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Cylindrical contact homology of subcritical Stein-fillable contact manifolds

Mei-Lin Yau

Department of Mathematics, Michigan State University
East Lansing, MI 48824, USA

Email: yau@math.msu.edu

Abstract

We use contact handle decompositions and a stabilization process to compute the cylindrical contact homology of a subcritical Stein-fillable contact manifold with vanishing first Chern class, and show that it is completely determined by the homology of a subcritical Stein-filling of the contact manifold.

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1 Introduction

A 1–form α on a $(2n-1)$–dimensional oriented manifold M is called a *contact* 1–form if it satisfies the contact condition:

$$\alpha \wedge (d\alpha)^{n-1} \neq 0 \quad \text{everywhere.} \tag{1}$$

Its kernel $\xi = \{\alpha = 0\}$ is called a (co-orientable) *contact structure.* ξ is a codimension 1 tangent distribution with maximal non-integrability. The pair (M,ξ) is called a *contact* manifold. Sometimes we write (M,α) to stress the contact 1–form α instead of the contact structure defined by α. Note that if α is a contact 1–form then so is $f\alpha$ for any $f \in C^\infty(M,\mathbb{R}_+)$, and $\ker(\alpha) = \ker(f\alpha)$. In this paper we assume $\xi = \ker\alpha$ to be *positive*, ie, $\alpha\wedge(d\alpha)^{n-1} > 0$ is a volume form of M. Two contact manifolds (M,ξ) and (M',ξ') are *contactomorphic* if there is a diffeomorphism $\phi\colon M \to M'$ such that $\phi_*\xi = \xi'$. ϕ is called a *contactomorphism.* Contact manifolds, which include many S^1–bundles and hypersurfaces of symplectic manifolds, and eventually every 3–manifold, were first introduced in [25] and [26], and has been under study for decades.

By the contact version of Darboux's theorem, all contact 1–forms are locally isomorphic, which implies that there is no local invariant for a contact structure. Moreover, it is proved by Gray in [16] that if two contact structures on a closed contact manifold are homotopic as contact structures, then they are isotopic as contact structures. Therefore there are also no local invariants of the space of contact structures on a closed manifold. Note that the contact condition (1) implies that $d\alpha$ restricts to a symplectic structure on ξ. The conformal class of such symplectic structures is independent of the choice of a defining contact 1–form for ξ. Thus we can endow ξ with a $d\alpha$–compatible almost complex structure and the first Chern class $c_1(\xi)$ is an invariant of ξ.

On the other hand, there are many contact structures which are homotopic as hyperplane distributions (hence have the same $c_1(\xi)$) but not homotopic as contact structures ([15], [22], [23], [32], [33], etc). This fact makes the classification of contact structures an interesting and challenging problem. For contact 3–manifolds, many nice partial results have been obtained ([7], [15], [22], [23], [24]). But much less is known for higher dimensional cases ([13], [32], [33]).

Contact Homology Theory ([10], see also [33], [1], [2]), introduced by Y Eliashberg and H Hofer in 1994 and has been expanded into a bigger framework *Symplectic Field Theory* ([12], [3]) provides Floer–Gromov–Witten type of invariants to distinguish non-isomorphic contact structures on closed manifolds: A contact 1–form α of M associates a unique vector field R_α which satisfies

$$\alpha(R_\alpha) = 1, \quad d\alpha(R_\alpha,\cdot) = 0.$$

R_α is called the *Reeb vector field (of α)*. $(M, \xi := \ker\alpha)$ also associates a symplectic manifold $(\mathbb{R}\times M, d(e^t\alpha))$, the *symplectization* of (M,ξ), whose symplectic structure $d(e^t\alpha)$ depends (up to an $\mathbb{R}$–invariant diffeomorphism of $\mathbb{R}\times M$) only on ξ. Then contact homology of (M,ξ) is defined by suitably counting in $\mathbb{R}\times M$ $(1+s)$–punctured pseudo-holomorphic spheres which converges exponentially to *good* periodic Reeb trajectories at $t=\pm\infty$ at punctures. In some favorable cases (see section 2) one can count only pseudo-holomorphic cylinders connecting good *contractible* Reeb orbits and define *cylindrical contact homology* $HC(M,\xi)$ of (M,ξ). In this paper we consider only the $c_1(\xi)=0$ case, then $HC(M,\xi)$ is graded by the *reduced Conley–Zehnder index* of Reeb orbits. The construction of $HC(M,\xi)$ involves choices of a contact 1–form α and an α–admissible almost complex structure. Yet the resulting contact homology is independent of all these extra choices and is truly an invariant of isotopy classes of contact structures. Though the full strength of contact homology is yet to be explored, some interesting classification results have been obtained in the spirit of (cylindrical) contact homology theory([4], [32], [33], see also [12], [1]).

Though contact homology is meant to distinguish non-isomorphic contact structures, itself is actually an subject of interest. One would like to know what contact homology tells about a contact manifold. Thus it is important to compute some concrete examples and develop computational mechanisms of contact homology.

This paper focuses on the computation of cylindrical contact homology of *subcritical Stein-fillable* contact manifolds. A complex n–dimensional Stein domain (V,J) is called *subcritical* if it admits a proper, strictly J–convex Morse function with finitely many critical points and all critical points have Morse index $< n$. Such a function is called *subcritical*. A contact manifold is called *subcritical Stein-fillable* if it is the boundary of some subcritical Stein domain and its contact structure is the corresponding CR–structure, ie, the field of maximal complex tangencies. Equivalently a subcritical Stein-fillable (M,ξ) can be identified with a regular level set of a subcritical strictly J–convex function on a Stein manifold. From now on we will often use the shorthand "*SSFC*" for "subcritical Stein-fillable contact" and simply call a subcritical Stein-fillable contact manifold a *SSFC* manifold, and similarly call a Stein-fillable contact manifold a *SFC* manifold. In this paper we obtain the following result.

Main Theorem *Let (M,ξ) be a $(2n-1)$–dimensional SSFC manifold with $n\geq 2$, $c_1(\xi)|_{\pi_2(M)}=0$, and (V,J) a subcritical Stein domain such that $\partial V = M$ and ξ is the maximal complex subbundle of TM. Then*

$$HC_i(M,\xi) \cong \bigoplus_{m\in\mathbb{N}\cup\{0\}} H_{2(n+m-1)-i}(V).$$

The Main Theorem results from the fact that, roughly speaking, counting pseudo-holomorphic cylinders is equivalent to counting gradient trajectories that connect critical points of consecutive indexes of a Morse function of a Stein filling of (M,ξ). Hence the theorem shows that the contact homology of a SSFC manifold (M,ξ) recovers in a way the homology of a Stein domain bounded by M.

Here is a brief outline of this paper: After introducing cylindrical contact homology in Section 2 we study in section 3 Reeb dynamics on subcritical contact handles, the building block of SSFC manifolds. Global dynamics on M is discussed in Section 4. It is shown there that, since (M,ξ) is subcritical, one gets enough room to maneuver attaching handles and hence contact 1–forms to show that contact homology of (M,ξ) is essentially generated by Reeb orbits contained in cocores of contact handles. To compute $HC(M,\xi)$ we introduce in Section 5 (M',ξ'), the *stabilization* of (M,ξ). (M',ξ') is a SSFC manifold containing (M,ξ) as a codimension 2 contact submanifold with $M'\setminus M\cong V\times S^1$ a trivial S^1–bundle over a Stein-filling V of M. By shaping contact handles of (M',ξ') one finds that cylindrical contact homologies of (M,ξ) and (M',ξ') can be represented by the same set of generators with degrees shifted by 2. In Section 6 we prove $HC_*(M,\xi)\cong HC_{*+2}(M',\xi')$. In Section 7 we prove that the counting of pseudo-holomorphic cylinders in (M',ξ') is equivalent to the counting of gradient trajectories in a subcritical Stein-filling V of M and hence deduce the Main Theorem. To this end we first show that for generic S^1–invariant admissible almost complex structure the linearized $\overline{\partial}$–operator at an S^1–invariant solution is surjective. This is done by identifying it with the corresponding surjectivity problem in Floer Theory. Then by applying branched covering maps on M' and the said surjectivity result to show that up to contact isotopies there are only S^1–invariant solutions to be counted.

2 Cylindrical contact homology

Before introducing the cylindrical contact homology we would like to give a brief account on the reduced Conley–Zehnder index of a contractible Reeb orbit at first.

Let $Sp(2n)=Sp(2n,\mathbb{R})$ denote the group of symplectic $2n\times 2n$–matrices. For a path $\Phi\colon [0,1]\to Sp(2n)$ a *Conley–Zehnder index* (also called μ–index) $\mu(\Phi)$ is defined in terms of crossing numbers ([30]). Here we refer readers to [30] for a precise definition of μ and to [6] for the original definition and general properties of μ. We point out here that if Φ' is a path in $Sp(2n')$ and Φ'' is a path in $Sp(2n'')$ then $\mu(\Phi' \oplus \Phi'') = \mu(\Phi') + \mu(\Phi'')$, here $Sp(2n') \oplus Sp(2n'')$ is identified as a subgroup of $Sp(2n' + 2n'')$ in the obvious way. The following example shows that when $n = 1$, $\mu/2$ is roughly the winding number of Φ.

Example Fix $T > 0$ and $A \in sp(2) = sl(2)$. Consider the path $\gamma\colon [0,T] \to e^{tA} \in Sp(2)$. Then

- $\mu(\gamma) = 0$ if $A = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}$ for some constants $a > 0$, $b > 0$;
- if $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ then $\mu(\gamma) = m$, $m = 2n+1$ if $n\pi < T < (n+1)\pi$, $m = 2n$ if $T = n\pi$.

For computational convenience we define the *reduced* Conley–Zehnder index (also called $\overline{\mu}$–index) of a path Φ in $Sp(2n-2)$ to be

$$\overline{\mu}(\Phi) = \mu(\Phi) + (n-3).$$

Fix a contact 1–form α on a $(2n-1)$–dimensional contact manifold (M, ξ). Let $\gamma\colon [o, \tau] \to M$ be a Reeb trajectory with $\dot{\gamma}(t) = R_\alpha(\gamma(t))$. Define the *action* $\mathcal{A}(\gamma)$ of γ to be the number

$$T = \mathcal{A}(\gamma) := \int_\gamma \alpha$$

The flow $(R_\alpha)^t$ of R_α preserves ξ. Thus the linearized Reeb flow $(R_\alpha)^t_*$, when restricted on γ, defines a path of symplectic maps

$$\Upsilon(t) = (R_\alpha)^t_*(\gamma(0))\colon\ \xi|_{\gamma(0)} \to \xi|_{\gamma(t)}.$$

When γ is periodic with period T, $\Upsilon(T)$ is called the *linearized Poincaré return map* along γ. We call γ *non-degenerate* if 1 is not an eigenvalue of $\Upsilon(T)$, *simple* if γ is not a nontrivial multiple cover of another Reeb orbit. A contact 1–form α is called *regular* if every (contractible) Reeb orbit of α is non-degenerate. It is well-known that generic contact 1–forms are regular. If we identify $\xi_{\gamma(T)}$ with $\mathbb{R}^{2n-2}$ then $\Upsilon(T) \in Sp(2n-2)$ is a symplectic matrix. The eigenvalues of a symplectic matrix comes in pairs ρ, ρ^{-1}.

Assume γ is a *contractible* periodic Reeb trajectory with action T. Let D be a spanning disc of γ and $\Psi\colon \xi|_D \to \mathbb{R}^{2n-2} \times D$ a symplectic trivialization of ξ over D. Then (γ, Φ) defines a path $(\Psi \circ \Upsilon \circ \Psi^{-1})|_\gamma\colon [0,T] \to Sp(2n-2)$ starting from Id. The μ–index of (γ, D) is defined to be

$$\mu(\gamma, \xi, D) := \mu(\Psi \circ \Upsilon \circ \Psi^{-1}|_\gamma),$$

and the corresponding $\overline{\mu}$–index is

$$\overline{\mu}(\gamma,\xi,D)=\mu(\gamma,\xi,D)+(n-3).$$

Since D is contractible, $\overline{\mu}(\gamma,\xi,D)$ does not depend on Ψ. Let D' be another spanning disc of γ. Then

$$\overline{\mu}(\gamma,\xi,D)-\overline{\mu}(\gamma,\xi,D')=2c_1(A) \tag{2}$$

where $c_1(A):=c_1(\xi)(A)$, $c_1(\xi)$ is the first Chern class of ξ and $A=[D\cup D']\in H_2(M,\mathbb{Z})$. In this paper we will only consider $c_1(\xi)=0$ case, therefore $\overline{\mu}(\gamma,\xi)=\overline{\mu}(\gamma,\xi,D)$ is independent of the choice of a spanning disc and is denoted as $\overline{\gamma}$ for notational simplicity.

For a Reeb orbit γ we denote by γ^m the m-th multiple of γ. Recall $\Upsilon(T)$ the Poincaré return map of γ. Let $n(\gamma)$ denote the number of real negative eigenvalues of $\Upsilon(T)$ from the interval $(-1,0)$. $n(\gamma)$ does not depend on the trivialization of $\xi_{\gamma(T)}$.

Definition 2.1 A Reeb orbit σ is said to be *good* if

$$\sigma\neq\gamma^{2m}\quad \text{for any } \gamma \text{ with } n(\gamma)=\text{odd},\ m\in\mathbb{N}. \tag{3}$$

For the rest of the paper we will use the notation $\mathcal{P}=\mathcal{P}(\alpha)$ to denote the set of all good contractible Reeb orbits of α. Good contractible Reeb orbits with any positive multiplicity are included in $\mathcal{P}$ as individual elements. Those contractible orbits not included in $\mathcal{P}$ are called *bad*. The exclusion of these bad orbits is necessary in order to define coherent orientations of moduli spaces of pseudo-holomorphic curves (see section 1.9 of [12] for more detail).

