ISSN 1364-0380 (on line) 1465-3060 (printed)

Geometry & Topology
Volume 8 (2004) 645–673
Published: 22 April 2004

Units of ring spectra and their traces in algebraic K-theory

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Abstract

Let $GL_1(R)$ be the units of a commutative ring spectrum R. In this paper we identify the composition

$$\eta_R\colon\ BGL_1(R) \to \mathrm{K}(R) \to \mathrm{THH}(R) \to \Omega^\infty(R),$$

where $\mathrm{K}(R)$ is the algebraic K-theory and $\mathrm{THH}(R)$ the topological Hochschild homology of R. As a corollary we show that classes in $\pi_{i-1}R$ not annihilated by the stable Hopf map $\eta \in \pi_1^s(S^0)$ give rise to non-trivial classes in $\mathrm{K}_i(R)$ for $i \geq 3$.

AMS Classification numbers Primary: 19D55, 55P43

Secondary: 19D10, 55P48

Keywords: Ring spectra, algebraic K-theory, topological Hochschild homology

Proposed: Thomas Goodwillie
Seconded: Ralph Cohen, Haynes Miller

Received: 25 November 2003
Revised: 21 April 2004

1 Introduction

Given a connective (symmetric) ring spectrum R, we follow Waldhausen and define the units $GL_1(R)$ to be the union of the components in $\Omega^\infty(R)$ that correspond to units in the discrete ring $\pi_0 R$. With this definition $GL_1(R)$ is a group-like monoid whose group of components equals $GL_1(\pi_0 R)$. As in the case of a discrete ring there is a natural map $BGL_1(R) \to \mathrm{K}(R)$ to the algebraic K-theory of R. If R is a commutative discrete ring this is split by the determinant, but the definition of the determinant does not generalize to the setting of ring spectra and the above map is in general not split, even if R is commutative. For example, Waldhausen shows [21] that this fails quite badly for the sphere spectrum. However, it turns out that the notion of traces of matrices does generalize to ring spectra. This gives rise to the trace map $\mathrm{tr}\colon \mathrm{K}(R) \to \mathrm{THH}(R)$, where the target is the topological Hochschild homology first defined by Bökstedt [6]. The purpose of the present paper is to identify the composition

$$\eta_R\colon BGL_1(R) \to \mathrm{K}(R) \xrightarrow{\mathrm{tr}} \mathrm{THH}(R) \xrightarrow{r} \Omega^\infty(R) \tag{1.1}$$

when R is a commutative ring spectrum. The first two arrows are defined for any (symmetric) ring spectrum, whereas the definition of the last map depends on R being commutative. By definition, $\mathrm{THH}(R)$ is the infinite loop space associated to the realization of the cyclic spectrum $[k] \mapsto R^{\wedge(k+1)}$ with Hochschild type structure maps. We shall use Bökstedt's explicit definition of the smash products $R^{\wedge(k+1)}$. If R is commutative, the degree-wise multiplication $R^{\wedge(k+1)} \to R$ defines a map to the constant cyclic spectrum. This gives rise to the infinite loop map r in the definition of η_R.

In order to state our main result, we need the fact that $GL_1(R)$ has the structure of an infinite loop space when R is commutative, ie, that there exists a spectrum $gl_1(R)$ such that $\Omega^\infty(gl_1(R)) \simeq GL_1(R)$. (We follow the convention to use small letters for the spectrum associated to an infinite loop space written in capital letters.) It will be convenient for our purpose to give an explicit construction of $gl_1(R)$ using Segal's notion of Γ-spaces. Let $\eta \in \pi_1^s(S^0)$ denote the stable Hopf map.

Theorem 1.2 *The composite map η_R admits a factorization*

$$BGL_1(R) \to GL_1(R) \to \Omega^\infty(R),$$

in which the second map is the natural inclusion and the first map is multiplication by η in the sense of the following commutative diagram in the homotopy *category of spaces,*

$$\begin{array}{ccc} BGL_1(R) & \longrightarrow & GL_1(R) \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ \Omega^\infty(gl_1(R)\wedge S^1) & \xrightarrow{\Omega^\infty(id\wedge\eta)} & \Omega^\infty(gl_1(R)). \end{array}$$

In the case where R equals the sphere spectrum this result is due to Bökstedt and Waldhausen [8] (with a completely different proof).

It is clear from the definition that there is an isomorphism of abelian groups $\pi_i gl_1(R) \cong \pi_i R$ for $i \geq 1$, but since the spectrum structures are different this is not an isomorphism of $\pi_*^s(S^0)$-modules. However, using that η is realized as an unstable map $\eta\colon S^3 \to S^2$, it is not difficult to check that the actions of η are compatible in degrees $i \geq 2$. The following is then an immediate corollary of Theorem 1.2.

Corollary 1.3 *For $i \geq 3$, the composition*

$$\pi_{i-1}R \cong \pi_i BGL_1(R) \to \pi_i \mathrm{K}(R) \to \pi_i \mathrm{THH}(R) \to \pi_i R$$

is multiplication by $\eta \in \pi_1^s(S^0)$.

It thus follows that classes in $\pi_{i-1}R$ not annihilated by η give rise to non-trivial elements in $\pi_i \mathrm{K}(R)$.

Example 1.4 Let $R = ko$, the real connective K-theory spectrum. In this case $GL_1(ko) \simeq \{\pm 1\} \times BO_\otimes$, where $\otimes$ indicates that the H-space structure is the one corresponding to tensor products of vector bundles. Using the cofibration sequence $\Sigma ko \xrightarrow{\eta} ko \to ku$, [18, V.5.15], we see that

$$\mathbb{Z} \cong \pi_{8k}(ko) \xrightarrow{\eta} \pi_{8k+1}(ko) \cong \mathbb{Z}/2$$

is surjective and that

$$\mathbb{Z}/2 \cong \pi_{8k+1}(ko) \xrightarrow{\eta} \pi_{8k+2}(ko) \cong \mathbb{Z}/2$$

is an isomorphism. We conclude that for $k \geq 1$,

- $\pi_{8k+1}BBO_\otimes \cong \mathbb{Z}$ maps non-trivially to $\pi_{8k+1}\mathrm{K}(ko)$;
- $\pi_{8k+2}BBO_\otimes \cong \mathbb{Z}/2$ injects as a direct summand in $\pi_{8k+2}\mathrm{K}(ko)$.

This example is interesting in view of the attempts [1], [2], to relate algebraic K-theory to elliptic cohomology and the chromatic filtration of homotopy theory. Another major source for the interest in algebraic K-theory in the non-linear setting is the relation to high dimensional manifold theory via Waldhausen's work on stable concordances [20].

Example 1.5 Let $R = \Sigma^\infty(G_+)$ be the suspension spectrum of a commutative (or E_∞) group-like monoid G. By definition, the algebraic K-theory of this spectrum is Waldhausen's $A(BG)$. In this case, $\pi_i BGL_1(R) \cong \pi^s_{i-1}(G_+)$, and thus classes in the stable homotopy that are not annihilated by η map non-trivially to $\pi_i A(BG)$ in degrees $i \geq 3$.

Remark 1.6 Given a discrete ring $\bar{R}$, the algebraic K-theory of the associated Eilenberg-MacLane spectrum $H\bar{R}$ reduces to Quillen's $\mathrm{K}(\bar{R})$. Starting with a ring spectrum R and $\bar{R} = \pi_0 R$, the linearization map $R \to H\bar{R}$ gives rise to a fibration sequence

$$F \to \mathrm{K}(R) \to \mathrm{K}(\bar{R}),$$

where by definition F is the homotopy fibre. Let $SL_1(R)$ be the unit component of $GL_1(R)$. Using that $BSL_1(\bar{R}) = *$ we get a map $BSL_1(R) \to F$ which is important in the understanding of how algebraic K-theory behaves under linearization.

The proof of Theorem 1.2 breaks up into two parts. The first part is to give a description of η_R in non-K-theoretical terms as the composition

$$BGL_1(R) \to L(BGL_1(R)) \xleftarrow{\sim} B^{cy}GL_1(R) \xrightarrow{r} GL_1(R) \subseteq \Omega^\infty(R).$$

Here $L(BGL_1(R))$ denotes the free loop space of $BGL_1(R)$ and $B^{\mathrm{cy}}GL_1(R)$ is Waldhausen's cyclic bar construction, see Section 3. The first map is the inclusion of the constant loops and the map $r\colon B^{\mathrm{cy}}GL_1(R) \to GL_1(R)$ is given by iterated multiplication in $GL_1(R)$. The fact that $GL_1(R)$ is an infinite loop space ensures that it is sufficiently homotopy commutative for the latter map to be well-defined.

The second part of the proof is then to show that the composite map $BGL_1(R) \to GL_1(R)$ is multiplication by η. This follows from a general analysis of how the free loop space of an infinite loop space relates to the cyclic bar construction. Let us say that a sequence of maps of based spaces $F \to X \to Y$ is a homotopy fibration sequence if (i) the composition is constant and (ii) the canonical map from F to the homotopy fiber of the second map is a weak homotopy equivalence. (This definition is most useful if Y is connected.) Given a well-pointed group-like topological monoid G, there is a commutative diagram of homotopy fibration sequences

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\sim"] & B^{\mathrm{cy}}G \arrow[r] \arrow[d, "\sim"] & BG \arrow[d, equal] \\ \Omega(BG) \arrow[r] & L(BG) \arrow[r] & BG, \end{tikzcd}$$

in which the lower sequence is split by the inclusion of the constant loops $BG \to L(BG)$. If furthermore G admits the structure of an infinite loop space, then the upper sequence has a natural splitting $B^{\mathrm{cy}}G \to G$ given by the iterated product in G. The failure of these splittings to be compatible is measured by the fact that the composition

$$BG \to L(BG) \simeq B^{\mathrm{cy}}G \to G$$

is multiplication by η in the sense described above for $GL_1(R)$.

The paper is as a whole fairly self-contained, and in particular we present in Section 4 a new explicit construction of the trace map $\mathrm{tr}\colon \mathrm{K}(R) \to \mathrm{THH}(R)$. This version of the trace map is used here to identify the action on $BGL_1(R)$, but there are many other applications of this combinatorial construction. In Section 2 we recall the definition of symmetric ring spectra and their units and in Section 3 we recall Waldhausen's definition of algebraic K-theory in this framework. The Sections 2-4 can be read as a self-contained account of the topological trace map.

