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Hodge integrals and invariants of the unknot

A Okounkov
R Pandharipande

Department of Mathematics, Princeton University
Princeton, NJ 08544, USA

Email: `okounkov@math.princeton.edu`, `rahulp@math.princeton.edu`

Abstract

We prove the Gopakumar–Mariño–Vafa formula for special cubic Hodge integrals. The GMV formula arises from Chern–Simons/string duality applied to the unknot in the three sphere. The GMV formula is a q–analog of the ELSV formula for linear Hodge integrals. We find a system of bilinear localization equations relating linear and special cubic Hodge integrals. The GMV formula then follows easily from the ELSV formula. An operator form of the GMV formula is presented in the last section of the paper.

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0 Introduction

0.1 Let $\overline{M}_{g,n}$ be the Deligne–Mumford moduli stack of stable curves of genus g with n marked points. We study here Hodge integrals over $\overline{M}_{g,n}$.

Let L_i be the line bundle over $\overline{M}_{g,n}$ with fiber over the moduli point

$$[C, p_1, \dots, p_n] \in \overline{M}_{g,n}$$

given by the cotangent space T^*_{C,p_i} of the curve C at p_i. The ψ classes are the first Chern classes of the cotangent line bundles,

$$\psi_i = c_1(L_i)$$

in $H^2(\overline{M}_{g,n}, \mathbb{Q})$.

Let $\pi : C \to \overline{M}_{g,n}$ be the universal curve. Let ω_π be the relative dualizing sheaf. Let $\mathbb{E}$ be the rank g Hodge bundle on the moduli space of curves,

$$\mathbb{E} = \pi_*(\omega_\pi).$$

The λ classes are defined by

$$\lambda_i = c_i(\mathbb{E})$$

in $H^{2i}(\overline{M}_{g,n}, \mathbb{Q})$. The Chern polynomial of the Hodge bundle,

$$\Lambda(t) = 1 + t\lambda_1 + \cdots + t^g \lambda_g,$$

will appear often in the paper.

By definition, the Hodge integrals are the integrals of the ψ and λ classes over $\overline{M}_{g,n}$.

0.2 The polynomial,

$$\mathsf{H}^\circ_g(z_1, \dots, z_n; t_1, \dots, t_s) = \prod_{i=1}^n z_i \int_{\overline{M}_{g,n}} \frac{\prod_{i=1}^s \Lambda(t_i)}{\prod_{i=1}^n (1 - z_i \psi_i)}, \tag{0.1}$$

generates Hodge integrals on $\overline{M}_{g,n}$ with at most s classes λ_i. In case $s = 0$, $\mathsf{H}^\circ_g(z)$ generates pure ψ integrals on $\overline{M}_{g,n}$. The prefactor $\prod z_i$ in the definition is introduced for later convenience.

The Hodge integral in (0.1) arises as a vertex term in the localization formula for the Gromov–Witten invariants of an s dimensional target [8].

0.3 Mumford's relations may be used to reduce Hodge integrals to ψ integrals containing no λ classes, see [4, 5, 21]. However, the reduction method often destroys the rich structure possessed by special classes of Hodge integrals.

0.4 Linear Hodge integrals, generated by $\mathsf{H}^\circ_g(z;t_1)$, are connected to Hurwitz theory and the related combinatorics of symmetric group characters. In case

$$z_i \in \mathbb{N}, \quad t_1 = -1\,, \tag{0.2}$$

the Ekedahl–Lando–Shapiro–Vainstein (ELSV) formula expresses $\mathsf{H}^\circ_g(z;t_1)$ in terms of a Hurwitz number [2]. For general evaluations, there exists an operator formula expressing $\mathsf{H}^\circ_g(z;t_1)$ in terms of vacuum expectations of products of explicit operators in Fock space [19].

The $s=0$ case of pure ψ integrals, studied by Witten and Kontsevich, is perhaps best understood as the $t_1 \to 0$ limit of the $s=1$ case of linear Hodge integrals, see [17]. In particular, the appearance of random matrices should be viewed as the continuous limit of random partitions, which play a central role in the $s=1$ theory.

0.5 We study here *special cubic Hodge integrals*: the Hodge integrals generated by $\mathsf{H}^\circ_g(z;t_1,t_2,t_3)$, with parameters t_i subject to the constraint

$$\boxed{\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}=0}\,. \tag{0.3}$$

In the context of the localization formulas, the constraint (0.3) is the local Calabi–Yau condition. The linear Hodge integrals can be recovered from the special cubic Hodge integrals by a limit:

$$\mathsf{H}^\circ_g(z;t_1) = \lim_{t_2,t_3\to 0} \mathsf{H}^\circ_g(z;t_1,t_2,t_3), \tag{0.4}$$

where t_1 is held fixed and the parameters are subject to the constraint (0.3).

We prove two formulas for special cubic Hodge integrals. The first, covering the evaluations (0.2), is a q–analog, or trigonometric deformation, of the ELSV formula. The formula is based upon the conjectural Chern–Simons/string duality of Gopakumar and Vafa [7] applied to the case of the unknot in S^3 [16]. Hodge integrals enter via a heuristic localization computation of S. Katz and C.-C. Liu of the corresponding open string Gromov–Witten theory [10]. An exposition for mathematicians of the physics behind the conjecture can be found in [15]. We will call the formula the Gopakumar–Mariño–Vafa (GMV) formula.

While a precise mathematical definition of open string Gromov–Witten theory is lacking at the moment, our results provide significant theoretical evidence for both the general Gopakumar–Vafa conjecture and the assumptions made in [10].

Our second formula for special cubic Hodge integrals is a q–analog of the operator formula of [19]. Our derivation of the operator formula from the GMV formula follows the steps taken in [19].

0.6 In order to state the GMV formula for Hodge integrals, we introduce the generating series

$$\mathsf{H}^\circ(z;t;u) = \sum_{g\geq 0} u^{2g-2}\, \mathsf{H}^\circ_g(z;t).$$

We follow the conventions of [19] regarding the unstable terms,

$$\mathsf{H}^\circ_0(z_1;t) = \frac{1}{z_1},$$
$$\mathsf{H}^\circ_0(z_1,z_2;t) = \frac{z_1 z_2}{z_1+z_2}.$$

Let $\mathsf{H}^\bullet(z;t;u)$ denote the disconnected n–point series. For example,

$$\mathsf{H}^\bullet(z_1,z_2;t;u) = \mathsf{H}^\circ(z_1,z_2;t;u) + \mathsf{H}^\circ(z_1;t;u)\, \mathsf{H}^\circ(z_2;t;u)\,.$$

Further discussion of these definitions can be found in [19].

