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A rational noncommutative invariant of boundary links

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Abstract

In 1999, Rozansky conjectured the existence of a rational presentation of the Kontsevich integral of a knot. Roughly speaking, this rational presentation of the Kontsevich integral would sum formal power series into rational functions with prescribed denominators. Rozansky's conjecture was soon proven by the second author. We begin our paper by reviewing Rozansky's conjecture and the main ideas that lead to its proof. The natural question of extending this conjecture to links leads to the class of boundary links, and a proof of Rozansky's conjecture in this case. A subtle issue is the fact that a 'hair' map which replaces beads by the exponential of hair is not 1-1. This raises the question of whether a rational invariant of boundary links exists in an appropriate space of trivalent graphs whose edges are decorated by rational functions in noncommuting variables. A main result of the paper is to construct such an invariant, using the so-called surgery view of boundary links and after developing a formal diagrammatic Gaussian integration.

Since our invariant is one of many rational forms of the Kontsevich integral, one may ask if our invariant is in some sense canonical. We prove that this is indeed the case, by axiomatically characterizing our invariant as a universal finite type invariant of boundary links with respect to the null move. Finally, we discuss relations between our rational invariant and homology surgery, and give some applications to low dimensional topology.

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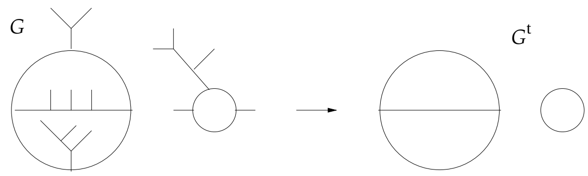
1 Introduction

1.1 Rozansky's conjecture on a rational presentation of the Kontsevich integral of a knot

The Kontsevich integral of a knot is a powerful invariant that can be interpreted to take values in a completed vector space of graphs. The graphs in question have univalent and trivalent vertices only (so-called *unitrivalent graphs*), and are considered modulo some well-known relations that include the AS and IHX relations.

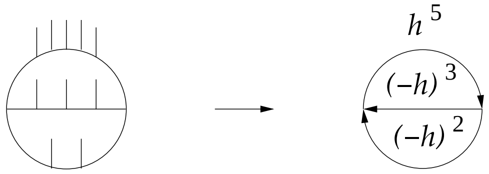
Every unitrivalent graph G is the union of a trivalent graph G^t together with a number of unitrivalent trees attached on the edges of G^t:



This is true provided that no component of G is a tree, and provided that we allow G^t to include circles in case G contains components with one loop (ie, with betti number 1).

The AS relation kills all trees with an internal trivalent vertex, which are possibly attached on an edge of a trivalent graph. Thus, the only trees that survive are the *hair*, that is the trees with one edge and two univalent vertices.

As a result, we need only consider trivalent graphs with a number of hair attached on their edges. (The exceptional case of a single hair unattached anywhere is excluded since we are silently assuming that the knot is zero-framed). The number of hair may be recorded by a monomial in a variable h attached on each edge of a trivalent graph, together with an orientation of the edge which keeps track of which side of the edge should the hair grow. For example, we have:



By linearity, we may decorate edges of trivalent graphs by polynomials, and even by formal power series in h.

In the summer of 1999 Rozansky made the bold conjecture that the Kontsevich integral of a knot can be interpreted to take values in a space of trivalent graphs with edges decorated by *rational functions in* e^h. Moreover, the denominators of these rational functions ought to be the Alexander polynomial of a knot.

Rozansky's conjecture did not come out of the blue. It was motivated by earlier work of his on the colored Jones function; see [42]. In that reference, Rozansky proved that the colored Jones function of a knot (a certain power series quotient of the Kontsevich integral) can be presented as a power series of rational functions whose denominators where powers of the Alexander polynomial.

1.2 Kricker's proof of Rozansky's Conjecture

Shortly after Rozansky's Conjecture appeared, the second author gave a proof of it in [33]. Since the proof contains several ideas that are key to the results of the present paper, we would like to summarize them here.

Fact 1 Untie the knot.

The key idea behind this is the fact that knots (or rather, knot projections) can be unknotted via a sequence of crossing changes, and that a crossing change can be achieved by surgery on a ± 1-framed unknot as follows:

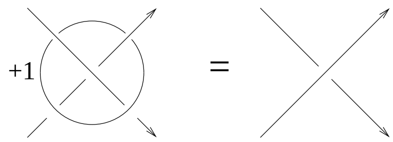
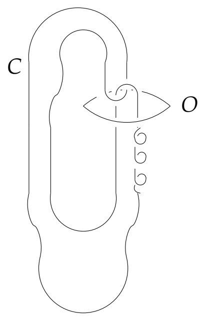


Figure 1: A crossing change can be achieved by surgery on a unit framed unknot.

Thus, every knot K in S^3 can be obtained by surgery on a framed link C in the complement $S^3 \smallsetminus \mathcal{O}$ of a standard unknot $\mathcal{O}$. We will call such a link C, an *untying link* for K. For example, an untying link for the Figure 8 knot is:



Observe that untying links are framed, and null homotopic in $S^3 \smallsetminus \mathcal{O}$ (in the sense that every component is contractible in $S^3 \smallsetminus \mathcal{O}$), the interior of a solid torus. Observe further that untying links exist for every knot K in an integral homology 3-sphere M.

Fact 2 Compute the Kontsevich integral of a knot from the Aarhus integral of an untying link.

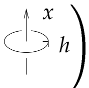
By this, we mean the following. Consider a knot K and an untying link C in $S^3 \smallsetminus \mathcal{O}$. Then, we may consider the (normalized) Kontsevich integral $\check{Z}(C \cup \mathcal{O})$ of the link $C \cup \mathcal{O}$, which can be interpreted to take values in a completed vector space of unitrivalent graphs with legs decorated by the components of $C \cup \mathcal{O}$. Then, the Kontsevich integral $Z(K)$ of K can be computed from $\check{Z}(C \cup \mathcal{O})$ by

$$Z(K) = \int dC\, (\check{Z}(C \cup \mathcal{O}))$$

Here $\int dC$ refers to a *diagrammatic formal Gaussian integration* which roughly speaking glues pairwise the C-colored legs of the graphs in $\check{Z}(C \cup \mathcal{O})$ using the negative inverse linking matrix of C; see [5].

Fact 3 Compute the Kontsevich integral of an untying link from the Kontsevich integral of a Long Hopf Link.

By this, we mean the following. The following formula for the Kontsevich integral of a *Long Hopf Link*, was conjectured in [4] (in conjunction with the so-called *Wheels* and *Wheeling* Conjectures) and proven in [7]:

$$Z \Bigg($$


$$\Bigg) =$$


$$e^h \sqcup \nu(h)$$

where $\nu(h)$ is the Kontsevich integral of the unknot, expressed in terms of graphs with legs colored by h. $\sqcup$ refers to the disjoint union multiplication of graphs. In this formula a *bead* e^h (that is, the exponential of hair), appears explicitly.

Using locality of the Kontsevich integral of a link, we may slice a planar projection of $C \cup \mathcal{O}$ into a tangle T by cutting $C \cup \mathcal{O}$ along the meridianal disk in the solid torus $S^3 \smallsetminus \mathcal{O}$ that $\mathcal{O}$ bounds. Then, we can compute $\check{Z}(C \cup \mathcal{O})$ from $\check{Z}(T)$ after we glue in the beads as instructed by the formula for the Long Hopf Link.

The result of this step is that we managed to write the Kontsevich integral of an untying link in terms of unitrivalent graphs with edges decorated by Laurent polynomials in e^h, and with univalent vertices colored by C.

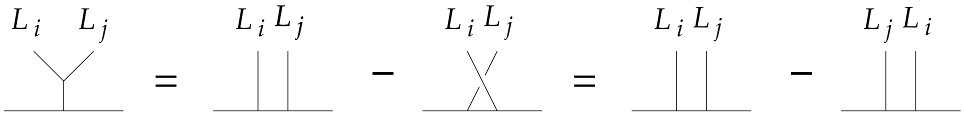
Fact 4 Commute the Aarhus integration with the above formula of the Kontsevich integral of an untying link.

Modulo some care with the 1-loop part of the Aarhus integral, this proves Rozansky's Conjecture for knots.

1.3 Rozansky's conjecture for boundary links

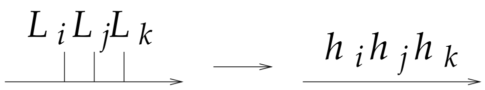
A question arises immediately after Kricker's proof. The Kontsevich integral is defined not just for knots, but for all links. Is there a more general class of links that this proof can be applied to?

In order to answer this, let us recall that the Kontsevich integral of a link takes values in a completed vector space of unitrivalent graphs, whose univalent vertices are labeled by the components of the link. The graphs are considered modulo some well-known relations, that include a 3-term relation (the IHX relation) depicted as follows:



Using this relation, it follows that every unitrivalent graph with no tree components is a sum of trivalent graphs with *hair* attached on their edges. Moreover, the hair is now labeled by the components of the link.

