor V^μ_λ. Assume that there exists an index pair (N, L) for S^μ_λ such that $K_1 \subset N$ and $K_2 \subset L$. Then we could define the pointed map we were looking for:

$$B(U'_n, R_0)/S(U'_n, R_0) \to (B(U_n, \epsilon_n) \times N)/(B(U_n, \epsilon_n) \times L \cup S(U_n, \epsilon_n) \times N),$$

by applying $\mathrm{pr}_{U_n \times V^\mu_\lambda} \circ SW^\mu$ to the elements of $\tilde{K}$ and sending everything else to the basepoint. Equivalently, via a homotopy equivalence we would get a map:

$$\Psi_{n,\mu,\lambda,\hat{A}} : (U'_n)^+ \to (U_n)^+ \wedge I^\mu_\lambda.$$

Of course, for this to be true we need to prove:

Proposition 6 *For $\mu, -\lambda$ sufficiently large and n sufficiently large compared to μ and $-\lambda$, there exists an index pair (N, L) for S^μ_λ such that $K_1 \subset N$ and $K_2 \subset L$.*

Let us first state an auxiliary result that will be needed. The proof follows from the same argument as the proof of Proposition 3, so we omit it.

Lemma 4 *Let $t_0 \in \mathbb{R}$. Suppose $\mu_n, -\lambda_n \to \infty$, and we have approximate Seiberg–Witten half-trajectories $x_n : [t_0, \infty) \to L^2_{k+1}(V^{\mu_n}_{\lambda_n})$ such that $x_n(t) \in \overline{B(2R)}$ for all $t \in [t_0, \infty)$. Then $x_n(t) \in B(R)$ for any $t > t_0$ and for any n sufficiently large. Also, for any $s > t_0$, a subsequence of $x_n(t)$ converges to some $x(t)$ in C^m norm, uniformly in t for $t \in [s, \infty)$ and for any $m > 0$.*

Proof of Proposition 6 We choose an isolating neighborhood for S^μ_λ to be $\overline{B(2R)} \cap V^\mu_\lambda$. Here R, the constant in Proposition 3, is chosen to be large enough so that $B(R)$ contains the image under i^μ of the ball of radius R_0 in $L^2_{k+3/2}(i\Omega^1_g(X) \oplus \Gamma(W^+))$. By virtue of Theorem 4, all we need to show is that K_1 and K_2 satisfy conditions (i) and (ii) in its hypothesis.

Step 1 Assume that there exist sequences $\mu_n, -\lambda_n \to \infty$ and a subsequence of U_n (denoted still U_n for simplicity) such that the corresponding K_1 do not satisfy (i) for any n. Then we can find $(\hat{a}_n, \hat{\phi}_n) \in B(U'_n, R_0)$ and $t_n \geq 0$ such that

$$\mathrm{pr}_{U_n \times V^{\mu_n}_{\lambda_n}} \circ SW^{\mu_n}(\hat{a}_n, \hat{\phi}_n) = (u_n, x_n)$$

with

$$\|u_n\|_{L^2_{k+1/2}} \leq \epsilon_n;\ (\varphi^{\mu_n}_{\lambda_n})_{[0,\infty)}(x_n) \subset \overline{B(2R)};\ (\varphi^{\mu_n}_{\lambda_n})_{t_n}(x_n) \in \partial \overline{B(2R)}.$$

We distinguish two cases: when $t_n \to \infty$ and when t_n has a convergent subsequence. In the first case, let

$$y_n : \mathbb{R} \to L^2_{k+1}(V^{\mu_n}_{\lambda_n})$$

be the trajectory of $\varphi^{\mu_n}_{\lambda_n}$ such that $y_n(-t_n) = x_n$. Then, because of our hypotheses, $\|y_n(0)\|_{L^2_{k+1}} = 2R$ and $y_n(t) \in \overline{B(2R)}$ for all $t \in [-t_n, \infty)$. Since $t_n \to \infty$, by Lemma 4 we have that $y_n(0) \in B(R)$ for n sufficiently large. This is a contradiction.

In the second case, by passing to a subsequence we can assume that $t_n \to t^* \geq 0$. We use a different normalization:

$$y_n : [0, \infty) \to L^2_{k+1}(V^{\mu_n}_{\lambda_n})$$

is the trajectory of $\varphi^{\mu_n}_{\lambda_n}$ such that $y_n(0) = x_n$. Then $\|y_n(t_n)\|_{L^2_{k+1}} = 2R$ and $y_n(t) \in \overline{B(2R)}$ for all $t \geq 0$. By the Arzela–Ascoli Theorem we know that y_n converges to some $y : [0, \infty) \to V$ in L^2_k norm, uniformly on compact sets of $t \in [0, \infty)$. This y must be the Coulomb projection of a Seiberg–Witten trajectory.