We now consider a class of almost complex structures on the symplectization

$$Symp(M,\alpha):=(\mathbb{R}\times M,d(e^t\alpha))$$

of $(M,\xi=\ker\alpha)$. An almost complex structure $J\colon\xi\to\xi$ on ξ is called *$d\alpha$–compatible* if

$$\begin{aligned} d\alpha(x,Jx) &> 0 \ \text{ for nonzero } x\in\xi,\\ d\alpha(Jx,Jy) &= d\alpha(x,y) \quad \text{for } x,y\in\xi. \end{aligned}$$

This compatibility property dose not depend on the choice of α. Note that $d\alpha(\cdot,J\cdot)$ is a Riemannian metric on ξ. A $d\alpha$–compatible J can be extended uniquely to a $d(e^t\alpha)$–compatible almost complex structure on $Symp(M,\alpha)$, also denoted by J by the abuse of language, such that

$$J(\frac{\partial}{\partial t})=R_\alpha,\quad J(R_\alpha)=-\frac{\partial}{\partial t}.$$

Such J is called an α*–admissible* almost complex structure on $Symp(M,\alpha)$. Observe that the Reeb vector field R_α satisfies $\omega(R_\alpha,\cdot)=-d(e^t)$, hence is the Hamiltonian vector field of of the function $H\colon \mathbb{R}\times M\to\mathbb{R}$, $H(t,p)=e^t$.

Fix a contact quadruple (M,ξ,α,J) so that J is α–admissible. We assume that α is regular. Fix a spanning disk $D_\gamma\subset M$ of γ for each $\gamma\in\mathcal{P}=\mathcal{P}(\alpha)$. Given two Reeb orbits γ_-, γ_+ we denote by $\mathcal{M}_J(M;\gamma_-,\gamma_+)$ the moduli space of maps $(\tilde{u},j)$ where

(1) j is an almost complex structure on $\dot{S}^2:=S^2\setminus\{0,\infty\}$ (here we identify S^2 with $\mathbb{C}\cup\{\infty\}$);
(2) $\tilde{u}=(a,u)\colon(\dot{S}^2,j)\to(\mathbb{R}\times M,J)$ is a proper map and is (j,J)–holomorphic, ie, $\tilde{u}$ satisfies $d\tilde{u}\circ j=J\circ d\tilde{u}$;
(3) $\tilde{u}$ is asymptotically cylindrical over γ_- at the negative end of $\mathbb{R}\times M$ at the puncture $0\in S^2$; and $\tilde{u}$ is asymptotically cylindrical over γ_+ at the positive end of $\mathbb{R}\times M$ at the puncture $\infty\in S^2$;
(4) $(\tilde{u},j)\sim(\tilde{v},j')$ if there is a diffeomorphism $f\colon\dot{S}^2\to\dot{S}^2$ such that $\tilde{v}\circ f=\tilde{u}$, $f_*j=j'$, and f fixes all punctures.

For generic choice of J, $\mathcal{M}(\gamma_-,\gamma_+)=\mathcal{M}_J(M;\gamma_-,\gamma_+)$, if not empty, is a smooth manifold,

$$\dim\mathcal{M}(\gamma_-,\gamma_+):=\overline{\gamma}_+-\overline{\gamma}_-$$

(recall that $c_1(\xi)=0$). Such a J is called *regular*. Note that since J is $\mathbb{R}$–invariant, the $\mathbb{R}$–translation along the $\mathbb{R}$–component of $\mathbb{R}\times M$ induces a free $\mathbb{R}$–action on $\mathcal{M}(\gamma_-,\gamma_+)$. If $\tilde{u}=(a,u)\in\mathcal{M}(M;\gamma_-,\gamma_+)$ then $u^*d\alpha\geq 0$ pointwise. We have

$$0\leq E(\tilde{u}):=\int_{\dot{S}^2}u^*d\alpha=\mathcal{A}_\alpha(\gamma_+)-\mathcal{A}_\alpha(\gamma_-).$$

$E(\tilde{u})$ is called the $d\alpha$–energy of $\tilde{u}$. $E(\tilde{u})=0$ iff $\gamma_-=\gamma_+$, and in this case the moduli space consists of a single element $\mathbb{R}\times\gamma_+$.

We now proceed to define the cylindrical contact homology of a contact manifold (M,ξ). For a regular contact 1–form α defining ξ we define the associated *cylindrical contact complex* $C(\alpha)=\underset{k\in\mathbb{Z}}{\oplus}C_k(\alpha)$ to be the graded vector space over $\mathbb{Q}$ generated by elements of $\mathcal{P}=\mathcal{P}(\alpha)$, where $C_k(\alpha)$ is the vector space spanned by elements $\gamma\in\mathcal{P}$ with $\overline{\gamma}=k$.

Now we fix a regular α–admissible almost complex structure and define the boundary map $\partial:C_*(\alpha)\to C_{*-1}(\alpha)$ as follows. Let $m(\gamma)$ denote the multiplicity of $\gamma\in\mathcal{P}$, then

$$\partial\gamma:=m(\gamma)\sum_{\sigma\in\mathcal{P},\overline{\sigma}=\overline{\gamma}-1}n_{\gamma,\sigma}\sigma$$

where $n_{\gamma,\sigma}$ is the algebraic number of elements of $\mathcal{M}(\sigma,\gamma)/\mathbb{R}$, each element $C \in \mathcal{M}(\sigma,\gamma)/\mathbb{R}$ is weighted by $\frac{1}{m(C)}$, where $m(C)$ is the multiplicity of C. Then extend ∂ $\mathbb{Q}$–linearly over $C(\alpha)$. Note that since α is regular, σ and γ are non-degenerate, $\mathcal{M}(\sigma,\gamma)/\mathbb{R}$ is compact and hence a finite set. Moreover, for any $\gamma \in \mathcal{P}$ there are only finitely many σ with $\mathcal{A}_\alpha(\sigma) < \mathcal{A}_\alpha(\gamma)$. Thus $\partial\gamma$ is a finite sum.

We have the following theorem (see [33] and Remark 1.9.2 of [12]).

Theorem 2.1 *Let (α, J) be a regular pair. Then $\partial \circ \partial = 0$ if $C_1(\alpha) = 0$.*

To prove $\partial \circ \partial = 0$ one wants to show that if a 2–dimensional moduli space $\mathcal{M}(\gamma_-,\gamma_+)$ has nonempty boundary, then its boundary consists of "broken cylinders" $C_1 \# C_2$, where $C_1 \in \mathcal{M}(\gamma,\gamma_+)/\mathbb{R}$, $C_2 \in \mathcal{M}(\gamma_-,\gamma)/\mathbb{R}$ for some $\gamma \in \mathcal{P}$ with $\overline{\gamma} = \overline{\gamma_+} - 1$. If this is not true then the boundary of $\mathcal{M}(\gamma_-,\gamma_+)$ will involve holomorphic curves with more than one negative ends. Such curves are elements of some *1–dimensional* moduli space $\mathcal{M}(\gamma_-,\gamma_1,\cdots,\gamma_j;\gamma_+)$ with $j \geq 1$, and γ_-, $\gamma_1,\ldots,\gamma_j$ are Reeb orbits that form the negative ends of the holomorphic curves. But

$$\dim \mathcal{M}(\gamma_-,\gamma_1,\cdots,\gamma_j;\gamma_+) = \overline{\gamma}_+ - \overline{\gamma}_- - \sum_{\nu=1}^{j} \overline{\gamma}_\nu = 2 - \sum_{\nu=1}^{j} \overline{\gamma}_\nu$$

which is less than 1 if $C_1(\alpha) = 0$. So if $C_1(\alpha) = 0$ then $\partial \circ \partial = 0$. We will see later that every SSFC manifold with dim > 3 and $c_1(\xi) = 0$ will have $C_*(\alpha) = 0$ for all $* \leq 1$.

When $\partial \circ \partial = 0$ we define the The *j-th cylindrical contact homology group* of the pair (α, J) to be

$$HC_j(\alpha, J) := \ker(\partial|_{C_j(\alpha)})/\partial(C_{j+1}(\alpha)).$$

The following theorem, analogous to its counterpart in Floer theory, asserts that $HC(\alpha, J)$ is independent of regular pairs (α, J) satisfying $C_*(\alpha) = 0$ for $* = -1, 0, 1$, hence is an invariant of of (M, ξ) (see [33]).

Theorem 2.2 *Let $(\alpha_0 = f_0\alpha, J_0)$, $(\alpha_1 = f_1\alpha, J_1)$ be two regular pairs. Assume $C_i(\alpha_0) = C_i(\alpha_1) = 0$ for $i = -1, 0, 1$. Then there exists a natural isomorphism*

$$\phi^{10}\colon HC_i(\alpha_0, J_0) \to HC_i(\alpha_1, J_1).$$

If (α_2, J_2) is a third regular pair then

$$\phi^{20} = \phi^{21} \circ \phi^{10}, \quad \phi^{00} = id.$$

The proof of Theorem 2.2 is similar to the proof of the corresponding theorem in Floer theory. Here the required chain homotopies are guaranteed by the existence of smooth functions f on $\mathbb{R}\times M$ such that $d(e^t f\alpha)$ is symplectic on $\mathbb{R}\times M$ and $e^t f\alpha$ interpolates $e^t\alpha_0$ and $e^t\alpha_1$. Moreover, in a similar fashion one can show that $HC(M,\xi_0)\cong HC(M,\xi_1)$ for isotopic contact structures ξ_0 and ξ_1 on M.

We remark here that though the condition $C_*(\alpha)=0$ for $*=1,0,-1$ looks artificial, it (or similar conditions on $\overline{\mu}$) may impose restrictions on the topology of M and even the type of ξ. For example, when $\dim M=3$ and $c_1(\xi)|_{\pi_2(M)}=0$ it is proved in [19] that if for some α $\overline{\gamma}\geq 2$ for all contractible Reeb orbits of α, then $\pi_2(M)=0$ and ξ is *tight*, ie, there exists no embedded disc D in M such that (i) ∂D is tangent to ξ, and (ii) D is transversal to ξ along ∂D (see for example [7]).

3 Contact handles

In this section we describe some basic models of contact handles. These basic models have been provided and discussed in detail in [34]. Since contact handles are building blocks of SFC manifolds we present a similar discussion here but with a focus on the dynamics of Reeb orbits.

The complex n–dimensional space $\mathbb{C}^n$ together with its standard complex structure i is a Stein manifold. Let (x,y,z) be the standard coordinates of $\mathbb{C}^n$ with respect to the decomposition $\mathbb{C}^n=\mathbb{R}^k\times\mathbb{R}^k\times\mathbb{C}^{n-k}$, $(k\leq n)$. $x=(x_1,\dots,x_k)$, $y=(y_1,\dots,y_k)$, $z=(z_{k+1},\dots,z_n)$, $z_l=x_l+iy_l$.

Fix $0\leq k\leq n$ and define

$$f_{\mathrm{st}}(x,y,z)=|x|^2-\frac{1}{2}|y|^2+\frac{1}{4}|z|^2.$$

f_{st} is a strictly i–convex function on $\mathbb{C}^n$. Note that the origin 0 is the only critical point of f_{st}, and its Morse index is k.

Define

$$Y_{\mathrm{st}}:=(2x,-y,\frac{z}{2})=\sum_{j=1}^{k}(2x_j\frac{\partial}{\partial x_j}-y_j\frac{\partial}{\partial y_j})+\sum_{l=k+1}^{n}\frac{1}{2}(x_l\frac{\partial}{\partial x_l}+y_l\frac{\partial}{\partial y_l}),$$

$Y_{\mathrm{st}}=\nabla f_{\mathrm{st}}$, the gradient vector field of f_{st} with respect to the Euclidean metric. Denote by ω_{st} the standard symplectic structure $\sum_{j=1}^n dx_j\wedge dy_j$ on $\mathbb{C}^n$. We have $L_{Y_{\mathrm{st}}}\omega_{\mathrm{st}}=\omega_{\mathrm{st}}$. Y_{st} is a complete Liouville vector field on the symplectic manifold $(\mathbb{C}^n,\omega_{\mathrm{st}})$.

Define $\alpha_{\mathrm{st}} := \omega_{\mathrm{st}}(Y_{\mathrm{st}}, \cdot)$. α_{st} restricts to a contact 1–form on $\mathcal{H}$ for any hypersurface $\mathcal{H} \subset \mathbb{C}$ transversal to Y_{st}. Note that $\alpha_{\mathrm{st}} = -df_{\mathrm{st}} \circ i$.

Consider a function $\mathsf{f}\colon \mathbb{C}^n \to \mathbb{R}$

$$\mathsf{f}(x, y, z) = b|x|^2 - b'|y|^2 + \sum_{k+1}^{n} \frac{|z_l|^2}{c_l^2}; \quad b > b', c_l : \text{ positive constants.} \tag{4}$$

0 is the only critical point of f.

Define $H_c := \{\mathsf{f} = c\}$. Then $H_c \pitchfork Y_{\mathrm{st}}$ when $c \neq 0$. Denote the punctured level set $H_{c=0} - \{0\}$ by $H_o^\times$. We have $H_o^\times \pitchfork Y_{\mathrm{st}}$. So α_{st} restricts to a contact 1–form on each of the level sets of f, except at the point of origin.

For $c > 0$ H_c contains two special submanifolds:

- a $(2n-k-1)$–dimensional coisotropic ellipsoid $S_c^+ := \{|y| = 0\} \cap H_c$; and
- a $(2n-2k-1)$–dimensional contact ellipsoid $S_c^* := \{|y| = |x| = 0\} \cap H_c$.