Thus, in order to formulate a conjecture for the Kontevich integral of links along the lines of Rozansky, we need to restrict attention to links whose Kontsevich integral has no tree part. For such links, the Kontsevich integral takes values in a completed vector space generated by trivalent graphs with hair. The hair are colored by the components of the link, and we can record this information by placing monomials in variables h_i (one variable per link component) on the edges of a trivalent graph as follows:



By linearity, we can place polynomials, and even formal power series, in the noncommuting variables h_i on the edges of trivalent graphs.

The question arises: are there any links whose Kontsevich integral has vanishing tree-part? Using Habegger-Masbaum (see [30]), this condition is equivalent to the *vanishing of all Milnor invariants* of a link. This class of links contains (and perhaps coincides with) the class of *sublinks of homology boundary links*, [36]. For simplicity, we will focus on the class of *boundary links*, namely those each component of which bounds a surface, such that the surfaces are pairwise disjoint.

Fact 5 Boundary links can be untied.

Indeed the idea is the following. Choose a Seifert surface as above for a boundary link L; that is, one surface per component, so that the surfaces are pairwise disjoint. Then, do crossing changes among the bands of the Seifert surface to unknot each band and unlink them. The result is a standard unlink, and a framed nullhomotopic link C, such that surgery on C transforms the unlink to the boundary link. We will call such a link C, an *untying link* for L.

Using this fact, and repeating Facts 2-4 for boundary links, proves Rozansky's conjecture for boundary links. In Fact 3, the Kontsevich integral of an untying link $C \cup \mathcal{O}$ takes values in a completed space of trivalent graphs whose edges are labeled by Laurent polynomials in noncommuting variables e^{h_i}, for $i = 1, \dots, g$ where g is the number of components of L. In Fact 4, care has to be taken to make sense of rational functions in noncommuting variables e^{h_i}.

1.4 Is there a rational form of the Kontsevich integral of boundary links?

Our success in proving a rational presentation for the Kontsevich integral of a boundary link suggests that

- we may try to define an invariant Z^{rat} of boundary links with values in a completed space $\mathcal{A}(\Lambda_{\mathrm{loc}})$ of trivalent graphs with beads, where the beads are rational functions in noncommuting variables t_i, for $i = 1, \dots, g$.
- Rational functions in noncommuting variables ought to form a ring Λ_{loc}, which should be some kind of localization of the *group ring* $\Lambda = \mathbb{Z}[F]$ of the free group F with generators $t_1, \dots, t_g$.
- There ought to be a Hair map from graphs with beads to unitrivalent graph with hair which replaces t_i by e^{h_i}, such that the Kontsevich integral Z is given by $\mathrm{Hair} \circ Z^{\mathrm{rat}}$.

Let us call the above statement a *strong form of Rozansky's Conjecture* for boundary links, and let us call any such invariant Z^{rat} *a rational form* of the Kontsevich integral.

Let us point out a subtlety (easy to miss) of this stronger conjecture, even in the case of knots with trivial Alexander polynomial (in which case we may work with graphs with beads in $\Lambda = \mathbb{Z}[t^{\pm}]$ only): although it is true that the map

$$\Lambda = \mathbb{Z}[t^{\pm}] \longrightarrow \hat{\Lambda} = \mathbb{Q}[h]$$

given by $t \mapsto e^h$ is 1-1, it does not follow in some obvious way that the Hair map from the space of graphs with beads to the space of unitrivalent graphs is 1-1. This may seem counterintuitive, however a recent paper of Patureau-Mirand (using Vogel's work on universal algebras and exotic weight systems that do not come from Lie algebras) proves that this Hair map is not 1-1; see [41].

If the Hair map were 1-1, then Rozansky's Conjecture for boundary links (which we proved in Section 1.3 would easily imply the existence and uniqueness of a rational form Z^{rat} of the Kontsevich integral. However, as we discussed above, this is not the case.

This raises two problems:

- Is there a rational form Z^{rat} of the Kontsevich integral of a boundary link?
- Assuming there is one, is there a canonical (in some sense) form?

The purpose of the paper is to solve both problems.

1.5 The main results of the paper

In the remainder of this introduction, let us discuss the main results of this paper, which we will explain in lengthy detail in the following sections.

Fact 6 A surgery view of boundary links.

We mentioned already in Fact 5 tat boundary links can be untied. Unfortunately, an untying link of a boundary link is not unique. In fact, Kirby moves on an untying link do not change the result of surgery on an untying link, and therefore give rise to the same boundary link. It turns out that Kirby moves preserve not only the boundary link but its F-structure as well.

By this we mean the following. The choice of Seifert surface is not part of a boundary link. A refinement of *boundary links* (abbreviated ∂-links) are F-*links*, ie, a triple (M, L, ϕ) of a link L in an integral homology 3-sphere M and an onto map $\phi : \pi_1(M - L) \twoheadrightarrow F$, where F is the free group on a set $T = \{t_1, \dots, t_g\}$ (in 1-1 correspondence with the components of L) such that the ith meridian is sent to the ith generator. Two F-links (M, L, ϕ) and (M', L', ϕ') are equivalent iff (M, L) and (M', L') are isotopic and the maps ϕ and ϕ' differ by an inner automorphism of F. The underlying link of an F-link is a ∂-link. Indeed, an F-link gives rise to a map $\tilde{\phi} : M \longrightarrow \vee^g S^1$ which induces the map ϕ of fundamental groups. Choose generic points p_i, one in each circle of $\vee^g S^1$. It follows by transversality that $\tilde{\phi}^{-1}(p_i)$ is a surface with boundary component the corresponding component of L. These surfaces are obviously pairwise disjoint, thus (M, L) is a boundary link.

There is an action

$$\mathrm{String}_g \times F\text{-links} \longrightarrow F\text{-links}$$

whose orbits can be identified with the set of boundary links. Here String_g is the *group of group of motions of an unlink in 3-space*, and can be identified with the automorphisms of the free group that map generators to conjugates of themselves:

$$\mathrm{String}_g \cong \{f \in \mathrm{Aut}(F) | f(t_i) = \alpha_i^{-1} t_i \alpha_i, \quad i = 1, \ldots, g\}.$$

The action of String_g on an F-link (M, L, ϕ) is given by composition with the map ϕ, as is explained in Section 6.1.

Let $\mathcal{N}(\mathcal{O})$ denote the set of nullhomotopic framed links C in the complement of a standard unlink $\mathcal{O}$ in S^3, such that the linking matrix of C is invertible over $\mathbb{Z}$.

Then, in [25] we prove that the surgery map induces a 1-1 correspondence

$$\mathcal{N}(\mathcal{O}) / \langle \text{Kirby moves}, \mathrm{String}_g \rangle \leftrightarrow \partial - \text{links}.$$

This is the so-called surgery view of boundary links.

Fact 7 Construct an invariant of $\mathcal{N}(\mathcal{O})$ in a space of graphs with beads.

By analogy with Fact 3, in Section 4 we define an invariant of links $C \in \mathcal{N}(\mathcal{O})$ as above. It takes values in a completed vector space of univalent graphs with beads. The beads are elements of $\Lambda = \mathbb{Z}[F]$, and record the winding of a tangle representative of C in $S^3 - \mathcal{O}$. The legs of the graphs are labeled by the components of C. The strut part of this invariant records the equivariant linking matrix of C.

Fact 8 Develop an equivariant version of the Aarhus integral.

Unfortunately, the above described invariant of links $C \in \mathcal{N}(\mathcal{O})$ is not invariant under Kirby moves of C. To accommodate for that, in Section 5 we construct an integration theory $\int^{\mathrm{rat}}$ in the spirit of the Aarhus Integral [5]. Let us mention that the Aarhus integration is a map that cares only about the univalent vertices of graphs, and not about their internal structure (such as beads on edges, or valency of internal vertices). By construction, Aarhus integration behaves like integration of functions in the sense that it behaves well with changes of variables.

In our integration theory $\int^{\mathrm{rat}}$, we separate the strut part of a diagram with beads, invert their martix (it is here that a suitable ring Λ_{loc} is needed) to construct new struts, and glue the legs of these new struts to the rest of the diagrams. The result is a formal linear combination of trivalent graphs whose beads are elements of Λ_{loc}. The main property of this integration theory, is that the resulting invariant is independent under Kirby moves and respects the String_g action. Thus it gives an invariant Z^{rat} of ∂-links which takes values in a completed vector space of trivalent graphs with edges decorated by Λ_{loc}, modulo certain natural relations; see Theorem 14.

Fact 9 The ring Λ_{loc} of rational functions in noncommuting variables.

Our integration theory $\int^{\mathrm{rat}}$ reveals the need for a ring Λ_{loc}. This ring should satisfy the property that all matrices W over Λ which are invertible over $\mathbb{Z}$ (that is, ϵW is invertible over $\mathbb{Z}$ where $\epsilon : \Lambda = \mathbb{Z}[F] \longrightarrow \mathbb{Z}$ is the map that sends t_i to 1) are in fact invertible over Λ_{loc}. This is precisely the defining property of the *noncommutative (Cohn) localization* of $\Lambda = \mathbb{Z}[F]$. Farber-Vogel identified Λ_{loc} with the ring of *rational functions in noncommuting variables*; see [14].

Fact 10 Z^{rat} is a rational form of the Kontsevich integral; see Theorem 14 in Section 7.4.

By analogy with Fact 2, we compare the Z^{rat} invariant of boundary links with their Kontsevich integral, via the Hair map. The comparison is achieved using the formula of the Kontsevich integral of the Long Hopf Link. Section 7.4 completes Fact 10.