Let $z_n = y_n - y$. From Lemma 4 we know that the convergence $z_n \to 0$ can be taken to be in C^∞, but only over compact subsets of $t \in (0, \infty)$. However, we can get something stronger than L^2_k for $t = 0$ as well. Since l is self-adjoint, there is a well-defined compact operator $e^l : L^2_{k+1}(V^0) \to L^2_{k+1}(V^0)$. We have the estimate:

$$\begin{aligned} \|p^0 z_n(0) - e^l p^0 z_n(1)\|_{L^2_{k+1}} &= \|\int_0^1 \frac{\partial}{\partial t}(e^{tl} p^0 z_n(t)) dt\|_{L^2_{k+1}} \\ &\leq \int_0^1 \|e^{tl} p^0 \big(\frac{\partial}{\partial t} z_n(t) + l z_n(t)\big)\|_{L^2_{k+1}} dt \end{aligned}$$

But since y_n and y are trajectories of the respective flows, if we denote $\pi_n = p^{\mu_n}_{\lambda_n}, \pi^n = 1 - \pi_n$ we have

$$\frac{\partial}{\partial t} z_n(t) + l z_n(t) = c(y(t)) - \pi_n c(y_n(t)),$$

so that

$$\begin{aligned} \|p^0 z_n(0) - e^l p^0 z_n(1)\|_{L^2_{k+1}} \leq \int_0^1 \|e^{tl} p^0 \pi_n \big(c(y(t)) - c(y_n(t))\big)\|_{L^2_{k+1}} dt + \\ + \int_0^1 \|e^{tl} p^0 (\pi^n c(y(t)))\|_{L^2_{k+1}} dt. \end{aligned} \tag{13}$$

Fix $\epsilon > 0$. We break each of the two integrals on the right hand side of (13) into $\int_0^\epsilon + \int_\epsilon^1$. Recall that $y_n(t)$ live in $\overline{B(2R)}$. This must also be true for $y(t)$ because of the weak convergence $y_n(t) \to y(t)$ in L^2_{k+1}. Since $e^{tl}p^0$ and c are continuous maps from L^2_{k+1} to L^2_{k+1}, there is a bound:

$$\int_0^\epsilon \|e^{tl}p^0\pi_n\big(c(y(t)) - c(y_n(t))\big)\|_{L^2_{k+1}}dt + \int_0^\epsilon \|e^{tl}p^0(\pi^n c(y(t)))\|_{L^2_{k+1}}dt \leq C_1 \cdot \epsilon, \tag{14}$$

where C_1 is a constant independent of ϵ.

On the other hand, on the interval $[\epsilon, 1]$ we have $\|e^{tl}p^0\| \leq \|e^{\epsilon l}p^0\|$ and $e^{\epsilon l}p^0$ is a compact map from L^2_{k+1} to L^2_{k+1}. We get

$$\int_\epsilon^1 \|e^{tl}p^0(\pi^n c(y(t)))\|_{L^2_{k+1}}dt \leq \int_\epsilon^1 \|\pi^n e^{\epsilon l}p^0 c(y(t))\|_{L^2_{k+1}}dt.$$

In addition, $e^{\epsilon l}p^0 c(y(t))$ live inside a compact set of $L^2_{k+1}(V)$ and we know that $\pi^n \to 0$ uniformly on such sets. Therefore,

$$\int_\epsilon^1 \|e^{tl}p^0(\pi^n c(y(t)))\|_{L^2_{k+1}}dt \to 0. \tag{15}$$

Similarly, using the fact that $y_n(t) \to y(t)$ in $L^2_{k+1}(V)$ uniformly in t for $t \in [\epsilon, 1]$, we get:

$$\int_\epsilon^1 \|e^{tl}p^0\pi_n\big(c(y(t)) - c(y_n(t))\big)\|_{L^2_{k+1}}dt \to 0. \tag{16}$$

Putting (13), (14), (15), and (16) together and letting $\epsilon \to 0$ we obtain:

$$\|p^0 z_n(0) - e^l p^0 z_n(1)\|_{L^2_{k+1}} \to 0.$$

Since $z_n(1) \to 0$ in L^2_{k+1}, the same must be true for $p^0 z_n(0)$. Recall that $z_n(0) = x_n$ is the boundary value of an approximate Seiberg–Witten solution on X :

$$\mathrm{pr}_{U_n \times V^{\mu_n}_{\lambda_n}} \circ SW^{\mu_n}(\hat{a}_n, \hat{\phi}_n) = (u_n, x_n),$$

with $\|u_n\|_{L^2_{k+1/2}} \leq \epsilon_n$. Equivalently,

$$\big(L^{\mu_n} + \mathrm{pr}_{U_n \times V^{\mu_n}_{\lambda_n}} \circ C^{\mu_n}\big)(\hat{a}_n, \hat{\phi}_n) = (u_n, x_n).$$

Since $\hat{x}_n = (\hat{a}_n, \hat{\phi}_n)$ are uniformly bounded in $L^2_{k+3/2}$ norm, after passing to a subsequence we can assume that they converge to some $\hat{x} = (\hat{a}, \hat{\phi})$ weakly in $L^2_{k+3/2}$. Changing everything by a gauge, we can assume without loss of generality that $i^*(\hat{a}) \in \ker d^*$. Now Proposition 5 says that:

$$\|\hat{x}_n - \hat{x}\|_{L^2_{k+3/2}} \leq C(0)\cdot\big(\|(d^+ \oplus D_{\hat{A}})(\hat{x}_n - \hat{x})\|_{L^2_{k+1/2}} \\ + \|p^0 \mathrm{pr}_{\ker d^*} i^*(\hat{x}_n - \hat{x})\|_{L^2_{k+1}} + \|\hat{x}_n - \hat{x}\|_{L^2}\big). \quad (17)$$

We already know that the last term on the right hand side goes to 0 as $n \to \infty$. Let us discuss the first term. First, it is worth seeing that $sw(\hat{x}) = 0$. Let $sw = \hat{l} + \hat{c}$ be the decomposition of sw into its linear and compact parts; $\hat{l}$ and $\hat{c}$ are direct summands of L^μ and C^μ, respectively. We have $\mathrm{pr}_{U_n} sw(\hat{x}_n) = u_n \to 0$ in $L^2_{k+1/2}$ (because $\epsilon_n \to 0$ by construction), and

$$sw(\hat{x}) - \mathrm{pr}_{U_n} sw(\hat{x}_n) = \hat{l}(\hat{x} - \hat{x}_n) + \mathrm{pr}_{U_n}(\hat{c}(\hat{x}) - \hat{c}(\hat{x}_n)) + (1 - \mathrm{pr}_{U_n})\hat{c}(\hat{x}).$$