In Section 5 we explain the infinite loop structure of $GL_1(R)$ used in the formulation of Theorem 1.2, and in Section 6 we construct the splitting $r\colon \mathrm{THH}(R) \to \Omega^\infty(R)$ and complete the first part of the proof. Finally, in Section 7 we consider the relationship between the free loop space and the cyclic bar construction of an infinite loop space and finish the second part of the proof.

1.1 Notation and conventions

Let $\mathcal{T}$ be the category of based spaces. In this paper this can be understood as either the category of compactly generated Hausdorff (or weak Hausdorff) topological spaces or the category of based simplicial sets. However, we will usually use the topological terminology and talk about topological monoids etc. In both cases equivalences mean weak homotopy equivalences. In the topological case we will sometimes have to assume that base points are non-degenerate in the usual sense of being neighborhood deformation retracts.

We let S^n denote the n-fold smash product of the circle $S^1 = I/\partial I$. By a spectrum E we understand a sequence $\{E_n\colon n \geq 0\}$ of based spaces together with based maps $\sigma\colon S^1 \wedge E_n \to E_{n+1}$. Again this may be interpreted either in the topological or simplicial category. A map of spectra $f\colon E \to F$ is a sequence of based maps $f_n\colon E_n \to F_n$ that commute with the structure maps. We say that f is an equivalence if it induces an isomorphism on spectrum homotopy groups, the latter being defined by $\pi_n E = \operatorname{colim}_k \pi_{n+k} E_k$. All spectra we consider will be connective, ie, $\pi_n E = 0$ for $n < 0$. We shall also assume that the spectra we consider are *convergent* in the sense that there exists an unbounded, non-decreasing sequence of natural numbers $\{\lambda_n\colon n \geq 0\}$ such that $S^1 \wedge E_n \to E_{n+1}$ is as least $n + \lambda_n$-connected for all n. This is not a serious restriction as any connective spectrum is equivalent to a convergent one.

2 Units of ring spectra

In this section we recall Waldhausen's definition of the space of units associated to a ring spectrum. We shall work in the framework of symmetric spectra and begin by recalling the relevant definitions from [12] and, for the version with topological spaces instead of simplicial sets, [16].

2.1 Symmetric spectra

A symmetric spectrum is a spectrum in which each of the spaces E_n is equipped with a base point preserving left Σ_n-action, such that the iterated structure maps

$$\sigma^m\colon S^m \wedge E_n \to E_{m+n}$$

are $\Sigma_m \times \Sigma_n$-equivariant. A symmetric ring spectrum is a symmetric spectrum equipped with Σ_n-equivariant maps $1_n\colon S^n \to E_n$ for $n \geq 0$, and $\Sigma_m \times \Sigma_n$-equivariant maps $\mu_{m,n}\colon E_m \wedge E_n \to E_{m+n}$ for $m, n \geq 0$. In order to formulate the axioms, let $\check{\sigma}^n$ be the composite

$$\check{\sigma}^n\colon E_m \wedge S^n \xrightarrow{\text{tw}} S^n \wedge E_m \xrightarrow{\sigma^n} E_{n+m} \xrightarrow{\tau_{n,m}} E_{m+n},$$

where tw twists the two factors, and $\tau_{n,m}$ is the (n,m)-shuffle $i \mapsto i+m$ for $i \leq n$, $i \mapsto i-n$ for $i > n$. Notice that $\check{\sigma}^n$ is $\Sigma_m \times \Sigma_n$-equivariant. Also, let $\sigma^0\colon S^0 \wedge E_n \to E_n$ and $\check{\sigma}^0\colon E_n \wedge S^0 \to E_n$ be the canonical identifications. These maps are required to satisfy the following relations for all $l, m, n \geq 0$:

(a) $1_{m+n} = \sigma^m \circ (S^m \wedge 1_n)$,

(b) $\sigma^m = \mu_{m,n} \circ (1_m \wedge E_n), \quad \check{\sigma}^n = \mu_{m,n} \circ (E_m \wedge 1_n)$,

(c) $\mu_{l+m,n} \circ (\mu_{l,m} \wedge E_n) = \mu_{l,m+n} \circ (E_l \wedge \mu_{m,n})$.

Here condition (a) states that the maps 1_n assemble to give a map of spectra $1\colon S \to E$, where S denotes the sphere spectrum. Notice that (b) and (c) imply that

$$\mu_{l,m+n} \circ (E_l \wedge \sigma^m) = \mu_{l+m,n} \circ (\check{\sigma}^m \wedge E_n)$$

as maps $E_l \wedge S^m \wedge E_n \to E_{l+m+n}$ and that

$$\sigma^l \circ (S^l \wedge \mu_{m,n}) = \mu_{l+m,n} \circ (\sigma^l \wedge E_n).$$

These are exactly the conditions for the maps $\mu_{m,n}$ to produce a map of spectra $\mu\colon E \wedge E \to E$, where the domain is the internal smash product in the category of symmetric spectra. Condition (b) then says that 1 is a two-sided unit, and (c) is the condition that the multiplication is associative. (These comments on the internal smash product are only to motivate the definitions; we shall not make explicit use of the internal smash product in this paper.) We say that R is commutative if the diagrams

$$\begin{array}{ccc} R_m \wedge R_n & \xrightarrow{\mu_{m,n}} & R_{m+n} \\ \downarrow{\scriptstyle \mathrm{tw}} & & \downarrow{\scriptstyle \tau_{m,n}} \\ R_n \wedge R_m & \xrightarrow{\mu_{m,n}} & R_{n+m} \end{array}$$

are commutative.

2.2 $\mathcal{I}$-spaces and $\mathcal{I}$-monoids

In order to define the units of a symmetric ring spectrum we need a combinatorial framework to keep track of the suspension coordinates. Let $\mathcal{I}$ be the category whose objects are the finite sets $\mathbf{n} = \{1, \dots, n\}$ and whose morphisms are the injective (not necessarily order preserving) maps. The empty set $\mathbf{0}$ is an initial object. The concatenation $\mathbf{m} \sqcup \mathbf{n}$ defined by letting $\mathbf{m}$ correspond to the first m elements and $\mathbf{n}$ to the last n elements of $\{1, \dots, m+n\}$ gives $\mathcal{I}$ the structure of a symmetric monoidal category. The symmetric structure is given by the shuffles $\tau_{m,n}\colon \mathbf{m} \sqcup \mathbf{n} \to \mathbf{n} \sqcup \mathbf{m}$.

We define an *$\mathcal{I}$-space* to be a functor $X\colon \mathcal{I} \to \mathcal{T}$. Given an $\mathcal{I}$-space X, we write $X_{h\mathcal{I}} = \operatorname{hocolim}_{\mathcal{I}} X$. The homotopy type of $X_{h\mathcal{I}}$ can be analyzed using the following lemma due to Bökstedt. For published versions see [15, 2.3.7] and [9, 2.5.1]. Let $\mathcal{F}_n\mathcal{I}$ be the full subcategory of $\mathcal{I}$ containing the objects of cardinality at least n.

Lemma 2.1 (Bökstedt) *Let X be an $\mathcal{I}$-space and suppose that each morphism $\mathbf{n}_1 \to \mathbf{n}_2$ in $\mathcal{F}_n\mathcal{I}$ induces a λ_n-connected map $X(n_1) \to X(n_2)$. Then, given any object $\mathbf{m}$ in $\mathcal{F}_n\mathcal{I}$, the natural map $X(m) \to X_{h\mathcal{I}}$ given by the inclusion in the 0-skeleton is at least $(\lambda_n - 1)$-connected.* □

Let us say that an $\mathcal{I}$-space X is *convergent* if there exists an unbounded, non-decreasing sequence of natural numbers $\{\lambda_n\colon n \geq 0\}$ such that any morphism $\mathbf{n}_1 \to \mathbf{n}_2$ in $\mathcal{F}_n\mathcal{I}$ induces a λ_n-connected map $X(n_1) \to X(n_2)$. It follows from Bökstedt's lemma that in this case $X_{h\mathcal{I}}$ is equivalent to the usual telescope of the sequence of spaces $X(n)$ obtained by restricting to the natural subset inclusions in $\mathcal{I}$. In particular, $\pi_* X_{h\mathcal{I}}$ is the usual directed colimit of the groups $\pi_* X(n)$ if X is convergent.

We say that an $\mathcal{I}$-space X is an $\mathcal{I}$*-monoid* if it comes equipped with an associative and unital natural transformation

$$\mu_{m,n}\colon\ X(m) \times X(n) \to X(m+n),$$

where both sides are considered functors on $\mathcal{I}^2$. The unital condition means that the basepoint in $X(0)$ acts as a unit and associativity means that the identity

$$\mu_{l,m+n} \circ (X_l \times \mu_{m,n}) = \mu_{l+m,n} \circ (\mu_{l,m} \times X_n)$$

holds for all $l, m, n \geq 0$. By definition an $\mathcal{I}$-monoid X is commutative if the diagrams

$$\begin{array}{ccc} X(m) \times X(n) & \xrightarrow{\mu_{m,n}} & X(m+n) \\ \Big\downarrow{\scriptstyle \mathrm{tw}} & & \Big\downarrow{\scriptstyle \tau_{m,n}} \\ X(n) \times X(m) & \xrightarrow{\mu_{n,m}} & X(n+m) \end{array}$$

are commutative. If X is an $\mathcal{I}$-monoid, then $X_{h\mathcal{I}}$ inherits the structure of a topological monoid. The product is given by the composition

$$X_{h\mathcal{I}} \times X_{h\mathcal{I}} = \operatorname*{hocolim}_{\mathcal{I}\times\mathcal{I}} X(m) \times X(n) \xrightarrow{\mu} \operatorname*{hocolim}_{\mathcal{I}\times\mathcal{I}} X(m+n) \to X_{h\mathcal{I}},$$

in which the last map is induced by the monoidal structure of $\mathcal{I}$. We say that X is group-like if this is the case for $X_{h\mathcal{I}}$, ie, if the monoid of components $\pi_0 X_{h\mathcal{I}}$ is a group. We will show in Section 5 that if X is commutative and group-like, then $X_{h\mathcal{I}}$ has the structure of an infinite loop space.