So far out efforts give a rational form of the Kontsevich integral of a link. As we mentioned before, there is potentially more than one rational form of the Kontsevich integral. How do we know that our construction is in some sense a natural one?

To answer this question, let us recall that the Kontsevich integral is a *universal* finite type invariant of links, with respect to the Goussarov-Vassiliev crossing change moves. This axiomatically characterizes the Kontsevich integral, up to a universal constant which is independent of a link.

Fact 11 The $Z^{\rm rat}$ invariant is a universal finite type invariant of F-links with respect to the null-move; see Theorem 15 in Section 8.1.

This so called *null-move* is described in terms of surgery on a nullhomotopic clasper in the complement of the F-link, and generalizes the null-move on knots. The latter was introduced and studied extensively by [18].

Section 8 is devoted to the proof of Theorem 15, namely the universal property of $Z^{\rm rat}$. This follows from the general principle of locality of the invariant $Z^{\rm rat}$ (ie, with the behavior of $Z^{\rm rat}$ under sublinks) together with the identification of the covariance of $Z^{\rm rat}$ with an equivariant linking function.

1.6 Some applications of the $Z^{\rm rat}$ invariant

The rational form $Z^{\rm rat}$ of the Kontsevich integral of a boundary link sums an infinite series of Vassiliev invariants into rational functions. An application of the universality property of the $Z^{\rm rat}$ invariant is a *realization theorem* for these rational functions in Section 8; see Proposition 8.5. Our realization theorem is in the same spirit as the realization theorem for the values of the Alexander polynomial of a knot that was achieved decades earlier by Levine; see [35].

An application of this realization theorem for the 2-loop part of the $Z^{\rm rat}$ invariant settles a question of low-dimensional topology (namely, to separate minimal rank knots from Alexander polynomial 1 knots), and fixes an error in earlier work of M. Freedman; see [26]. This is perhaps the first application of finite type invariants in a purely 3-dimensional question.

Another application of the 2-loop part is a formula for the Casson-Walker invariant of cyclic branched coverings of a knot in terms of the signature of the knot and residues of the Q function, obtained in joint work [24].

1.7 Future directions

Rozansky has recently introduced a Rationality Conjecture for a class of *algebraically connected* links, that is links with nonvanishing Alexander polynomial, [45]. This class is disjoint from ours, since a boundary link with more than one component has vanishing Alexander polynomial. It is an interesting question to extend our results in this setting of Rozansky. We plan to address this issue in a later publication.

1.8 Acknowledgements

The results of this work were first announced by the second named author in a meeting on "Influence of Physics on Topology" on August 2000, in the sunny San Diego. We thank the organizers, M Freedman and P Teichner for arranging such a fertile meeting. We also wish to express our gratitude to J Levine who shared with the first author his expertise on topology, and to J Hillman and T Ohtsuki for encouragement and support of the second author. Finally, the authors wish to thank D Bar-Natan and the referee for numerous comments who improved the presentation of the paper.

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2 A surgery view of F-links

2.1 A surgery description of F-links

In this section we recall the surgery view of links, introduced in [25], which is a key ingredient in the construction of the rational invariant Z^{rat}. Let us recall some important notions. Fix once and for all a based unlink $\mathcal{O}$ of g components in S^3.

Definition 2.1 Let $\mathcal{N}(\mathcal{O})$ denote the set of *nullhomotopic links C with $\mathbb{Z}$-invertible linking matrix* in the complement of $\mathcal{O}$.

Surgery on an element C of $\mathcal{N}(\mathcal{O})$ transforms $(S^3, \mathcal{O}\,\phi)$ to a F-link (M, L, ϕ). Indeed, since C is nullhomotopic the natural map $\pi_1(S^3 \smallsetminus \mathcal{O}) \to F$ gives rise to a map $\pi_1(M \smallsetminus L) \to F$. Alternatively, one can construct disjoint Seifert surfaces for each component of L by tubing the disjoint disks that $\mathcal{O}$ bounds, which is possible, since each component of C is nullhomotopic.

Since the linking matrix of C is invertible over $\mathbb{Z}$, M is an integral homology 3-sphere. Let $\overset{\kappa}{\sim}$ denote the equivalence relation on $\mathcal{N}(\mathcal{O})$ generated by the moves of *handle-slide* κ_1 (ie, adding a parallel of a link component to another component) and *stabilization* κ_2 (ie, adding to a link an unknot away from the link with framing ± 1). It is well-known that κ-equivalence preserves surgery. A main result of [25] is the following:

Theorem 1 [25] *The surgery map gives a 1-1 and onto correspondence*

$$\mathcal{N}(\mathcal{O})/\langle\kappa\rangle \to F\text{-links.}$$

As an application of the above theorem, in [25] we constructed a map:

$$W : F\text{-links} \longrightarrow \mathcal{B}(\Lambda \to \mathbb{Z})$$

where $\mathcal{B}(\Lambda \to \mathbb{Z})$ is the set of simple stably congruence class of Hermitian matrices A, invertible over $\mathbb{Z}$. Here, a matrix A over Λ is *Hermitian* if $A^\star = A$ where $A^\star$ denotes the conjugate transpose of A with respect to the involution $\Lambda = \mathbb{Z}[F] \to \Lambda = \mathbb{Z}[F]$ which sends g to $\bar{g} := g^{-1}$. Moreover, two Hermitian matrices A, B are *simply stably congruent* iff $A \oplus S_1 = P(B \oplus S_2)P^\star$ for some diagonal matrices S_1, S_2 with ± 1 entries and some *elementary* matrix P (ie, one which differs from the identity matrix on a single non-diagonal entry) or a diagonal matrix with entries in $\pm F \subset \Lambda$. The map W sends an F-link to the equivariant linking matrix of $\widetilde{C}$, a lift of a surgery presentation C, to the free cover of $S^3 \smallsetminus \mathcal{O}$. It was shown in [25] that the map W determines the Blanchfield form of the F-link, as well as a noncommutative version of the Alexander polynomial defined by Farber [15].

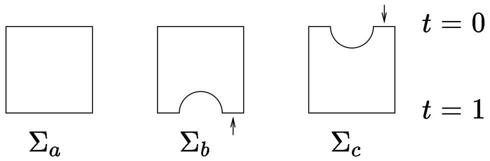
2.2 A tangle description of F-links

In this section we give a tangle diagram description of the set $\mathcal{N}(\mathcal{O})$. Before we proceed, a remark on notation:

Remark 2.2 Throughout the paper, given an equivalence relation ρ on a set X, we will denote by $X/\langle\rho\rangle$ the set of *equivalence classes*. Occasionally, an equivalence relation on X will be defined by one of the following ways:

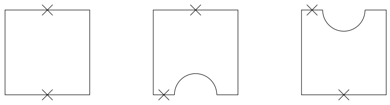
- Either by the *action* of a group G on X, in which case $X/\langle\rho\rangle$ coincides with the set of orbits of G on X.
- Or by a *move* on X, ie, by a subset of $X \times X$, in which case the equivalence relation is the smallest one that contains the move.

Consider the following surfaces in $\mathbb{R}^2$

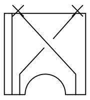


together with a distinguished part of their boundary (called a *gluing site*) marked by an arrow in the figure above. By a *tangle diagram on a surface* Σ_x (for $x = a, b, c$) we mean an oriented, framed, smooth, proper immersions of an oriented 1-manifold into the surface, up to isotopies rel boundary of the surface, double points being equipped with crossing information, with the boundary points of the tangle lying on standard points at $t = 0$ and $t = 1$.

A *crossed tangle diagram on a surface* Σ_x is a tangle diagram on Σ_x possibly with some crosses (placed away from the gluing sites) that mark those boundary points of the diagram. Here are the possible places for a cross on the surfaces Σ_x for $x = a, b$ or c:

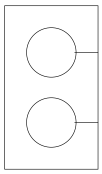


The following is an example of a crossed tangle diagram on Σ_b:



The cross notation evokes a small pair of scissors  that will be sites where the skeleton of the tangle will be cut.

Let D_g denote a standard disk with g holes ordered from top to bottom and g gluing sites, shown as horizontal segments below. For example, when $g = 2$, D_g is:



Definition 2.3 A *sliced crossed link* in D_g is

(a) a nullhomotopic link L in $D_g \times I$ with $\mathbb{Z}$-invertible linking matrix such that

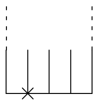
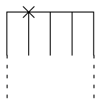
(b) L is in general position with respect to the gluing sites of D_g times I in $D_g \times I$.

(c) Every component of L is equipped with a point (depicted as a cross $\times$) away from the gluing sites times I.

Let $\mathcal{L}^\times(D_g)$ denote the set of isotopy classes of sliced crossed links.

Definition 2.4 A *sliced crossed diagram* in D_g is a sequence of crossed tangles, $\{T_1, \dots, T_k\}$ such that

(a) the boundaries of the surfaces match up, that is, their shape, distribution of endpoints, and crosses.

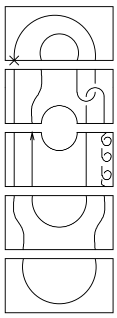
For example, if T_i is  then T_{i+1} is 

(b) the top and bottom boundaries of T_1 and T_k look like:

(c) After stacking the tangles $\{T_1, \dots, T_k\}$ from top to bottom, we obtain a crossed link in $D_g \times I$, where D_g is a disk with g holes.