Using the fact that $\hat{x}_n \to \hat{x}$ weakly in $L^2_{k+3/2}$ we get that each term on the right hand side converges to 0 weakly in $L^2_{k+1/2}$. Hence $sw(\hat{x}) = 0$. Now the first term on the right hand side of (17) is

$$\hat{l}(\hat{x}_n - \hat{x}) = u_n + \mathrm{pr}_{U_n}(\hat{c}(\hat{x}) - \hat{c}(\hat{x}_n)) + (1 - \mathrm{pr}_{U_n})\hat{c}(\hat{x}).$$

It is easy to see that this converges to 0 in $L^2_{k+1/2}$ norm. We are using here the fact that $\mathrm{pr}_{U_n} \to 1$ uniformly on compact sets.

Similarly one can show that the second term in (17) converges to 0. We already know that $p^0 p^{\mu_n}_{\lambda_n} \Pi i^*(\hat{x}_n) = p^0 x_n$ converges to $p^0 x$. This was proved starting from the boundedness of the y_n on the cylinder on the right. In the same way, using the boundedness of the $\hat{x}_n$ on the manifold X on the left (which has a cylindrical end), it follows that $p_0 x_n \to p_0 x$ in L^2_{k+1}. Thus, $x_n \to x$ in L^2_{k+1}. Let $i^*(\hat{x}_n) = (a_n + db_n, \phi_n)$ with $a_n \in \ker d^*$. We know that $x_n = p^{\mu_n}_{\lambda_n}(a_n, e^{ib_n}\phi_n)$ converges. Also, $\hat{x}_n \to \hat{x}$ weakly in $L^2_{k+3/2}$, hence strongly in $L^2_{k+1/2}$. This implies that $db_n \to 0$ in L^2_k and $b_n \to 0$ in L^2_{k+1}. Since $p^0 \mathrm{pr}_{ker d^*} i^* \hat{x}_n = p^0(a_n, \phi_n) \in V^0_{\lambda_n}$, they must converge in L^2_{k+1} just like the x_n. We are using the Sobolev multiplication $L^2_{k+1} \times L^2_{k+1} \to L^2_{k+1}$.

Putting all of these together, we conclude that the expression in (17) converges to 0. Thus $\hat{x}_n \to \hat{x}$ in $L^2_{k+3/2}$. We also know that $sw(\hat{x}) = 0$. In addition, since $i^*(\hat{x}_n) \to i^*(\hat{x})$ in L^2_{k+1} and using $p^{\mu_n}_{\lambda_n} \Pi i^*(\hat{x}_n) = x_n$ we get that $x_n = y_n(0) \to y(0)$ in L^2_{k+1}. This implies that $\Pi i^*(\hat{x}) = y(0)$.

Now it is easy to reach a contradiction: by a gauge transformation $\hat{u}$ of $\hat{x}$ on X we can obtain a solution of the Seiberg–Witten equations on X with $\Pi i^*(\hat{u}\cdot\hat{x}) = y(0)$. Recall that $y(0)$ was the starting point of $y : [0, \infty) \to \overline{B(2R)}$, the Coulomb projection of a Seiberg–Witten half-trajectory of finite type. By gluing this half-trajectory to $\hat{u} \cdot \hat{x}$ we get a C^0 monopole on the complete manifold $X \cup (\mathbb{R}_+ \times Y)$. From Proposition 2 we know that there are "universal" bounds on the C^m norms of the monopole (in some gauge) restricted to any compact set, for any m. These bounds are "universal" in the sense that they depend only on the metric on X. In particular, since Coulomb projection is continuous, we obtain such a bound B on the L^2_{k+1} norm of $y(t)$ for all t. Recall that $y_n(t_n) \to y(t^*)$ because $t_n \to t^*$, and that $\|y_n(t_n)\|_{L^2_{k+1}} = 2R$. When we chose the constant R, we were free to choose it as large as we wanted. Provided that $2R > B$, we get the desired contradiction.

Step 2 The proof is somewhat similar to that in Step 1.

Assume that there exist sequences $\mu_n, -\lambda_n \to \infty$ and a subsequence of U_n (denoted still U_n for simplicity) such that the corresponding K_2 do not satisfy condition (ii) in Theorem 4 for any n. Then we can find $\hat{x}_n \in S(U'_n, R_0)$ such that

$$\mathrm{pr}_{U_n \times V^{\mu_n}_{\lambda_n}} \circ SW^{\mu_n}(\hat{x}_n) = (u_n, x_n),$$

with

$$\|u_n\|_{L^2_{k+1/2}} \leq \epsilon_n; \ \ (\varphi^{\mu_n}_{\lambda_n})_{[0,\infty)}(x_n) \subset \overline{B(2R)}.$$

Let $y_n : [0, \infty) \to L^2_{k+1}(V^{\mu_n}_{\lambda_n})$ be the half-trajectory of $\varphi^{\mu_n}_{\lambda_n}$ starting at $y_n(0) = x_n$. Repeating the argument in Step 1, after passing to a subsequence we can assume that $y_n(t)$ converges to some $y(t)$ in $L^2_k(V^{\mu_n}_{\lambda_n})$, uniformly over compact sets of t. Also, this convergence can be taken to be in C^∞ for $t > 0$, while for $t = 0$ we get that $p^0(y_n(0) - y(0)) \to 0$ in L^2_{k+1}. Observe that y is the Coulomb projection of a Seiberg–Witten half-trajectory of finite type, which we denote by y'. We can assume that $y'(0) = y(0)$.