The 0–point function $\mathsf{H}^\circ(;t;u)$ is *not* included in the disconnected n–point series. For linear Hodge integrals, the 0–point function vanishes. For cubic Hodge integrals, the 0–point function is determined by the results of [5]:

$$\begin{aligned}\mathsf{H}^\circ_g(;t_1,t_2,t_3) &= \sum_{\sigma\in S_3} t^g_{\sigma(1)} t^{g-1}_{\sigma(2)} t^{g-2}_{\sigma(3)} \int_{\overline{M}_g} \lambda_g\lambda_{g-1}\lambda_{g-2} + t_1^{g-1}t_2^{g-1}t_3^{g-1}\int_{\overline{M}_g}\lambda^3_{g-1}\\ &= \Big(\sum_{\sigma\in S_3} \frac{t^g_{\sigma(1)} t^{g-1}_{\sigma(2)} t^{g-2}_{\sigma(3)}}{2} + t_1^{g-1}t_2^{g-1}t_3^{g-1}\Big)\frac{|B_{2g}|}{2g}\,\frac{|B_{2g-2}|}{2g-2}\,\frac{1}{(2g-2)!},\end{aligned}$$

for $g\geq 2$. Here, B_m denotes the Bernoulli number.

0.7 A partition μ is a monotone sequence $(\mu_1\geq\mu_2\geq\cdots\geq 0)$ of natural numbers such that $\mu_i=0$ for sufficiently large i. Let $|\mu|$ denote the size of the partition,

$$|\mu| = \sum \mu_i,$$

and let $\ell(\mu)$ denote the length, the number of nonzero parts of μ. Let $\operatorname{Aut}\mu$ be the group permuting equal parts of μ.

The ELSV formula is equivalent to the following equality relating Hodge integrals to the representation theory of the symmetric group:

$$\left(\prod \frac{\mu_i^{\mu_i}}{\mu_i!}\right) \mathsf{H}^\bullet(\mu;-1;u) = \frac{u^{-|\mu|-\ell(\mu)}}{|\operatorname{Aut}\mu|} \sum_{|\lambda|=|\mu|} \left(\frac{\dim\lambda}{|\lambda|!}\right) e^{uf_2(\lambda)}\,\chi^\lambda_\mu\,. \tag{0.5}$$

The summation is over all partitions λ of size $|\mu|$.

Here, $\dim\lambda$ is the dimension of the corresponding representation of the symmetric group, χ^λ_μ is the irreducible character of the symmetric group corresponding to representation λ and conjugacy class μ, and $f_2(\lambda)$ is the central character

$$f_2(\lambda) = \binom{|\lambda|}{2} \frac{\chi^\lambda_{(2,1,\dots,1)}}{\dim\lambda}$$

of a transposition in the representation λ. Explicitly, the central character is given by the following formula

$$f_2(\lambda) = \frac{1}{2}\sum_{i\geq 1}\left[(\lambda_i - i + \frac{1}{2})^2 - (-i+\frac{1}{2})^2\right].$$

0.8 The GMV formula is a q–deformation of the ELSV formula (0.5). In fact, the only term of the right side of formula (0.5) which is deformed is the dimension $\dim\lambda$. The dimension $\dim\lambda$ is replaced by the q–dimension of λ, a well-known notion in the theory of quantum groups and the corresponding theory of knot invariants, see for example [9].

The q–dimension of λ is the rational function of the variable $q^{\frac{1}{2}}$ defined by

$$\dim_q\lambda = q^{-\frac{1}{2}f_2(\lambda)-\frac{1}{2}|\lambda|}\, s_\lambda(1,q^{-1},q^{-2},q^{-3},\dots), \tag{0.6}$$

where s_λ is the Schur function. Alternatively, the q–dimension may be defined by

$$\frac{\dim_q\lambda}{|\lambda|!} = \prod_{\square\in\lambda}\frac{1}{q^{h(\square)/2}-q^{-h(\square)/2}}, \tag{0.7}$$

where the product is over all squares $\square$ in the diagram of λ and $h(\square)$ is the corresponding hook-length. The standard hook-length formula for $\dim\lambda$ arises as the coefficient of the leading term in (0.7) as $q\to 1$.

0.9 The first result of the paper is the Gopakumar–Mariño–Vafa formula. The proof is presented in Section 1.

Theorem 1 (Gopakumar–Mariño–Vafa formula) *For any number a and any partition μ we have*

$$\left(\prod\binom{(a+1)\mu_i}{\mu_i}\right)\mathsf{H}^\bullet\left(\mu;-1,-\frac{1}{a},\frac{1}{a+1};\sqrt{a(a+1)}\,u\right) = \frac{(au)^{-\ell(\mu)}}{|\operatorname{Aut}\mu|}\sum_{|\lambda|=|\mu|}\left(\frac{\dim_q\lambda}{|\lambda|!}\right)e^{(a+\frac{1}{2})uf_2(\lambda)}\chi^\lambda_\mu, \tag{0.8}$$

where $q=e^u$.

The q–dimension of λ may be rewritten after the substitution $q = e^u$ as

$$\frac{\dim_{e^u} \lambda}{|\lambda|!} = \prod_{\square \in \lambda} \left(2 \sinh \frac{u h(\square)}{2}\right)^{-1} . \tag{0.9}$$

A straightforward analysis shows the GMV formula specializes to the ELSV formula as $a \to \infty$ and $u \to 0$ while keeping the product au constant.

0.10 Following the treatment of linear Hodge integrals in [19], the GMV formula can be expressed in operator form,

$$\mathsf{H}^\bullet\left(z; -1, -\frac{1}{a}, \frac{1}{a+1}; \sqrt{a(a+1)}\, u\right) = \left\langle \prod \mathsf{A}(z_i; a) \right\rangle . \tag{0.10}$$

Here, $\mathsf{A}(z_i; a)$ is an explicit operator in a fermionic Fock space, and the angle brackets denote the vacuum expectation. The operator formula, Theorem 2, is fully stated and proven in Section 2.

The operators $\mathsf{A}(z_i; a)$ are q–analogs of the operators studied in [19] in connection with linear Hodge integrals. The operator formula for linear Hodge integrals played a fundamental role in the study of the equivariant Gromov–Witten of $\mathbf{P}^1$ in [19]. Similarly, the operator formula for special cubic Hodge integrals is applicable to the study of several local Calabi–Yau geometries.

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1 Proof of the GMV formula

1.1 Strategy

We find a system of bilinear localization equations which relates linear Hodge integrals to special cubic Hodge integrals with the following two properties:

(i) given the linear Hodge integrals, the system has a unique solution for the complete set of special cubic Hodge integrals,

(ii) the ELSV and GMV formulas satisfy the system.

Localization equations of a similar form were used successfully in [6]. Our strategy here was motivated by [6].

In fact, bilinear localization equations uniquely constrain *all* cubic Hodge integrals. A further study of cubic Hodge integrals will be presented in a future paper.

1.2 Integrals

1.2.1 Our localization relations are obtained from the analysis of integrals over the moduli space of stable maps to $\mathbf{P}^1$.

Let $\overline{M}_{g,n}(\mathbf{P}^1, d)$ be the moduli space of degree $d > 0$ stable maps from genus g, n–pointed *connected* curves to $\mathbf{P}^1$. The virtual dimension of the moduli space $\overline{M}_{g,n}(\mathbf{P}^1, d)$ is $2g + 2d - 2$. Let

$$\mathrm{ev}_i : \overline{M}_{g,n}(\mathbf{P}^1, d) \to \mathbf{P}^1$$

denote the evaluation map at the ith marked point, and let

$$\pi : C \to \overline{M}_{g,n}(\mathbf{P}^1, d),$$
$$f : C \to \mathbf{P}^1$$

denote the universal curve and universal map. The bundles

$$\mathbb{A} = R^1\pi_* f^*\mathcal{O}, \quad \mathbb{B} = R^1\pi_* f^*\mathcal{O}(-1),$$

will play an important role. The ranks of $\mathbb{A}$ and $\mathbb{B}$ are g and $g + d - 1$ respectively.