Remark 2.2 For $\mathcal{I}$-spaces X that are not convergent, the homotopy type of $X_{h\mathcal{I}}$ may well differ from that of the usual telescope. Consider for example the $\mathcal{I}$-monoid $\mathbf{n} \mapsto B\Sigma_n$. In this case the associated homotopy colimit is equivalent to the base point component of $Q(S^0)$. To see this one uses that the natural map $B\Sigma_\infty \to \mathrm{hocolim}_{\mathcal{I}} B\Sigma_n$ induces an isomorphism on integral homology. By the universal property of Quillen's plus-construction and the fact that the target is a connected H-space, it follows that the latter is equivalent to $B\Sigma_\infty^+$. The conclusion then follows from the Barratt-Priddy-Quillen-Segal Theorem. As a second example, let R be a discrete ring and consider the $\mathcal{I}$-monoid defined by the classifying spaces $BGL_n(R)$. By an argument similar to the above, the associated homotopy colimit is equivalent to the base point component of the algebraic K-theory space $\mathrm{K}(R)$. In these examples (and many more), evaluating the homotopy colimit over $\mathcal{I}$ thus has the same effect as Quillen's plus-construction.

2.3 Units of ring spectra

Given a symmetric ring spectrum R, the sequence of spaces $\Omega^n(R_n)$ defines an $\mathcal{I}$-space as follows. A morphism $\alpha\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$ induces a map $\Omega^m(R_m) \to \Omega^n(R_n)$ by taking $f \in \Omega^m(R_m)$ to the composition

$$S^n \xrightarrow{\bar{\alpha}^{-1}} S^n = S^l \wedge S^m \xrightarrow{S^l \wedge f} S^l \wedge R_m \xrightarrow{\sigma^l} R_n \xrightarrow{\bar{\alpha}} R_n. \tag{2.3}$$

Here $\bar{\alpha}\colon \mathbf{n} = \mathbf{l} \sqcup \mathbf{m} \to \mathbf{n}$ is the unique permutation that is order preserving on the first $l = n - m$ elements and acts as α on the last m elements. The action on S^n is the usual left action. The multiplication in R gives a multiplicative structure

$$\mu_{m,n}\colon \Omega^m(R_m) \times \Omega^n(R_n) \to \Omega^{m+n}(R_{m+n}),$$
$$\mu_{m,n}(f,g)\colon S^m \wedge S^n \xrightarrow{f \wedge g} R_m \wedge R_n \xrightarrow{\mu_{m,n}} R_{m+n},$$

which is commutative if R is. We let $\Omega^n(R_n)^*$ be the union of the components in $\Omega^n(R_n)$ that have stable multiplicative homotopy inverses in the following sense: For each f in $\Omega^n(R_n)^*$ there exists an element $g \in \Omega^m(R_m)$ such that $\mu_{n,m}(f,g)$ and $\mu_{m,n}(g,f)$ are homotopic to the unit 1_{m+n} in $\Omega^{m+n}(R_{m+n})$. We consider $\Omega^n(R_n)^*$ as a based space with base point 1_n and restricting the above structure maps gives an $\mathcal{I}$-monoid $\Omega^n(R_n)^*$. We define

$$GL_1(R) = \operatorname*{hocolim}_{\mathcal{I}} \Omega^n(R_n)^*$$

with the monoid structure explained above. If R is convergent so is the $\mathcal{I}$-space $\Omega^n(R_n)^*$, and by Lemma 2.1, $\pi_0(GL_1(R)) = GL_1(\pi_0(R))$. If furthermore R is commutative, the general construction in section 5 will produce a spectrum $gl_1(R)$ such that $\Omega^\infty(gl_1(R)) \simeq GL_1(R)$.

3 K-theory and cyclic K-theory of ring spectra

In this section we recall the definition of the algebraic K-theory $\mathrm{K}(R)$ and the cyclic algebraic K-theory $\mathrm{K}^{\mathrm{cy}}(R)$ of a symmetric ring spectrum R. We also recall the inclusion of the units $BGL_1(R) \to \mathrm{K}(R)$. This material is due mainly to Waldhausen. Let $M_n(R)$ be the symmetric ring spectrum whose mth space is $\mathrm{Map}_*(\mathbf{n}_+, \mathbf{n}_+ \wedge R_m)$. The multiplication resembles multiplication of $n \times n$ matrices over an ordinary ring. (In this case the "matrices" in question have at most one non-base point entry in each column.) We let $GL_n(R) = GL_1(M_n(R))$ with the monoid structure coming from the multiplication in $M_n(R)$. Using the natural maps

$$\begin{aligned}&\mathrm{Map}_*(\mathbf{m}_+ \wedge S^k, \mathbf{m}_+ \wedge R_k) \times \mathrm{Map}_*(\mathbf{n}_+ \wedge S^l, \mathbf{n}_+ \wedge R_l)\\ &\to \mathrm{Map}_*((\mathbf{m} \sqcup \mathbf{n})_+ \wedge S^{k+l}, (\mathbf{m} \sqcup \mathbf{n})_+ \wedge R_{k+l})\end{aligned}$$

we have a notion of block sum of matrices and corresponding monoid homomorphisms

$$GL_m(R) \times GL_n(R) \to GL_{m+n}(R).$$

These homomorphisms are associative in the obvious sense and thus the induced maps of classifying spaces give $\coprod_{n\geq 0} BGL_n(R)$ the structure of an associative topological monoid. By definition $\mathrm{K}(R)$ is the group completion

$$\mathrm{K}(R) = \Omega B\Big(\coprod_{n\geq 0} BGL_n(R)\Big).$$

Notice that this is the version of algebraic K-theory with $\pi_0 \mathrm{K}(R) = \mathbb{Z}$.

The classifying space of the units $BGL_1(R)$ embeds in the 1-simplices of $B_\bullet(\coprod_{n\geq 0} BGL_n(R))$ and since there is just a single 0-simplex there is an induced map

$$S^1 \wedge BGL_1(R)_+ \to B\Big(\coprod_{n\geq 0} BGL_n(R)\Big)$$

whose adjoint is the requested map $BGL_1(R) \to \mathrm{K}(R)$. The image is contained in the 1-component of $\mathrm{K}(R)$.

There is a variant of all this using the cyclic bar construction $B^{\mathrm{cy}}GL_n(R)$. Recall that for a topological monoid G, $B^{\mathrm{cy}}G$ is the realization of the cyclic space $[k] \mapsto G^{k+1}$ with simplicial operators

$$d_i(g_0, \dots, g_k) = \begin{cases}(g_0, \dots, g_i g_{i+1}, \dots, g_k), & \text{for } 0 \leq i < k\\ (g_k g_0, \dots, g_{k-1}), & \text{for } i = k\end{cases}$$

$$s_i(g_0, \dots, g_k) = (g_0, \dots, g_i, 1, \dots, g_k), \quad \text{for } 0 \leq i \leq k,$$

and cyclic operator $t_k(g_0, \dots, g_k) = (g_k, g_0, \dots, g_{k-1})$. We refer the reader to [13] for background material on cyclic spaces. The degree-wise projections $(g_0, \dots, g_k) \mapsto (g_1, \dots, g_k)$ define a simplicial map $p\colon B^{\mathrm{cy}}_\bullet G \to B_\bullet G$, and if G is group-like and has a non-degenerate unit there results a homotopy fibration sequence

$$G \to B^{\mathrm{cy}} G \to BG.$$

Since $B^{\mathrm{cy}}_\bullet G$ is a cyclic space its realization has a canonical action of the circle group $\mathbb{T}$. Consider the composite map $\mathbb{T} \times B^{\mathrm{cy}} G \to B^{\mathrm{cy}} G \xrightarrow{p} BG$, where the first map is given by the $\mathbb{T}$-action. Letting $L(-)$ denote the free loop space, the adjoint is a map $B^{\mathrm{cy}} G \to L(BG)$. It is immediate from the definition that this is a $\mathbb{T}$-equivariant map when the action on $L(BG)$ is by multiplication in $\mathbb{T}$. The following proposition is well-known and follows easily from the definition of the $\mathbb{T}$-action. We shall prove a related result in Proposition 7.1 with a proof that can easily be adapted to the present situation.

Proposition 3.1 *There is a commutative diagram*

$$\begin{tikzcd} G \arrow[r, "i"] \arrow[d] & B^{\mathrm{cy}} G \arrow[r, "p"] \arrow[d] & BG \arrow[d, equal] \\ \Omega(BG) \arrow[r, "i"] & L(BG) \arrow[r, "ev"] & BG \end{tikzcd}$$

in which the lower sequence is the usual fibration sequence associated to the evaluation at the unit element of $\mathbb{T}$. If G is group-like and has a non-degenerate unit, then the upper sequence is a homotopy fibration sequence and the vertical maps are equivalences. □

Notice that the lower sequence in (3.1) is split by the inclusion of BG in $L(BG)$ as the constant loops. By definition the cyclic K-theory of a symmetric ring spectrum R is given by

$$\mathrm{K}^{\mathrm{cy}}(R) = \Omega B\Big(\coprod_{n \geq 0} B^{\mathrm{cy}} GL_n(R)\Big).$$

The projections $p\colon B^{\mathrm{cy}} GL_n(R) \to BGL(R)$ induce a map $p\colon \mathrm{K}^{\mathrm{cy}}(R) \to \mathrm{K}(R)$ which has a section in the homotopy category. The quickest way to see this is to consider the diagram of monoid homomorphisms

$$\coprod_{n \geq 0} BGL_n(R) \longrightarrow \coprod_{n \geq 0} L(BGL_n(R)) \xleftarrow{\sim} \coprod_{n \geq 0} B^{\mathrm{cy}} GL_n(R),$$

where the equivalence is a consequence of Proposition 3.1. After group completion there results a well-defined homotopy class $\mathrm{K}(R) \to \mathrm{K}^{cy}(R)$, giving a section of p up to homotopy.