Then, just as in Step 1, we deduce that $\hat{x}_n$ converges in $L^2_{k+3/2}$ to $\hat{x}$, a solution of the Seiberg–Witten equations on X with $\Pi i^*(\hat{x}) = y(0)$. By gluing $\hat{x}$ to y' we obtain a C^0 monopole on $X \cup (\mathbb{R}_+ \times Y)$. By Proposition 2, this monopole must be smooth in some gauge, and when restricted to compact sets its C^m norms must be bounded above by some constant which depend only on the metric on X. Since the four-dimensional Coulomb projection from $i\Omega^1(X) \oplus \Gamma(W^+)$ to $i\Omega^1_g(X) \oplus \Gamma(W^+)$ is continuous, we get a bound B' on the $L^2_{k+3/2}$ norm of $\hat{x}$. But $\hat{x}_n \to \hat{x}$ in $L^2_{k+3/2}$ and $\|\hat{x}_n\|_{L^2_{k+3/2}} = R_0$. Provided that we have chosen the constant R_0 to be larger than B', we obtain a contradiction. □

Thus we have constructed some maps

$$\Psi_{n,\mu,\lambda,\hat{A}} : (U'_n)^+ \to (U_n)^+ \wedge I^\mu_\lambda$$

for any $\mu, -\lambda$ sufficiently large and for all n sufficiently large compared to μ and $-\lambda$. In other words, we get such maps from $(U')^+$ to $U^+ \wedge I_\lambda^\mu$ for any μ and λ sufficiently large and for any finite dimensional subspace $U \subset L^2_{k+1/2}(i\Omega_2^+(X) \oplus \Gamma(W^-))$ which contains a fixed subspace U_0 (depending on μ and λ).

For $\mu \gg 0$, the linear map L^μ is injective, because in the limit $\mu \to \infty$ there are no nonzero solutions to an elliptic equation on X which vanish on the boundary. For $U \times V_\lambda^\mu$ transversal to coker L^μ and for $U' = (L^\mu)^{-1}(U \times V_\lambda^\mu)$, we get a natural identification:

$$U \oplus V_\lambda^\mu \cong U' \oplus \mathrm{coker} L^\mu.$$

It is not hard to see that there is another natural identification:

$$\mathrm{coker} L^\mu \cong \mathrm{coker} L^0 \oplus \mathrm{coker}(p_0^\mu(\mathrm{pr}_{\ker d^*} i^*) : \ker L^0 \to V_0^\mu).$$

Using the fact that $p_0^\mu(\mathrm{pr}_{\ker d^*} i^*) : \ker L^0 \to V_0^\mu$ is injective, we get:

$$\ker L^0 \oplus \mathrm{coker} L^\mu \cong \mathrm{coker} L^0 \oplus V_0^\mu.$$

Consequently, the map

$$(U')^+ \to U^+ \wedge I_\lambda^\mu \cong U^+ \wedge (V_\lambda^0)^+ \wedge \Sigma^{-V_\lambda^0} I_\lambda^\mu$$

is stably the same as a map:

$$(\ker L^0)^+ \to (\mathrm{coker} L^0)^+ \wedge \Sigma^{-V_\lambda^0} I_\lambda^\mu. \tag{18}$$

The real part of L^0 is the $(d^+, p^0 i^*)$ operator restricted to $\mathrm{Im}(d^*)$. This has zero kernel, and cokernel isomorphic to $H^2_+(X;\mathbb{R})$. Using our homology orientation, we can identify the latter with $\mathbb{R}^{b_2^+(X)}$. The complex part of L^0 is $D^+_{\hat{A}}$, which may have nontrivial kernel and cokernel. Assuming that all our constructions have been done S^1–equivariantly, (18) produces a stable equivariant morphism:

$$(\ker D^+_{\hat{A}})^+ \to (\mathrm{coker} D^+_{\hat{A}} \oplus \mathbb{C}^{n(Y,\mathfrak{c},g)} \oplus \mathbb{R}^{b_2^+(X)})^+ \wedge \mathrm{SWF}(Y,\mathfrak{c}). \tag{19}$$

We can put these maps together for all classes $[\hat{A}] \in Pic^0(X)$ as follows. We started our construction from a bundle map between two Hilbert bundles over the Picard torus $Pic^0(X)$. Such bundles are trivial by Kuiper's theorem, so we can choose subbundles of the form $U \times Pic^0(X)$ when doing the finite dimensional approximation. The maps $(U')^+ \to U^+ \wedge I_\lambda^\mu$ can be grouped into an S^1–map from the Thom space of the vector bundle over $Pic^0(X)$ with fibers U'. In the process of stabilization, these U'–bundles differ from each other only by taking direct sums with trivial bundles. In the end the collection of maps (19) produces an S^1–stable equivariant homotopy class:

$$\Psi \in \{(T(Ind), b_2^+(X), 0), \mathrm{SWF}(Y,\mathfrak{c})\}_{S^1},$$

where $T(Ind)$ is the Thom space of the virtual index bundle over $Pic^0(X)$ of the Dirac operator D^+, with a shift in complex degree by $n(Y,\mathfrak{c},g)$.