When $c < 0$ there is a $(k-1)$–dimensional isotropic sphere $S_c^- := \{|x| = 0 = |z|\} \cap H_c$ on H_c. The normal bundle of S_c^- has the decomposition

$$\mathcal{N}(S_c^-, H_c) = \mathcal{CSN}(S_c^-, H_c) \oplus T^*S_c^- \oplus \mathbb{R}R_{\mathsf{f}}$$

where $\mathcal{CSN}(S_c^-, H_c)$ is the *conformal symplectic normal* bundle (see [34]) of $S_c^- \subset H_c$, R_{f} is the Reeb vector field of $(H_c, \alpha_{\mathrm{st}})$.

Note that the vector bundle $T^*S_c^- \oplus \mathbb{R}R_{\mathsf{f}} \cong S_c^- \times \mathbb{R}^k$ is trivial and has a natural framing $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_k}\}$. The vector bundle $\mathcal{CSN}(S_c^-, H_c)$ is also trivial (of rank $2(n-k)$), and has a natural framing $\mathcal{F}_k = \{\frac{\partial}{\partial x_{k+1}}, \frac{\partial}{\partial y_{k+1}}, ..., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\}$.

When $c \to 0$, S_c^+ (or S_c^-) degenerates to the point 0.

We denote by ξ_c (resp. ξ_o) the corresponding contact structure on $(H_c, \alpha_{\mathrm{st}})$ (resp. $(H_o^\times, \alpha_{\mathrm{st}})$).

Proposition 3.1 *Let f and $\bar{\mathsf{f}}$ be two quadratic (up to an addition of a constant) functions of index k with respective coefficients (b, b', c_j) and $(\bar{b}, \bar{b}', \bar{c}_j)$ satisfying conditions in (4). Then (12) implies that for any level sets H_c of f and any level set $\bar{H}_{\bar{c}}$ of $\bar{\mathsf{f}}$ the flow of Y_{st} will induce*

(1) *a contact isotopy between $(H_c, S_c^+, S_c^*, \xi_c)$ and $(\bar{H}_{\bar{c}}, \bar{S}_{\bar{c}}^+, \bar{S}_{\bar{c}}^*, \bar{\xi}_{\bar{c}})$ when $c > 0$ and $\bar{c} > 0$;*

(2) *a contact isotopy between (H_c, S_c^-, ξ_c) and $(\bar{H}_{\bar{c}}, \bar{S}_{\bar{c}}^-, \bar{\xi}_{\bar{c}})$ when $c < 0$ and $\bar{c} < 0$;*

(3) *a contact isotopy between* $(H_c - S_c^+, \xi_c)$ *and* $(\bar{H}_{\bar{c}} - \bar{S}_{\bar{c}}^-, \bar{\xi}_{\bar{c}})$ *when* $c > 0$ *and* $\bar{c} < 0$.

In particular up to contact isotopy, the contact structures on $(H_c^\pm, \alpha_{\mathrm{st}})$ do not depend on the choice of the coefficients b, b', and c_j; the flow of Y_{st} will produce the required contact isotopies that even preserve submanifolds like $S_c^\pm$ and S_c^*. We hence have the freedom to adjust the values of b, b' and c_j to get Reeb vector fields with desired dynamical behavior.

For notational simplicity we will from time to time use the following symbols: (H_+, S_+, S_*, ξ_+) to represent $(H_c, S_c^+, S_c^*, \xi_c)$ when $c > 0$; and (H_-, S_-, ξ_-) to represent (H_c, S_c^-, ξ_c) when $c < 0$. (H_+, ξ_+) is called (a standard model of) a *contact* k*-handle.* It is *subcritical* if $k < n$.

We now study the Hamiltonian and Reeb dynamics on level sets $H_\pm$, H_o, of $(\mathbb{C}^n, \omega_{\mathrm{st}}, Y_{\mathrm{st}}, \mathsf{f})$. Again α_{st} is used as the preferred contact 1–form. Let X_{f} denote the Hamiltonian vector field of f with respect to ω_{st},

$$X_{\mathsf{f}} = \sum_{j=1}^{k}(2bx_j\frac{\partial}{\partial y_j} + 2b'y_j\frac{\partial}{\partial x_j}) + \sum_{l=k+1}^{n}\frac{2}{c_l^2}(x_l\frac{\partial}{\partial y_l} - y_l\frac{\partial}{\partial x_l}). \tag{5}$$

The Reeb vector fields on $(H_o^\times, \alpha_{\mathrm{st}})$ and $(H_\pm, \alpha_{\mathrm{st}})$ are

$$R_{\mathsf{f}} := \frac{X_{\mathsf{f}}}{\alpha_{\mathrm{st}}(X_{\mathsf{f}})} \tag{6}$$

where

$$\alpha_{\mathrm{st}}(X_{\mathsf{f}}) = 4b|x|^2 + 2b'|y|^2 + \sum_{k+1}^{n}|z_l|^2/c_l{}^2 = 3b|x|^2 + 3b'|y|^2 + c \tag{7}$$

is positive away from the point of origin. R_{f} and X_{f} have the same integral trajectories up to a reparametrization. Let $\gamma\colon [0, T_h] \to \mathbb{C}^n$ be a periodic X_{f}–trajectory such that $\dot{\gamma}(t) = X_{\mathsf{f}}(\gamma(t))$. T_h is then called the *Hamiltonian period* of γ. The *Reeb period* of γ can be defined similarly, and is actually its *action* $\int_\gamma \alpha_{\mathrm{st}}$.

Lemma 3.1 (i) *There is no periodic Reeb trajectory on* $H_o^\times$ *and* H_-.

(ii) *On* H_+ *all periodic Reeb trajectories are contained in* S_*.

(iii) *If* $c_{k+1}^2, ..., c_n^2$ *are linearly independent over* $\mathbb{Q}$ *then the Hamiltonian period of a simple periodic trajectory on* H_c *with* $c > 0$ *is* πc_l^2 *for some* $k < l \le n$, *while its action is* $\pi c_l^2 c$.

Proof Let $\varphi_t(w) = \varphi(t,w)\colon \mathbb{R}\times\mathbb{C}^n \to \mathbb{C}^n$, $\varphi(0,w)=id$ be the flow of $X_{\mathfrak{f}}$, $\gamma(t) = (x(t), y(t), z(t))$ an integral trajectory of $X_{\mathfrak{f}}$ with $\dot\gamma(t) = X_{\mathfrak{f}}(\gamma(t))$. Then on γ

$$\dot x_j = 2b' y_j, \quad \dot y_j = 2b x_j, \qquad \text{for} \quad 1\le j\le k \tag{8}$$

$$\dot x_l = 2y_l/c_l^2, \quad \dot y_l = -2x_l/c_l^2, \quad \text{for} \quad k+1\le l\le n \tag{9}$$

We have

$$\begin{cases} x_j(t) = x_j(0)\cosh(2\sqrt{bb'}t) + y_j(0)\sqrt{\frac{b'}{b}}\sinh(2\sqrt{bb'}t) \\ y_j(t) = y_j(0)\cosh(2\sqrt{bb'}t) + x_j(0)\sqrt{\frac{b}{b'}}\sinh(2\sqrt{bb'}t) \\ x_l(t) = x_l(0)\cos(2t/c_l^2) - y_l(0)\sin(2t/c_l^2) \\ y_l(t) = y_l(0)\cos(2t/c_l^2) + x_l(0)\sin(2t/c_l^2) \end{cases}$$

for $1\le j\le k$ and $k+1\le l\le n$. $(x_j(t), y_j(t))$, if not identically zero, is hyperbolic, while $|z_l(t)|$ is a constant along any γ. So γ is contained in S_* if it is periodic. Hence (i) and (ii) are true.

Assume $\gamma\subset S_*$. The Hamiltonian period of the z_l–component of γ is πc_l^2. Hence if $c_{k+1}^2, ..., c_n^2$ are linearly independent over $\mathbb{Q}$ then for any $c>0$ there are only $n-k$ simple periodic trajectories on H_c. They are $\sigma_l := \{|z_l|^2 = c_l^2 c\}$, $l = k+1, ..., n$. The Hamiltonian period of σ_l is πc_l^2, which is independent of the value of c, while the action of σ_l is $\pi c_l^2 c$. This proves (iii).

Note that actions of simple Reeb orbits can be made as small as we want by choosing c_l^2 to be small enough. □

Theorem 3.1 *Let H_+ be as above.*

(I) *All periodic Reeb orbits of H_+ are "good" as defined in (3).*

(II) *If c_{k+1}^2,...,c_n^2 are linearly independent over $\mathbb{Q}$ then all Reeb orbits of H_c are non-degenerate.*

(III) $C_*(H_+, \alpha_{\rm st}) = 0$ *if* $* < 2n-k-2$ *or* $* - (2n-k-2)$ *is odd.*

(IV) *Can choose (or deform) H_+ for any given $m_o > 0$ such that for $* \le m_o$*

$$rk(C_*(H_+, \alpha_{\rm st})) = \begin{cases} 1 & if\ * = 2n-k-4+2i\ for\ some\ i\in\mathbb{N}; \\ 0 & otherwise. \end{cases}$$

We start with the following lemma:

Lemma 3.2 *Let γ be a contractible Reeb orbit of a contact manifold (M,ξ) with contact 1–form α. Let $D \subset M$ be a spanning disc of γ. Then*

$$\mu(\gamma, \xi|_D) = \mu(\gamma, (\mathbb{C}R_\alpha \oplus \xi)|_D)$$

where R_α is the Reeb vector field of α.

Proof of Lemma 3.2 Let $\varphi\colon \mathbb{C}R_\alpha|_D \tilde{\to} \mathbb{C} \times D$ be the vector bundle isomorphism $\varphi(\lambda R_\alpha, p) = (\lambda, p)$ for $\lambda \in \mathbb{C}$ and $p \in D$. φ is a symplectic trivialization of the vector bundle $\mathbb{C}R_\alpha|_D$. The action of the linearized Reeb flow on $\mathbb{C}R_\alpha|_\gamma$ is a constant path (ie, a point) in $Sp(2)$ with respect to φ. Let Φ be any symplectic trivialization of ξ over D, then $\varphi\oplus\Phi$ is a symplectic trivialization of $\mathbb{C}R_\alpha \oplus \xi$ over D. The definition of φ implies that $\mu(\gamma, \xi|_D; \Phi) = \mu(\gamma, (\mathbb{C}R_\alpha \oplus \xi)|_D; \varphi\oplus\Phi)$. Since the μ–index is independent of the choice of a symplectic trivialization over a fixed spanning disc, we conclude that $\mu(\gamma, \xi|_D) = \mu(\gamma, (\mathbb{C}R_\alpha \oplus \xi)|_D)$. □

Proof of Theorem 3.1 Let $\Phi := T\mathbb{C}^n \tilde{\to} \mathbb{C}^n \times \mathbb{C}^n$ be the standard trivialization of the tangent bundle $T\mathbb{C}^n$ of $\mathbb{C}^n$. When restricted on H_+, Φ is a trivialization of the stabilized contact bundle $\mathbb{C}R \oplus \xi = T_{H_+}\mathbb{C}^n$. Here $R = R_{\mathfrak{f}}$ is the Reeb vector field of α_{st} on H_+.

By Lemma 3.2 we can use Φ to compute the μ–index of any $R_{\mathfrak{f}}$–orbit σ in H_+. Moreover, $\pi_2(H_+) = 0$ if $\dim H_+ > 3$. When $\dim H_+ = 3$, $\pi_2(H_+)$ is generated by S_+. The inclusion $H_+ \hookrightarrow \mathbb{C}^2$ implies that $(\xi_+ \oplus \mathbb{C}R)|_{S_+}$ and $T_{S_+}\mathbb{C} \cong \mathbb{C}^2 \times S_+$ are isomorphic vector bundles over S_+. Since $\mathbb{C}R|_{S_+}$ is a trivial bundle over S_+, so is $\xi_+|_{S_+}$, which implies that $c_1(\xi_+) = 0$. Therefore the index $\mu(\sigma, D_\sigma)$ is independent of the choice of the spanning disk D_σ in H_+.

Extend the linearized Reeb flow R^t_* of R to $T_{H_+}\mathbb{C}^n$ by assigning $R^t_*(Y_{\mathrm{st}}) = Y_{\mathrm{st}} \circ R^t$. We have (with respect to Φ)

$$R^t_*|_{\sigma_l} = e^{tD} = \begin{pmatrix} e^{tD_1} & & \\ & \ddots & \\ & & e^{tD_n} \end{pmatrix} \in Sp(2n)$$

where

$$D_j = \begin{pmatrix} 0 & 2b' \\ 2b & 0 \end{pmatrix} \quad , \quad D_\ell = \frac{2}{cc_\ell^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in Gl(2, \mathbb{R})$$

for $j = 1, ...k$, $\ell = k+1, ...n$.

By easy computation one finds that for $j = 1, ..., k$,

$$e^{tD_j} = \begin{pmatrix} \cosh 2\sqrt{2bb'}t & \sqrt{\frac{b'}{b}} \sinh 2\sqrt{bb'}t \\ \sqrt{\frac{b}{b'}} \sinh 2\sqrt{2bb'}t & \cosh 2\sqrt{bb'}t \end{pmatrix}$$

with $\det(I - e^{tD_j}) < 0$ for all $t \neq 0$. Moreover, each e^{tD_j} has two real positive eigenvalues $\cosh 2\sqrt{bb'}t \pm \sinh 2\sqrt{bb'}t$ for all t. Therefore e^{tD_j} makes no rotations to the (x_j, y_j)–plane, hence has no contribution to $\mu(\sigma_l)$.

For $\ell = k+1, ..., n$, we have

$$e^{tD_\ell} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \text{where} \ \ \theta = \frac{2t}{cc_\ell^2}.$$

Note that $det(I - e^{tD_\ell}) \geq 0$, the equality holds if and only if t is an integral multiple of πcc_ℓ^2.