Let $\nu = (\nu_1, \ldots, \nu_n)$ be a partition such that $|\nu| \leq d-1$. Here, we allow $\nu_i = 0$. Consider the integral

$$I_{g,d}(\nu) = \int_{[\overline{M}_{g,n}(\mathbf{P}^1,d)]^{vir}} \mathrm{ev}_1^*(\omega)^{d-1-|\nu|} \prod_{i=1}^{n} \psi_i^{\nu_i}\, \mathrm{ev}_i^*(\omega)\ c_{top}(\mathbb{A})\, c_{top}(\mathbb{B}), \qquad (1.1)$$

where $\omega \in H^2(\mathbf{P}^1, \mathbb{C})$ is the Poincaré dual of the point class. The dimension of the integrand equals the the virtual dimension of $\overline{M}_{g,n}(\mathbf{P}^1, d)]$.

The integrand in (1.1) involves the class $\mathrm{ev}_1^*(\omega)^{d-|\nu|}$. Since

$$\omega^2 = 0,$$

$I_{g,d}(\nu)$ vanishes if $|\nu| < d-1$.

If $|\nu| = d-1$, the above vanishing does not apply. Let

$$I_d(\nu;u) = \sum_{g\geq 0} u^{2g-2}\, I_{g,d}(\nu)\,.$$

The nonvanishing values of the series $I_d(\nu;u)$ are determined by the following result proven in Section 1.5.

Proposition 1 *If $|\nu| = d-1$, then*

$$I_d(\nu;u) = (-1)^{d+1} \frac{d^{n-2}}{\prod \nu_i!}\, \frac{1}{2u\sin\frac{du}{2}}\,.$$

1.3 Virtual localization

1.3.1 Let $\mathbb{C}^*$ act diagonally on a two dimensional vector space V via the trivial and standard representations,

$$\xi\cdot(v_1,v_2) = (v_1, \xi\cdot v_2). \tag{1.2}$$

Let $\mathbf{P}^1 = \mathbf{P}(V)$. Let $0,\infty$ be the fixed points $[1,0],[0,1]$ of the corresponding $\mathbb{C}^*$–action on $\mathbf{P}(V)$.

An equivariant lifting of $\mathbb{C}^*$ to a line bundle L over $\mathbf{P}(V)$ is uniquely determined by the fiber representations L_0 and L_∞ at the fixed points. We represent the data of the $\mathbb{C}^*$ lifting to L by the integral weights $[l_0, l_\infty]$ of the representations L_0 and L_∞. The canonical lifting of $\mathbb{C}^*$ to the tangent bundle $T_{\mathbf{P}^1}$ has weights $[1,-1]$.

Let a be an integer. Equivariant lifts of $\mathbb{C}^*$ to the line bundles $\mathcal{O}$ and $\mathcal{O}(-1)$ are given by the weights

$$[-a,-a],\quad [a,a+1], \tag{1.3}$$

respectively.

The Poincaré dual of a point, $\omega \in H^2(\mathbf{P}^1,\mathbb{C})$, can lifted to the equivariant cohomology of $\mathbf{P}^1$ by selecting the class of either fixed point $0,\infty\in\mathbf{P}^1$. We will first define the lift by

$$\omega = [0] \in H^2_{\mathbb{C}^*}(\mathbf{P}^1,\mathbb{C}). \tag{1.4}$$

The second choice will be used in Section 1.5.

The representation (1.2) canonically induces a $\mathbb{C}^*$–action on the moduli space $\overline{M}_{g,n}(\mathbf{P}^1, d)$ by translation of the map. The $\mathbb{C}^*$–action lifts canonically to the cotangent lines of the moduli space of maps. A localization formula for the virtual fundamental class is proven in [8].

A $\mathbb{C}^*$–equivariant lift of the integrand of $I_{g,d}(\nu)$ is defined for each parameter a by the lifts (1.3) and (1.4) . The virtual localization formula provides an evaluation of $I_{g,d}(\nu)$ in terms of Hodge integrals.

1.3.2 The localization analysis of $I_{g,d}(\nu)$ is uniform for $a \neq -1, 0$. The localized equivariant integral, $\tilde{I}_{g,d}(\nu)$, defined by

$$\int_{[\overline{M}_{g,n}(\mathbf{P}^1,d)]^{vir}} \mathrm{ev}_1^*(\omega)^{d-1-|\nu|} \prod_{i=1}^{n} \psi_i^{\nu_i}\, \mathrm{ev}_i^*(\omega)\, \frac{c_{top}(R^1\pi_* f^*\mathcal{O})}{c_{top}(R^0\pi_* f^*\mathcal{O})} \frac{c_{top}(R^1\pi_* f^*\mathcal{O}(-1))}{c_{top}(R^0\pi_* f^*\mathcal{O}(-1))},$$

is more convenient to study in case a is nonzero. Since,

$$-\frac{1}{a} I_{g,d}(\nu) = \tilde{I}_{g,d}(\nu),$$

the difference is quite minor.

1.4 The virtual localization formula expresses equivariant integrals over the moduli space $\overline{M}_{g,d}(\mathbf{P}^1, d)$ as a sum over localization graphs Γ. The graphs Γ corresponds bijectively to components of the locus of the $\mathbb{C}^*$–fixed points in $\overline{M}_{g,n}(\mathbf{P}^1, d)$. A complete discussion of the graph structure of the virtual localization formula for $\overline{M}_{g,n}(\mathbf{P}^1, d)$ can be found in [17].

Let $[f] \in \overline{M}_{g,n}(\mathbf{P}^1, d)$ represent a generic point of a component of the $\mathbb{C}^*$–fixed locus,

$$[f : (C, p_1, \ldots, p_n) \to \mathbf{P}^1].$$

The components of C are either collapsed by f to $\{0, \infty\} \in \mathbf{P}^1$ or are unmarked rational Galois covers of $\mathbf{P}^1$ fully ramified over 0 and ∞.

The vertex set V of Γ corresponds to the connected components of the set,

$$f^{-1}(\{0, \infty\}).$$

Excepting degenerate issues, the vertices correspond to the collapsed components of C. The vertices carry a genus g_v, a marking set, and an assignment to 0 or ∞ in $\mathbf{P}^1$.

By our choice of the equivariant lift of the class ω, only vertices lying over $0 \in \mathbf{P}^1$ are allowed to carry markings. Graphs Γ with markings on vertices lying over ∞ do not contribute to the localization calculation of $\tilde{I}_{g,d}(\nu)$.

The edge set E of Γ corresponds to the non-collapsed components C_i of C. The edges carry the degree μ_i of the map $f\big|_{C_i}$. The edge degrees μ_i form a partition of the number d.