Let $\mathcal{D}^\times(D_g)$ denote the set of sliced crossed diagrams on D_g such that each component of the associated link is nullhomotopic in $D_g \times I$ and marked with precisely one cross.

Here is an example of a sliced crossed diagram in D_1 whose corresponding link in $\mathcal{N}(\mathcal{O})$ is a surgery presentation for the Figure 8 knot:

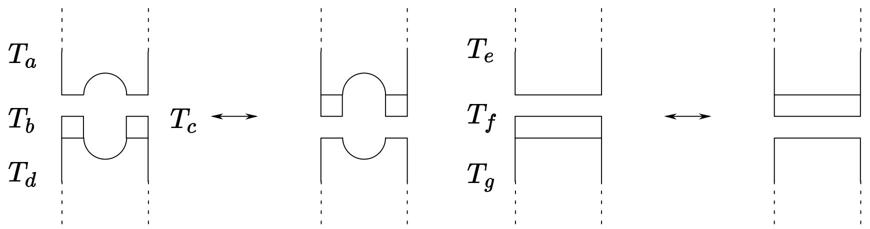


Clearly, there is a map

$$\mathcal{D}^\times(D_g) \longrightarrow \mathcal{L}^\times(D_g).$$

We now introduce some equivalence relations on $\mathcal{D}^\times(D_g)$ that are important in comparing $\mathcal{D}^\times(D_g)$ to $\mathcal{N}(\mathcal{O})$.

Definition 2.5 *Regular isotopy* $\overset{r}{\sim}$ of sliced crossed diagrams is the equivalence relation generated by regular isotopy of individual tangles and the following moves:



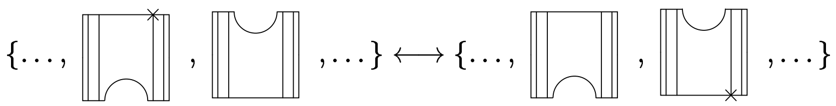
where the glued edge of T_e must have no crosses marking it.

Proposition 2.6 *There is a 1-1 correspondence:*

$$\mathcal{D}^{\times}(D_g)/\langle r\rangle \longleftrightarrow \mathcal{L}^{\times}(D_g).$$

Proof This follows from standard transversality arguments; see for example [1] and [48, Figure 7]. □

Definition 2.7 *Basing relation* $\overset{\beta}{\sim}$ of sliced crossed diagrams is the equivalence relation generated by moving a cross of each component of the associated link in some other admissible position in the sliced crossed tangle. Moving the cross of a sliced crossed link can be obtained (in an equivalent way) by the local moves β_1 and β_2, where β_1 (resp. β_2) moves the cross across consecutive tangles in such a way that a gluing site is not (resp. is) crossed. Here is an example of a β_2 move:



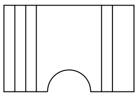
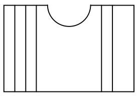
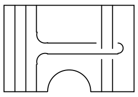
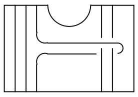
A *link* L in D_g is one that satisfies condition (a) of Definition 2.3 only. Let $\mathcal{N}(D_g)$ denote the set of isotopy classes of sliced links in D_g. Proposition 2.6 implies that

Proposition 2.8 *There is a 1-1 correspondence:*

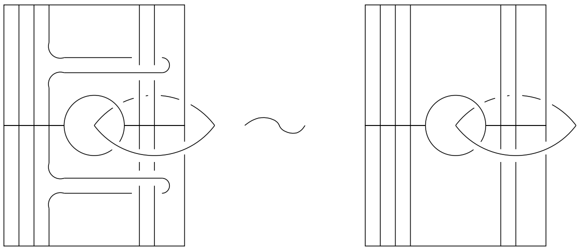
$$\mathcal{D}^{\times}(D_g)/\langle r,\beta\rangle \longleftrightarrow \mathcal{N}(D_g).$$

Observe that $D_g \times I$ contained in the complement of an unlink $S^3 \smallsetminus \mathcal{O}$. Moreover, up to homotopy we have that $S^3 \smallsetminus \mathcal{O}$ is obtained from $D_g \times I$ by attaching $g-1$ 2-spheres. The next equivalence relation on sliced crossed diagrams is precisely the move of sliding over these 2-spheres.

Definition 2.9 The *wrapping relation* $\overset{\omega}{\sim}$ of sliced crossed diagrams is generated by the following move

$\{\ldots,$  ,  $,\ldots\} \longleftrightarrow \{\ldots,$  ,  $,\ldots\}$

It is easy to see that a wrapping relation of sliced crossed diagrams implies that the corresponding links in $S^3 \setminus \mathcal{O}$ are isotopic. Indeed,



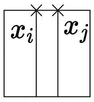
The above discussion implies that:

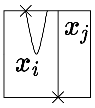
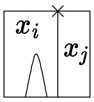
Proposition 2.10 *There is a 1-1 correspondence:*

$$\mathcal{D}^{\times}(D_g)/\langle r, \beta, \omega\rangle \longleftrightarrow \mathcal{N}(\mathcal{O}).$$

Recall that Theorem 1 identifies the set of F-links with a quotient of $\mathcal{N}(\mathcal{O})$. Since we are interested to give a description of the set of F-links in terms of a quotient of $\mathcal{D}^{\times}(D_g)$, we need to introduce analogs of the κ-relation on $\mathcal{D}^{\times}(D_g)$.

Definition 2.11 The *handle-slide move* $\overset{\kappa}{\sim}$ is generated by the move:

$\{\ldots, T_{i-1}, T_i,$  $, T_{i+2}, \ldots\}$

$\longleftrightarrow \quad \{\ldots, T'_{i-1}, T'_i,$  ,  $, T'_{i+2}, \ldots\}.$

where each tangle in $\{T'_k\}$ is obtained from the corresponding tangle in $\{T_k\}$ by taking a framed parallel of every component contributing to the component x_j,.

We end this section with our final proposition:

Proposition 2.12 *There is a 1-1 correspondence*

$$\mathcal{D}^{\times}(D_g)/\langle r, \beta, \omega, \kappa\rangle \longleftrightarrow F\text{-links}.$$

Proof F-links are defined modulo inner automorphisms of the free group. One need only observe that inner automorphisms follow from the basing relation. □

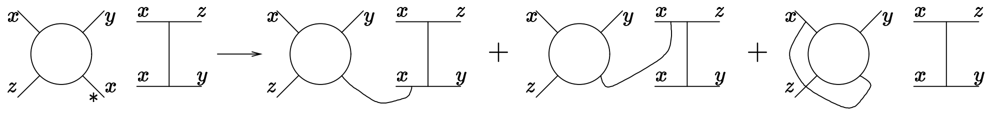
3 An assortment of diagrams and their relations

3.1 Diagrams for the Kontsevich integral of a link

In this section we explain what are the spaces of diagrams in which our invariants will take values. Let us begin by recalling well-known spaces of diagrams that the Kontsevich integral of a knot takes values.

The words "diagram" and "graph" will be used throughout the paper in a synonymous way. All graphs will have unitrivalent vertices. The univalent vertices are attached to the *skeleton* of the graphs, which will consist of a disjoint union of oriented segments (indicated by $\uparrow_X$), circles (indicated by $\circlearrowleft_X$) or the empty set (indicated by $\star_X$). The skeleton is labeled by a set X in 1-1 correspondence with the components of a link. The *Vassiliev degree* of a diagram is half the number of vertices.

Definition 3.1 Let $\mathcal{A}(\circlearrowleft_X, \uparrow_Y, \star_T, \circledast_S)$ denote the quotient of the completed graded $\mathbb{Q}$-vector space spanned by diagrams with the prescribed skeleton, modulo the AS, IHX relations, the STU relation on X and Y and the S-colored *infinitesimal basing relation* (the latter was called S-colored link relation in [5, Part II, Sec. 5.2]) shown by example for $S = \{x\}$:



where the right hand side, by definition of the relation, equals to zero.

When X, Y, T or S are the empty set, then they will be ommitted from the discussion. In particular, $\mathcal{A}(\phi)$ denotes the completed vector space spanned by trivalent diagrams, modulo the AS and IHX relations.

By its definition, the Kontsevich integral $Z(L)$ of a link L takes values in $\mathcal{A}(\circlearrowleft_H)$, where H is a set in 1-1 correspondence with the components of L.

There are several useful vector space isomorphisms of the spaces of diagrams which we now recall.

There is a *symmetrization map*

$$\chi_X : \mathcal{A}(\star_X) \longrightarrow \mathcal{A}(\uparrow_X) \tag{1}$$

which is the average of all ways of placing the symmetric legs of a diagram on a line, whose inverse is denoted by

$$\sigma_X : \mathcal{A}(\uparrow_X) \longrightarrow \mathcal{A}(\star_X). \tag{2}$$

Moreover, χ_X induces an isomorphism:

$$\mathcal{A}(\circledast_X) \longrightarrow \mathcal{A}(\circlearrowleft_X) \tag{3}$$

Thus, the Kontsevich integral of a link may be interpreted to take values the quotient of $\mathcal{A}(\star_H)$ modulo the H-colored infinitesimal basing relations.