We have for a simple Reeb orbit σ in H_+:

(1) $\overline{\sigma} \geq (n-k-1) + 2 + (n-3) = 2n - k - 2$. The minimum is always achieved by some σ.

(2) For $m, m' \in \mathbb{N}$ with $m > m'$, $\overline{\sigma^m} - \overline{\sigma^{m'}}$ is a positive even number, and is $\geq 2(m - m')$. Here σ^m denotes the m-th multiple of σ.

This proves Part (I) and (III) of the theorem.

If we choose to have c_{k+1}^2, c_{k+2}^2, ..., c_n^2 linearly independent over $\mathbb{Q}$, then there are exactly $n-k$ simple Reeb orbits σ_{k+1},...,σ_n as defined before. From the computations above it is easy to see that these σ_l and their multiple covers are all non-degenerate. So Part (II) is true.

To prove Part (IV) we consider the following perturbation of c_l to compute indexes. For any (large) integer $n_o \in \mathbb{N}$, choose c_{k+1}, ..., c_n such that c_{k+1}^2, ..., c_n^2 are linearly independent over $\mathbb{Q}$, and $n_o c_n^2 < c_l^2$ for $l = k+1$, ..., $n-1$. Then

$$\begin{aligned} &\overline{\sigma_n^m} = 2n - k - 4 + 2m \text{ for } 1 \leq m \leq n_o, \\ &\overline{\sigma_n^m} \geq 2n - k - 4 + 2m \text{ for } m > n_o, \\ &\overline{\sigma_l^m} \geq 2n - k - 4 + 2(n_o + m) \text{ for } l > k+1, \ m \geq 1. \end{aligned} \tag{10}$$

Now choose n_o so that $n_o > m_o$. This completes the proof of Part (IV). □

Remark 3.1 We call the Reeb orbit σ_n corresponding to c_n the *principal (Reeb) periodic trajectory* or the *principal (Reeb) orbit* of $(H_+, \alpha_{\text{st}})$ if (10) is satisfied. When $m_o \to \infty$ the contact complex $C_*(H_+, \alpha_{\text{st}})$ is essentially generated by σ_n and its positive multiples. We call each positive multiple of σ_n a *principal generator* of $C_*(H_+, \alpha_{\mathrm{st}})$. We see that $C(H_+, \alpha_{\mathrm{st}})$ stabilizes as $m_o \to \infty$, where $C_*(H_+, \alpha_{\mathrm{st}})$ is a vector space of rank 1 precisely when $* = 2n-k-4+2j$ for some $j \in \mathbb{N}$, ..., otherwise it is 0.

We now proceed to study the local index of a non-periodic Reeb trajectory γ on H_+. Recall that our contact k–handle is modelled on the following hypersurface in $\mathbb{C}^n$:

$$b|x|^2 - b'|y|^2 + \sum_{k+1}^{n} \frac{|z_l|^2}{c_l^2} = c > 0$$

Since later we will see that a SSFC manifold can be constructed by attaching thin subcritical contact handles to a tiny tubular neighborhood of attaching isotropic spheres, we are mainly interested in the domain $U_+ \subset H_+$ (a tubular neighborhood of the belt sphere of H_+) where $b'|y|^2 \leq C$ for some constant $C > 0$, $|x|^2 + |z|^2$ is small.

Recall the Hamiltonian vector field $X_{\mathfrak{f}}$ and Reeb vector field $R_{\mathfrak{f}} = X_{\mathfrak{f}}/\alpha_{\mathrm{st}}(X_{\mathfrak{f}})$ from (5), (6) and (7). Recall that the standard trivialization of the tangent bundle $T\mathbb{C}^n$ induces a symplectic trivialization (with respect to ω_{st}) Φ of the stabilized bundle $\mathbb{C}R_{\mathfrak{f}} \oplus \xi_+$ of ξ_+.

View γ as a non-periodic $X_{\mathfrak{f}}$–trajectory on U_+ with Hamiltonian period T_h, and reparametrize γ so that $\dot\gamma(t) = X_{\mathfrak{f}}(\gamma(t))$. Since H_+ is subcritical, given any positive number N_o we can have

$$\mu(\gamma(t), \tilde{\xi}, \Phi_{\mathrm{st}}) > N_o T_h$$

by thinning U_+, ie, by choosing to have c_l^2 small enough.

Now we parametrize γ as an $R_{\mathfrak{f}}$–trajectory, ie, $\dot\gamma(\tau) = R_{\mathfrak{f}}(\gamma(\tau))$. Since on U_+

$$\alpha_{\mathrm{st}}(X_{\mathfrak{f}}) = 3b|x|^2 + 3b'|y|^2 + c \leq 6C + 4c$$

the action T of γ satisfies

$$T \leq C_o T_h, \qquad C_o = 6C + 4c.$$

Denote by ψ the flow of $R_{\mathfrak{f}}$, and φ the flow of $X_{\mathfrak{f}}$. We have

$$\psi(\tau, w) = \varphi(t(\tau, w), w)$$

Both flows preserve ξ and have the following relation between their linearized flows:

$$d\psi(\tau, w) = d\varphi(t(\tau, w), w) + \frac{d\varphi}{dt}(t(\tau, w), w) \otimes dt \tag{11}$$

where $\frac{d\varphi}{dt}(t,w) = X_{\mathrm{f}} \circ \varphi(t,w)$. The term $\frac{d\varphi}{dt} \otimes dt = X_{\mathrm{f}} \otimes dt$ is a path of $2n \times 2n$ matrices of rank 1 and has μ–index equal to 0. Hence by [32] we have

$$|\mu(\gamma(\tau), \tilde{\xi}, \Phi_{\mathrm{st}}) - \mu(\gamma(t), \tilde{\xi}, \Phi_{\mathrm{st}})| \leq 2n,$$
$$\mu(\gamma(\tau), \xi, \Phi_{\mathrm{st}}) = \mu(\gamma(\tau), \tilde{\xi}, \Phi_{\mathrm{st}}) \geq \mu(\gamma(t), \tilde{\xi}, \Phi_{\mathrm{st}}) - 2n > N_o T_h - 2n.$$

So we obtain a linear inequality relating the action T of a Reeb trajectory γ and its μ–index:

$$\mu(\gamma(\tau), \xi, \Phi_{\mathrm{st}}) > NT - 2n, \quad N = C_o^{-1} N_o$$

N can be made very large by thinning the *subcritical* handle. Here the fact that H_+ is subcritical is essential to the largeness of N. We summarize the above discussion about non-periodic trajectories in the following lemma:

Lemma 3.3 *Let U_+ be a tubular neighborhood of the belt sphere of a $(2n-1)$–dimensional subcritical contact k–handle H_+. Let N be any positive number. Then by thinning U_+, ie, by choosing to have c_l^2 $(l = k+1, ..., n)$ small enough, we have*

$$\mu(\gamma, \xi, \Phi_{\mathrm{st}}) > NT - 2n$$

for any non-periodic Reeb trajectory of α_{st} on U_+ with action T.

4 Reeb dynamics on SSFC manifolds

A closed, orientable $(2n-1)$–dimensional contact manifold (M, ξ) is called *Stein-fillable* if there is a $2n$–dimensional Stein domain (V, J) such that $\partial V = M$ and ξ is the maximal complex subbundle of TM. (V, J) is called a *Stein filling* of (M, ξ). Let f be a strictly J–convex function on V that extends smoothly to the boundary $\partial V = M$ as a constant function, then $\omega_f := -dd^J f = -d(df \circ J)$ is a symplectic 2–form on $\overline{V}$ (the nondegeneracy of ω_f is ensured by the strict J–convexity of f), and ξ is the kernel of the restriction of the 1–form $\alpha_f := \omega(\nabla f, \cdot)$ on TM. Here ∇f is the gradient vector field of f with respect to the Riemannian metric $g(\cdot,\cdot) := \omega_f(\cdot, J\cdot)$, and hence is a Liouville vector field of ω_f. Without loss of generality we may assume that f is also a Morse function. We call a Stein-fillable (M, ξ) *subcritical* if the corresponding f is *subcritical*, ie, has no critical points of index $\geq n$. Notice that $(V, \omega_f, \nabla f, f)$ is actually an open domain of a *Weinstein* manifold. In the following we will study the Reeb dynamics of SSFC manifolds in the setup of Weinstein manifolds.

A *Weinstein* manifold is a quadruple (W, ω, Y, f) where

- (W, ω) is a symplectic manifold,
- Y is a complete smooth vector field on W, and Y is a *Liouville* vector field of (W, ω), ie,
$$L_Y \omega = \omega$$
where $L_Y \omega$ denotes the Lie derivative of ω with respect to Y,
- f is an exhausting Morse function on W, and Y is gradient-like with respect to f, ie, $df(Y) > 0$ except at critical points of f.

In this paper we are interested in Weinstein manifolds of *finite type* where the function f has only finitely many critical points, and Y has only finitely many zeros accordingly.

A Weinstein manifold (W, ω, Y, f) associates a 1–form $\alpha := \omega(Y, \cdot)$ which is a primitive of ω. Let $S \subset W$ be a hypersurface transversal to Y, then α restricts to a contact 1–form on S. Let X be a nonvanishing vector field which span the line field $\mathcal{L}_S \subset TS$ on which ω degenerates. Then the Reeb vector field of $(S, \alpha|_{TS})$ is $R := X/\alpha(X)$. If S is also a level set of a function h, then $R = X_h/\alpha(X_h)$, where X_h, satisfying $\omega(X_h, \cdot) = -dh$, is the Hamiltonian vector field associated to h.

Let $S' \subset W$ be another hypersurface transversal to Y. Let ζ and ζ' be contact structures on S and S' defined by α respectively. If a reparametrized flow of Y induces a diffeomorphism $\varphi\colon S \to S'$ then we have $\varphi_* \zeta = \zeta'$, hence (S, ζ) and (S', ζ') are contactomorphic. This is because for any two smooth functions h_1, $h_2 > 0$ on W, we have
$$L_{h_1 Y} h_2 \alpha = h_1 (dh_2(Y) + h_2)\alpha. \tag{12}$$
Note that we have $\alpha(Y) = 0$ by definition.

Now let f be a subcritical strictly J_W–convex Morse function on a Stein manifold (W, J_W). If W is of finite type, then $\omega := \omega_f$ is independent of the choice of f up to a diffeomorphism of W. Let $Y_f = \nabla f$. The quadruple (W, ω_f, Y_f, f) is then a Weinstein manifold. It is easy to see that a contact manifold (M, ξ) is subcritical Stein-fillable (up to contact isotopy) iff it can be realized as a hypersurface in some subcritical (W, ω_f, Y_f, f) that is also transversal to Y_f, or equivalently, a regular level set of f. For any level set Q of f, $\alpha_f := \omega(Y_f, \cdot)$ restricted to a contact 1–form on Q away from critical points of f. Moreover the Reeb vector field associated to α and the Hamiltonian vector field of f have the same integral trajectories.

We now proceed to study the Reeb dynamics on level sets of (W, ω_f, Y_f, f). First of all, a theorem of Eliashberg [9] states that one can manipulate critical points f as freely as in the smooth case. Thus f can be assumed to has only one critical point of index 0 (we assume that W is connected), and $f(p) < f(q)$ for $p, q \in Crit(f)$ if the Morse index of p is less than the Morse index of q; $f(p) = f(q)$ if $p, q \in Crit(f)$ are of the same index. Also, following [8] a subcritical Stein manifold W of dimension $2n \geq 4$ can reconstructed by attaching handles of index $< n$. Therefore, once a subcritical J–convex Morse function f on (W, J_W) (of finite type) is chosen, (W, J_W) can be decomposed into a (finite) union of handlebodies of subcritical indexes accordingly. A critical point of Morse index k corresponds to exactly a handlebody of of index k, and (W, J_W) can be constructed by attaching back these handlebodies along isotropic spheres with specified framings in the order of handle indexes.

A $2n$–dimensional handlebody of index k is diffeomorphic to $D^k \times D^{2n-k}$ with boundary $S^{k-1} \times D^{2n-k} \cup D^k \times D^{2n-k-1}$. $S^{k-1} \times D^{2n-k}$ is to be glued, while $D^k \times S^{2n-k-1}$ is a contact k–handle. Thus a SSFC manifold can be constructed by attaching subcritical contact handles modelled on a tubular neighborhood U_+ of the belt sphere of $H_+ = \{b|x|^2 - b'|y|^2 + \sum_{l=k+1}^{n} |z_l|^2/c_l^2 = c\}$ (see [5] for more detail). We may assume that contact handles of (M, ξ) of the same index are pairwise disjoint.

Recall that each subcritical contact k–handle has only $n-k$ simple Reeb orbits. We may assume that all attaching $(k-1)$–spheres miss all the simple Reeb orbits in the middle of contact handles of lower indexes. Thus a SSFC manifold has two types of contractible Reeb orbits. *Type I* Reeb orbits are those contained in the middle of subcritical contact handles; Reeb orbits which are not of type I are called *Type II*. Type II orbits run through different handles.

Lemma 4.1 (See Lemma 3 of [5]) *Let (M, ξ) be a SSFC manifold with a contact handle decomposition. Let T be any positive number. Then up to a contact isotopy there is a defining contact 1–form of (M, ξ) so that any Reeb trajectory which leaves a contact k–handle and return to possibly another contact k–handle has action $\geq T$.*

Here is a brief explanation of why Lemma 4.1 is true. The attaching isotropic spheres of subcritical contact k–handles are of dimension less than $(\dim M - 1)/2$, hence after isotopy we may assume that there are no Reeb chords connecting these spheres. So for any $T > 0$, there is a neighborhood $\mathcal{U}_k$ of these spheres such that any Reeb trajectory leaves $\mathcal{U}_k$ at time 0 will not meet $\mathcal{U}_k$ again before time T. Now we glue the contact k–handles to the interior of $\mathcal{U}_k$.