The class Ψ is independent of μ, λ, U and the other choices made in the construction, such as ϵ_n and R_0. This can be seen using standard homotopy arguments analogous to those in the proof of Theorem 1. To interpolate between different $U, U' \supset U_0$, it suffices to consider the case $U \subset U'$ and use a linear homotopy $t\mathrm{pr}_U + (1-t)\mathrm{pr}_{U'}$. Similar arguments show that Ψ does not depend on the metric on X either, up to composition with canonical isomorphisms. Therefore we have constructed an invariant of X and its *spin*^{c} structure, which we denote $\Psi(X,\hat{\mathfrak{c}})$. This ends the proof of Theorem 2. □

Remark 1 If we restrict Ψ to a single fiber of Ind we get an element in an equivariant stable homotopy group:

$$\psi(X,\hat{\mathfrak{c}}) \in \tilde{\pi}^{S^1}_{-b,d}(\mathrm{SWF}(Y,\mathfrak{c})),$$

where $b = b_2^+(X)$ and

$$d = \mathrm{ind}_{\mathbb{C}}(D^+_{\hat{A}}) - n(Y,\mathfrak{c},g) = \frac{c_1(\hat{L})^2 - \sigma(X)}{8}.$$

(This is in fact given by the morphism (19) above.)

Since $\tilde{\pi}^{S^1}_*$ is the universal equivariant homology theory, by composing with the canonical map we obtain an invariant of X in $\tilde{h}_{-b,d}(\mathrm{SWF}(Y,\mathfrak{c}))$ for every reduced equivariant homology theory $\tilde{h}$.

Remark 2 We can reinterpret the invariant ψ in terms of cobordisms. If Y_1 and Y_2 are 3–manifolds with $b_1 = 0$, a *cobordism* between Y_1 and Y_2 is a 4–manifold X with $\partial X = \bar{Y}_1 \cup Y_2$. Let us omit the *spin*^{c} structures from notation for simplicity. We have an invariant

$$\psi(X) \in \tilde{\pi}^{S^1}_{-b,d}\big(\mathrm{SWF}(\bar{Y}_1) \wedge \mathrm{SWF}(Y_2)\big) = \{(S^0, b, -d), (\mathrm{SWF}(\bar{Y}_1) \wedge \mathrm{SWF}(Y_2)\}_{S^1}.$$

In [8], Cornea proves a duality theorem for the Conley indices of the forward and reverse flows in a stably parallelizable manifold. This result (adapted to the equivariant setting) shows that the spectra $\mathrm{SWF}(Y_1)$ and $\mathrm{SWF}(\bar{Y}_1)$ are equivariantly Spanier-Whitehead dual to each other. According to [19], this implies the equivalence:

$$\{(S^0, b, -d), (\mathrm{SWF}(\bar{Y}_1) \wedge \mathrm{SWF}(Y_2)\}_{S^1} \cong \{\mathrm{SWF}(Y_1), (\mathrm{SWF}(Y_2), -b, d)\}_{S^1}.$$

Therefore, a cobordism between Y_1 and Y_2 induces an equivariant stable homotopy class of S^1-maps between $\mathrm{SWF}(Y_1)$ and $\mathrm{SWF}(Y_2)$, with a possible shift in degree:

$$\mathcal{D}_X \in \{\mathrm{SWF}(Y_1), (\mathrm{SWF}(Y_2), -b, d)\}_{S^1}.$$

10 Four-manifolds with negative definite intersection form

In [4], Bauer and Furuta give a proof of Donaldson's theorem using the invariant $\Psi(X, \hat{\mathfrak{c}})$ for closed 4–manifolds. Along the same lines we can use our invariant to study 4–manifolds with boundary with negative definite intersection form. The bound that we get is parallel to that obtained by Frøyshov in [12].

If Y is our 3–manifold with $b_1(Y) = 0$ and *spin*^{c} structure $\mathfrak{c}$, we denote by $s(Y, \mathfrak{c})$ the largest s such that there exists an element

$$[f] \in \{(S^0, 0, -s), \mathrm{SWF}(Y, \mathfrak{c})\}_{S^1}$$

which is represented by a pointed S^1–map f whose restriction to the fixed point set has degree 1. Then we set

$$s(Y) = \max_{\mathfrak{c}} s(Y, \mathfrak{c}).$$

The first step in making the invariant $s(Y)$ more explicit is the following lemma (which also appears in [5]):

Lemma 5 *Let $f : (\mathbb{R}^m \oplus \mathbb{C}^{n+d})^+ \to (\mathbb{R}^m \oplus \mathbb{C}^n)^+$ be an S^1–equivariant map such that the induced map on the fixed point sets has degree 1. Then $d \leq 0$.*

Proof Let f_c be the complexification of the map f. Note that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = V(1) \oplus V(-1)$, where $V(j)$ is the representation $S^1 \times \mathbb{C} \to \mathbb{C}, (q, z) \to q^j z$. Using the equivariant K-theory mapping degree, tom Dieck proves in [9, II.5.15] the formula:

$$d(f_c) = \lim_{q \to 1} d(f_c^{S^1}) \cdot \mathrm{tr}\, \lambda_{-1}([nV(1) \oplus nV(-1)] - [(n+d)V(1) \oplus (n+d)V(-1)])(q),$$

where $q \in S^1$, d is the usual mapping degree, and $\lambda_{-1}([nV(1) \oplus nV(-1)] - [(n+d)V(1) \oplus (n+d)V(-1)])$ is the K_{S^1}–theoretic Euler class of f_c; in our case its character evaluated at q equals $(1-q)^{-d}(1-q^{-1})^{-d}$. Since $d(f_c^{S^1}) = 1$, the limit only exists in the case $d \leq 0$. □