Notice that $\mathcal{A}(\uparrow_X)$ is an algebra (with respect to the stacking of diagrams that connects their skeleton), and that $\mathcal{A}(\star_X)$ is an algebra with respect to the disjoint union multiplication. However, the map χ_X is not an algebra map.

3.2 Group-like elements and relations

In this short section we review the notion of *group-like elements* in the spaces $\mathcal{A}(\circlearrowleft_X)$, $\mathcal{A}(\uparrow_X)$, $\mathcal{A}(\star_X)$ and $\mathcal{A}(\circledast_X)$. All these spaces have natural Hopf algebra structures, which are cocommutative and completed.

Definition 3.2 An element g in a completed Hopf algebra is *group-like* if $g = \exp(x)$ for a *primitive* element x.

In case of the Hopf algebras of interest to us, the primitive elements are the span of the *connected graphs.*

We we denote the group like elements of

$$\begin{cases} \mathcal{A}(\circlearrowleft_X) \\ \mathcal{A}(\uparrow_X) \\ \mathcal{A}(\star_X) \\ \mathcal{A}(\circledast_X) \end{cases} \quad \text{by} \quad \begin{cases} \mathcal{A}^{\mathrm{gp}}(\circlearrowleft_X) \\ \mathcal{A}^{\mathrm{gp}}(\uparrow_X) \\ \mathcal{A}^{\mathrm{gp}}(\star_X) \\ \mathcal{A}^{\mathrm{gp}}(\circledast_X) \end{cases}$$

The isomorphisms of Equations (1), (2) and (3) induce isomorphisms of the corresponding sets of group-like elements.

Note that the Kontsevich integral of a link (or more generally, a tangle) takes values in the set of group-like elements.

Let us now present a group-like version of the infinitesimal basing relation. Since this does not appear in the literature, and since it will be the prototype for basing relations of group-like elements which include beads, we will discuss it more extensively.

In [5, part II, Sec.5.2], the following map

$$\vec{m}_x^{y,z} : \mathcal{A}^{\mathrm{gp}}(\uparrow_{\{y,z\}} \cup \uparrow_X, \Lambda_{\mathrm{loc}}) \to \mathcal{A}^{\mathrm{gp}}(\uparrow_{\{x\}} \cup \uparrow_X, \Lambda_{\mathrm{loc}}) \tag{4}$$

was introduced, that glues the end of the y-skeleton to the beginning of the z-skeleton and then relabeling the skeleton by x, for $x, y, z \notin X$.

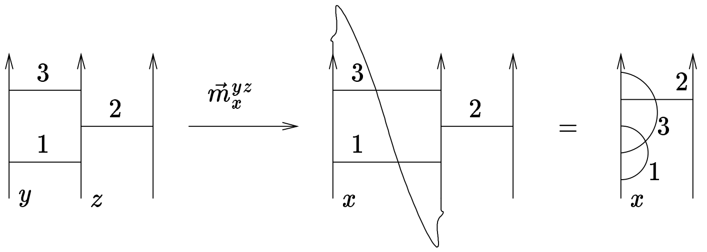


Figure 2: The map $\vec{m}_x^{yz}$ in degree 2: Connect the strands labeled x and y in a diagram in $\mathcal{A}(\uparrow_x\uparrow_y\uparrow)$, to form a new "long" strand labeled z, without touching all extra strands.

Definition 3.3 Given $s_1, s_2 \in \mathcal{A}^{\mathrm{gp}}(\uparrow_X)$, we say that

- $s_1 \overset{\beta^{\mathrm{gp}}}{\sim} s_2$, if there exists an element $s \in \mathcal{A}^{\mathrm{gp}}(\uparrow_{\{y\}}\uparrow_{\{z\}}\uparrow_{X-\{x\}})$ with the property that

$$\vec{m}_x^{y,z}(s) = s_1 \quad \text{and } \vec{m}_x^{z,y}(s) = s_2. \tag{5}$$

Using the isomorphism σ_X, we can define $\overset{\beta^{\mathrm{gp}}}{\sim}$ on $\mathcal{A}^{\mathrm{gp}}(\star_X)$. We define

$$\mathcal{A}^{\mathrm{gp}}(\circledast_X) = \mathcal{A}^{\mathrm{gp}}(\star_X)/\langle\beta^{\mathrm{gp}}\rangle. \tag{6}$$

The group-like basing relation can be formulated in terms of pushing an exponential e^h of hair to group-like diagrams. Since this was not noticed in the work of [5], and since in our paper, pushing exponential of hair is a useful operation, we will give the following reformulation of the group-like basing relation.

Definition 3.4 Let

$$\mathrm{con} : \mathcal{A}(\star_{X\cup\partial X\cup Y}) \to \mathcal{A}(\star_Y)$$

denote the *contraction map* defined as follows. For $s \in \mathcal{A}(\star_{X\cup\partial X\cup Y})$, $\mathrm{con}_X(s)$ denotes the sum of all ways of pairing *all* legs labeled from ∂X with *all* legs labeled from X. The contraction map preserves group-like elements.

Definition 3.5 For $s_1, s_2 \in \mathcal{A}^{\mathrm{gp}}(\star_X)$, we say that

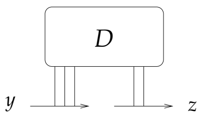
- $s_1 \overset{\beta^{\mathrm{gp}\prime}}{\sim} s_2$ iff there exists $s \in \mathcal{A}^{\mathrm{gp}}(\star_{X\cup\{\partial h\}})$ such that

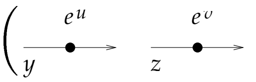
$$s_1 = \mathrm{con}_{\{h\}}(s) \quad \text{ and } \quad s_2 = \mathrm{con}_{\{h\}}(s|_{X\to Xe^h})$$

where $s|_{X\to Xe^h}$ is by definition the result of pushing e^h to each X-labeled leg of s_{12}.

Lemma 3.6 *The equivalence relations $\beta^{\mathrm{gp}\prime}$ and β^{gp} are equal on $\mathcal{A}^{\mathrm{gp}}(\star_X)$.*

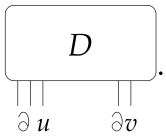
Proof Consider $s_1, s_2 \in \mathcal{A}^{\mathrm{gp}}(\star_X)$ so that $\chi_X(s_1) \overset{\beta^{\mathrm{gp}}}{\sim} \chi_X(s_2)$ and consider the corresponding element $s \in \mathcal{A}^{\mathrm{gp}}(\uparrow_{\{y\}}\uparrow_{\{z\}}\uparrow_{X-\{x\}})$ satisfying Equation (5). We start by presenting s as a result of a contraction. Inserting $\chi_{\{y,z\}}$ and $\sigma_{\{z,y\}}$ to s, we have that

$s =$ 

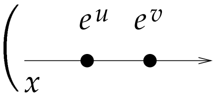
$= \mathrm{con}_{\{u,v\}}\Big($ 

$\alpha\Big)$

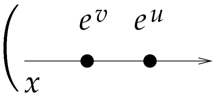
where

$\alpha = (\sigma_{\{y,z\}}(s))|_{y\to\partial u, z\to\partial v} =$ 

Morevoer, we have:

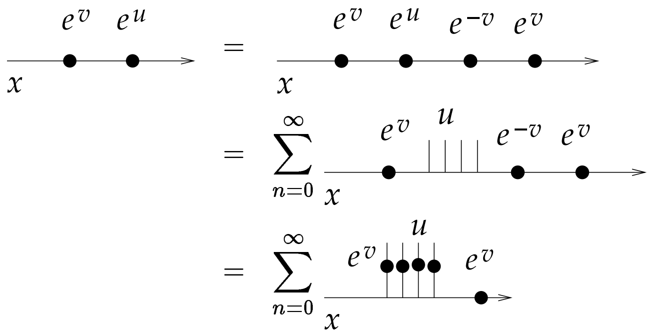
$\chi_X(s_1) = \mathrm{con}_{\{u,v\}}\Big($ 

$\alpha\Big)$

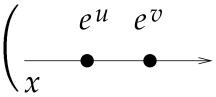
$\chi_X(s_2) = \mathrm{con}_{\{u,v\}}\Big($ 

$\alpha\Big)$

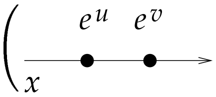
Now, we bring $e^v e^u$ to $e^u e^v$, at the cost of pushing e^v hair on the u-colored legs. Indeed, we have:



Remembering that we need to contract $(u, \partial u)$ legs and $(v, \partial v)$ legs, we can push the e^v-hair on α. Let β denote the result of pushing e^v hair on each ∂u-colored leg of α. Then, we have:

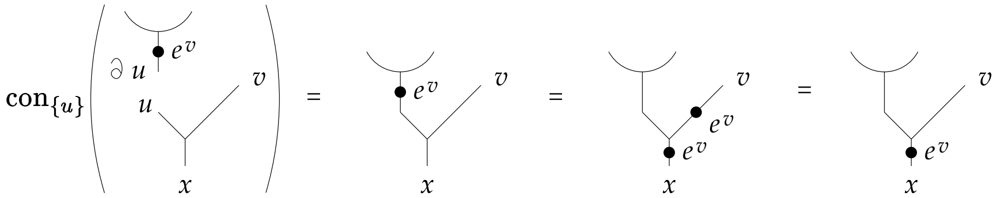
$\chi_X(s_1) = \mathrm{con}_{\{u,v\}}\Big($ 

$\alpha\Big)$

$\chi_X(s_2) = \mathrm{con}_{\{u,v\}}\Big($ 

$\beta\Big)$

Now, we may get s_2 from $\chi_X(s_2)$ by attaching (u,v)-rooted labeled forests $T(u,v)$ whose root is colored by x; see [5, part II, prop.5.4]. Since



it follows that

$$\begin{aligned} s_2 &= \mathrm{con}_{\{u,v\}}(\alpha\ T(u,v)|_{x\to xe^v}) \\ s_1 &= \mathrm{con}_{\{u,v\}}(\alpha\ T(u,v)) \end{aligned}$$

Letting $\gamma = \mathrm{con}_{\{u\}}(\alpha\ T(u,v))$, it follows that

$$\begin{aligned} s_2 &= \mathrm{con}_{\{v\}}(\gamma|_{X\to Xe^v}) \\ s_1 &= \mathrm{con}_{\{v\}}(\gamma) \end{aligned}$$

In other words, $s_1 \overset{\beta^{\mathrm{gp}'}}{\sim} s_2$.