4 Topological Hochschild homology and the trace map

In this section we present an explicit construction of the topological trace map $\mathrm{tr}\colon \mathrm{K}(R) \to \mathrm{THH}(R)$, where the target is the topological Hochschild homology. In order to motivate the construction we first recall the linear trace map for an ordinary discrete ring $\mathrm{tr}\colon \mathrm{K}(R) \to \mathrm{HH}(R)$ with target the Hochschild homology of R. The latter is the realization of a cyclic abelian group $\mathrm{HH}_\bullet(R)\colon [k] \mapsto R^{\otimes k+1}$ with cyclic structure maps similar to $B^{\mathrm{cy}}_\bullet G$. The multi-trace $\mathrm{tr}\colon \mathrm{HH}_\bullet(M_n(R)) \to \mathrm{HH}_\bullet(R)$ is the cyclic map given in degree k by

$$\mathrm{tr}(A^0 \otimes \cdots \otimes A^k) = \sum_{s_0,\dots,s_k} a^0_{s_k,s_0} \otimes \cdots \otimes a^k_{s_{k-1},s_k}, \quad \text{where } A^i = (a^i_{s,t}).$$

Composing with the obvious inclusions $B^{\mathrm{cy}}_\bullet GL_n(R) \to \mathrm{HH}_\bullet(M_n(R))$ we get a cyclic map

$$\coprod_{n\geq 0} B^{\mathrm{cy}}_\bullet GL_n(R) \to \coprod_{n\geq 0} \mathrm{HH}_\bullet(M_n(R)) \to \mathrm{HH}_\bullet(R).$$

This is a monoid homomorphism with respect to block-sums of matrices on the domain and the abelian group structure on the target. After realization and group completion we get maps

$$\mathrm{K}^{\mathrm{cy}}(R) \to \Omega B\, \mathrm{HH}(R) \xleftarrow{\sim} \mathrm{HH}(R).$$

The linear trace map $\mathrm{tr}\colon \mathrm{K}(R) \to \mathrm{HH}(R)$ is the homotopy class obtained by composing with the homotopy section $\mathrm{K}(R) \to \mathrm{K}^{\mathrm{cy}}(R)$.

4.1 Topological Hochschild homology

Topological Hochschild homology is obtained by replacing the tensor products in $\mathrm{HH}_\bullet(R)$ by smash products of spectra. We shall follow Bökstedt [6] in making this precise. Given a sequence of symmetric spectra $E^1,\dots,E^r$, we consider the smash product as a multi-indexed spectrum in the natural way,

$$(E^1 \wedge \cdots \wedge E^r)_{n_1,\dots,n_r} = E^1_{n_1} \wedge \cdots \wedge E^r_{n_r}.$$

In general an r-fold multi-indexed symmetric spectrum $E = \{E_{n_1,\dots,n_r}\}$ has an associated infinite loop space

$$\Omega^\infty(E) = \operatorname*{hocolim}_{\mathcal{I}^r} \Omega^{n_1+\cdots+n_r}(E_{n_1,\dots,n_r}).$$

The functoriality underlying this definition is analogous to that in (2.3). We shall always use the symbol $\Omega^\infty(E)$ in this precise way. Notice that the monoidal structure of $\mathcal{I}^r$ makes Ω^∞ a functor from multi-indexed spectra to topological monoids. The topological Hochschild homology of a ring spectrum R is the topological realization of the cyclic spectrum $\mathrm{TH}_\bullet(R)$, defined in spectrum degree n by

$$\mathrm{TH}_\bullet(R,n)\colon\ [k] \mapsto \Omega^\infty(\underbrace{R \wedge \cdots \wedge R}_{k+1} \wedge S^n).$$

The spectrum structure maps are defined in the obvious way involving only the S^n-factor. This construction represents Bökstedt's solution to the problem of how to turn the multi-indexed spectrum $R^{\wedge(k+1)}$ into an equivalent singly-indexed spectrum. The cyclic structure maps are analogous to those in $B^{\mathrm{cy}}_\bullet(G)$ and $\mathrm{HH}_\bullet(R)$. Thus for example $d_0\colon\ \mathrm{TH}_1(R) \to \mathrm{TH}_0(R)$ is the composition

$$\begin{aligned}\operatorname*{hocolim}_{\mathcal{I}^2} \Omega^{n_0+n_1}(R_{n_0} \wedge R_{n_1} \wedge S^n) &\to \operatorname*{hocolim}_{\mathcal{I}^2} \Omega^{n_0+n_1}(R_{n_0+n_1} \wedge S^n)\\ &\to \operatorname*{hocolim}_{\mathcal{I}} \Omega^{n_0}(R_{n_0}),\end{aligned}$$

where the first map uses the multiplication in R and the second map is induced by the monoidal structure $\sqcup\colon\ \mathcal{I} \times \mathcal{I} \to \mathcal{I}$. It follows from the version of Bökstedt's approximation Lemma 2.1 with $\mathcal{I}^{k+1}$ instead of $\mathcal{I}$ that $\mathrm{TH}(R)$ is an Ω-spectrum, and we let $\mathrm{THH}(R)$ be the 0th space.

In order to define the spectrum level multi-trace, we need to model the additive structure of a spectrum in a very precise way. We next explain how this can be done.

4.2 The cyclic Barratt-Eccles construction

Let $E_\bullet\Sigma_n$ be the cyclic set $[k] \mapsto \Sigma_n^{k+1}$ with simplicial operators

$$\begin{aligned} d_i(\sigma_0,\ldots,\sigma_k) &= (\sigma_0,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_k), \quad 0 \le i \le k,\\ s_i(\sigma_0,\ldots,\sigma_k) &= (\sigma_0,\ldots,\sigma_{i-1},\sigma_i,\sigma_i,\ldots,\sigma_k), \quad 0 \le i \le k,\end{aligned}$$

and cyclic operator $t_k(\alpha_0,\ldots,\alpha_k) = (\alpha_k,\alpha_0,\ldots,\alpha_{k-1})$. We let $\mathbb{E}^\infty$ be the cyclic Barratt-Eccles operad with nth space $E_\bullet\Sigma_n$, see [3], [17, 6.5]. This is an E_∞ operad in the sense that the realization $E\Sigma_n$ of the nth space is Σ_n-free and contractible. We use the notation $\mathbb{E}^\infty_\bullet$ for the associated functor from based spaces to simplicial based spaces,

$$\mathbb{E}^\infty_\bullet(X) = \Big(\coprod_{n\ge 0} E_\bullet\Sigma_n \times X^n\Big)/\sim, \tag{4.1}$$

where the equivalence relation $\sim$ is defined as follows. Notice first that the correspondence $\mathbf{n} \mapsto E_\bullet\Sigma_n$ defines a contravariant functor from $\mathcal{I}$ to simplicial sets: Given a morphism $\alpha\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$ and $\sigma \in \Sigma_n$, the composition $\sigma\alpha$ has a unique factorization $\sigma\alpha = \sigma_*(\alpha)\alpha^*(\sigma)$ with $\sigma_*(\alpha)\colon \mathbf{m} \to \mathbf{n}$ injective and order preserving and $\alpha^*(\sigma) \in \Sigma_m$. In this way α induces a simplicial map,

$$\alpha^*\colon E_\bullet\Sigma_n \to E_\bullet\Sigma_m, \quad (\sigma_0, \dots, \sigma_k) \mapsto (\alpha^*(\sigma_0), \dots, \alpha^*(\sigma_k))$$

and given $\beta\colon \mathbf{l} \to \mathbf{m}$ it is clear that $(\alpha\beta)^* = \beta^*\alpha^*$. Secondly, given a based space X the correspondence $\mathbf{n} \mapsto X^n$ defines a covariant functor on $\mathcal{I}$ by letting a morphism $\alpha\colon \mathbf{m} \to \mathbf{n}$ act on $\mathbf{x} \in X^m$ by $\alpha_*(\mathbf{x}) = \mathbf{y}$, where

$$y_j = \begin{cases} x_i, & \text{if } \alpha(i) = j \\ *, & \text{if } j \notin \alpha(\mathbf{m}). \end{cases}$$

With this notation the equivalence relation in (4.1) is generated by the relations

$$(e, \alpha_*(\mathbf{x})) \sim (\alpha^*(e), \mathbf{x}) \quad \text{for } e \in E_k\Sigma_n,\ \mathbf{x} \in X^m \text{ and } \alpha\colon \mathbf{m} \to \mathbf{n}.$$

In other words, $\mathbb{E}^\infty_\bullet(X)$ is the tensor product of the functors $\mathbf{n} \mapsto E_\bullet\Sigma_n$ and $\mathbf{n} \mapsto X^n$ over $\mathcal{I}$, ie, the coend of the $\mathcal{I}^{\mathrm{op}} \times \mathcal{I}$-diagram $E\Sigma_m \times X^n$, cf. [14, IX.6]. We let $\mathbb{E}^\infty(X)$ be the realization. (Barratt and Eccles use the notation $\Gamma^+(X)$, but we want to avoid this since we also use Γ-spaces in the sense of Segal.) We write the elements of $\mathbb{E}^\infty_\bullet(X)$ as $[\sigma, \mathbf{x}]$ where $\sigma \in E_\bullet\Sigma_k$ and $\mathbf{x} \in X^k$. Block sums of permutations give $\mathbb{E}^\infty_\bullet(X)$ the structure of a simplicial topological monoid,

$$[\sigma, \mathbf{x}] \cdot [\sigma', \mathbf{x}'] = [\sigma \oplus \sigma', (\mathbf{x}, \mathbf{x}')].$$

The homotopy theoretical significance of the functor $\mathbb{E}^\infty(X)$ is that it provides a combinatorial model of $\Omega^\infty\Sigma^\infty(X)$ for non-degenerately based connected X. In more detail, it is proved in [3] that in the diagram

$$\mathbb{E}^\infty(X) \to \operatorname{colim} \Omega^n \mathbb{E}^\infty(S^n \wedge X) \leftarrow \operatorname{colim} \Omega^n(S^n \wedge X),$$

the left hand arrow is an equivalence for connected X and the right hand arrow is an equivalence in general.