Example Let us consider the case when Y is the Poincaré homology sphere P, oriented as the link of the E_8 singularity. There is a unique *spin*^{c} structure $\mathfrak{c}$ on P, and P admits a metric g of positive scalar curvature. The only solution of the Seiberg–Witten equations on P with the metric g is the reducible $\theta = (0, 0)$. In addition, the Weitzenböck formula tells us that the operator $\not{\partial}$ is injective, hence so is l. We can choose R as small as we want in Proposition 3. Taking the L^2_{k+1} norms, we get a bound

$$\|p^\mu_\lambda c(v)\| \leq \|c(v)\| \leq \alpha\|v\|^2$$

for all $v \in V$ sufficiently close to 0. Also, if λ_0 is the eigenvalue of l of smallest absolute value, then

$$\|l(v)\| \geq |\lambda_0| \cdot \|v\|.$$

Putting the two inequalities together, we get that for $R > 0$ sufficiently small and $-\lambda, \mu$ sufficiently large, the only zero of the map $l + p^\mu_\lambda c$ in $\overline{B(2R)}$ is 0. It follows that $S^\mu_\lambda = \mathrm{Inv}\left(\overline{B(2R)} \cap V^\mu_\lambda\right) = \{0\}$. Its Conley index is $(\mathbb{R}^{m'_\lambda})^+ \wedge (\mathbb{C}^{n'_\lambda})^+$. In [12], K. Frøyshov computed $n(P, \mathfrak{c}, g) = -1$, so that we can conclude:

$$\mathrm{SWF}(P, \mathfrak{c}) = \mathbb{C}^+$$

up to isomorphism. We get that $s(P) = 1$ as a simple consequence of Lemma 5.

Let us come back to the general case and try to obtain a bound on $s(Y)$. Recall the notations from section 8. Choose a metric g and a good perturbation ν. We seek to find s so that there is no element in $\{(S^0, 0, -s-1), \mathrm{SWF}(Y, \mathfrak{c})\}$ representable by a map which has degree 1 on the fixed point sets. Equivalently, for μ and $-\lambda$ sufficiently large, there should not be any S^1–map $\tilde{f}$ of that kind between $(\mathbb{R}^{m'_\lambda} \oplus \mathbb{C}^{n'_\lambda + r+1})^+$ and $I(S)$, where $r = s + n(Y, \mathfrak{c}, g)$. Assume that $r \geq 0$.

Suppose that there exists $\tilde{f}$ as above and denote $N = m'_\lambda + 2n'_\lambda$. Consider S^1–equivariant cell decompositions of $I(S), I(S^{\mathrm{irr}}_{>0}), I(S^{\mathrm{irr}}_{\leq 0}), I(S_{\leq 0}), I(\Theta)$ compatible with the coexact sequences (10) and (11) from section 8 in the sense that all maps are cellular. We can assume that all the S^1–cells of $I(S)$ of cellular dimension $\geq N$ and all the S^1–cells of $I(S^{\mathrm{irr}}_{>0})$ and $I(S^{\mathrm{irr}}_{\leq 0})$ are free. Also note that $(\mathbb{R}^{m'_\lambda} \oplus \mathbb{C}^{n'_\lambda + r+1})^+$ has an S^1–cell structure with no equivariant cells of cellular dimension greater than $N + 2r + 1$. Thus we can homotope $\tilde{f}$ equivariantly relative to the fixed point set so that its image is contained in the $(N + 2r + 1)$–skeleton of $I(S)$. Assuming that there exists an S^1–map

$$f : I(S)_{N+2r+1} \to (\mathbb{R}^{m'_\lambda} \oplus \mathbb{C}^{n'_\lambda + r})^+$$

whose restriction to the fixed point set has degree 1, by composing $\tilde{f}$ with f we would get a contradiction with Lemma 5.

Therefore, our job is to construct the map f. Start with the inclusion:

$$I(\Theta) \cong (\mathbb{R}^{m'_\lambda} \oplus \mathbb{C}^{n'_\lambda})^+ \hookrightarrow (\mathbb{R}^{m'_\lambda} \oplus \mathbb{C}^{n'_\lambda + r})^+$$

By composing with the second map in (11) and by restricting to the $(N+2r+1)$–skeleton we obtain a map f_0 defined on $I(S_{\leq 0})_{N+2r+1}$. Let us look at the sequence (10). Since $I(S^{\mathrm{irr}}_{>0})$ is S^1–free, we could obtain the desired f once we are able to extend f_0 from $I(S_{\leq 0})_{N+2r+1}$ to $I(S)_{N+2r+1}$. This is an exercise in equivariant obstruction theory. First, it is easy to see that we can always extend f_0 up to the $(N+2r)$–skeleton. Proposition II.3.15 in [9] tells us that the extension to the $(N+2r+1)$–skeleton is possible if and only if the corresponding obstruction

$$\tilde{\gamma}_r \in \mathfrak{H}^{N+2r+1}_{S^1}\Big(I(S), I(S_{\leq 0}); \pi_{N+2r}((\mathbb{R}^{m'_\lambda} \oplus \mathbb{C}^{n'_\lambda + r})^+)\Big) \cong H^{N+2r+1}((I^{\mathrm{irr}}_{>0})^\mu_\lambda; \mathbb{Z})$$

vanishes. Here $\mathfrak{H}$ denotes the Bredon cohomology theory from [9, Section II.3].

After stabilization, the obstruction $\tilde{\gamma}_r$ becomes an element

$$\gamma_r \in H^{2r+1}(\mathtt{swf}^{\mathrm{irr}}_{>0}(Y, \mathfrak{c}, g, \nu)).$$

Thus, we have obtained the following bound:

$$s(Y) \leq \max_{\mathfrak{c}} \inf_{g,\nu}\big(-n(Y, \mathfrak{c}, g) + \min\{r \in \mathbb{Z}_+ | \gamma_r = 0\}\big). \tag{20}$$

We have now developed the tools necessary to study four-manifolds with negative definite intersection forms.