The localization graphs Γ for $\overline{M}_{g,n}(\mathbf{P}^1, d)$ must be connected and satisfy the global genus condition

$$\sum_{\text{vertices } v} g_v + h^1(\Gamma) = g\,,$$

where $h^1(\Gamma)$ is the first Betti number of Γ

$$h^1(\Gamma) = 1 - |V| + |E|.$$

1.4.3 Let Γ be a localization graph for the moduli space $\overline{M}_{g,n}(\mathbf{P}^1, d)$. The localization contribution of the graph Γ to $\tilde{I}_{g,d}(\nu)$ factors into vertex and edge contributions.

- Let v be a vertex lying over $0 \in \mathbf{P}^1$ carrying genus g_v and s marked points $p_1, \dots, p_s$. Let $\mu_1, \dots, \mu_r$ be the degrees associated to the edges incident to v. The contribution of v to the localization formula for $\tilde{I}_{g,d}(\nu)$ is:

$$\begin{aligned}\mathrm{Cont}(v) &= (-1)^{g_v-1} a^{2g_v-2} \int_{\overline{M}_{g_v,s+r}} \frac{\Lambda(-1)\,\Lambda(1/a)\,\Lambda(-1/a)\,\prod_1^s \psi_i^{\nu_i}}{\prod_{i=1}^r \left(1/\mu_i - \psi_{s+i}\right)} \\ &= \left[u^{2g_v-2} \prod z_i^{\nu_i+1}\right] \mathsf{H}^\circ(z_1, \dots, z_s, \mu_1, \dots, \mu_r; -1; iau)\,.\end{aligned}$$

Here, $\left[u^{2g_v-2} \prod z_i^{\nu_i+1}\right] \mathsf{H}^\circ$ denotes the coefficient of the monomial in the function H°. The second equality above follows from the relation,

$$\Lambda(t)\,\Lambda(-t) = 1\,, \tag{1.5}$$

proven by Mumford [21].

- Let v be a vertex lying over ∞ carrying genus g_v. Let $\mu_1, \dots, \mu_r$ be the degrees associated to the edges incident to v. The contribution of v to the localization formula for $\tilde{I}_{g,d}(\nu)$ is:

$$\mathrm{Cont}(v) = \left[u^{2g_v-2}\right] \mathsf{H}^\circ(\mu_1, \dots, \mu_r; -1, -\frac{1}{a}, \frac{1}{a+1}; i\sqrt{a(a+1)}u)$$

As previously noted, no markings are allowed on vertices over ∞.

- Let e be an edge of degree μ. The contribution of e to the localization formula for $\tilde{I}_{g,d}(\nu)$ is:

$$\mathrm{Cont}(e) = (-1)^\mu\, a^2\, \frac{\mu^\mu}{\mu!} \binom{(a+1)\mu}{a\mu}\,.$$

The vertex and edge contributions are obtained directly from the virtual localization formula, see [8, 17].

Let $G_{g,n}(\mathbf{P}^1, d)$ denote the set of localization graphs for the moduli space $\overline{M}_{g,n}(\mathbf{P}^1, d)$. The localization formula for $\tilde{I}_{g,d}(\nu)$ is:

$$\tilde{I}_{g,d}(\nu) = \sum_{\Gamma} \frac{1}{|\mathrm{Aut}\ \Gamma|} \prod_{v \in V} \mathrm{Cont}(v) \prod_{e \in E} \mathrm{Cont}(e).$$

We have proven the following result.

Lemma 2 *For* $a \neq -1, 0$,

$$-\frac{1}{a} I_{g,d}(\nu) = \sum_{\Gamma \in G_{g,n}(\mathbf{P}^1, d)} \frac{1}{|\mathrm{Aut}\ \Gamma|} \prod_{v \in V} \mathrm{Cont}(v) \prod_{e \in E} \mathrm{Cont}(e).$$

1.4.4 Define the disconnected bilinear Hodge integral function $Z_d^\bullet(\nu; u)$ by the following formula:

$$Z_d^\bullet(\nu; u) = (-1)^d \left[\prod z_i^{\nu_i+1}\right] \sum_{|\mu|=d} \frac{(au)^{2\ell(\mu)}}{\mathfrak{z}(\mu)} \prod \frac{\mu_i^{\mu_i}}{\mu_i!} \binom{(a+1)\mu_i}{a\mu_i} \times \\ \mathsf{H}^\bullet(z, \mu; -1; iau)\, \mathsf{H}^\bullet\left(\mu; -1, -\frac{1}{a}, \frac{1}{a+1}; i\sqrt{a(a+1)}u\right), \quad (1.6)$$

where

$$\mathfrak{z}(\mu) = |\mathrm{Aut}(\mu)| \prod_{i=1}^{\ell(\mu)} \mu_i \,.$$

Let $Z_d^\circ(\nu; u)$ denote the connected part of $Z_d^\bullet(\nu; u)$.

Lemma 2 together with the formula for the vertex and edge contributions in Section 1.3 yields the following result.

Lemma 3 *For* $a \neq -1, 0$,

$$-\frac{1}{a} I_d(\nu; u) = Z_d^\circ(\nu; u) \,. \quad (1.7)$$

1.5 Proof of Proposition 1

1.5.1 We will evaluate the integral $I_{g,d}(\nu)$ for $|\nu| = d-1$ by virtual localization. A *new* lift of the $\mathbb{C}^*$-action to the integrand will be used to evaluate the localization graph sum.

A lift of the $\mathbb{C}^*$–action to the integrand is chosen as follows. The lift of the $\mathbb{C}^*$–action to the bundles $\mathbb{A}$ and $\mathbb{B}$ is defined by the parameter value $a = 0$ in (1.3). The class ω is lifted by

$$\omega = [\infty] \in H^2_{\mathbb{C}^*}(\mathbf{P}^1, \mathbb{C}).$$

Let Γ be a localization graph with nonvanishing contribution to the integral $I_{g,d}(\nu)$ with the specified lift:

(i) The weight 0 linearization of $\mathcal{O}(-1)$ over 0 implies each vertex of Γ of 0 is of valence 1.

(ii) Each vertex v over 0 carries the class $c_{g(v)}(\mathbb{E})^2$ obtained weight zero linearizations of $\mathcal{O}$ and $\mathcal{O}(-1)$ over 0. Since
$$c_{g(v)}(\mathbb{E})^2 = 0$$
for $g(v) > 0$, all vertices over 0 must have genus 0.

(iii) Since Γ is connected, there is a unique vertex v_∞ over ∞. The vertex v_∞ carries the full genus g.

(iv) All the markings of Γ lie on v_∞.

The contributing graphs Γ are therefore indexed uniquely by the degree partition μ specified by the edges.

The localization graph sum directly yields a formula for $I_{g,d}(\nu)$ in terms of a sum over partitions.