The converse follows by reversing the steps in the above proof. □

3.3 Diagrams for the rational form of the Kontsevich integral

Let us now introduce diagrams with beads which will be useful in our paper. The notation will generalize the notion of the $\mathcal{A}$-groups introduced in [19].

Definition 3.7 Consider a ring R with involution and a distinguished group of units U. An admissible labeling of a diagram D with prescribed skeleton is a labeling of the edges of D and the edges of its skeleton so that:

- the labelings on the $\circlearrowleft_X$ and $\uparrow_Y$ lie in U, and satisfy the condition that the product of the labelings of the edges along each component of the skeleton is 1.
- the labelings on the rest of the edges of D lie in R.

Labels on edges or part of the skeleton will be called *beads*.

Definition 3.8 Consider a ring R with involution and a distinguished group of units U, and (possibly empty sets) X, Y, T. Then,

$$\mathcal{A}(\circlearrowleft_X, \uparrow_Y, \star_T, R, U) = \frac{\mathcal{D}(\circlearrowleft_X, \uparrow_Y \star_T, R, U)}{(\mathrm{AS}, \mathrm{IHX}, \mathrm{STU}, \mathrm{Multilinear}, \mathrm{Vertex\ Invariance})}$$

where:

- $\mathcal{D}(\circlearrowleft_X, \uparrow_Y \star_T, R, U)$ is the completed graded vector space over $\mathbb{Q}$ of diagrams of prescribed skeleton with oriented edges, and with admissible labeling.
- The *degree* of a diagram is the number of trivalent vertices.
- AS, IHX and Multilinear are the relations shown in Figure 3 and Vertex Invariance is the relation shown in Figure 4. Note that all relations are homogeneous, thus the quotient is a completed graded vector space.

Some remarks on the notation. Empty sets will be omitted from the notation, and so will U, the selected group of units of R. For example, $\mathcal{A}(\star_Y, R)$, $\mathcal{A}(R)$ and $\mathcal{A}(\phi)$ stands for $\mathcal{A}(\uparrow_\phi, \star_Y, \circledast_\phi, R, U)$, $\mathcal{A}(\uparrow_\phi, \star_\phi, \circledast_\phi, R, U)$ and $\mathcal{A}(\uparrow_\phi, \star_\phi, \circledast_\phi, \mathbb{Z}, 1)$ respectively. Univalent vertices of diagrams will often be called *legs*. Special diagrams, called *struts*, labeled by a, c with bead b are drawn as follows

$$\overset{a}{\underset{c}{\updownarrow}}\, b.$$

oriented from bottom to top. Multiplication of diagrams D_1 and D_2, unless otherwise mentioned, means their disjoint union and will be denoted by $D_1 D_2$ and occasionally by $D_1 \sqcup D_2$.

We will be interested only in the rings

- $\Lambda = \mathbb{Z}[F]$ (the group ring of the free group F on some fixed set $T = \{t_1, \dots, t_g\}$ of generators),
- its completion $\hat{\Lambda}$ (with respect to the powers of the augmentation ideal)
- and its *Cohn localization* Λ_{loc} (ie, the localization with respect to the set of matrices over Λ that are invertible over $\mathbb{Z}$).

For all three rings $\Lambda, \hat{\Lambda}$ and Λ_{loc} the selected group of units is F. All three are rings with involution induced by $\bar{g} = g^{-1}$ for $g \in F$, with augmentation over $\mathbb{Q}$, and with a commutative diagram

$$\begin{tikzcd} \Lambda \arrow[rr] \arrow[dr] & & \Lambda_{\mathrm{loc}} \arrow[dl] \\ & \hat{\Lambda} & \end{tikzcd}$$

where $\Lambda \to \hat{\Lambda}$ is given by the exponential version of the Magnus expansion $t_i \to e^{h_i}$. $\hat{\Lambda}$ can be indentified with the ring of noncommuting variables $\{h_1, \dots, h_g\}$ and Λ_{loc} with the ring of rational functions in noncommuting

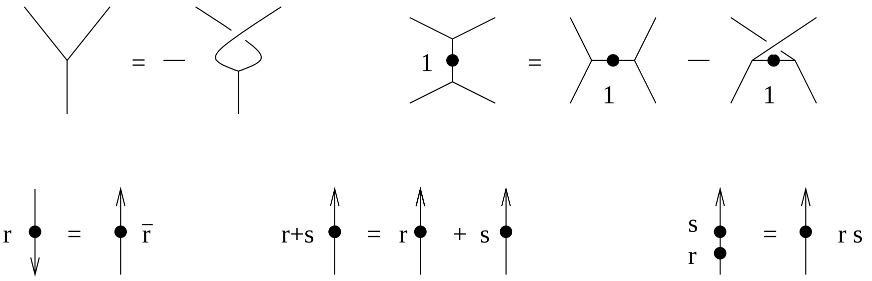


Figure 3: The AS, IHX (for arbitrary orientations of the edges), Orientation Reversal and Linearity relations. Here $r \to \bar{r}$ is the involution of R, $r, s \in R$.

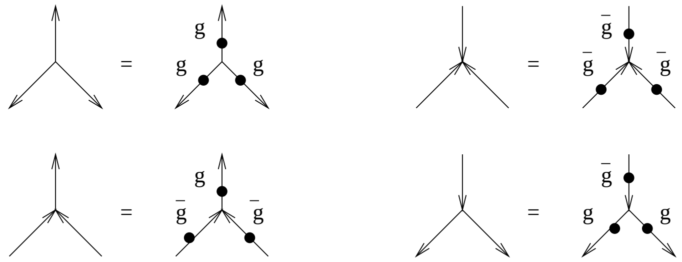


Figure 4: The Vertex Invariance relation that pushes a unit $g \in U$ past a trivalent vertex.

variables $\{t_1, \dots, t_g\}$, [14, 9]. In all cases, the distinguished group of units is F.

The following is a companion to Remark 2.2:

Remark 3.9 Throughout the paper, given a subspace W of a vector space V, we will denote by V/W the quotient of V modulo W. W will be defined by one of the following ways:

- By a subset S of V, in which case $W = (S)$ is the subspace spanned by S.
- By a *linear action* of a group G on V, in which case $S = \{gv - v\}$ defines W and V/W coincides with the space of *coinvariants* of G on V.

Since labels of the edges of the skeleton are in U, and the product of the labels around each skeleton component is 1, the the Vertex Invariance Relation implies the following analogue of Equations (1), (2):

Lemma 3.10 *For every X and $R = \Lambda, \Lambda_{\mathrm{loc}}$ or $\hat{\Lambda}$, there are inverse maps*

$$\chi_X : \mathcal{A}(\star_X, R) \longrightarrow \mathcal{A}(\uparrow_X, R) \quad \text{and} \quad \sigma_X : \mathcal{A}(\uparrow_X, R) \longrightarrow \mathcal{A}(\star_X, R).$$

Our next task is to introduce analogs for the basing and the wrapping relations for diagrams with beads. Before we do that, let us discuss in detail the beads that we will be considering.

3.4 Noncommutative localization

In this self-contained section we review several facts about the Cohn localization that are used in the text. Consider the free group F_g on generators $\{t_1, \dots, t_g\}$. The Cohn localization Λ_{loc} of its group-ring $\Lambda = \mathbb{Z}[F_g]$ is characterized by a universal property,

$$\begin{tikzcd} \Lambda \arrow[r] \arrow[dr, "\alpha"'] & \Lambda_{\mathrm{loc}} \arrow[d, dashed, "\bar{\alpha}"] \\ & R \end{tikzcd}$$

namely that for every Σ-*inverting* ring homomorphism $\alpha : \Lambda \to R$ there exists a unique ring homomorphism $\bar{\alpha} : \Lambda_{\mathrm{loc}} \to R$ that makes the above diagram commute, [12]. Recall that a homomorphism α is Σ-inverting if αM is invertible over R for every matrix M over Λ which is invertible over $\mathbb{Z}$.