Proof of Theorem 3 A characteristic element c is one that satisfies $c \cdot x \equiv x \cdot x \bmod 2$ for all $x \in H_2(X)/\mathrm{Torsion}$. Given such a c, there is a *spin*^{c} structure $\hat{\mathfrak{c}}$ on X with $c_1(\hat{L}) = c$.

Let $d = (c^2 - \sigma(X))/8$. In section 9 we constructed an element:

$$\psi(X, \hat{\mathfrak{c}}) \in \{(S^0, 0, -d), \mathrm{SWF}(Y, \mathfrak{c})\}_{S^1}.$$

The restriction to the fixed point set of one of the maps $\Psi_{n,\mu,\lambda,\hat{A}}$ which represents $\psi(X, \hat{\mathfrak{c}})$ is linear near 0 and has degree ± 1 because $b_2^+(X) = 0$. Hence

$$d \leq s(Y, \mathfrak{c}) \leq s(Y).$$

Together with the inequality (20), this completes the proof. □

Corollary 1 (Donaldson) *Let X be a closed, oriented, smooth four-manifold with negative definite intersection form. Then its intersection form is diagonalizable.*

Proof If we apply Theorem 3 for $Y = \emptyset$, we get $b_2(X) + c^2 \leq 0$ for all characteristic vectors c. By a theorem of Elkies from [10], the only unimodular forms with this property are the diagonal ones. □

Corollary 2 (Frøyshov) *Let X be a smooth, compact, oriented 4–manifold with boundary the Poincaré sphere P. If the intersection form of X is of the form $m\langle -1\rangle \oplus J$ with J even and negative definite, then $J = 0$ or $J = -E_8$.*

Proof Since J is even, the vector c whose first m coordinates are 1 and the rest are 0 is characteristic. We have $c^2 = -m$ and we have shown that $s(P) = 1$. Rather than applying Theorem 3, we use the bound $d = b_2(X) + c^2 \leq 8s(P)$ directly. This gives that $\mathrm{rank}(J) \leq 8$. But the only even, negative definite form of rank at most 8 is $-E_8$. □

A Existence of index pairs

This appendix contains the proof of Theorem 4, which is an adaptation of the argument given in [7], pages 46-48.

The proof is rather technical, so let us first provide the reader with some intuition. As a first guess for the index pair, we could take N to be the complement in A of a small open neighborhood of $\partial A \cap A^+$ and L to be the complement in N of a very small neighborhood of A^+. (This choice explains condition (ii) in the statement of Theorem 3.) At this stage (N, L) satisfies conditions 1 and 2 in the definition of the index pair, but it may not satisy the relative positive invariance condition. We try to correct this by enlarging N and L with the help of the positive flow. More precisely, if $B \subset A$, we denote

$$P(B) = \{x \in A : \exists y \in B, t \geq 0 \text{ such that } \varphi_{[0,t]}(y) \subset A, x = \varphi_t(y)\}.$$

We could replace N and L by $P(N)$ and $P(L)$, respectively. (This explains the condition (i) in the satement of Theorem 3, which can be rewritten $P(K_1) \cap \partial A \cap A^+ = \emptyset$.) We have taken care of positive invariance, but a new problem appears: $P(N)$ and $P(L)$ may no longer be compact. Therefore, we need to find conditions which guarantee their compactness:

Lemma 6 *Let B be a compact subset of A which either contains A^- or is disjoint from A^+. Then $P(B)$ is compact.*

Proof Since $P(B) \subset A$ and A is compact, it suffices to show that for any $x_n \in P(B)$ with $x_n \to x \in A$, we have $x \in P(B)$. Let x_n be such a sequence, $x_n = \varphi_{t_n}(y_n), y_n \in B$ so that $\varphi_{[0,t_n]}(y_n) \subset A$. Since B is compact, by passing to a subsequence we can assume that $y_n \to y \in B$. If t_n have a convergent subsequence as well, say $t_{n_k} \to t \geq 0$, then by continuity $\varphi_{t_{n_k}}(y_{n_k}) \to \varphi_t(y) = x$ and $\varphi_{[0,t]}(y) \subset A$. Thus $x \in P(B)$, as desired.

If t_n has no convergent subsequences, then $t_n \to \infty$. Given any $m > 0$, for n sufficiently large $t_n > m$, so $\varphi_{[0,m]}(y_n) \subset A$. Letting $n \to \infty$ and using the comapctness of A we obtain $\varphi_{[0,m]}(y) \subset A$. Since this is true for all $m > 0$, we have $y \in A^+$. This takes care of the case $A^+ \cap B = \emptyset$, since we obtain a contradiction. If $A^- \subset B$, we reason differently: $\varphi_{[0,t_n]}(y_n) \subset A$ is equivalent to $\varphi_{[-t_n,0]}(x_n) \subset A$; letting $n \to \infty$, we get $\varphi_{(-\infty,0]}(x) \subset A$, so $x \in A^-$. Thus $x \in B \subset P(B)$, as desired. □

Proof of Theorem 4 Choose C a small compact neighborhood of $A^+ \cap \partial A$ such that $A^- \cap C = \emptyset$. We claim that if we choose C sufficiently small, we have $P(K_1) \cap C = \emptyset$. Indeed, if there were no such C, we could find $x_n \in P(K_1)$ with $x_n \to x \in A^+ \cap \partial A$. Let $x_n = \varphi_{t_n}(y_n)$ for some $y_n \in K_1$ such that $\varphi_{[0,t_n]}(y_n) \subset A$. By passing to a subsequence we can assume $y_n \to y \in K_1$. If t_n has a subsequence converging to some $t \in [0,\infty)$, then by taking the limit $\varphi_{[0,t]}(y) \subset A$ and $\varphi_t(y) = x$, which contradicts $P(K_1) \cap A^+ \cap \partial A = \emptyset$. If t_n has no such subsequence, then $t_n \to \infty$. Since $\varphi_{[-t_n,0]}(x_n) \subset A$, by taking the limit we get $\varphi_{(-\infty,0]}(x) \subset A$. Thus $x \in A^-$. On the other hand $x \in A^+ \cap \partial A$, which contradicts the fact that $A^+ \cap A^- =$ Inv $A \subset \mathrm{int}(A)$.