Lemma 4 *If $|\nu| = d-1$, then*

$$I_{g,d}(\nu) = \sum_{|\mu|=d} \frac{(-1)^{\ell(\mu)+1}}{|\operatorname{Aut}\mu|} \prod \frac{\mu_i^{\mu_i-1}}{\mu_i!} \int_{\overline{M}_{g,n+\ell(\mu)}} \lambda_g \frac{\prod_{i=1}^n \psi_i^{\nu_i}}{\prod_{i=1}^{\ell(\mu)} (1-\mu_i\psi_{i+n})}. \tag{1.8}$$

1.5.2 The value of the λ_g–integral on right side of (1.8) can be computed using the following formula [6],

$$\int_{\overline{M}_{g,m}} \lambda_g \prod_{i=1}^m \psi_i^{\gamma_i} = \binom{2g-3+m}{\gamma_1,\ldots,\gamma_m} [u^{2g-2}] \frac{1}{2u\sin u/2}. \tag{1.9}$$

Since $|\nu| = d-1$ and $|\mu| = d$, we obtain

$$\int_{\overline{M}_{g,n+\ell(\mu)}} \lambda_g \frac{\prod_{i=1}^n \psi_i^{\nu_i}}{\prod_{i=1}^{\ell(\mu)}(1-\mu_i\psi_{i+n})} = \\ d^{2g-2+n-d+\ell(\mu)} \binom{d-1}{\nu_1,\dots,\nu_n} \binom{2g-3+n+\ell(\mu)}{d-1} [u^{2g-2}] \frac{1}{2u\sin u/2}. \quad (1.10)$$

1.5.3

Lemma 5 *Let t be a variable, and let $k \geq 0$ be an integer. Then,*

$$\sum_{|\mu|=d} \frac{(-t)^{\ell(\mu)}}{|\operatorname{Aut}\mu|} \binom{\ell(\mu)}{k} \prod \frac{\mu_i^{\mu_i-1}}{\mu_i!} = (-1)^k \frac{t^k(k-t)(-t+d)^{d-k-1}}{k!(d-k)!}. \quad (1.11)$$

Evaluation at $t = d$ yields

$$\begin{cases} \dfrac{(-d)^d}{d!}, & k = d-1, \\ 0, & k = 0,1,\dots,d-2. \end{cases} \quad (1.12)$$

Proof Consider the following function

$$T(x) = \sum_{n>0} \frac{n^{n-1}}{n!} x^n, \quad (1.13)$$

which enumerates rooted trees with n vertices and solves the functional equation

$$x\exp(T(x)) = T(x), \quad (1.14)$$

as shown, in particular, in [3]. More generally,

$$\exp(t\,T(x)) = \sum_{n\geq 0} \frac{t(t+n)^{n-1}}{n!} x^n, \quad (1.15)$$

see [20]. The left side of (1.11) equals the coefficient of x^d in the expansion of

$$\sum_{l\geq k} \frac{(-t\,T(x))^l}{k!(l-k)!} = \frac{(-1)^k}{k!} t^k T(x)^k \exp(-t\,T(x)) \\ = \frac{(-1)^k}{k!} t^k \sum_n \frac{(k-t)(k-t+n)^{n-1}}{n!} x^{n+k},$$

where the second equality follows from (1.14) and (1.15). □

1.5.4 We view the binomial coefficient,

$$\binom{2g-3+n+\ell(\mu)}{d-1}, \tag{1.16}$$

as a polynomial in the variable $\ell(\mu)$. The polynomial (1.16) agrees in highest order with the polynomial

$$\binom{\ell(\mu)}{d-1}$$

The lower order terms can be expressed as a linear combination of the polynomials

$$\binom{\ell(\mu)}{k}, \quad k=0,1,\ldots,d-2\,.$$

After substituting the evaluation (1.10) into the partition sum (1.8) and applying Lemma 5, we obtain

$$I_{g,d}(\nu) = (-1)^{d+1}\frac{d^{2g-2+n-1}}{\prod \nu_i!}\,[u^{2g-2}]\frac{1}{2u\sin u/2}\,. \tag{1.17}$$

Hence,

$$I_d(\nu;u) = (-1)^{d+1}\frac{d^{n-2}}{\prod \nu_i!}\,\frac{1}{2u\,\sin\frac{du}{2}}\,,$$

concluding the proof of Proposition 1. □

1.6 Bilinear relations

Let $a \neq -1, 0$ be a fixed integer. We have found homogeneous and inhomogeneous bilinear localization equations relating linear Hodge integrals to the set of special cubic Hodge integrals,

$$\mathsf{H}^\circ(\mu_1,\ldots,\mu_r;-1,-\frac{1}{a},\frac{1}{a+1};i\sqrt{a(a+1)}u),$$

indexed by partitions μ.

Let $|\nu| < d-1$. Since $I_d(\nu;u)$ vanishes, the homogeneous bilinear equation,

$$Z^\circ_d(\nu;u) = 0, \tag{1.18}$$

is obtained from Lemma 3.

Let $|\nu| = d-1$. By Lemma 3 and Proposition 1, we obtain the inhomogeneous bilinear equation,

$$Z^\circ_d(\nu;u) = -\frac{1}{a}(-1)^{d+1}\frac{d^{n-2}}{\prod \nu_i!}\,\frac{1}{2u\,\sin\frac{du}{2}}, \tag{1.19}$$

Lemma 6 *The bilinear equations* (1.18) *and* (1.19) *uniquely determine the special cubic Hodge integrals*

$$[u^{2g-2}]\ \mathsf{H}^\circ(\mu_1,\ldots,\mu_r;-1,-\frac{1}{a},\frac{1}{a+1};i\sqrt{a(a+1)}u) \tag{1.20}$$

from linear Hodge integrals.

Proof We proceed by induction on the genus g and the degree $|\mu|$. The base case and the induction step are proven simultaneously.

Assume the Lemma is true for all $g' < g$ and all partitions μ' of $d' < d$. Consider first the genus g homogeneous equations,

$$[u^{2g-2}]\ Z^\circ_d(\nu;u) = 0, \tag{1.21}$$

for $|\nu| < d-1$. We need only consider the *principal terms* corresponding localization graphs with a single genus g vertex over ∞ incident to all edges. All non-principal terms are determined by the induction hypothesis.

As the partition ν varies, we obtain linear equations for the scaled vertex integrals of the principal terms,

$$[u^{2g-2}]\ \frac{(-1)^{d+\ell(\mu)}}{\mathfrak{z}(\mu)}\prod\frac{\mu_i^{\mu_i}}{\mu_i!}\binom{(a+1)\mu_i}{a\mu_i}\mathsf{H}^\circ\left(\mu;-1,-\frac{1}{a},\frac{1}{a+1};i\sqrt{a(a+1)}u\right).$$

We view the above scaled vertex integrals as a set of variables indexed by partitions μ of d. Since lower terms appear, the equation are *not* homogeneous in the scaled vertex integrals of the principal terms.

We now consider the homogeneous localization equations obtained in case the parameter a is set to 0 in the integrand of $I_{g,d}(\nu)$ with lift

$$\omega = [0] \in H^2_{\mathbb{C}^*}(\mathbf{P}^1,\mathbb{C}).$$

The $a=0$ equations were studied in [6] to calculate λ_g integrals. The coefficients of the scaled vertex integrals in the linear equations obtained from (1.21) for $a \neq -1, 0$ are identical to the coefficients of the scaled λ_g integrals,

$$\frac{(-1)^{d+\ell(\mu)}}{\mathfrak{z}(\mu)}\prod\frac{\mu_i^{\mu_i}}{\mu_i!}\prod\mu_i\int_{\overline{M}_{g,\ell(\mu)}}\frac{\lambda_g}{\prod(1-\mu_i\psi_i)},$$

in the linear equations considered in [6]. By the main result of [6], the system of linear equations has a rank 1 solution space for the scaled vertex integrals.