We extend $\mathbb{E}^\infty$ to a functor on (symmetric) spectra by applying it in each spectrum dimension, ie, $\mathbb{E}^\infty(E)_n = \mathbb{E}^\infty(E_n)$ with structure maps

$$S^1 \wedge \mathbb{E}^\infty(E_n) \to \mathbb{E}^\infty(S^1 \wedge E_n) \to \mathbb{E}^\infty(E_{n+1}).$$

Since we assume spectra to be connective and convergent, it easily follows that the natural map $E \to \mathbb{E}^\infty(E)$ is an equivalence. Similarly, given a simplicial spectrum, we may apply $\mathbb{E}^\infty_\bullet$ degree-wise to get a bisimplicial spectrum and then restrict to the simplicial diagonal. This is in effect what we shall do when defining the spectrum level multi-trace.

4.3 The spectrum level multi-trace

The multi-trace for a symmetric ring spectrum R is a natural map of multi-indexed spectra

$$\mathrm{tr}\colon\ \underbrace{M_n(R)\wedge\cdots\wedge M_n(R)}_{k+1} \to \mathbb{E}_k^\infty(\underbrace{R\wedge\cdots\wedge R}_{k+1}). \tag{4.2}$$

Let us first explain how to define this when R is a spectrum of based simplicial sets. In this case tr is based on a natural transformation

$$\mathrm{tr}\colon\ M_n(X_0)\wedge\cdots\wedge M_n(X_k) \to \mathbb{E}_k^\infty(X_0\wedge\cdots\wedge X_k),$$

where $X_0,\ldots,X_k$ are based sets and $M_n(X_i)=\mathrm{Map}_*(\mathbf{n}_+,\mathbf{n}_+\wedge X_i)$. Suppose given an element $(A^0,\ldots,A^k)$ in the domain and use matrix notation to write $A^i=(x^i_{s,t})$. Let D be the set of multi-indices corresponding to the non-trivial summands in the multi-trace formula, ie,

$$D=\{(s_0,\ldots,s_k)\colon\ x^0_{s_k,s_0}\neq *,\ldots,x^k_{s_{k-1},s_k}\neq *\}.$$

Since by definition the matrices have at most one non-base point entry in each column, the projections $(s_0,\ldots,s_k)\mapsto s_i$ give rise to injective maps $p_i\colon D\to \mathbf{n}$. Suppose that D has cardinality m and order the elements by choosing a bijection $\gamma\colon \mathbf{m}\to D$. The composition $p_i\gamma\colon \mathbf{m}\to\mathbf{n}$ is injective for each i and admits a unique factorization $p_i\gamma=\alpha_i\sigma_i$, where α_i is injective and order preserving and $\sigma_i\in\Sigma_m$. Consider the natural map

$$D\to X_0\wedge\cdots\wedge X_k,\quad (s_0,\ldots,s_k)\mapsto (x^0_{s_k,s_0},\ldots,x^k_{s_{k-1},s_k})$$

and let $\mathbf{x}$ be the composition

$$\mathbf{x}\colon\ \mathbf{m}\to D\to X_0\wedge\cdots\wedge X_k.$$

The first observation is that the element

$$[(\sigma_0,\ldots,\sigma_k);\mathbf{x}]\in \Sigma_m^{k+1}\times_{\Sigma_m}(X_0\wedge\cdots\wedge X_k)^m$$

is independent of the ordering γ used to define it. By definition the multi-trace is the image in $\mathbb{E}_k^\infty(X_0\wedge\cdots\wedge X_k)$,

$$\mathrm{tr}(A^0,\ldots,A^k)=[(\sigma_0,\ldots,\sigma_k);\mathbf{x}]\in\mathbb{E}_k^\infty(X_0\wedge\cdots\wedge X_k). \tag{4.3}$$

The second observation is that because of the base point relations in the target this construction is natural with respect to based maps in $X_0,\ldots X_k$.

Example 4.4 As an example to illustrate the construction we calculate

$$\mathrm{tr}\left(\begin{bmatrix} * & x^0_{12}\\ x^0_{21} & *\end{bmatrix},\begin{bmatrix} * & x^1_{12}\\ x^1_{21} & *\end{bmatrix}\right)=[(1_2,\tau);(x^0_{21},x^1_{12}),(x^0_{12},x^1_{21})],$$

where $\tau\in\Sigma_2$ is the non-identity element.

The spectrum level multi-trace (4.2) is defined by degree-wise extending the above natural transformation to a natural transformation between functors of simplicial sets. We then extend this to a natural transformation of multi-indexed spectra by applying it in each multi-degree. This gives the required maps

$$\mathrm{tr}\colon\ M_n(R_{n_0}) \wedge \cdots \wedge M_n(R_{n_k}) \to \mathbb{E}_k^\infty(R_{n_0} \wedge \cdots \wedge R_{n_k}).$$

In the case where R is a spectrum of topological spaces we observe that the expression in (4.3) also makes sense if $X_0, \ldots, X_k$ are (non-degenerately) based topological spaces, and we define tr by the same formula.

4.4 The topological trace map

We define a combinatorially enriched version $\mathrm{TH}^+(R)$ of topological Hochschild homology by applying Bökstedt's construction to the multi-indexed spectrum on the right hand side of (4.2), ie,

$$\mathrm{TH}_\bullet^+(R,n)\colon\ [k] \mapsto \Omega^\infty(\mathbb{E}_k^\infty(\underbrace{R \wedge \cdots \wedge R}_{k+1} \wedge S^n)).$$

This is in a natural way the cyclic diagonal of a bicyclic spectrum. The $+$ decoration indicates that $\mathrm{TH}_\bullet^+(R,n)$ is a homotopy commutative cyclic monoid. Using the natural inclusion $X \mapsto \mathbb{E}_\bullet^\infty(X)$ we get a degree-wise equivalence $\mathrm{TH}_\bullet(R) \to \mathrm{TH}_\bullet^+(R)$ and thus an equivalence of realizations $\mathrm{TH}(R) \xrightarrow{\sim} \mathrm{TH}^+(R)$. The spectrum level multi-trace has formal properties similar to the linear multi-trace and in particular there results a cyclic map

$$\mathrm{tr}\colon\ \mathrm{TH}_\bullet(M_n(R)) \to \mathrm{TH}_\bullet^+(R).$$

(One can show that the realization can be extended to give an equivalence of $\mathbb{T}$-equivariant spectra, but we shall not use this here.) The definition of the topological trace map is now completely analogous to the linear case. There is an obvious embedding of cyclic spaces $B_\bullet^{\mathrm{cy}}GL_n(R) \to \mathrm{THH}_\bullet(M_n(R))$ induced by the natural transformation

$$\begin{aligned}&\Omega^{n_0}(M_n(R_{n_0}))^* \times \cdots \times \Omega^{n_k}(M_n(R_{n_k}))^* \\ &\quad \to \Omega^{n_0+\cdots+n_k}(M_n(R_{n_0}) \wedge \cdots \wedge M_n(R_{n_k}))\end{aligned} \tag{4.5}$$

that sends a tuple of maps to their smash product. Composing with the multi-trace we get a cyclic map

$$\coprod_{n\geq 0} B_\bullet^{\mathrm{cy}}GL_n(R) \to \coprod_{n\geq 0} \mathrm{THH}_\bullet(M_n(R)) \to \mathrm{THH}_\bullet^+(R).$$

This is a monoid homomorphism with respect to block sums of matrices on the domain and the simplicial monoid structure on the target. After realization and group completion we get maps

$$\mathrm{K}^{\mathrm{cy}}(R) \to \Omega B(\mathrm{THH}^{+}(R)) \xleftarrow{\simeq} \mathrm{THH}^{+}(R) \xleftarrow{\simeq} \mathrm{THH}(R).$$

The topological trace map $\mathrm{tr}\colon \mathrm{K}(R) \to \mathrm{THH}(R)$ is the homotopy class obtained by composing with the homotopy section $\mathrm{K}(R) \to \mathrm{K}^{\mathrm{cy}}(R)$.

Remark 4.6 It is not difficult to extend this definition of the trace map to a map of spectra or to refine it to a version of the cyclotomic trace $\mathrm{trc}\colon \mathrm{K}(R) \to \mathrm{TC}(R)$, cf. [7]. However, this is not the purpose of the present paper. A construction of the trace map from a more categorical point of view has been given by Dundas and McCarthy [11] and Dundas [10].

Letting $n = 1$ in (4.5) gives a map $B^{\mathrm{cy}}GL_1(R) \to \mathrm{THH}(R)$. The following proposition is immediate from the definitions

Proposition 4.7 *There is a strictly commutative diagram of spaces*

$$\begin{tikzcd} B^{\mathrm{cy}}GL_1(R) \arrow[r] \arrow[d] & \mathrm{THH}(R) \arrow[d, "\sim"] \\ \mathrm{K}^{\mathrm{cy}}(R) \arrow[r, "tr"] & \Omega B\,\mathrm{THH}^{+}(R). \end{tikzcd} \tag{4.8}$$

□

5 Γ-spaces and units of commutative ring spectra

In this section we show that if R is a commutative (and convergent) ring spectrum, then $GL_1(R)$ is the 0th space of an Ω-spectrum. The same is true for the group-like monoid $X_{h\mathcal{I}}$ associated to a commutative and group-like $\mathcal{I}$-monoid X, and we formulate the construction in this generality.

5.1 Γ-spaces

We first recall Segal's notion of Γ-spaces and the Anderson-Segal method for constructing the associated homology theory. The paper by Bousfield and Friedlander [4] is the basic reference for this material. Let Γ^o denote the category of finite pointed sets and pointed maps. A Γ-space is a functor $A\colon \Gamma^o \to \mathcal{T}$ such that $A(*) = *$. We say that a Γ-space is *special* if given pointed sets S and T the natural map $A(S \vee T) \to A(S) \times A(T)$ is an equivalence. This implies in particular that $A(S^0)$ has the structure of a homotopy associative and commutative H-space with multiplication

$$A(S^0) \times A(S^0) \simeq A(S^0 \vee S^0) \to A(S^0).$$

We say that A is *very special* if $A(S^0)$ is group-like, ie, if the monoid of components is a group. A Γ-space A extends to a functor on the category of pointed simplicial sets in a two stage procedure. First A is extended to the category of all pointed sets by forcing it to commute with colimits. Given a simplicial set X we then apply A degree-wise to get a simplicial space $[k] \mapsto A(X_k)$ with realization $A(X)$. The main result is that if A is very special then the resulting functor is a homology theory: Applying A to a cofibration sequence of pointed simplicial sets $X \to Y \to Y/X$ gives a homotopy fibration sequence

$$A(X) \to A(Y) \to A(Y/X)$$

in the sense that the inclusion of $A(X)$ in the homotopy fiber of the second map is an equivalence. In particular, a very special Γ-space gives rise to a symmetric Ω-spectrum $\{A(S^n)\colon n \geq 0\}$, in which the structure maps are the realizations of the obvious (multi)-simplicial maps $S^1_\bullet \wedge A(S^n_\bullet) \to A(S^{n+1}_\bullet)$.