Farber and Vogel identified Λ_{loc} with the ring of *rational functions in noncommuting variables*, [14, 9]. An example of such a rational function is

$$(3 - t_1^2 t_2^{-1}(t_3 + 1)t_1^{-1})^{-1} t_2 - 5.$$

There is a close relation between rational functions in noncommuting variables and *finite state automata*, sedcribed in [9, 14]. Farber-Vogel showed that every element $s \in \Lambda_{\mathrm{loc}}$ can be represented as a solution to a system of equations $M\vec{x} = \vec{b}$ where M is a matrix over Λ, invertible over $\mathbb{Z}$, and $\vec{b}$ is a column vector over Λ, [14, Proposition 2.1]. More precisely, we have that

$$s = (1, 0, \dots, 0) M^{-1} \vec{b},$$

which we will call a *matrix presentation* of $s \in \Lambda_{\mathrm{loc}}$.

In the text, we often use substitutions of the form $\varphi_{t_i \to t_i e^h}$, or $\varphi_{t_i \to e^h t_i}$ or $\varphi_{t_i \to e^{-h} t_i e^h}$, where h does not commute with F_g. In order to make sense of these substitutions, we need to enlarge our ring Λ_{loc}. This can be achieved in the following way. F_g can be included in the free group F with one additional generator t. Let $\mathcal{L}$ denote the group-ring of F, $\mathcal{L}_{\mathrm{loc}}$ denote its Cohn localization and $\widehat{\mathcal{L}_{\mathrm{loc}}}$ denote the completion of $\mathcal{L}_{\mathrm{loc}}$ with respect to the the ideal generated by $t-1$. There is an identification of $\widehat{\mathcal{L}_{\mathrm{loc}}}$ with the following ring $\mathcal{R}$ whose elements consist of formal sums of the form $\sum_{\mathsf{m},\mathsf{f}} \prod_{i=1}^{\infty} h^{m_i} f_i$ where $f_i \in \Lambda_{\mathrm{loc}}$ and

- $\mathsf{m} = (m_1, m_2, \dots) : \mathbb{N} \to \mathbb{N}$ is eventually 0 and $\mathsf{f} = (f_1, f_2, \dots) : \mathbb{N} \to \Lambda_{\mathrm{loc}}$ is eventually 1.
- In the above sum, for each fixed k, there are finitely many sequences m with $|\mathsf{m}| = \sum_i m_i \leq k$.

It is easy to see that $\mathcal{R}$ is a ring with involution (in fact a subring of the completion of $\mathcal{L}$ with respect to the augmentation ideal) such that t_i and e^h are units, where $h = \log t$. In particular, $\widehat{\mathcal{L}_{\mathrm{loc}}}$ is an h-graded ring. Let $\deg_h^n : \widehat{\mathcal{L}_{\mathrm{loc}}} \to \widehat{\mathcal{L}_{\mathrm{loc}}}^n$ denote the projection on the h-degree n part of $\widehat{\mathcal{L}_{\mathrm{loc}}}$.

Definition 3.11 There is a map $\varphi_i : \Lambda \to \widehat{\mathcal{L}_{\mathrm{loc}}}$ given by substituting t_i for $e^{-h} t_i e^h$ for $1 \leq i \leq g$. It extends to a map $\varphi_i : \Lambda_{\mathrm{loc}} \to \widehat{\mathcal{L}_{\mathrm{loc}}}$.

Indeed, using the defining property of the Cohn localization, it suffices to show that φ_i is Σ-*inverting*, [12, 14]. In other words, we need to show that if a matrix W over Λ is invertible over $\mathbb{Z}$, then $\varphi_i(W)$ is invertible over $\widehat{\mathcal{L}_{\mathrm{loc}}}$. Since $\widehat{\mathcal{L}_{\mathrm{loc}}}$ is a completion of $\mathcal{L}_{\mathrm{loc}}$, it suffices to show that $\deg_h^0(\varphi_i(W))$ is invertible over $\mathcal{L}_{\mathrm{loc}}$. However, $\deg_h^0(\varphi_i(W)) = W$, invertible over Λ_{loc} and thus also over $\mathcal{L}_{\mathrm{loc}}$. The result follows.

Definition 3.12 Let

$$\eta_i : \Lambda_{\mathrm{loc}} \to \widehat{\mathcal{L}_{\mathrm{loc}}}^1$$

denote the h-degree 1 part of φ_i.

The following lemma gives an axiomatic definition of η_i by properties analogous to a *derivation*. Compare also with the derivations of Fox differential calculus, [14].

Lemma 3.13 (a) *η_i is characterized by the following properties:*

$$\begin{aligned} \eta_i(t_j) &= \delta_{ij}[t_i, h] = \delta_{ij}(t_i h - h t_i) \\ \eta_i(ab) &= \eta_i(a) b + a \eta_i(b) \\ \eta_i(a+b) &= \eta_i(a) + \eta_i(b). \end{aligned}$$

(b) *Moreover, η_i can be extended to matrices with entries in Λ_{loc}. In that case, η_i satisfies*

$$\eta_i(AB) = \eta_i(A)B + A\eta_i(B)$$

for matrices A and B that can be multiplied, as well as

$$\eta_i(A^{-1}) = -A^{-1}\eta_i(A)A^{-1}$$

for invertible matrices A.

Proof For (a), it is easy to see that η_i satisfies the above properties, by looking at the h-degree 0 and 1 part in $\varphi_i(a)$. This determines η_i uniquely on Λ. By the universal property, we know that η_i extends over Λ_{loc}; a priori the extension need not be unique. Since however, η_i extends uniquely to matrices over Λ, and since every element of Λ_{loc} has a matrix presentation (see the discussion in the beginning of the section), it follows that η_i is characterized by the derivation properties.

Part (b) follows easily from part (a). □

3.5 The infinitesimal wrapping relations

We now have all the required preliminaries to introduce an infinitesimal wrapping relation on

$$\mathcal{A}^0(\star_X, \Lambda_{\mathrm{loc}}) := \mathcal{B}(\Lambda_{\mathrm{loc}} \to \mathbb{Z}) \times \mathcal{A}(\star_X, \Lambda_{\mathrm{loc}}). \tag{7}$$

Given a vector $v \in \mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$ and a diagram $D \in \mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$ with a *single* special leg colored by ∂h and $M \in \mathrm{Herm}(\Lambda_{\mathrm{loc}} \to \mathbb{Z})$, the *infinitesimal wrapping relation* ω_i at the ith place in $\mathrm{Herm}(\Lambda_{\mathrm{loc}} \to \mathbb{Z}) \times \mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$ is generated by the move

$$(M, v) \overset{\omega_i}{\sim} \left(M, v + \mathrm{con}_{\{h\}}\left(\eta_i(D) - \frac{1}{2} D \sqcup \mathrm{tr}(M^{-1}\eta_i(M))\right)\right).$$

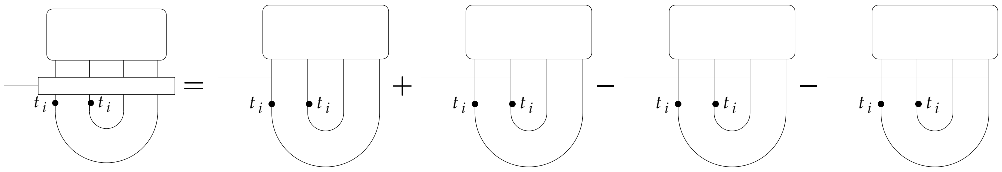
It is easy to see that the above definition depends only on the image of M in $\mathcal{B}(\Lambda_{\mathrm{loc}} \to \mathbb{Z})$. We define

$$\mathcal{A}^0(\star_X, \Lambda_{\mathrm{loc}})/\langle\omega\rangle = \mathcal{A}^0(\star_X, \Lambda_{\mathrm{loc}})/\langle\omega_1, \ldots, \omega_g\rangle.$$

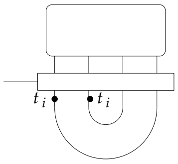
We give two examples of infinitesimal wrapping relations:

Example 3.14 An important special case of the infinitesimal wrapping relation is when M is the empty matrix and the beads of a diagram D lie in

$\Lambda = \mathbb{Z}[F]$. Without loss of generality, we may assume that the beads of D are monomials t_j. If *all* the edges labeled by t_i are shown, and we define



then we have:



$= 0.$

Note that in the ω relation, we allow for D to be a diagram of a circle (with a bead and with a special ∂h leg). In addition, we allow the ∂h leg to have a bead in Λ_{loc}.

The next example illustrates the infinitesimal wrapping relation in its full complication.