Next, we study the genus g inhomogeneous equations for $a \neq -1, 0$,

$$[u^{2g-2}]\ Z^\circ_d(\nu;u) = [u^{2g-2}]\ \Big(-\frac{1}{a}(-1)^{d+1}\frac{d^{n-2}}{\prod\nu_i!}\frac{1}{2u\,\sin\frac{du}{2}}\Big), \tag{1.22}$$

where $|\nu| = d-1$. Again we consider the linear equations for the scaled vertex integrals of the principal terms.

The coefficients of the linear equation for the scaled vertex integrals obtained from (1.22) for $a \neq -1, 0$ match the corresponding coefficients of scaled λ_g integrals in the $a = 0$ calculation of $I_{g,d(\nu)}$ with lift

$$\omega = [0] \in H^2_{\mathbb{C}^*}(\mathbf{P}^1, \mathbb{C}).$$

The $a = 0$ calculation *consists only of principal terms.* The scaled λ_g integrals in the $a = 0$ are *not* annihilated by the $|\nu| = d-1$ equations since the coefficient

$$[u^{2g-2}] \left(-\frac{1}{a}(-1)^{d+1} \frac{d^{n-2}}{\prod \nu_i!} \frac{1}{2u \sin \frac{du}{2}} \right)$$

is proportional a nonvanishing Bernoulli number, see [5].

The linear equations for the scaled vertex integrals of the principal terms obtained from (1.22) for $a \neq -1, 0$ are therefore *not* dependent upon the linear equations obtained from (1.21) for $a \neq -1, 0$. Therefore, the full set of linear equations determines the genus g degree d special cubic Hodge integrals (1.20).

□

For fixed genus, the a dependence of the special cubic Hodge integral

$$[u^{2g-2}] \; \mathsf{H}^\circ(\mu_1, \ldots, \mu_r; -1, -\frac{1}{a}, \frac{1}{a+1}; i\sqrt{a(a+1)}u) \tag{1.23}$$

is a rational function. Rational functions are specified by values taken on integers not equal to $-1, 0$.

To prove the GMV formula for special cubic Hodge integrals, we need only show the ELSV and GMV formulas satisfy the homogeneous and inhomogeneous localization equations for all parameters $a \neq -1, 0$.

1.7 Operator formalism

1.7.1 We will prove the ELSV and GMV formulas satisfy the bilinear localization equations by a calculation in the infinite wedge representation $\Lambda^{\frac{\infty}{2}} V$.

We refer the reader to [19] for a discussion of the vector space $\Lambda^{\frac{\infty}{2}} V$. We will use several fundamental operators on $\Lambda^{\frac{\infty}{2}} V$ defined in [19].

1.7.2 The first step is to rewrite $Z^\bullet_d(\nu, u)$ defined by formula (1.6) as a vacuum expectation in $\Lambda^{\frac{\infty}{2}} V$.

An operator formula for Hodge integrals of the form $\mathsf{H}^\bullet(z;-1;u)$ was obtained in [19]. Applied to the present setting, we find:

$$\mathsf{H}^\bullet(z_1,\dots,z_n,\mu;-1;iau) = \\ (iau)^{-n-d-\ell(\mu)} \left(\prod \frac{\mu_i!}{\mu_i^{\mu_i}}\right) \left\langle \prod \mathcal{A}(z_i, iauz_i)\, e^{\alpha_1}\, e^{iau\mathcal{F}_2} \,\middle|\, \mu \right\rangle , \quad (1.24)$$

where

$$|\mu\rangle = \prod \alpha_{-\mu_i}\, v_\emptyset \,. \quad (1.25)$$

The GMV formula (0.8) can also be recast in an operator form. The Schur function entering the definition of the q–dimension can be written as the following matrix element:

$$(\Gamma_+(u) v_\lambda, v_\emptyset) = s_\lambda(1, e^{iu}, e^{2iu}, \dots)\,, \quad (1.26)$$

where $\Gamma_+(u)$ is the following operator

$$\Gamma_+(u) = \exp\left(\sum_{n>0} \frac{1}{n}\frac{1}{1-e^{iun}}\,\alpha_n\right) . \quad (1.27)$$

The coefficients inside the exponential in (1.27) are the power-sum symmetric functions in the variables $1, e^{iu}, e^{2iu}, \dots$, that is,

$$\frac{1}{1-e^{iun}} = \sum_{k=0}^{\infty} e^{iukn} \,.$$

Using the $u \mapsto -u$ symmetry of the right side of the GMV formula (0.8), we obtain:

$$\left(\prod \binom{(a+1)\mu_i}{\mu_i}\right) \mathsf{H}^\bullet\left(\mu; -1, -\frac{1}{a}, \frac{1}{a+1}; i\sqrt{a(a+1)}\,u\right) = \\ (-iau)^{-\ell(\mu)} \left\langle \Gamma_+(u)\, e^{-iau\mathcal{F}_2 + \frac{iu}{2}H} \,\middle|\, \mu \right\rangle , \quad (1.28)$$

where H is the energy operator.

We perform the summation (1.6) defining $Z_d^\bullet(\nu,u)$ using the operator formulas (1.24) and (1.28) for the occurring Hodge integrals and the formula

$$P_d = \sum_{|\mu|=d} \frac{1}{\mathfrak{z}(\mu)}\, |\mu\rangle\, \langle\mu| \ ,$$

for the orthogonal projection P_d onto the space of vectors of energy d. The steps here exactly follow Section 3.1 of [19].

The operator P_d commutes with the operator $\mathcal{F}_2$. It follows that the terms $\exp(\pm aiu\mathcal{F}_2)$ of (1.24) and (1.28) cancel. We obtain the following all degree generating function in the auxiliary degree variable Q,

$$\sum_d Q^d Z^\bullet_d(\nu;u) = (iau)^{-n}\left[\prod z_i^{\nu_i+1}\right] \\ \left\langle \prod \mathcal{A}(z_i, iauz_i)\, e^{\alpha_1} \left(-\frac{Qe^{\frac{iu}{2}}}{iau}\right)^H \Gamma_-(u)\right\rangle . \quad (1.29)$$

Here,

$$\Gamma_-(u) = \Gamma_+(u)^* = \exp\left(\sum_{n>0} \frac{1}{n}\frac{1}{1-e^{iun}}\,\alpha_{-n}\right) .$$

is the transpose of $\Gamma_+(u)$.