Let C be as above and let V be an open neighborhood of A^+ such that $\mathrm{cl}(V \setminus C) \subset \mathrm{int}(A)$. Since $K_2 \cap A^+ = \emptyset$ and K_2 is compact, by making V sufficiently small we can assume that $K_2 \cap V = \emptyset$.

Let us show that there exists $t^* \geq 0$ such that $\varphi_{[-t^*,0]}(y) \not\subset A$ for any $y \in C$. If not, we could find $y_n \in C$ with $\varphi_{[-n,0]}(y_n) \subset A$. Since C is compact, there is a subsequence of y_n which converges to some $y \in C$ such that $\varphi_{(-\infty,0]}(y) \subset A$ or, equivalently, $y \in A^-$. This contradicts the fact that A^- and C are disjoint.

Let t^* be as above. For each $x \in A^-$, either $\varphi_{[0,t^*]}(x) \subset A^-$ or there is $t(x) \in [0,t^*]$ so that $\varphi_{[0,t(x)]}(x) \cap C = \emptyset$ and $\varphi_{t(x)}(x) \notin A$. In the first case we choose $K(x)$ a compact neighborhood of x such that $\varphi_{[0,t^*]}(K(x)) \cap C = \emptyset$. In the second case we choose $K(x)$ to be a compact neighborhood of x with $\varphi_{[0,t(x)]}(K(x)) \cap C = \emptyset$ and $\varphi_{t(x)}(K(x)) \cap A = \emptyset$. Since A^- is compact, it is covered by a finite collection of the sets $K(x)$. Let B' be their union and let $B = B' \cup K_1$. Then B is compact, and we can assume that it contains a neighborhood of A^-.

We choose the index pair to be

$$L = P(A \setminus V); \quad N = P(B) \cup L.$$

Clearly $K_1 \subset B \subset N$ and $K_2 \subset A \setminus V \subset L$. It remains to show that (N, L) is an index pair. First, since $A \setminus V$ is compact and disjoint from A^+, by Lemma 6 above L is compact. Since $A^- \subset B, N = P(B \cup (A \setminus V))$ is compact as well.

We need to check the three conditions in the definition of an index pair. Condition 1 is equivalent to $S \subset \mathrm{int}(N \setminus L) = \mathrm{int}(N) \setminus L$. We have $S \subset \mathrm{int}(N)$ because $S \subset A^-$ and $B \subset N$ contains a neighborhood of A^-. We have $S \cap L = \emptyset$ because if $x \in S \subset A^+$ is of the form $x = \varphi_t(y)$ for $y \in A \setminus V, t \geq 0$ such that $\varphi_{[0,t]}(y) \subset A$, then $y \in A^+$, which contradicts $A^+ \subset V$.

Condition 3 can be easily checked from the definitions: L is positively invariant in A by construction, and this implies that it is positively invariant in N as well.

Condition 2 requires more work. Let us first prove that $P(B) \cap C = \emptyset$. We have $P(B) = P(B') \cup P(K_1)$ and we already know that $P(K_1) \cap C = \emptyset$. For $y' \in P(B')$, there exists $y \in B'$ such that $\varphi_{[0,t]}(y) \subset A$ and $\varphi_t(y) = y'$. Recall that we chose $t^* \geq 0$ so that $\varphi_{[-t^*,0]}(x) \not\subset A$ for any $x \in C$.If $t \geq t^*$, this implies $y' \notin C$. If $t < t^*$ then, because y is in some $K(x)$ for $x \in A^-$, the fact that $\varphi_{[0,t]}(y) \subset A$ implies again $y' = \varphi_t(y) \notin C$. Therefore $P(B') \cap C = \emptyset$, so $P(B) \cap C = \emptyset$.

To prove that L is an exit set for N, pick $x \in N \setminus L$ and let $\tau = \sup\{t | \varphi_{[0,t]}(x) \subset N \setminus L\}$. It suffices to show that $\varphi_\tau(x) \in L$. Assume this is false; then $\varphi_\tau(x) \subset N \setminus L$. Note that

$$N \setminus L \subset (A \setminus P(A \setminus V)) \subset V.$$

Also $N \setminus L \subset P(B) \subset (A \setminus C)$, so $N \setminus L$ is contained in $V \setminus C \subset \mathrm{int}(A)$. It follows that for $\epsilon > 0$ sufficiently small, $\varphi_{[\tau,\tau+\epsilon]}(x) \subset A \setminus L$. Since N is positively invariant in A and $\varphi_\tau(x) \in N$, we get $\varphi_{[\tau,\tau+\epsilon]}(x) \subset N \setminus L$. This contradicts the definition of τ. Therefore, $\varphi_\tau(x) \in L$.

We conclude that (N, L) is a genuine index pair, with $K_1 \subset N$ and $K_2 \subset L$. $\square$

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