5.2 Γ-spaces associated to commutative $\mathcal{I}$-monoids

In order to motivate the construction we recall the definition of the Γ-space associated to a commutative topological monoid G. Given a finite based set S, let $\bar{S}$ be the subset obtained by excluding the base point. Then $G(S) = G^{\bar{S}}$, and a based map $\alpha\colon S \to T$ induces a map $G(S) \to G(T)$ by multiplying the elements in G indexed by $\alpha^{-1}\{t\}$ for each $t \in \bar{T}$.

Implementing this idea for a commutative $\mathcal{I}$-monoid requires some preparation. Given S as above, let $\mathcal{P}(\bar{S})$ be the category of subsets and inclusions in $\bar{S}$. A based map $\alpha\colon S \to T$ induces a functor $\alpha^*\colon \mathcal{P}(\bar{T}) \to \mathcal{P}(\bar{S})$ by letting $\alpha^*(U) = \alpha^{-1}(U)$ for $U \subseteq \bar{T}$. We define a category $\mathcal{D}(S)$ of $\bar{S}$-indexed sum diagrams in $\mathcal{I}$ as follows. An object is a functor $\theta\colon \mathcal{P}(\bar{S}) \to \mathcal{I}$ that takes disjoint unions to coproducts of finite sets, ie, if $U \cap V = \emptyset$, then the diagram $\theta_U \to \theta_{U\cup V} \leftarrow \theta_V$ represents $\theta_{U\cup V}$ as a coproduct of finite sets. (The category $\mathcal{I}$ itself does of course not have coproducts.) Notice in particular that $\theta_\emptyset = \mathbf{0}$. A morphisms in $\mathcal{D}(S)$ is a natural transformations of functors (not necessarily an isomorphism). This construction is clearly functorial in Γ^o: A based map $\alpha\colon S \to T$ induces a functor $\alpha_*\colon \mathcal{D}(S) \to \mathcal{D}(T)$ by letting $\alpha_*\theta = \theta \circ \alpha^*$. Notice that an object θ in $\mathcal{D}(S)$ is determined by

- its values θ_s for $s \in \bar{S}$;

- a choice of injective map $\theta_s \to \theta_U$ whenever $s \in U$ such that the induced map $\sqcup_{s\in U}\theta_s \to \theta_U$ (with any ordering of the summands) is a bijection.

Restricting to the one-point subsets of $\bar{S}$ gives a functor $\pi_S\colon \mathcal{D}(S) \to \mathcal{I}^{\bar{S}}$, where the latter denotes the product category indexed by $\bar{S}$ (we let $\mathcal{I}^{\emptyset}$ denote the one-point category). This is an equivalence of categories, and specifying an ordering of $\bar{S}$ gives a canonical choice of an inverse functor $\mathcal{I}^{\bar{S}} \to \mathcal{D}(S)$, using the monoidal structure of $\mathcal{I}$. Notice however, that $\mathcal{I}^{\bar{S}}$ is not functorial in Γ^o as is the case for $\mathcal{D}(S)$.

Lemma 5.1 *Given a functor $Y\colon \mathcal{I}^{\bar{S}} \to \mathcal{T}$, the natural map*

$$\operatorname*{hocolim}_{\mathcal{D}(S)} Y \circ \pi_S \to \operatorname*{hocolim}_{\mathcal{I}^{\bar{S}}} Y$$

induced by π_S is an equivalence.

Proof By the cofinality criterion in [5, XI.9.2] (or rather its dual version) it suffices to check that for any object $\mathbf{a} \in \mathcal{I}^{\bar{S}}$, the category $(\mathbf{a} \downarrow \pi_S)$ of objects under $\mathbf{a}$ is contractible. But this is clear since this category has an initial object. □

Let now X be a commutative $\mathcal{I}$-monoid and consider the $\mathcal{I}^{\bar{S}}$-diagram $X^{\bar{S}}$ defined by

$$\{\mathbf{n}_s\colon\ s \in \bar{S}\} \mapsto \prod_{s\in\bar{S}} X(n_s).$$

For $\bar{S} = \emptyset$ this should be interpreted as the one-point space. We use the notation $X(S)$ for the $\mathcal{D}(S)$-diagram obtained by composing with π_S. With this definition $X(S)$ is functorial in S is the sense that a based map $\alpha\colon\ S \to T$ gives rise to a natural transformation of $\mathcal{D}(S)$ diagrams

$$X(S) \to X(T) \circ \alpha_*.$$

In order to see this fix an object θ in $\mathcal{D}(S)$ and choose an ordering of the subsets $U_t = \alpha^{-1}(t)$ for each $t \in \bar{T}$. The map in question is then a product over $\bar{T}$ of maps of the form

$$\prod_{s\in U_t} X(\theta_s) \to X(\bigsqcup_{s\in U_t} \theta_s) \to X(\theta_{U_t}),$$

where the first arrow comes from the multiplication in X and the second arrow is induced by the bijection $\sqcup_{s\in U_t}\theta_s \to \theta_{U_t}$ determined by the sum diagram θ. The main point is that since X is commutative the composite map does not depend on the ordering of U_t used to define it.

By definition the Γ-space associated to X is given by

$$X_{h\mathcal{I}}(S) = \operatorname*{hocolim}_{\mathcal{D}(S)} X(S).$$

Given a based map $\alpha\colon S \to T$, the induced map $\alpha_*\colon X_{h\mathcal{I}}(S) \to X_{h\mathcal{I}}(T)$ is the composition

$$\operatorname*{hocolim}_{\mathcal{D}(S)} X(S) \to \operatorname*{hocolim}_{\mathcal{D}(T)} X(T) \circ \alpha_* \to \operatorname*{hocolim}_{\mathcal{D}(T)} X(T),$$

where the first map is induced by the above natural transformation and the second map is the map of homotopy colimits determined by the functor α_*.

It follows immediately from the definition that $X_{h\mathcal{I}}(S^0) = X_{h\mathcal{I}}$. In order to compare $X_{h\mathcal{I}}(S^1)$ to the usual bar construction of $X_{h\mathcal{I}}$ we specify an ordering of the k-simplices in $S^1_\bullet$ by letting

$$u_j = (\underbrace{0, \dots, 0}_{j}, 1, \dots, 1), \quad \text{for } j = 0, \dots, k. \tag{5.2}$$

Then u_0 is the base point and $\bar{S}^1_k = \{u_1, \dots u_k\}$.

Proposition 5.3 *The Γ-space associated to a commutative $\mathcal{I}$-monoid X is always special and is very special if and only if the underlying monoid $X_{h\mathcal{I}}$ is group-like. In general there is a natural equivalence $BX_{h\mathcal{I}} \xrightarrow{\sim} X_{h\mathcal{I}}(S^1)$.*

Proof Using Lemma 5.1 we get an equivalence

$$X_{h\mathcal{I}}(S) = \operatorname*{hocolim}_{\mathcal{D}(S)} X(S) \xrightarrow{\sim} \operatorname*{hocolim}_{\mathcal{I}^{\bar{S}}} X^{\bar{S}} \cong \prod_{s \in \bar{S}} X_{h\mathcal{I}},$$

which is the condition for $X_{h\mathcal{I}}$ to be special. The statement about being very special follows from the definition. In order to define the equivalence we use the ordering of the simplices of $S^1_\bullet$ given by (5.2). As noted earlier this ordering determines an equivalence

$$\operatorname*{hocolim}_{\mathcal{I}^k} X(n_1) \times \cdots \times X(n_k) \to \operatorname*{hocolim}_{\mathcal{D}(S^1_k)} X(S^1_k)$$

using the monoidal structure of $\mathcal{I}$. Identifying the left hand side with the k-simplices of $B_\bullet X_{h\mathcal{I}}$ we get a simplicial map $B_\bullet X_{h\mathcal{I}} \to X_{h\mathcal{I}}(S^1_\bullet)$. Since this is an equivalence in each simplicial degree its realization is also an equivalence as required. $\square$

Remark 5.4 This construction of Γ-spaces based on the category $\mathcal{D}(S)$ differs from that of Segal [19, §2] in that we allow *all* natural transformations, not only the natural isomorphisms. Consequently, our definition of the Γ-space associated to a commutative $\mathcal{I}$-monoid takes into account all the maps $X(m) \to X(n)$ induced by morphisms in $\mathcal{I}$. As an example, consider the commutative $\mathcal{I}$-monoid given by the classifying spaces $BO(n)$ of the orthogonal groups. In this case Segal's construction produces a special Γ-space with underlying space $\coprod BO(n)$, whereas our construction produces a very special Γ-space with underlying space $\operatorname{hocolim}_{\mathcal{I}} BO(n) \simeq BO$. The last equivalence follows from Bökstedt's Lemma 2.1. Thus the two constructions respectively produce models of the (-1)-connected and 0-connected topological K-theory spectrum.

Definition 5.5 Given a commutative (and convergent) symmetric ring spectrum R, let $GL_1(R)$ be the Γ-space associated to the $\mathcal{I}$-monoid $\Omega^n(R_n)^*$ considered in Section 2.3, and let $gl_1(R)$ be the associated spectrum.

It will always be clear from the context whether $GL_1(R)$ denotes a Γ-space as above or the underlying group-like monoid as in Section 2.

6 Commutative ring spectra and splittings

Let R be a commutative symmetric ring spectrum. In this section we show that the natural inclusions $GL_1(R) \to B^{\mathrm{cy}}GL_1(R)$ and $\Omega^\infty(R) \to \mathrm{THH}(R)$ have compatible left inverses in the homotopy category, where by compatible we mean that these splittings are related by a homotopy commutative diagram

$$\begin{array}{ccc} B^{\mathrm{cy}}GL_1(R) & \xrightarrow{r} & GL_1(R) \\ \downarrow & & \downarrow \\ \mathrm{THH}(R) & \xrightarrow{r} & \Omega^\infty(R). \end{array} \tag{6.1}$$

We then define η_R to be the composite homotopy class

$$\eta_R\colon\ BGL_1(R) \to K(R) \xrightarrow{\mathrm{tr}} \mathrm{THH}(R) \xrightarrow{r} \Omega^\infty(R).$$

Using the diagrams (4.8) and (6.1) we get an alternative description as follows.