1.8 Extracting the connected part

1.8.1 We must extract the degree d connected part from the matrix element (1.29). By definition, $\Gamma_-(u)\, v_\emptyset$ is a linear combination of vectors of the form (1.25). We can also expand the vector

$$e^{\alpha_{-1}} \prod \mathcal{A}(z_i, iauz_i)^*\, v_\emptyset \quad (1.30)$$

in the basis (1.25). The matrix element (1.29) is then obtained via the canonical pairing

$$\langle \mu\,|\lambda\rangle = \mathfrak{z}(\mu)\,\delta_{\lambda,\mu} . \quad (1.31)$$

The pairing (1.31) may be interpreted as an enumeration of branched coverings of $\mathbf{P}^1$ ramified over two points. Of course, a cover of $\mathbf{P}^1$ is connected only if the partition

$$\mu = \lambda$$

has exactly one part. We will see the connected part of (1.29) corresponds to the contributions of the connected covers of $\mathbf{P}^1$.

1.8.2 We introduce a weight filtration on operators on $\Lambda^{\frac{\infty}{2}}V$, or, more precisely, on the universal enveloping algebra of the Lie algebra $\mathfrak{gl}(\infty)$ which acts on $\Lambda^{\frac{\infty}{2}}V$. Define the weights of the spanning elements $H^k\alpha_m$ by:

$$\operatorname{wt} H^k\alpha_m = m+k-1\,, \quad k=0,1,\dots, \quad m\in\mathbb{Z}\,, \quad (1.32)$$

where the multiplication is in the associative algebra $\mathrm{End}(\infty)$. Since

$$[H^a\alpha_b, H^c\alpha_d] = (bc - ad)\, H^{a+c-1}\alpha_{b+d}\,, \tag{1.33}$$

we obtain a well-defined filtration on the Lie algebra $\mathfrak{gl}(\infty)$ and the associated universal enveloping algebra.

To simplify notation, let the coefficients of $\mathcal{A}(z, iauz)$ be denoted by $\mathcal{A}_k$,

$$\mathcal{A}_k = [z^k]\, \mathcal{A}(z, iauz)\,. \tag{1.34}$$

The weight,

$$\mathrm{wt}\ \mathcal{A}_k = k - 1\,, \tag{1.35}$$

is obtained directly from the definition of $\mathcal{A}(z, iauz)$. Similarly,

$$\mathcal{A}_k = [z^k]\, \mathcal{A}(0, iauz) + \dots\,, \tag{1.36}$$

where dots stand for terms of weight smaller than $k-1$.

1.8.3 The Lie algebra $\mathfrak{gl}(\infty)$ acts on $\Lambda^{\frac{\infty}{2}}V$ and the filtration (1.32) is compatible with the following filtration on $\Lambda^{\frac{\infty}{2}}V$. We set, by definition,

$$\mathrm{wt}\ |\mu\rangle = |\mu| - \ell(\mu)\,, \tag{1.37}$$

Equation (1.33) implies that indeed

$$\mathrm{wt}\ X^*\,|\mu\rangle \le \mathrm{wt}\, X + \mathrm{wt}\ |\mu\rangle\ . \tag{1.38}$$

From the definitions, we have

$$\mathcal{A}_k^*\, v_\emptyset = \frac{(iau)^k}{k!}\ |k\rangle + \dots\,, \tag{1.39}$$

where dots stand for term of energy at most $k-1$ and, hence, of weight at most $k-2$. More generally, from the commutation relation

$$[\alpha_n, \mathcal{E}_m(s)] = \varsigma(ns)\, \mathcal{E}_{m+n}(s)\,, \tag{1.40}$$

where

$$\varsigma(x) = e^{x/2} - e^{-x/2}\,, \tag{1.41}$$

we obtain

$$[\alpha_{\mu_1}, [\alpha_{\mu_2}, \dots [\alpha_{\mu_r}, \mathcal{A}_k]]]^*\ v_\emptyset = \frac{(iau)^k}{(k-r)!} \prod_{j=1}^{r} \mu_j\, \left| k - r + \sum \mu_i \right\rangle + \dots\,, \tag{1.42}$$

where the dots stand for terms of lower energy and weight. In particular, the leading term vanishes if $r > k$.

1.8.4 We now apply the operators $\mathcal{A}(z_i, iauz_i)^*$ in (1.30) in order. After each application, we expand the result in the basis $|\mu\rangle$. The action of the operator $\mathcal{A}_k^*$ on the vector $|\mu\rangle$ is obtained by summing over all subsets of the μ_i's with which $\mathcal{A}_k^*$ interacts. Here, *interaction* denotes commutation before application to the vacuum. Such an interaction is described by equation (1.42). Interaction histories can be recorded as diagrams. As usual, extracting the connected part means extracting the contribution of the connected diagrams. Since the operator $e^{\alpha_{-1}}$ does not interact with the operators $\mathcal{A}_k^*$, the operator $e^{\alpha_{-1}}$ is a spectator in the extraction of the connected part.

From equation (1.42), we find the leading contribution of a connected diagram to the expansion of $\prod \mathcal{A}_{\nu_i+1}^* \, v_\emptyset$ is a constant multiple of the vector $|1+|\nu|\rangle$. In particular, to obtain a connected degree d contribution, the size of the partition ν must be at least $d-1$. In other words, the degree d connected part of (1.29) vanishes if $|\nu| < d-1$.

1.8.5 For the computation in the first nonvanishing case,

$$|\nu| = d - 1\,, \tag{1.43}$$

we use formula (1.36). By equation (3.12) in [19],

$$\mathcal{A}(0, iauz_i) = e^{\alpha_1} \, \mathcal{E}(iauz) \, e^{-\alpha_1}\,. \tag{1.44}$$

Clearly,

$$\left[z^{\nu_i+1}\right] \, \mathcal{E}(iauz) = (iau)^{\nu_i+1} \, \frac{\mathcal{P}_{\nu_i+1}}{(\nu_i+1)!} \tag{1.45}$$

After substituting (1.45) into (1.7) and (1.29) and simplifying, we obtain

$$I_d(\nu; u) = \frac{(-1)^{d+1}}{2du \sin\frac{du}{2}} \left\langle e^{\alpha_1} \prod \frac{\mathcal{P}_{\nu_i+1}}{(\nu_i+1)!} \,\middle|\, d \right\rangle^{\circ}\,, \tag{1.46}$$

where the superscript $\circ$ denotes the connected part of the matrix element.

The following Lemma finishes the computation and completes the proof of Theorem 0.8.

Lemma 7 *We have*

$$\left\langle e^{\alpha_1} \prod \frac{\mathcal{P}_{\nu_i+1}}{(\nu_i+1)!} \,\middle|\, d \right\rangle^{\circ} = \frac{d^{n-1}}{\prod \nu_i!}\,. \tag{1.47}$$

Proof This matrix element is a relative degree d Gromov–Witten invariant of $\mathbf{P}^1$, or equivalently, a Hurwitz number with completed cycles insertions [18]. By condition (1.43), the invariant has genus zero and, therefore, also equals, up to a factor, the corresponding ordinary Hurwitz number. The value of the the genus zero Hurwitz number is well-known, see for example [1]. Alternatively, the matrix element can be easily computed by using the commutation relations among the operators involved. □

2 Operator formula for Hodge integrals

2.1 Operators $\mathsf{A}(z,a)$

2.1.1 We begin with an operator form of the GMV formula equivalent to equation (1.28):

$$\left(\prod \binom{(a+1)\mu_i}{\mu_i}\right) \mathsf{H}^\bullet\left(\mu; -1, -\frac{1}{a}, \frac{1}{a+1}; \sqrt{a(a+1)}\, u\right) = \\ e^{-u|\mu|/2}\,(au)^{-\ell(\mu)} \left\langle \Gamma_+(iu)\, e^{au\mathcal{F}_2} \,\middle|\, \mu \right\rangle . \quad (2.1)$$

We will transform the above formula by commuting the operators $\Gamma_+(iu)$ and $e^{au\mathcal{F}_2}$, which fix the vacuum vector, through the operators $\alpha_{-\mu_i}$. Our strategy here follows Section 2.2 of [19].