By combining the proof of Proposition 1 in [5] and an estimate of $\bar{\mu}$–index based on Lemma 4.4 and the analysis on handles in the previous section we can derive the following lemma concerning the $\bar{\mu}$–index of contractible Type II Reeb orbits.

Lemma 4.2 *Let (M,ξ) be a SSFC manifold with a subcritical contact handle decomposition. Let K be any positive number. Then up to contact isotopy (by thinning handles) every Type II contractible Reeb orbit has $\bar{\mu}$–index greater than K.*

The rest of this section is devoted to proving Lemma 4.2.

We have shown that over each subcritical contact handle there is a linear inequality relating the action T of a Reeb trajectory γ and its μ–index. Namely $\mu(\gamma) \geq N \cdot T - 2n$, C is independent of γ, N can be made very large by shaping the *subcritical* handle. In the following we will estimate the actual μ–index of a contractible Type II orbit, which is related to local indexes, the number of times the orbit crosses different handles, the framings of the symplectic normal bundles of the attaching isotropic spheres, and the gluing process. We will prove that there is a linear relation between the action of a contractible Type II orbit and the number of times it crosses different handles. This linear relation, together with the said linear inequality and the largeness of the action of any Type II orbit, enable us to prove Lemma 4.2.

We now make a digression here to prepare for the statement of the inequality that links all local estimates together and guarantees the largeness of μ– (and hence $\bar{\mu}$–) indexes of Reeb orbits of Type II.

Let $S_n := \{0, 1, 2, ..., n-1\}$ be a set of n "letters", $n \geq 2$. Define $\mathcal{W}_n$ to be the set of "words" of finite length whose letters are elements of S_n.

Definition 4.1 Given $w = l_1 \cdots l_m \in \mathcal{W}_n$, $l_i \in S_n$, w is called *jumpy* if $l_i \neq l_{i+1}$ for all $1 \leq i < m$.

Definition 4.2 Given $w \in \mathcal{W}_n$, $w = l_1 l_2 l_3 \cdots l_m$, w *contains a basin* or *has a basin* if there is an k, $0 < k < n$ and a subword $w' \subset w$, $w' = l_i \cdots l_j$, $1 < i \leq j < m$, such that $l_{i-1} = l_{j+1} = k > l_\nu$ for $i \leq \nu \leq j$. w' is called a *basin* of w.

Lemma 4.3 (Word Lemma) *Any jumpy word $w \in \mathcal{W}_n$ of length 2^n must contain a basin ($n \geq 2$).*

Corollary 4.1 *Any jumpy word $w \in \mathcal{W}_n$ of length m must contain at least $[\frac{m}{2^n}]$ disjoint basins.*

Proof of the Word Lemma By mathematical induction. When $n = 2$, there are only two jumpy words of length 4: 0101 and 1010. Both contain a basin "0" with $k = 1$. Hence the lemma is true for $n = 2$. Assume the lemma holds for $n = s$. Let $w \in \mathcal{W}_{s+1}$ be jumpy of length 2^{s+1}. If the largest letter appearing in w is less than s, then it reduces to the case $n = s$ and the statement holds again. If the letter s appears in W at least twice then we are done. If not, then $w{=}w_1 s w_2$. w_1, $w_2 \in \mathcal{W}_s$ are jumpy. Observe that one of them is of length $\geq 2^s$ and hence contains a basin by assumption. This basin is also a basin of w. So the lemma is true for $n = s + 1$. By induction we conclude that the lemma is true for all n greater than 1. □

Now let k_o be the highest index of contact handles of (M, ξ). We may assume that Lemma 4.1 holds true for (M, ξ).

Let $H(k)$ denote th union of all contact k–handles of (M, ξ). Define

$$\mathcal{H}(k) := H(k) \setminus \bigcup_{k'>k} H(k').$$

Let γ be a simple Reeb orbit of Type II. γ associates a jumpy word $w(\gamma) \in \mathcal{W}_n$ constructed as follows.

The codimension 1 boundaries of $\mathcal{H}(k)$, $k = 0, ..., k_o$, cut γ into $m-1$ connected curves with boundaries. Let $\bar{k} > 0$ be the maximal value of k such that $\gamma \cap \mathcal{H}(k)$ is not empty. Fix a connected component of $\gamma \cap \mathcal{H}(\bar{k})$ and call it γ_1. Following the Hamiltonian flow starting from γ_1 we write γ as the ordered union $\gamma = \cup_{j=1}^{m-1} \gamma_j$ of these connected curves γ_j. Define for $j = 1, .., m-1$ that $l_j := k$ if $\gamma_j \subset \mathcal{H}(k)$, and $l_m := l_1$. Then define $w(\gamma)$ to be $w(\gamma){:=}l_1 l_2 \cdots l_m$. $w(\gamma)$ is well-defined up to a choice of γ_1. Clearly $w(\gamma)$ is jumpy and contains at least one basin. By the Word Lemma $w(\gamma)$ contains at least $[\frac{m}{2^n}]$ disjoint basins. Also, following Lemma 4.1 we have the property that, if $l_i \cdots l_j$ is a basin of $w(\gamma)$ then the action of $\gamma_i \cup \cdots \cup \gamma_j$ must be greater that T for any prescribed number $T > 0$. We then have the following lemma:

Lemma 4.4 *For any $T > 0$, one can modify a subcritical Stein manifold (W, ω, Y, f) such that any simple Type II Hamiltonian orbit γ with a word $w(\gamma)$ of length m must have action $\mathcal{A}(\gamma) > C_m \cdot T$, where $C_m = \max\{1, [\frac{m}{2^n}]\}$.*

Recall that each handle is attached along an isotropic $(k-1)$–sphere modelled on S_- with a specified framing $\mathcal{F}$ of the normal bundle of the sphere. According to Weinstein [34] there is a neighborhood U of S in W, a neighborhood U_- of S_- in $\mathbb{C}^n$, and an isomorphism of isotropic setups

$$\phi\colon\ (U, \omega, Y, M \cap U, S) \to (U_-, \omega_{\mathrm{st}}, Y_{\mathrm{st}}, H_- \cap U_-, S_-).$$

This isomorphism identifies the chosen framing $\mathcal{F}$ of $\mathcal{N}(S,M)$ with the standard framing of $\mathcal{N}(S_-,H_-)$. The reparamertized flow of Y_{st} induces a contactomorphism

$$\eta\colon U_-\setminus S_-\to U_+\setminus S_+\subset H_+,$$

H_+ is a contact k–handle as described in Proposition 3.1. Via the contactomorphism η the framing $\mathcal{F}$ induces a symplectic trivialization $\Phi_{\mathcal{F}}$ of ξ over $H_+\setminus S_+$. Recall also that ξ on H_+ has a symplectic trivialization Φ_{st} induced by the standard symplectic trivialization of $T\mathbb{C}^n$. We can choose c_l so that for any Hamiltonian trajectory γ in some $\mathcal{H}(p)$ with $\mathcal{A}(\gamma)=\tau$,

$$\mu(\gamma,\tilde{\xi}_k,\Phi_{\mathcal{F}})>(N_1(\mathbf{c})+N_2(\mathbf{c}))\tau-2n \tag{13}$$

where $N(\mathbf{c})$ is a constant depending only on c_l's and is big enough so that $N(\mathbf{c})\tau$ exceeds any of the Conley–Zehnder indexes rising from the ambiguities caused by the different symplectic trivializations $\Phi_{\mathcal{F}}$ and Φ_{st}. $N_2(\mathbf{c})$ is also a very large constant.

Now let γ be a simple contractible Type II orbit of M. Let $D\subset M$ be a closed spanning disc of γ. By perturbing the interior of D we may assume the following condition on D.

Condition 4.1 *Each connected component of the intersection $D\cap\mathcal{H}(k)$ contains a part of γ if it is not empty.*

Let $w(\gamma)=l_1\cdots l_m$ be the word associated to γ. Write $D=\cup_{j=1}^{m-1}D_j$, where D_j is the intersection of D with the j–th handle that γ crosses. On each D_j we use the symplectic trivialization $\Phi_j=\Phi_{\mathrm{st}}$ on $\tilde{\xi}$ and compute the local μ–index $\mu_j=\mu(\gamma_j,\tilde{\xi},\Phi_j)$ of γ_j. We denote by $\mu'(\gamma)$ the sum of these local indexes. Unfortunately, $\mu'(\gamma)$ is not the Conley–Zehnder index that we want because the local trivializations of $\tilde{\xi}$ by Φ_j do not match up to a symplectic trivialization of $\tilde{\xi}|_D$. There are two types of factors which cause mismatches of these local trivializations:

(1) The choices of a framing of the normal bundle of the attaching isotropic spheres.

(2) The gluing of a k–handle (using the flow of Y_{st}).

Type 1 can be overcome by choosing suitable $\mathbf{c}$'s (to produce large $N_1(\mathbf{c})$). Type 2 happens each time γ crosses from one $\mathcal{H}(k)$ to another. The gluing map η preserves contact structure ξ but not contact forms. Let $\alpha':=\eta^*\alpha=e^{-h}\alpha$, and let R denote the Reeb vector field of α, then the Reeb vector field R' of α' is

$$R' = e^h(R + X_h^\xi)$$

where X^ξ is the vector field tangent to ξ and satisfying

$$d\alpha(X_h^\xi, \cdot) = -dh|_\xi$$

Since the actual gluing take place in a thin collar of ∂U_-, we may assume that $h \sim const$ and $R' \sim e^h R$ on the collar. Then by mimicking the comparison of Hamiltonian flow and Reeb flow in the previous section, we conclude that each Type 2 error is bounded by $\pm 2n$.

Let τ_j be the action of γ_j and $\tau = \sum_{j=1}^{m-1} \tau_j$ be the action of γ. By Lemma 4.4 we have

$$\tau > C_m T$$

which together with (13) shows

$$\mu'(\gamma) > \sum_{j=1}^{m-1} N_1(\mathbf{c}) \cdot \tau_j + N_2(\mathbf{c}) \cdot C_m T - 2nm.$$

Then the actual Conley–Zehnder index $\mu(\gamma) = \mu(\gamma, \tilde{\xi}, D)$ satisfies the inequality

$$\mu(\gamma) > \left(\sum_{j=1}^{m-1} N_1(\mathbf{c}) \cdot \tau_j - (\text{error of Type 1}) \right) + N_2(\mathbf{c}) \cdot C_m T - 4mn \tag{14}$$

The first term on the right hand side of (14) can be made positive and very large by choosing suitable c_l as discussed before. For the second term, recall $C_m = \max\{1, [\frac{m}{2^n}]\}$, then $N_2(\mathbf{c}) \cdot C_m T - 4mn$ can be very large if we choose to have $N_2(\mathbf{c}) \gg 2^n$ (by choosing suitable c_l) and $T \gg 4n$. Note that none of these $N_1(\mathbf{c})$, $N_2(\mathbf{c})$ and T depend on m or on γ. This completes the proof of Lemma 4.2.

5 Stabilization of (M, ξ)

Let (M, ξ) be a $(2n-1)$–dimensional contact manifold. Let (W, ω, Y, f) be an Weinstein manifold associated to (M, ξ) as discussed in the previous section. We many assume that $M = \{f = c\}$ for some suitable constant c. We also assume that $\dim M > 3$ for the moment. This condition on dimension is to ensure that $\partial \circ \partial = 0$ because $C_*(M, \alpha) = 0$ for $* \leq 1$ when $\dim M > 3$. Later we will show that $\partial \circ \partial = 0$ also holds true when $\dim M = 3$ despite of the fact that $C_1(M, \alpha) \neq 0$ when $\dim M = 3$.

By now we have seen that simple Type I Reeb orbits of a SSFC manifold (M, ξ) are in one-one correspondence with critical points of f on W. One might expect that the counting of 1–dimensional moduli of holomorphic cylinders here is equivalent to the counting of gradient trajectories of f connecting critical points of consecutive Morse indexes. Let $p \in Crit(f)$ be of index $k < n$ $(2n = \dim W)$, and let

$$S_c(p) \cong \{\mathsf{f} = c\} \cap \{|y| = 0\} \quad (\text{ see } (4))$$

be the $(2n-k-1)$–dimensional coisotropic ellipsoid in the corresponding contact k–handle. We may identify γ_p with $\{|z_n|^2 = c_n^2 c\} \cap S_c(p)$. S^1 acts on $S_c(p)$ by rotating the z_n–plane, giving $S_c(p)$ an open book structure with binding $B := S_c(p) \cap \{z_n = 0\} \cong S^{2n-k-3}$, pages diffeomorphic to a $(2n-k-2)$–dimensional disc D^{2n-k-2}, and $S_c(p) \setminus B \cong S^1 \times D^{2n-k-2}$. Then by following the discussion in Section 7 on S^1–invariant holomorphic curves, one can see that $S_c(p) \setminus \gamma_p$ is foliated by 2–dimensional discs bounding γ_p and all such discs are images of some element of $\mathcal{M}(\gamma_p)/\mathbb{R}$ before contact handles of higher indexes are attached. In particular B is the parameter space of a connected component of $\mathcal{M}(\gamma_p)/\mathbb{R}$.

Suppose now that a contact $(k+1)$–handle (assuming $k+1 < n$) corresponding to $q \in Crit(f)$ is attached along an isotropic k–sphere which intersects transversally with $S_c(p) \setminus \gamma_p$ at finitely many points. Assume that these intersection points are on distinct elements of $\mathcal{M}(S_c(p); \gamma_p)/\mathbb{R}$. Intuition suggests these "marked" (by the intersection points) elements in $\mathcal{M}(S_c(p); \gamma_p)/\mathbb{R}$ may correspond to elements in $\mathcal{M}(\gamma_q, \gamma_p)/\mathbb{R}$ of the resulting manifold. This is where we get the speculation that perhaps the counting of holomorphic cylinders is equivalent to the counting of gradient trajectories. Of course many works have to be done to verify (or disprove) such a naive speculation.