Proposition 6.2 *The homotopy class η_R is represented by the composition*

$$BGL_1(R) \to L(BGL_1(R)) \xleftarrow{\sim} B^{\mathrm{cy}}GL_1(R) \xrightarrow{r} GL_1(R),$$

(where the first map is the inclusion of the constant loops), followed by the inclusion of $GL_1(R)$ in $\Omega^\infty(R)$. □

This concludes the first part of the proof of Theorem 1.2. In order to motivate the construction, consider the cyclic bar construction of a commutative monoid G. In this case the inclusion $G \to B^{\mathrm{cy}}(G)$ is split by degree-wise multiplication in G. This can also be expressed in terms of the Γ-space associated to G: The sequence $G \to B^{\mathrm{cy}}G \to BG$ is the effect of evaluating G on the cofibration sequence $S^0 \to S^1_{\bullet+} \to S^1_{\bullet}$, and the splitting is induced by the projection $S^1_{\bullet+} \to S^0$ that maps $S^1_{\bullet}$ to the non-base point in S^0. Let now $GL_1(R)$ be the Γ-space defined in Definition 5.5. The next Lemma shows that we may replace $B^{\mathrm{cy}}GL_1(R)$ by $GL_1(R)(S^1_+)$ up to homotopy.

Proposition 6.3 *Let X be a convergent and commutative $\mathcal{I}$-monoid. Then there exists a space W^{cy} and equivalences*

$$B^{\mathrm{cy}}X_{h\mathcal{I}} \xrightarrow{\sim} W^{\mathrm{cy}} \xleftarrow{\sim} X_{h\mathcal{I}}(S^1_+).$$

Proof Let $W^{\mathrm{cy}}_{\bullet}$ be the cyclic space

$$[k] \mapsto \operatorname*{hocolim}_{\mathcal{I}^{k+1}\times\mathcal{D}(S^1_{k+})} X(n_0 \sqcup \theta_0) \times \cdots \times X(n_k \sqcup \theta_k),$$

where θ is an object of $\mathcal{D}(S^1_{k+})$ and we write $\theta_i = \theta_{u_i}$. We have functors $\mathcal{I}^{k+1} \to \mathcal{I}^{k+1} \times \mathcal{D}(S^1_{k+}) \leftarrow \mathcal{D}(S^1_{k+})$ obtained by fixing the initial object in one of the factors. The cyclic structure of $W^{\mathrm{cy}}_{\bullet}$ is the obvious one such that the induced maps

$$B^{\mathrm{cy}}_{\bullet}X_{h\mathcal{I}} \to W^{\mathrm{cy}}_{\bullet} \leftarrow X_{h\mathcal{I}}(S^1_{\bullet+})$$

become maps of cyclic spaces. Since we assume that X is convergent, it follows from Bökstedt's approximation lemma 2.1 (with $\mathcal{I}^{k+1}$ instead of $\mathcal{I}$), that these maps are equivalences in each simplicial degree. After realization we thus get a pair of equivalences relating $B^{\mathrm{cy}}X_{h\mathcal{I}}$ and $X_{h\mathcal{I}}(S^1_+)$. $\square$

Remark 6.4 The condition that X be convergent is necessary for the argument in Proposition 6.3, since otherwise the map

$$\operatorname*{hocolim}_{\mathcal{I}} X(m) \to \operatorname*{hocolim}_{\mathcal{I}\times\mathcal{I}} X(m \sqcup n)$$

induced by the functor $I \to I^2$, $\mathbf{m} \mapsto (\mathbf{m}, \mathbf{0})$ need not be an equivalence. The $\mathcal{I}$-monoid $X(n) = X^n$ considered in Section 4 provides a counter example. It should also be noted that the construction of the simplicial map $B_{\bullet}X_{h\mathcal{I}} \to X_{hI}(S^1_{\bullet})$ in the proof of Proposition 5.3 cannot be applied to give a cyclic map $B^{\mathrm{cy}}_{\bullet}X_{h\mathcal{I}} \to X_{h\mathcal{I}}(S^1_{\bullet+})$.

Using the above equivalences, we define the splitting r to be the composite homotopy class

$$r\colon\ B^{\mathrm{cy}}GL_1(R) \simeq GL_1(R)(S^1_+) \to GL_1(R),$$

where the last map is induced by the projection $S^1_{\bullet+} \to S^0$.

We next consider a version $\mathrm{TH}'(R)$ of topological Hochschild homology that relates to $\mathrm{TH}(R)$ as $GL_1(R)(S^1_+)$ relates to $B^{\mathrm{cy}}GL_1(R)$. By definition this is the realization of the cyclic spectrum

$$\mathrm{TH}'_\bullet(R,n)\colon\ [k] \mapsto \operatorname*{hocolim}_{\mathcal{D}(S^1_{k+})} \Omega^{\theta_0\sqcup\cdots\sqcup\theta_k}(R_{\theta_0} \wedge \cdots \wedge R_{\theta_k} \wedge S^n),$$

where θ denotes an object in $\mathcal{D}(S^1_{k+})$ and we again write $\theta_i = \theta_{u_i}$. The cyclic structure maps are defined as for $GL_1(R)(S^1_{\bullet+})$. For example, in spectrum degree zero, $d_0\colon\ \mathrm{THH}'_1(R) \to \mathrm{THH}'_0(R)$ is the composition

$$\operatorname*{hocolim}_{\theta\in\mathcal{D}(S^1_{1+})} \Omega^{\theta_0\sqcup\theta_1}(R_{\theta_0} \wedge R_{\theta_1}) \to \operatorname*{hocolim}_{\theta\in\mathcal{D}(S^1_{1+})} \Omega^{\theta_{01}}(R_{\theta_{01}}) \to \operatorname*{hocolim}_{\psi\in\mathcal{D}(S^1_{0+})} \Omega^{\psi_0}(R_{\psi_0}).$$

Here $\theta_0 \to \theta_{01} \leftarrow \theta_1$ denotes an object in $\mathcal{D}(S^1_{1+})$ and the first map is induced by the natural transformation that takes $f \in \Omega^{\theta_0\sqcup\theta_1}(R_{\theta_0} \wedge R_{\theta_1})$ to the element in $\Omega^{\theta_{01}}(R_{\theta_{01}})$ given by the composition

$$S^{\theta_{01}} \xrightarrow{\sigma^{-1}} S^{\theta_0\sqcup\theta_1} = S^{\theta_0} \wedge S^{\theta_1} \xrightarrow{f} R_{\theta_0} \wedge R_{\theta_1} \xrightarrow{\mu_{\theta_0,\theta_1}} R_{\theta_0\sqcup\theta_1} \xrightarrow{\sigma} R_{\theta_{01}},$$

where $\sigma\colon\ \theta_0 \sqcup \theta_1 \to \theta_{01}$ is the bijection determined by θ. The second map is induced by the natural transformation $d_0\colon\ \mathcal{D}(S^1_{1+}) \to \mathcal{D}(S^1_{0+})$. With this definition we have the equality

$$d_0 = d_1\colon\ \mathrm{TH}'_1(R) \to \mathrm{TH}'_0(R)$$

and consequently the iterated boundary maps give a well-defined cyclic map $r\colon\ \mathrm{TH}'_\bullet(R) \to \mathrm{TH}'_0(R)$, where the target is considered a constant cyclic spectrum. In spectrum degree zero we thus get a cyclic map of spaces

$$r\colon\ \mathrm{THH}'_\bullet(R) \to \Omega^\infty(R).$$

The next proposition is the analogue of Proposition 6.3.

Proposition 6.5 *The spectra* $\mathrm{TH}(R)$ *and* $\mathrm{TH}'(R)$ *are related by a pair of equivalences.*

Proof Letting $W^{\mathrm{cy}}_\bullet = \{W^{\mathrm{cy}}_\bullet(n)\colon\ n \geq 0\}$ denote the cyclic spectrum

$$[k] \mapsto \operatorname*{hocolim}_{\mathcal{I}^{k+1}\times\mathcal{D}(S^1_{k+})} \Omega^{\mathbf{n}_0\sqcup\theta_0\sqcup\cdots\sqcup\mathbf{n}_k\sqcup\theta_k}(R_{\mathbf{n}_0\sqcup\theta_0} \wedge \cdots \wedge R_{\mathbf{n}_k\sqcup\theta_k} \wedge S^n),$$

the proof proceeds exactly like the proof of Proposition 6.3. □

As in the definition of the trace map we consider the transformation

$$\Omega^{\theta_0}(R_{\theta_0})^* \times \cdots \times \Omega(R_{\theta_k})^* \to \Omega^{\theta_0 \sqcup \cdots \sqcup \theta_k}(R_{\theta_0} \wedge \cdots \wedge R_{\theta_k})$$

that sends a tuple of maps to their smash product. Viewing these maps as natural transformations of $\mathcal{D}(S^1_{k+})$-diagrams we get a cyclic map

$$GL_1(R)(S^1_{\bullet+}) \to \mathrm{THH}'_\bullet(R).$$

It follows immediately from the definitions that this map is compatible with the splittings of $B^{\mathrm{cy}}GL_1(R)$ and $\mathrm{THH}'(R)$ in the sense of the following proposition.

Proposition 6.6 *There is a strictly commutative diagram of spaces*

$$\begin{array}{ccc} GL_1(R)(S^1_+) & \longrightarrow & GL_1(R) \\ \downarrow & & \downarrow \\ \mathrm{THH}'(R) & \longrightarrow & \Omega^\infty(R). \end{array}$$ □

The homotopy commutative diagram (6.1) in the beginning of this section is derived from this using the equivalences in Proposition 6.3 and Proposition 6.5.