2.1.2 The first conjugation

$$e^{au\mathcal{F}_2}\, \alpha_{-m}\, e^{-au\mathcal{F}_2} = \mathcal{E}_{-m}(aum) \quad (2.2)$$

follows easily from definitions, see Section 2.2.2 of [19].

The computation of the operator

$$\Gamma_+(iu)\, \mathcal{E}_{-m}(aum)\, \Gamma_+(iu)^{-1} \quad (2.3)$$

requires more work. We have

$$\Gamma_+(iu) = \prod_{n>0} \exp\left(\frac{1}{n}\,\frac{1}{1-e^{-un}}\,\alpha_n\right), \quad (2.4)$$

where the factors commute. After exponentiating the relation (1.40), we obtain

$$\exp\left(\frac{1}{n}\,\frac{1}{1-e^{-un}}\,\alpha_n\right)\, \mathcal{E}_{-m}(aum)\, \exp\left(-\frac{1}{n}\,\frac{1}{1-e^{-un}}\,\alpha_n\right) = \\ \sum_{k\geq 0} \frac{1}{k!\, n^k} \left(\frac{\varsigma(aunm)}{1-e^{-un}}\right)^k \mathcal{E}_{-m+kn}(aum)\,. \quad (2.5)$$

We find,

$$\Gamma_+(iu)\,\mathcal{E}_{-m}(aum)\,\Gamma_+(iu)^{-1} = \sum_{k\geq 0}\mathcal{E}_{-m+k}(aum)\sum_{k_1+2k_2+3k_3+\cdots=k}\frac{1}{\prod k_n!\,n^{k_n}}\prod\left(\frac{\varsigma(aumn)}{1-e^{-un}}\right)^{k_n}. \quad (2.6)$$

2.1.3 Let p_k and h_k denote the power sum and complete homogeneous symmetric functions. Let Φ be the specialization of the algebra of the symmetric functions defined by:

$$\Phi(p_n) = \frac{e^{aumn/2}-e^{-aumn/2}}{1-e^{-un}}, \quad n=1,2,\ldots.$$

The inner sum in the right-hand side of (2.6) equals $\Phi(h_k)$ by standard results in the theory of symmetric functions, see [14], formula (2.14′). Moreover, the equation,

$$\Phi(h_k) = \prod_{j=1}^{k}\frac{e^{aum/2}-e^{-aum/2-(j-1)u}}{1-e^{-uj}},$$

is a restatement of the q–binomial theorem, see [14], Example I.2.5. We conclude

$$\Gamma_+(iu)\,\mathcal{E}_{-m}(aum)\,\Gamma_+(iu)^{-1} = \sum_{k\geq 0}\left(\prod_{j=1}^{k}\frac{e^{aum/2}-e^{-aum/2-(j-1)u}}{1-e^{-uj}}\right)\mathcal{E}_{-m+k}(aum). \quad (2.7)$$

2.1.4 For a natural number m, introduce the function

$$\mathcal{R}(m,a,u) = \prod_{j=1}^{m}\frac{\mathcal{S}((am+j-1)u)}{\mathcal{S}(ju)}, \quad (2.8)$$

where

$$\mathcal{S}(z) = \frac{\sinh z/2}{z/2}.$$

The definition can be extended to nonintegral values of m by the following absolutely converging infinite product

$$\mathcal{R}(z,a,u) = \prod_{j=1}^{\infty}\frac{\mathcal{S}((az+j-1)u)\,\mathcal{S}((j+z)u)}{\mathcal{S}((az+z+j-1)u)\,\mathcal{S}(ju)}. \quad (2.9)$$

A series expansion of this function will be discussed below in Section 2.2.

2.1.5 Introduce the operator

$$\mathsf{A}(z;a) = \frac{1}{(a+1)\,u}\,\mathcal{R}(z,a,u)\times \sum_{l\in\mathbb{Z}} \left(\prod_{j=1}^{l} \frac{e^{auz/2} - e^{-auz/2-(z+j-1)u}}{1-e^{-u(z+j)}} \right) \mathcal{E}_l(auz)\,. \tag{2.10}$$

Taking into account all prefactors, equation (2.7) can be recast in the following form.

Theorem 2 *For positive integral values of the variables z_i, we have*

$$\mathsf{H}^\bullet\left(z;-1,-\frac{1}{a},\frac{1}{a+1};\sqrt{a(a+1)}\,u\right) = \left\langle \prod \mathsf{A}(z_i;a)\right\rangle \tag{2.11}$$

After suitable interpretation, we expect equality (2.11) to hold for all values of the variables z_i. An approach along the lines of [19] would involve establishing the commutation relations of the operators $\mathsf{A}(z;a)$. We plan to address the topic in the future.

The right side of (2.11) is less symmetric than the left side. For example, the symmetry with respect to

$$a \mapsto -a-1$$

is not obvious from the operator formula.

2.2 Series expansion of the function $\mathcal{R}$

2.2.1 Recall,

$$\ln \mathcal{S}(x) = \sum_{k>0} \frac{B_{2k}}{2k\,(2k)!}\,x^{2k}\,, \tag{2.12}$$

where B_m are the Bernoulli numbers defined by

$$\frac{x}{e^x-1} = \sum_{m\geq 0} \frac{B_m}{m!}\,x^m\,.$$

From (2.12), we have

$$\ln \mathcal{R}(m,a,u) = \sum_{k>0} \frac{B_{2k}\,u^{2k}}{2k\,(2k)!} \sum_{j=1}^{m} \left[(am+j-1)^{2k} - j^{2k} \right]\,. \tag{2.13}$$

The inner sum in (2.13) can be in turn computed in terms of the Bernoulli numbers. We obtain the following result.

Proposition 8 *We have*

$$\ln \mathcal{R}(z,a,u) = \sum_{0\le l\le 2k+1} \frac{B_{2k}\, B_{2k-l+1}\, u^{2k}}{(2k)\, l!\, (2k-l+1)!} \Big[(az+z)^l - (az)^l - (z+1)^l + 1\Big] \,. \quad (2.14)$$

2.2.2 Theorem 2 implies a formula for the 1–point function,

$$\mathsf{H}^\bullet\left(z_1; -1, -\frac{1}{a}, \frac{1}{a+1}; \sqrt{a(a+1)}\, u\right) = \frac{1}{(a+1)u}\, \frac{\mathcal{R}(z_1,a,u)}{\varsigma(az_1u)}\,, \quad (2.15)$$

since only the constant term of the operator $\mathsf{A}(z_1,u)$ contributes to the vacuum expectation $\langle \mathsf{A}(z_1,u)\rangle$.

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