On the other hand, if the above guess is true for (M, ξ) viewed as a regular level set of a subcritical Weinstein manifold (W, ω, Y, f), then it is also true for (M', ξ') which is the corresponding regular level set of $f + \kappa|z|^2 \colon W \times \mathbb{C} \to \mathbb{R}$ with $\kappa > 0$. It turns out that (M', ξ') has several nice features which allow an alternative approach of computing $HC_*(M, \xi)$ and establishing a relation between $HC_*(M, \xi)$ and $H_*(W)$ as one has expected. The rest of this section consists of more discussion on (M', ξ'), which serves as preparation for the next two sections.

Recall that from (W, ω, Y, f) we can define a new Weinstein manifold

$$(W', \omega', Y', f') := (W \times \mathbb{C}, \omega + \omega_o, Y + Y_o, f + \kappa|z|^2)$$

where $\omega_o = dx \wedge dy$ is the standard symplectic structure on $\mathbb{C}$; $Y_o := \frac{1}{2}(x\partial_x + y\partial_y)$ is a Liouville vector field with respect to ω_o; and $\kappa > 0$ is a constant. Here $z = x + iy$ is the complex coordinate of $\mathbb{C}$. (W', ω', Y', f') is called a *stabilization* of (W, ω, Y, f).

Consider on W' the hypersurface

$$M' := \{f' = c\}.$$

M' is a regular level set of f'. The 1–form $\alpha' := \omega'(Y', \cdot)$ restricts to a contact 1–form on M'. Denote by ξ' the associated contact structure on M'. It is easy to see that (M, ξ) is a codimension 2 contact submanifold of (M', ξ'). If M is subcritical, then so is M'. Note that the rotation in $\mathbb{C}$ centered at $z = 0$ induces an S^1–action on M' that acts freely on $M' \setminus M$ and fixes M. Indeed, we can view M' as an open book with binding M, pages diffeomorphic to V, and trivial monodromy $id\colon V \to V$. Here $V := \{f < c\} \subset W$ is called a *subcritical Stein-filling* of M.

We can smoothly embed $\mathbb{R} \times M'$ into W' by identifying $\{0\} \times M'$ with $M' \subset W'$, and the vector field $\frac{\partial}{\partial t}$ with Y'. The image of $\mathbb{R} \times M'$ in W' is then $W'_o := W' \setminus \mathcal{L}$, where $\mathcal{L}$ is the closure of the stable submanifolds of the flow of Y'. The image of $\mathbb{R} \times M$ is $W_o := W \setminus \mathcal{L}$.

Lemma 5.1 *$c_1(\xi; M)|_{\pi_2(M)} = 0$ if and only if $c_1(\xi'; M')|_{\pi_2(M')} = 0$.*

Proof Let $\iota\colon M \hookrightarrow M'$ be the inclusion map and S represent an element of $\pi_2(M')$. Since $M' \cong M \times D^2 \cup_{M \times S^1} V \times S^1$, $\dim V = 2n \geq 4$ and $H_{2n-2}(W) = 0$, S can be pushed into M, ie, S represents an element of $\pi_2(M)$. On the other hand an element of $\pi_2(M)$ is also an element of $\pi_2(M')$. Since $\iota^* c_1(\xi') = c_1(\xi)$ we conclude that $c_1(\xi; M)|_{\pi_2(M)} = 0 \Leftrightarrow c_1(\xi'; M')|_{\pi_2(M')} = 0$. □

Although α' may not be regular in the usual sense (α' may have S^1–families of Reeb orbits), the above properties ensure that cylindrical contact homology of (ξ', α') is still defined.

The Reeb vector field of (M', α') is

$$R' := \frac{X_f + 4\kappa i Y_o}{\alpha(X_f) + \kappa |z|^2}.$$

We may assume that each critical point of f is standard, then so are the critical points of f' (note that f and f' have the same set of critical points with the same Morse indexes). Since there are only finitely many critical points, κ can be chosen so that $\{c_{k+1}^2, ..., c_n^2, \kappa^{-1}\}$ is linearly independent over $\mathbb{Q}$ for any $\{c_{k+1}^2, ..., c_n^2\}$ associated to some critical point of index k of f.

When κ is much smaller than any of those c_j then the principal Reeb orbits of (M', α') are exactly principal Reeb orbits of (M, α). On the other hand, if κ is much bigger than any of those c_l then the simple principal Reeb orbits of (M', α') are

$$\gamma_p := \{(p, z) \in M' | p \in Crit(f)\}$$

they are in one-one correspondence with critical points of f. Certainly the contact homology of (M', ξ') does not depend on the choice of κ. In fact, we will show that, up to a degree shifting by 2, $HC(M, \xi)$ and $HC(M', \xi')$ are isomorphic. Thus we know about $HC_*(M, \xi)$ once we know about $HC_*(M', \xi')$.

6 $HC_*(M, \xi) = HC_{*+2}(M', \xi')$

Recall that $Y_o = \frac{1}{2}(x\partial_x + y\partial_y)$. Let $Y \subset TW$ be a gradient-like vector field with respect to f and let $b > 0$ be a constant. The vector field $Y' := Y + bY_o \subset TW'$ is gradient-like with repsect to f'. Let Y'^t denote the time t map of the flow of Y'. We can embed $\mathbb{R} \times M'$ into W' by identifying $\{t\} \times M' \subset \mathbb{R} \times M'$ with $Y'^t(M') \subset W'$. In particular the vector field $\partial_t \subset T(\mathbb{R} \times M')$ is identified with Y'. Note that $\mathbb{R} \times M \subset W \times \{0\}$ under this identification.

We need to know how to count pseudo-holomorphic curves in $Symp(M) = \mathbb{R} \times M$ and $Symp(M') = \mathbb{R} \times M'$. To achieve our goal, we first choose a class of admissible almost complex structures. First of all observe that the group S^1 acts on $W \times \mathbb{C}$ by rotations on $\mathbb{C}$, sending (p, z) to $(p, e^{i\theta}z)$ for $\theta \in S^1 \cong \mathbb{R}/(2\pi\mathbb{Z})$. It restricts to an S^1–action on M' that fixes M, acts freely on $M' \setminus M$, and preserves α'. Let $\Pi_{\mathbb{C}}$ denote the projection $W \times \mathbb{C} \to \mathbb{C}$, and Π the the projection $W \times \mathbb{C} \to W$. One might expect to find an α'–admissible almost complex structure which splits and preserves the subbundles $\Pi^* TW$ and $\Pi_{\mathbb{C}}^* T\mathbb{C}$ of TW'. This is however, not true in general.

On the other hand, since

$$\xi'|_M = \xi \oplus \underline{\mathbb{C}} \tag{15}$$

we consider an $d\alpha'$–admissible almost complex structure J' on $\xi' \subset TM'$ such that J' preserves the decomposition (15), and $J' = i$ when restricted to the second factor of (15), here i denote the standard complex structure on $\mathbb{C}$. It is easy to see that there are plenty of $d\alpha'$–compatible almost complex structures satisfying the above condition. Then we extend J' to become an admissible almost complex structure on $Symp(M') \subset W'$ via the flow of Y'^t. In particular, $J'(Y') = Y'^t_* R'$ on $Y'^t(M')$. We have

$$J' = \begin{bmatrix} J & O \\ O & i \end{bmatrix} + Q, \quad Q \to O \text{ as } |z| \to 0,\ z \in \mathbb{C}. \tag{16}$$

Theorem 6.1 $HC_*(M, \xi; \alpha, J) \cong HC_{*+2}(M', \xi'; \alpha', J')$.

Proof Fix any large positive integer m_o. M can be constructed as a level set of the Weinstein manifold (W, ω, Y, f) such that the following action condition is satisfied: Let γ_p^m denote the principal generator corresponding to the critical point p of f and with multiplicity m. Then $\mathcal{A}(\gamma_p^1) > \mathcal{A}(\gamma_q^m)$ when the index of p is greater than the index of q and $m \leq m_o$.

Let κ be small enough then the principal generators of the contact complex $C_*(M, \alpha)$ are principal generators of $C_*(M', \alpha')$. Let γ_+ and γ_- be two such principal generators, then

$$\mathcal{M} := \mathcal{M}'_J(M; \gamma_-, \gamma_+) \subset \mathcal{M}' := \mathcal{M}_{J'}(M'; \gamma_-, \gamma_+).$$

Given $u' \in \mathcal{M}'$, write $u' = (u_1, u_2)$ according to the splitting $W' = W \times \mathbb{C}$. Assume $u' \notin \mathcal{M}$, then $u_2 \not\equiv 0$. The u_2–component of u' associates two winding numbers (recall that $Symp(M) \subset W$)

$$\begin{aligned} n_- &:= \mathrm{wind}(u' \cap \{t \ll 0\}, Symp(M)), \\ n_+ &:= \mathrm{wind}(u' \cap \{t \gg 0\}, Symp(M)). \end{aligned}$$

Since $u'(\mathbb{C}^*)$ and $Symp(M)$ are pseudo-holomorphic submanifolds of complement dimensions, $u'(\mathbb{C}^*)$ intersects with $Symp(M)$ positively at every point of the intersection $u'(\mathbb{C}^*) \cap Symp(M)$. Thus we have

$$n_+ - n_- = \#\Big(u'(\mathbb{C}^*) \cap Symp(M)\Big) \geq 0. \tag{17}$$

Write $u_2 = u_2(w)$ where w denotes the complex coordinate of $\mathbb{C}$. $u_2\colon \mathbb{C}^* \to \mathbb{C}$ is a smooth function. Recall that we embed $Symp(M') = \mathbb{R} \times M'$ into $W' = W \times \mathbb{C}$ by identifying ∂_t with $Y' = Y + bY_o$ for some constant $b > 0$. The integral trajectories of Y' perserves the value $|z|^{-b/2}$. Since $u'(w)$ approaches $Symp(M) \subset W \times \{0\}$ asymptotically as w approaches either 0 or ∞ we have

$$u_2(w) \text{ is asymptotically holomorphic as } |w| \to 0 \text{ or } \infty, \tag{18}$$

$$|u_2(w)|^{-b/2} \to 0 \quad \text{as } |w| \to 0 \text{ or } \infty. \tag{19}$$

u_2 can be continuously extended to $\mathbb{C}$ by defining $u_2(0) := 0$. The extended function is still denoted by u_2 for simplicity.

Now by (18) and (19) we have that near $w = 0$,

$$u_2(w) \sim w^{n_-} \quad \text{for some } n_- \in \mathbb{N},\ n_- > b/2.$$

Here n_- is exactly the earlier defined winding number of $u'(\mathbb{C}^*)$ to $Symp(M)$ near $t = -\infty$. Similarly, near $w = \infty$ we have

$$u_2(w) \sim w^{n_+} \quad \text{for some } n_+ \in \mathbb{N},\ n_+ < b/2,$$

where n_+ is the winding number of $u'(\mathbb{C}^*)$ to $Symp(M)$ near $t = \infty$. Then we have $n_+ < b/2 < n_-$ and in particular $n_+ - n_- < 0$, which contradicts with (17). So $u_2 \equiv 0$. We conclude that $\mathcal{M} = \mathcal{M}'$. The degree 2 shift is an easy observation. This completes the proof. $\square$

7 Finding $HC(M, \xi)$

We now proceed to compute the cylindrical contact homology of (M', ξ') (again we assume that $\dim M > 3$). This time we choose to have $\kappa \gg 1$ so that the principal Reeb orbits are in $M' \setminus M$ and they are in one-one correspondence with elements of $Crit(f)$. More precisely they are

$$\gamma_p := \{(p, z)\} \subset M', \quad p \in Crit(f)$$

with index $\overline{\gamma}_p = 2n - \operatorname{ind}(p)$.

Let J' be the same as in the previous section. To determine the boundary operator of the contact complex we first need to characterize all moduli of the form $\mathcal{M}'(\gamma_-, \gamma_+)$ of formal dimension equal to 1.

Lemma 7.1 *Fix $m_o > 0$ then there are contact 1–forms on (M', ξ') with nondegenerate Type I Reeb orbits such that if are of Type I with multiplicity $\leq m_o$ and if the formal dimension of $\mathcal{M}'_{J'}(\gamma_-, \gamma_+)$ is 1 then $\gamma_\pm = \gamma_{p_\pm}^m$ for some $p_\pm \in Crit(f)$ with $ind(p_-) = ind(p_+) + 1$ and $\gamma_\pm$ have the same multiplicity m.*

Proof Let $\mathcal{M}' = \mathcal{M}'_{J'}(\gamma_-, \gamma_+)$ be nonempty and its formal dimension is $\overline{\gamma}_+ - \overline{\gamma}_- = 1$. $\gamma_+ = \gamma_{p_+}^{m_+}$ for some $p_+ \in Crit(f)$ with $\operatorname{ind}(p_+) = k_+$, $\gamma_- = \gamma_{p_-}^{m_-}$, where γ_{p_-}, m_-, and k_- are defined similarly, and $m_\pm \leq m_o$. We have

$$\begin{aligned} 1 &= 2n - k_+ + 2(m_+ - 1) - (2n - k_- + 2(m_- - 1)) \\ &= k_- - k_+ + 2(m_+ - m_-), \end{aligned}$$

therefore $k_- \neq k_+$. If $k_- < k_+$ then $\mathcal{A}(\gamma_+) < \mathcal{A}(\gamma_-)$ which is impossible, so $k_- > k_+$.