7 The Hopf map and free loops on infinite loop spaces

In this section we finish the proof of Theorem 1.2 by showing that the composite homotopy class

$$BGL_1(R) \to L(BGL_1(R)) \xleftarrow{\sim} B^{\mathrm{cy}}GL_1(R) \xrightarrow{r} GL_1(R)$$

is multiplication by η in the sense explained in the introduction. More generally, let G be a very special Γ-space and let $g = \{G(S^n)\colon n \geq 0\}$ be the associated Ω-spectrum. By [4, 4.1], the Γ-space G is determined by g in the sense that the diagram

$$G(X) \xrightarrow{\sim} \operatorname{hocolim} \Omega^n G(S^n \wedge X) \xleftarrow{\sim} \operatorname{hocolim} \Omega^n(G(S^n) \wedge X)) = \Omega^\infty(g \wedge X)$$

specifies a natural equivalence $G(X) \simeq \Omega^\infty(g \wedge X)$. Evaluating G on the based cyclic set $S^1_{\bullet+}$ gives a cyclic space $G(S^1_{\bullet+})$. The realization $G(S^1_+)$ then has a $\mathbb{T}$-action and, as in the case of the cyclic bar construction, we consider the composite map

$$\mathbb{T} \times G(S^1_+) \to G(S^1_+) \to G(S^1)$$

with adjoint $G(S^1_+) \to L(G(S^1))$. In the next proposition we analyze the homotopy fibration sequence obtained by evaluating G on the cofibration sequence $S^0 \to S^1_{\bullet+} \to S^1_\bullet$.

Proposition 7.1 *There is a commutative diagram of homotopy fibration sequences*

$$\begin{tikzcd} G(S^0) \arrow[r] \arrow[d, "\sim"] & G(S^1_+) \arrow[r] \arrow[d, "\sim"] & G(S^1) \arrow[d, equal] \\ \Omega(G(S^1)) \arrow[r] & L(G(S^1)) \arrow[r] & G(S^1) \end{tikzcd} \tag{7.2}$$

in which the the vertical maps are equivalences.

Proof The commutativity of the right hand square is immediate since we evaluate a loop at the unit element of $\mathbb{T}$. In order to prove commutativity of the left hand square we recall that for any cyclic space $X_\bullet$, the $\mathbb{T}$-action on the zero simplices $X_0 \subseteq |X_\bullet|$ has the following description. If x is an element of X_0 and $u \in \mathbb{T}$,

$$u \cdot x = [t_1 s_0 x, u] \in |X_\bullet|.$$

Here t_1 is the cyclic operator in degree one and we make the identification $\mathbb{T} = \Delta^1/\partial\Delta^1$. Using this, it is easy to check that the composition

$$\mathbb{T} \times G(S^0) \to \mathbb{T} \times G(S^1_+) \to G(S^1_+) \to G(S^1)$$

is given by $(u, x) \mapsto [x', u]$, where $x' \in G(S^1_1)$ is the image of x under the homeomorphism induced by the based bijection $S^0 \to S^1_1$. The above composition clearly equals the composition

$$\mathbb{T} \times G(S^0) \to S^1 \wedge G(S^0) \to G(S^1),$$

which shows that the left hand square in the diagram is also commutative. □

In Diagram (7.2) the map $L(G(S^1_+)) \to G(S^1)$ in the lower sequence is split by the inclusion of the constant loops, and the map $G(S^0) \to G(S^1_+)$ in the upper sequence is split by evaluating G on the projection $r\colon S^1_{\bullet+} \to S^0$ that maps $S^1_\bullet$ to the non-base point of S^0. The next proposition expresses the fact that these splittings are not compatible in general. As usual $\eta \in \pi^s_1(S^0)$ denotes the stable Hopf map.

Proposition 7.3 *Using the natural equivalences $\Omega^\infty(g \wedge S^1) \simeq G(S^1)$ and $\Omega^\infty(g) \simeq G(S^0)$, the composite homotopy class*

$$G(S^1) \to L(G(S^1)) \xleftarrow{\sim} G(S^1_+) \xrightarrow{r} G(S^0),$$

is given by $\Omega^\infty(g \wedge \eta)$.

Proof We first observe that (7.2) is in fact a diagram of infinite loop spaces and infinite loop maps, and that as such it is equivalent to the following diagram of spectra

$$\begin{array}{ccccc} g \wedge S^0 & \longrightarrow & g \wedge S^1_+ & \longrightarrow & g \wedge S^1 \\ \simeq\downarrow & & \simeq\downarrow\Phi & & \| \\ F(S^1, g \wedge S^1) & \longrightarrow & F(S^1_+, g \wedge S^1) & \longrightarrow & F(S^0, g \wedge S^1). \end{array}$$

Here $F(-, g \wedge S^1)$ is the obvious function spectrum and the upper and lower cofibration sequences are both induced from $S^0 \to S^1_+ \to S^1$. These cofibration sequences have canonical stable splittings induced by the projection $r\colon S^1_+ \to S^0$ and the associated stable section $s\colon S^1 \to S^1_+$. The vertical map Φ in the middle is the adjoint of

$$S^1_+ \wedge g \wedge S^1_+ \to g \wedge S^1_+ \to g \wedge S^1,$$

where the first map uses the action of S^1 on itself given by the group structure. It is clear that the above diagram is equivalent to the one obtained by smashing g with

$$\begin{array}{ccccc} S^0 & \longrightarrow & S^1_+ & \longrightarrow & S^1 \\ \simeq\downarrow & & \simeq\downarrow\Phi & & \| \\ F(S^1, S^1) & \longrightarrow & F(S^1_+, S^1) & \longrightarrow & F(S^0, S^1). \end{array}$$

Here the definition of Φ is analogous to the definition given above. We must prove that the stable map

$$S^1 = F(S^0, S^1) \xrightarrow{r^*} F(S^1_+, S^1) \overset{\Phi}{\simeq} S^1_+ \xrightarrow{r} S^0$$

represents η. Using the canonical splittings to represent Φ^{-1} as a 2×2 matrix, this composition represents the off diagonal term. It is therefore the negative of the composite stable map starting in the upper right corner of the diagram,

$$S^1 \xrightarrow{s} S^1_+ \xrightarrow{\Phi} F(S^1_+, S^1) \xrightarrow{s^*} F(S^1, S^1).$$

The adjoint of this is the stable map

$$S^1 \wedge S^1 \xrightarrow{s \wedge s} S^1_+ \wedge S^1_+ \to S^1_+ \to S^1,$$

where the second map is the group multiplication in S^1. It is well-known that this composition represents η. For example, one can see this by considering the equivariant splitting $\widetilde{\Sigma} S^1 \to S^1 \wedge S^1_+$, whose domain is the unreduced suspension of S^1, and then use that the map of homotopy colimits induced by the diagram

$$(* \leftarrow S^1 \times S^1 \to *) \to (* \leftarrow S^1 \to *)$$

represents a generator of $\pi_3(S^2)$, cf. [22, XI.4]. The result now follows since η has order two. □

Proof of Theorem 1.2 The only thing left to prove is that the homotopy class $BGL_1(R) \to GL_1(R)$ considered in Proposition 6.2 agrees with the one in Proposition 7.3 when the Γ-space in question is $GL_1(R)$ and we use the canonical equivalence $BGL_1(R) \xrightarrow{\sim} GL_1(R)(S^1)$, cf. Proposition 5.3. Let W be the realization of the simplicial space

$$[k] \mapsto \operatorname*{hocolim}_{\mathcal{I}^k \times \mathcal{D}(S^1_k)} X(\mathbf{n}_1 \sqcup \theta_1) \times \cdots \times X(\mathbf{n}_k \sqcup \theta_k).$$

Since the space W^{cy} considered in Proposition 6.3 is the realization of a cyclic space it has a $\mathbb{T}$-action, and the adjoint of the composition

$$\mathbb{T} \times W^{\mathrm{cy}} \to W^{\mathrm{cy}} \to W$$

gives an equivalence $W^{\mathrm{cy}} \to L(W)$. It easily follows that we have a commutative diagram of equivalences

$$\begin{tikzcd}
B^{cy}GL_1(R) \arrow[r, "\sim"] \arrow[d, "\sim"] & W^{cy} \arrow[d, "\sim"] & GL_1(R)(S^1_+) \arrow[l, "\sim"'] \arrow[d, "\sim"] \\
L(BGL_1(R)) \arrow[r, "\sim"] & L(W) & L(GL_1(R)(S^1)). \arrow[l, "\sim"']
\end{tikzcd}$$

It thus suffices to check that the homotopy class defined by the diagram

$$BGL_1(R) \xrightarrow{\sim} W \xleftarrow{\sim} GL_1(R)(S^1)$$

is compatible with the canonical equivalence $BGL_1(R) \xrightarrow{\sim} GL_1(R)(S^1)$. We do this by exhibiting an explicit homotopy inverse of the equivalence $GL_1(R)(S^1) \to W$. Let W' be the realization of the simplicial space

$$[k] \mapsto \operatorname*{hocolim}_{\mathcal{D}(S^1_k) \times \mathcal{D}(S^1_k)} X(\psi_1 \sqcup \theta_1) \times \cdots \times X(\psi_k \sqcup \theta_k).$$

Ordering the k-simplices S^1_k as in the proof of 5.3 gives an equivalence $W \xrightarrow{\sim} W'$. Moreover, using the monoidal structure of $\mathcal{I}$ we get a functor $\mathcal{D}(S^1_k) \times \mathcal{D}(S^1_k) \to \mathcal{D}(S^1_k)$. Varying k this is a transformation of simplicial categories and consequently there is an induced simplicial map $W'_\bullet \to GL_1(R)(S^1_\bullet)$. It is easy to check that the composition $W \to W' \to GL_1(R)(S^1)$ is a homotopy inverse of the map in question and that the composition with $BGL_1(R) \to W$ is the canonical equivalence. This completes the proof. □

Remark 7.4 The discussion in the above proof can be generalized to any commutative $\mathcal{I}$-monoid X. If X is convergent, $B^{\mathrm{cy}} X_{h\mathcal{I}}$ and $X_{h\mathcal{I}}(S^1_+)$ are related by cyclic equivalences as in Proposition 6.3, and if X is group-like we get equivalences relating the same spaces by comparing them to the relevant free loop spaces. In case X is both convergent and group-like, the two equivalences agree by an argument similar to the above.

Acknowledgement The author was partially supported by a grant from the NSF.

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