Assume $k_- > k_+ + 1$ then $m_+ < m_-$. Note that $m_\pm = n_\pm$ are the winding numbers of u' around $Symp(M) = R \times M$ near $t = \pm\infty$ respectively. Since $n_+ \geq n_-$ it cannot happen that $k_- > k_+ + 1$. So we must have $k_- = k_+ + 1$ and hence $m_+ = m_-$. □

By using $n_+ - n_- = \#(u'(\mathbb{C}^*) \cap Symp(M))$ and the positivity of the intersection $u'(\mathbb{C}^*) \cap Symp(M)$ we have the following simple lemma:

Lemma 7.2 *Let $\mathcal{M}' = \mathcal{M}'_{J'}(\gamma_-, \gamma_+)$ be as in the previous lemma and let $u' \in \mathcal{M}'$. Then the curve $u'(\mathbb{C}^*)$ does not intersect $\mathbb{R} \times M$.*

Since all curves that we are going to count are in the symplectization of $M' \setminus M$ we can use the diffeomorphism $M' \setminus M \cong V \times S^1$ to simplify the computation.

Consider the diffeomorphism :

$$\Phi\colon\ V \times S^1 \to M' \setminus M, \quad \Phi(x,\theta) = (x, \sqrt{\frac{c-f}{\kappa}}\theta)$$

Then $\Phi^*(\alpha') = e^{-h} d\theta + \alpha$. Write $\Phi^*(\alpha') = e^{-h}(d\theta + e^h\alpha)$ then by using the fact that $e^{-h}(d\theta + e^h\alpha)$ is contact one sees that $d(e^h\alpha)$ is symplectic on V. So by abusing notations we redenote $e^h\alpha$ as α and denote $\lambda := d\theta + \alpha$.

λ is a connection 1–form on the trivial principal bundle

$$V \times S^1 \overset{\pi}{\to} V,$$

$$d\lambda = \pi^*\omega, \qquad \omega := d\alpha$$

and h is a smooth Morse function on V with $Crit(h) = Crit(f)$ and the same corresponding Morse indexes.

With the above isomorphism understood we will from now on work with the contact manifold $(V \times S^1, e^{-h}\lambda)$. We denote the corresponding contact structure by ξ'. ξ' is the horizontal lifting of TV with respect to the connection 1–form λ. Let X_h be the Hamiltonian vector field of h with respect to the symplectic 2–form ω, ie, $\omega(X_h, \cdot) = -dh$. Then the Reeb vector field of $e^{-h}\lambda$ is

$$R' = \frac{e^h}{1 + \alpha(X_h)}(\partial_\theta + X_h)$$

with $1 + \alpha(X_h) > 0$ on $V \times S^1$.

S^1 acts freely on $V \times S^1$ by rotation along S^1 fibers. Let J' be an S^1–invariant $e^{-h}\lambda$–admissible almost complex structure. Since ξ' is transversal to the fibers, $J_{\xi'} := J'|_{\xi'}$ induces an ω–compatible almost complex structure $\bar{J}$ on V by

$$\bar{J}(\pi_*\eta) := \pi_* J'\eta \quad \eta \in \xi'.$$

Conversely an ω–compatible almost complex $\bar{J}$ structure on V induces an S^1–invariant $d\lambda$–compatible almost complex structure $J_{\xi'}$ on ξ' which extends to be an S^1–invariant $e^{-h}\lambda$–admissible almost complex structure J' on the symplectization of $V \times S^1$.

Let $\gamma_\pm = \gamma^m_{p_\pm}$ for some $p_\pm \in Crit(h)$ and $u' \in \mathcal{M}_{J'}(\gamma_-, \gamma_+)$ be with S^1–invariant image. Let

$$u = (\bar{u}, \theta)\colon \mathbb{R} \times S^1 \to V \times S^1$$

be the corresponding map into $V \times S^1$. Since $C := u(\mathbb{R} \times S^1)$ is S^1–invariant and ξ' is transversal to the S^1–fibers, ξ' induces a nonsingular foliation on C generated by $\xi' \cap TC$. We can reparametrize u' so that

$$\lambda(u_s) = 0, \quad \lambda(u_t) = m. \tag{20}$$

Here (s,t) are coordinates for $\mathbb{R} \times S^1$ (so $z = s + it$ is the complex coordinate), and $u_s := \dfrac{\partial u}{\partial s}$, $u_t := \dfrac{\partial u}{\partial t}$.

Let $\pi_1\colon T(V \times S^1) \to \xi'$ be the projection along the Reeb vector field R'. Since u' is J'–holomorphic we have

$$\pi_1 u_s + J' \pi_1 u_t = 0, \tag{21}$$

ie,

$$u_s - \frac{\lambda(u_s)}{1 + \alpha(X_h)}(\partial_\theta + X_h) + J'\Big(u_t - \frac{\lambda(u_t)}{1 + \alpha(X_h)}(\partial_\theta + X_h)\Big) = 0,$$

which by (20) is reduced to

$$u_s + J'\Big(u_t - m\rho(\partial_\theta + X_h)\Big) = 0, \tag{22}$$

where $\rho = (1 + \alpha(X_h))^{-1}$. Write $u_s = (\bar{u}_s, \theta_s)$, $u_t = (\bar{u}_t, \theta_t)$. Apply π_* to (22) and we have

$$\bar{u}_s + \bar{J}(\bar{u}_t - m\rho X_h) = 0. \tag{23}$$

Hence $\bar{u}$ is a finite-energy solution to the Cauchy-Riemann type equation as in Floer Theory with

$$\bar{u}_s = m\bar{J}(\rho X_h), \quad \bar{u}_t = 0.$$

Note that the flow of the vector field $m\rho\bar{J}X_h$ is of Morse-Smale type for generic $\bar{J}$ hence by [31] the linearization of (23) at an S^1–invariant solution $\bar{u}$

$$F_{\bar{u}}\bar{\eta} = \bar{\nabla}_s\bar{\eta} + \bar{J}\bar{\nabla}_t\bar{\eta} - m\bar{\nabla}_{\bar{\eta}}(\bar{J}(\rho X_h)) \tag{24}$$

is onto for generic $\bar{J}$. Here $\bar{\nabla}$ is the Levi-Civita connection associated to the Riemannian metric $\bar{g} := d\alpha \circ (Id \times \bar{J})$.

Conversely an S^1–invariant solution $\bar{u}$ to (23) can be "lifted" to an S^1–invariant J'–holomorphic map $u' = (a, u)$ into $Symp(V \times S^1)$ as follows. First we lift $\bar{u}$ to a map $u = (\bar{u}, \theta)$ into $V \times S^1$ with $\theta = \theta(s,t)\colon \mathbb{R} \times S^1 \to S^1$ satisfying

$$\theta_s = -\alpha(\bar{u}_s), \quad \theta_t = m - \alpha(\bar{u}_t) = m.$$

Such θ exists and is unique up to the addition of a constant rotation. The resulting map u satisfies (21) and (20).

Now solve for the function $a = a(s,t)\colon \mathbb{R} \times S^1 \to \mathbb{R}$ which satisfies

$$a_s = e^{-h}\lambda(u_t) = me^{-h}, \quad a_t = -e^{-h}\lambda(u_s) = 0. \tag{25}$$

Since $\mathbb{R} \times S^1$ is a noncompact Riemann surface, there exists a complex-valued function $b\colon \mathbb{R} \times S^1 \to \mathbb{C}$ such that

$$b_s = e^{-h}\lambda(u_t) = me^{-h}, \quad b_t = -e^{-h}\lambda(u_s) = 0.$$

b is unique up to an addition of a holomorphic function. Write b_1 for the real part of b, and b_2 for the imaginary part of b. We have

$$(b_2)_{ss} + (b_2)_{tt} = -u_s(e^{-h}\lambda(u_s)) - u_t(e^{-h}\lambda(u_t)) = 0$$

ie, b_2 is harmonic, hence the imaginary part of a holomorphic function $\tilde{b}$. Define $a := b - \tilde{b}$. a is a real-valued function on $\mathbb{R} \times S^1$ and satisfies (25). Moreover a is unique up to the addition of a real constant. The resulting map u' is S^1–invariant, J'–holomorphic with multiplicity m, unique up to the rotation by a constant angle and the addition of a real constant and satisfies

$$u(s, \cdot) \to \gamma_\pm \quad \text{as} \quad s \to \pm\infty,$$

$$0 < \int_{\mathbb{R} \times S^1} u^* d(e^{-h}\lambda) = \int_{\gamma_+} e^{-h}\lambda - \int_{\gamma_-} e^{-h}\lambda = m(e^{-h(p_+)} - e^{-h(p_-)}) < \infty.$$

Hence $u' \in \mathcal{M}'_{J'}(\gamma_-, \gamma_+)$.

Now that we have establishes for each fixed $m \in \mathbb{N}$ (and $m \leq m_o$ for some large m_o) a one-one correspondence between (i) the (gradient-like) $\bar{J}X_h$–trajectories in V that connecting critical points $p_\pm$ with $\mathrm{ind}(p_+) = \mathrm{ind}(p_-) = 1$ and (ii) the S^1–invariant elements in the moduli $\mathcal{M}'_{J'}(\gamma_-, \gamma_+)/\mathbb{R}$ with $\gamma_\pm = \gamma^m_{p_\pm}$. We proceed to show that the linearization of the operator $\overline{\partial} := \overline{\partial}_{J'}$ at an S^1–invariant solution $u' \in \mathcal{M}'_{J'}(\gamma_-, \gamma_+)$ is surjective for generic J'.

Recall that

$$\overline{\partial}(u') = u'_s + J'u'_t.$$

Let π_1, π_2 be the projections with respect to the orthogonal decomposition $T(Symp(V \times S^1)) \to \xi' \oplus E$, E is the vector bundle spanned by ∂_t and R'. Write $\overline{\partial} = \overline{\partial}_1 + \overline{\partial}_2$ where

$$\overline{\partial}_1(u') = \pi_1 u'_s + J'\pi_1 u'_t, \tag{26}$$

$$\overline{\partial}_2(u') = \pi_2 u'_s + J'\pi_2 u'_t. \tag{27}$$

Let D_1, D_2 denote the linearizations of $\overline{\partial}_1$ and $\overline{\partial}_2$ at u' respectively.

Lemma 7.3 *$\pi_1 D_1\colon W^{1,2}(\mathbb{R} \times S^1, u'^*\xi') \to L^2(\mathbb{R} \times S^1, u'^*\xi')$ is surjective for generic S^1–invariant J'.*

Proof Let $\eta \in W^{1,2}(\mathbb{R} \times S^1, u'^*\xi')$. Let ∇ be the Levi-Civita connection on $Symp(V \times S^1)$ with respect to the Riemannian metric $g' := e^{-t+h}d(e^{t-h}\lambda) \circ (Id \times J')$. Note that $g'|_{\xi'} = g|_{\xi'}$ where g is the Riemannian metric on $V \times S^1$ induced by $\bar{g}$ and the connection 1–form λ of the S^1–bundle $V \times S^1$ over V. Then

$$\begin{aligned} D_1(\eta) = \pi_1\nabla_s\eta - (\nabla_\eta\lambda)(u'_s)e^{-h}R' + J'\pi_1\nabla_t\eta \\ + \pi_1(\nabla_\eta J')\pi_1 u'_t - mJ'\nabla_\eta(e^{-h}R'). \end{aligned} \tag{28}$$

Write $e^{-h}R' = \dfrac{\partial_\theta + X_h}{1+\alpha(X_h)} = \zeta + \partial_\theta$ with $\zeta \in \xi'$,

$$\zeta = \frac{-\alpha(X_h)\partial_\theta + X_h}{1+\alpha(X_h)} = \frac{-1}{m}\cdot \pi_1 u'_t$$

and apply π_1 to (28) we get

$$\pi_1 D_1(\eta) = \pi_1\nabla_s\eta + J'\pi_1\nabla_t\eta - m\pi_1(\nabla_\eta J')\zeta - mJ'\pi_1(\nabla_\eta(\zeta + \partial_\theta)).$$

Let $\bar{\eta} := \pi_*\eta$, $\bar{\zeta} := \pi_*\zeta$. Since $\pi_1(\nabla_\eta(J'|_E))\zeta = 0$ and

$$\pi_1\nabla_\eta\zeta = \pi^*\bar{\nabla}_{\bar{\eta}}\bar{\zeta} \qquad (\pi^* \text{ means “horizontal lifting”})$$
$$\nabla_\eta\partial_\theta = \nabla_{\partial_\theta}\eta + [\eta, \partial_\theta] = \nabla_{\partial_\theta}\eta - L_{\partial_\theta}\eta = 0$$

and $\bar{\zeta} = \rho X_h$ with $\rho = (1+\alpha(X_h))^{-1}$ we have

$$\begin{aligned} \pi_*\pi_1 D_1(\eta) &= \bar{\nabla}_s\bar{\eta} + \bar{J}\bar{\nabla}_t\bar{\eta} - m(\bar{\nabla}_{\bar{\eta}}\bar{J})\bar{\zeta} - m\bar{J}\bar{\nabla}_{\bar{\eta}}\bar{\zeta} \\ &= \bar{\nabla}_s\bar{\eta} + \bar{J}\bar{\nabla}_t\bar{\eta} - m\bar{\nabla}_{\bar{\eta}}(\bar{J}(\rho X_h)) \\ &= F_{\bar{u}}(\bar{\eta}) \qquad (\text{see } (24)). \end{aligned} \tag{29}$$

Since $F_{\bar{u}}\colon W^{1,2}(\mathbb{R}\times S^1, \bar{u}^*TV)$ is surjective for generic $\bar{J}$ [31] we conclude that $\pi_1 D_1$ is surjective for generic S^1–invariant J'. □

Recall that E denote the vector bundle over $Symp(V \times S^1)$ spanned by ∂_t and the Reeb vector field R'.

Lemma 7.4 $D_2\colon W^{1,2}(\mathbb{R} \times S^1, u'^*E) \to L^2(\mathbb{R} \times S^1, u'^*E)$ *is surjective for all* J'.

Proof Let $\eta \in W^{1,2}(\mathbb{R} \times S^1, u'^*E)$ and let ∇ be the Levi-Civita connection defined in proof of Lemma 7.3. Then

$$D_2(\eta) = \pi_2 \nabla_s \eta + J' \pi_2 \nabla_t \eta + m(\nabla_\eta J')(e^{-h}R') + mJ' \nabla_\eta (e^{-h}R').$$

Since $\nabla_\eta J' \in \mathrm{End}(\xi')$, $\nabla_{R'} R' = 0$, $\nabla_{R'} e^{-h} = 0$ and $\nabla_\eta (e^{-h}R') = 0$ we have

$$D_2(\eta) = \pi_2(\nabla_s \eta + J' \nabla_t \eta).$$

Write $\eta = \eta_1 \partial_t + \eta_2 R'$ then

$$D_2((\eta_1, \eta_2)) = (\nabla_s \eta_1 - \nabla_t \eta_2, \nabla_s \eta_2 + \nabla_t \eta_1)$$

is the standard d–bar operator on $W^{1,2}(\mathbb{R} \times S^1, u'^*E)$ with respect to the almost complex structure $u'^*(J'|_E)$ on the trivial complex line bundle u'^*E, hence is surjective because $\mathbb{R} \times S^1$ is a noncompact Riemann surface. □

Lemma 7.3 and Lemma 7.4 together imply the following:

Lemma 7.5 *Let J' be an S^1–invariant α'–admissible almost complex structure on $Symp(M')$. Then for generic J' the linearized operator $D = D_1 + D_2$ of $\overline{\partial}_{J'}$ is surjective at every S^1–invariant element of $\mathcal{M}_{J'}(\gamma_-, \gamma_+)$ provided that the multiplicity of $\gamma_\pm$ is small.*

In the following we would like to show that up to a homotopy of contact 1–forms there are no elements of $\mathcal{M}_{J'}(\gamma_-, \gamma_+)$ which are not S^1–invariant. Our proof is based on results from [31] concerning finite energy solutions with small periods of Cauchy-Riemann type equations and the following construction.

Recall that the stabilization M' can be identified with the following hypersurface in $W \times \mathbb{C}$:

$$\{f + \kappa |z|^2 = c\}, \quad \kappa \gg 1 \text{ fixed},$$

with contact structure defined by the 1–form $\alpha' = \alpha + \alpha_o$, where $\alpha_o := \omega_o(Y_o, \cdot)$, ω_o is the standard symplectic 2–form on $\mathbb{C}$. For each $k \in \mathbb{N}$ with $k > 1$ the finite group

$$\mathbb{Z}_k := \{\theta \in S^1 | \theta^k = 1\} \subset S^1$$

acts on (M',α') via rotation in the $\mathbb{C}$–plane. The action preserves α' and induces a branched k–covering map

$$\Phi_k\colon\ M' \to M'.$$

$\Phi_k\colon\ M'\setminus M \to M'\setminus M$ is a k–covering map, and Φ_k fixes M pointwise. Φ_k induces a contact 1–form $\alpha_k := (\Phi_k)_*\alpha' = \alpha + \alpha_o/k$ on M'. Let ξ_k denote the contact structure defined by α_k. α_k and α' can be included into a smooth family of contact 1–forms of M' so ξ_k and ξ' are isotopic as contact structures. Moreover $(\Phi_k)_*\xi' = \xi_k$.

Let J_k denote the S^1–invariant α_k–admissible almost complex structure on $Symp(V\times S^1)$ such that

$$J_k = (\Phi_k)_* J' (\Phi_k^{-1})_* \quad \text{on } \xi_k.$$

Clearly Lemma 7.5 also holds for generic J_k, $1<k\in\mathbb{N}$.

For $v_k'\in\mathcal{M}_{J_k}(M';\gamma_-,\gamma_+)$ let $\tilde v_k\in\mathcal{M}_{J_k}(\gamma_-^k,\gamma_+^k)$ be the k–fold cover of v_k'. Then $\tilde v_k(s,t+\frac{1}{k}) = \tilde v_k(s,t)$ and the pullback by Φ_k of $\tilde v_k$ is a J'–holomorphic map $u_k'\in\mathcal{M}_{J'}(M';\gamma_-,\gamma_+)$ which also satisfies

$$u_k'(s,t+\frac{1}{k}) = \vartheta\cdot u_k'(s,t), \tag{30}$$

where ϑ is the generator of $\mathbb{Z}_k$ which represents the $2\pi/k$–rotation. Conversely if $u_k'\in\mathcal{M}_{J'}(M';\gamma_-,\gamma_+)$ satisfies (30) then $\Phi_k(u_k')\in\mathcal{M}_{J_k}(M';\gamma_-^k,\gamma_+^k)$ is a k–cover of some $v_k'\in\mathcal{M}_{J_k}(M';\gamma_-,\gamma_+)$.

Lemma 7.6 *Assume that $ind(p_-)-ind(p_+)=1$ and $\gamma_\pm=\gamma_{p_\pm}^m$. Then there exists $k_o\in\mathbb{N}$ such that for all $k\geq k_o$ all elements of $\mathcal{M}_{J_k}(M';\gamma_-,\gamma_+)$ are S^1–invariant.*

Proof Suppose not. Then there exists an infinite sequence k_ν of positive integers, $\lim\limits_{\nu\to\infty} k_\nu=\infty$, such that for each k_ν the moduli space $\mathcal{M}_{J_{k_\nu}}(M';\gamma_-,\gamma_+)$ has an element say v_{k_ν}' which is not S^1–invariant. Let $u_{k_\nu}' := \Phi_{k_\nu}^*\tilde v_{k_\nu}$ where Φ_{k_ν} is defined as before, $\tilde v_{k_\nu}$ is a k_ν–cover of v_{k_ν}'. Then $u_{k_\nu}'\in\mathcal{M}_{J'}(M';\gamma_-,\gamma_+)$. Since all u_{k_ν}' have the same contact energy $\mathcal{A}_{\alpha'}(\gamma_+)-\mathcal{A}_{\alpha'}(\gamma_-)$, there is an infinite subsequence of u_{k_ν}', also denoted by u_{k_ν}', such that up to translations in $\mathbb{R}$–direction, u_{k_ν}' converge to a J'–holomorphic curve u' as $\nu\to\infty$. u' is S^1–invariant.

All u_{k_ν}' and u' have the same winding numbers $n_\pm$ around $Symp(M)$ near $t=\pm\infty$. Moreover we have $n_+=n_-$ because none of the u_{k_ν}''s intersect with $Symp(M)$, so neither does u'. u' is therefore a finite union of S^1–invariant curves so that the image in $V \times S^1$ of each connected component is either a cylinder bounding a pair of type I Reeb orbits. The closure of the image of u' in V is a connected tree formed by trajectories of a gradient-like vector field. This tree contains a (perhaps broken) trajectory with endpoints $p_\pm = \pi(\gamma_\pm)$. Moreover, $p_\pm$ are vertices of valent 1 of the tree, here the valent of a vertex is the number of edges coming out from this vertex as an endpoint. Suppose that the this trajectory contains other critical points. Then there is a critical point $p \neq p_\pm$ such that a trajectory between p and p_+ is contained in the said broken trajectory from p_- to p_+. The corresponding preimage of p in $V \times S^1$ is the Reeb orbit $\gamma := \gamma_p^m$. Since we must have the action inequalities

$$\mathcal{A}(\gamma_-) < \mathcal{A}(\gamma) < \mathcal{A}(\gamma_+),$$

and since γ, γ_-, γ_+ have the same multiplicity we have

$$\mathrm{ind}(p_-) > \mathrm{ind}(p) > \mathrm{ind}(p_+).$$

But $\mathrm{ind}(p_-) - \mathrm{ind}(p_+) = 1$, there exists no such p. So the trajectory between $p_\pm$ is unbroken and hence is equal to the tree.

Now that the projection of the image of u' in V is a trajectory of a gradient-like vector field connecting critical points $p_- = \pi(\gamma_-)$ to $p_+ = \pi(\gamma_+)$, so the linearized operator $D_{u'}$ is surjective for generic J', hence u' is an isolated element of $\mathcal{M}_{J'}(M'; \gamma_-, \gamma_+)$. Thus u'_{k_ν} and hence v'_{k_ν} have to be S^1–invariant for all ν large enough, which contradicts with the assumption that there are non-S^1–invariant u'_{k_ν} for infinitely many k_ν. Hence the lemma holds. □

Lemma 7.7 *Let (α', J') be a regular pair. For $p \in Crit(f)$ with $ind(p) = k$, denote by $\gamma_p^m \in C_{2(n+m-1)-k}(\alpha')$ the corresponding principal generator with multiplicity $m \leq m_o$. Then*

$$\partial \gamma_p^m = m \sum_{ind(q)=k+1} \frac{a_q}{m} \gamma_q^m,$$

where a_q is the algebraic number of trajectories of $\bar{J}X_h$ running from q and p.

Now let $m_o \to \infty$. After an easy computation on index we obtain the following:

Theorem 7.1 *Let (M', ξ') be a stabilization of a $(2n+1)$–dimensional subcritical Stein-fillable contact manifold (M, ξ), $n > 2$, and (V, J) a subcritical Stein-filling of (M, ξ). Then*

$$HC_i(M', \xi') \cong \bigoplus_{m \in \mathbb{N} \cup \{0\}} H_{2(n+m)-i}(V).$$

Combining Theorem 7.1 with Theorem 6.1 we have the following:

Theorem 7.2 *Let (M,ξ) be a $(2n-1)$–dimensional subcritical Stein-fillable contact manifold with $n>2$, and V a subcritical Stein-filling of (M,ξ). Then*

$$HC_i(M,\xi) \cong \bigoplus_{m\in\mathbb{N}\cup\{0\}} H_{2(n+m-1)-i}(V).$$

When $n=2$ (M,ξ) is the union of S^3 (3–dimensional contact 0-handle) and a finite number of 3–dimensional contact 1–handles diffeomorphic to $\mathbb{R}\times S^2$. We write s for the number of contact 1–handles of M. Let γ_0 denote the principal Reeb orbit in the 0-handle, and γ_1, γ_2, ..., γ_s the principal Reeb orbits in each of the s' contact 1–handles. These 1–handles can be attached to S^3 pairwise disjoint. Recall that when $n=2$ $c_1(\xi)=0$ so the $\bar{\mu}$–index of contractible Reeb orbits are independent of the spanning discs and hence are well-defined.

Let $m_o \gg 1$ be a fixed positive integer, then by deforming 0- and 1–handles we can obtain a suitable regular contact 1–form α and assume the following:

Condition 7.1

$$\overline{\gamma_j^m} = \begin{cases} 2m & j=0, \\ 2m-1 & j=1,2,...,s \end{cases} \qquad \text{for } m\le m_o \tag{31}$$

$$\mathcal{A}(\gamma_1)=\mathcal{A}(\gamma_2)=\cdots=\mathcal{A}(\gamma_s)\ll\mathcal{A}(\gamma_0) \tag{32}$$

Let J be a regular α–admissible almost complex structure.

Proposition 7.1 *Assume $\mathcal{M} := \mathcal{M}_J(\gamma_-,\gamma_+)$ is not empty and $\overline{\gamma}_\pm \le 2m_o$.*

(1) *If $\dim\mathcal{M}=1$ then $\gamma_- = \gamma_j^m$, $\gamma_+ = \gamma_0^m$ for some $m\le m_o$, $1\le j\le s$.*

(2) *If $\dim\mathcal{M}=2$ then $\gamma_- = \gamma_o^{m-1}$, $\gamma_+ = \gamma_0^m$ for some $m\le m_o$.*

Note that $C_1(\alpha)$ is nontrivial, it is generated by γ_j, $1\le j\le s$. Nevertheless Proposition 7.1 and Condition 7.1 together imply that for $1<m\le m_o$, the boundary of $\mathcal{M}(\gamma_0^{m-1},\gamma_0^m)$ does not contain any element of $\mathcal{M}(\gamma_j^m,\gamma_0^m)$. Also when $m=1$ the boundary of $\mathcal{M}(\gamma_0)$, where $\mathcal{M}(\gamma_0)$ consists of holomorphic planes converging exponentially to γ_0 at $t=\infty$ at $z=\infty$, contains no holomorphic curves with more than one negative ends. Moreover, we have the following result:

Lemma 7.8 *The boundary operator $\partial\colon C_*(\alpha)\to C_{*-1}(\alpha)$ satisfies $\partial\circ\partial=0$, at least when $*\le 2m_o$.*

Proof Condition 7.1 implies that for $1 < m \leq m_o$,

$$\partial\gamma_j^m = 0 \quad \forall j = 1, 2, ..., s,$$

and hence

$$\partial^2\gamma_0^m = \partial(\sum_{j=1}^{s} a_j\gamma_j^m) = 0 \qquad \forall j = 1, 2, ..., s.$$

Now consider the $m = 1$ case. Observe that $\partial^2(\gamma_j) = 0$ for $j = 1, ..., s$ because $C_{-1}(\alpha) = 0$. Also $\partial^2(\gamma_0) = 0$ since $\partial\gamma_o = \sum_{j=1}^{s} b_j\gamma_j$ and $\partial\gamma_j = 0$ for $j = 1, ..., s$. Thus $\partial^2 = 0$ at least on $C_*(\alpha)$ with $* \leq 2m_o$. □

Therefore we can apply to M the stabilization technique as before and obtain the $n = 2$ version of Theorem 7.2.

Theorem 7.3 *Let (M, ξ) be a 3–dimensional subcritical Stein-fillable contact manifold, and (V, J) a subcritical Stein domain such that $\partial V = M$ and ξ is the maximal complex subbundle of TM. Then*

$$HC_i(M, \xi) \cong \bigoplus_{m\in\mathbb{N}\cup\{0\}} H_{2(n+m-1)-i}(V).$$

This completes the proof of the Main Theorem.

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