Example 3.15 Consider the matrix

$$M = \begin{bmatrix} 1 & t_3 - t_2 t_1^{-1} \\ t_3^{-1} - t_1 t_2^{-1} & 1 \end{bmatrix}$$

which is Hermitian over Λ, and invertible over $\mathbb{Z}$. Part (b) of Lemma 3.13 implies that

$$\begin{aligned} \eta_1(M) &= \begin{bmatrix} 0 & -t_2(t_1^{-1}h - ht_1^{-1}) \\ -(t_1 h - ht_1)t_2^{-1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ ht_1 t_2^{-1} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -t_2 t_1^{-1} h \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & t_2 h t_1^{-1} \\ -t_1 h t_2^{-1} & 0 \end{bmatrix} \end{aligned}$$

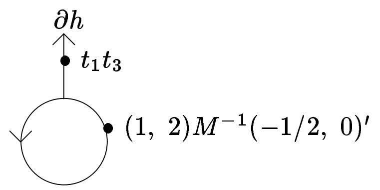
and

$$\begin{aligned} \eta_1(M^{-1}) &= M^{-1}(t_2 t_1^{-1}, 0)' h (0,1) M^{-1} + M^{-1}(-t_2, 0)' h (0, t_1^{-1}) M^{-1} \\ &+ M^{-1}(0, t_1)' h (t_2^{-1}, 0) M^{-1} + M^{-1}(0,-1)' h (t_1 t_2^{-1}, 0) M^{-1}. \end{aligned}$$

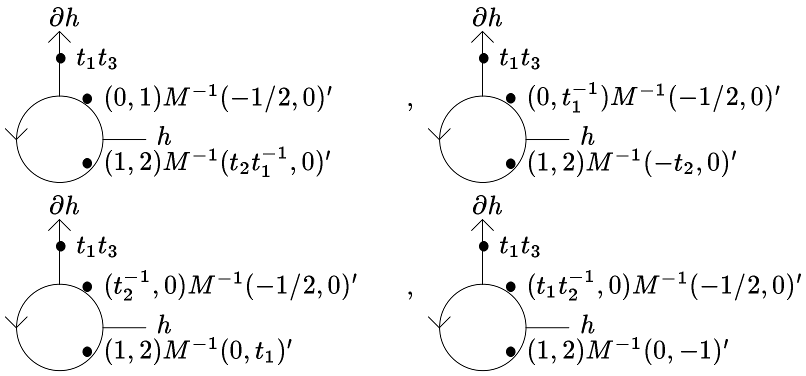
where A' denotes the transpose of A. Further, observe that

$$\begin{aligned} -1/2\,\mathrm{tr}(M^{-1}\eta_1(M)) &= (1/2, 0) M^{-1}(0, t_1)' h t_2^{-1} + (1/2, 0) M^{-1}(0,-1)' h t_1 t_2^{-1} \\ &+ (0, 1/2) M^{-1}(t_2 t_1^{-1}, 0)' h + (0, 1/2) M^{-1}(-t_2, 0)' h t_1^{-1}. \end{aligned}$$

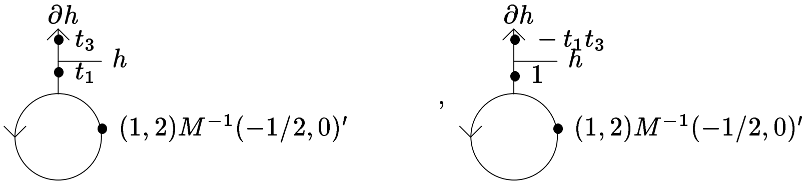
Now, consider the infinitesimal wrapping relation arising from the diagram

$D =$ 

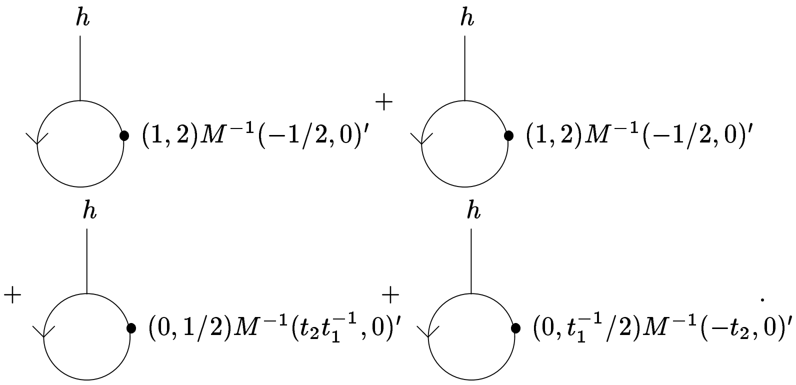
That is, we have $(M, v) \overset{\omega_1}{\sim} (M, v + x)$, where $x = x_1 + x_2$, and x_1 is given by gluing the $(h, \partial h)$ legs in each of the following diagrams,



as well as these diagrams



and summing the result up, and x_2 is given by gluing the $(h, \partial h)$ legs of the disjoint union of D with the following sum:



Notice that x is a sum of diagrams of degree 2 of the form Θ and OO. It is an interesting problem to understand the infinitesimal wrapping relation in degree 2.

3.6 Group-like relations

Our invariants (to be constructed) will take place in quotients of group-like elements, modulo appropriate relations. In this section we discuss these relations.

In analogy with Section 3.2, the subscript $^{\mathrm{gp}}$ indicates the set of group-like elements of the appropriate Hopf algebra.

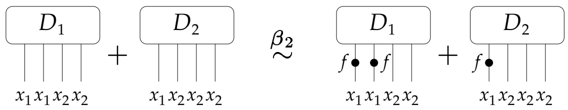
We now define an analog of the basing relations for group-like elements. Recall the map $\vec{m}_x^{yz}$ of Equation (4).

Definition 3.16 Given $s_1, s_2 \in \mathcal{A}^{\mathrm{gp}}(\star_X, \Lambda_{\mathrm{loc}})$, we say that

- $s_1 \overset{\beta_1^{\mathrm{gp}}}{\sim} s_2$, if there exists an element $s \in \mathcal{A}^{\mathrm{gp}}(\uparrow_{\{y\}}\uparrow_{\{z\}}\uparrow_{X-\{x\}}, \Lambda_{\mathrm{loc}})$ with the property that

$$\sigma_{\{x\}}(\vec{m}_x^{y,z}(s)) = s_1 \quad \text{and} \quad \sigma_{\{x\}}(\vec{m}_x^{z,y}(s)) = s_2. \tag{8}$$

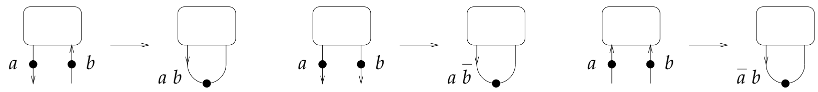
- $s_1 \overset{\beta_2^{\mathrm{gp}}}{\sim} s_2$ iff s_2 is obtained from s_1 by pushing an element of F on each of the X-labeled legs of s_1. For example, if all x_1 legs are shown below, we have:



We define

$$\mathcal{A}^{\mathrm{gp}}(\circledast_X, \Lambda_{\mathrm{loc}}) = \mathcal{A}^{\mathrm{gp}}(\star_X, \Lambda_{\mathrm{loc}})/\langle \beta_1^{\mathrm{gp}}, \beta_2^{\mathrm{gp}} \rangle. \tag{9}$$

Remark 3.17 When we push an element of the free group f in the legs of a diagram, depending on the orientation of the leg, we add f or $\bar{f}$ to the leg. See also Figure 4 for a convention for all choices of orientations. Similarly, when we glue two legs of diagrams together, the label on the bead depends on the orientations of the legs. For example:



We now define a useful reformulation of the β_1^{gp} relation for diagrams in $\mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$, which makes obvious that:

- The X-flavored group-like basing relations push a bead, or an exponential of hair on all X-colored legs.

Definition 3.18 For $s_1, s_2 \in \mathcal{A}^{\mathrm{gp}}(\star_X, \Lambda_{\mathrm{loc}})$, we say that

- $s_1 \overset{\beta_1^{\mathrm{gp}'}}{\sim} s_2$ iff there exists $s \in \mathcal{A}^{\mathrm{gp}}(\star_{X\cup\{\partial h\}}, \Lambda_{\mathrm{loc}})$ such that

$$s_1 = \mathrm{con}_{\{h\}}(s) \quad \text{and} \quad s_2 = \mathrm{con}_{\{h\}}(s|_{X\to Xe^h})$$

where $s|_{X\to Xe^h}$ is by definition the result of pushing e^h to each X-labeled leg of s.

Lemma 3.19 *The equivalence relations $\beta_1^{\mathrm{gp}'}$ and β_1^{gp} are equal on $\mathcal{A}^{\mathrm{gp}}(\star_X, \Lambda_{\mathrm{loc}})$.*

Proof It is identical to the proof of Lemma 3.6 and is omitted. □

Now, we define a group-like wrapping relation on group-like elements. First, we will enlarge the set of group-like elements by a quotient of a set of Hermitian matrices. This enlargement might sound artificial, however there are several good reasons for doing so. For example, the null move on the set of F-links and its associated filtration predicts that in degree 0, the universal invariant is given by the S-equivalence class (of F-links), or equivalently, by the set of Blanchfield pairings of F-links. $\mathcal{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}})$, defined below, maps onto $\mathcal{B}(\Lambda_{\mathrm{loc}} \to \mathbb{Z})$ which in turn maps onto the set of Blanchfield pairings of F-links.

In addition, the $\mathcal{A}$-groups, studied in relation to Homology Surgery in [19], are secondary obstructions to the vanishing of a Witt-type obstruction which lies in $\mathcal{B}(\mathbb{Z}[\pi] \to \mathbb{Z})$. In the case of F-links, this motivates the fact that the set $\mathcal{B}(\Lambda_{\mathrm{loc}} \to \mathbb{Z})$ should be taken together with $\mathcal{A}(\Lambda_{\mathrm{loc}})$. With these motivations,

Definition 3.20 Let us define

$$\mathcal{A}^{\mathrm{gp},0}(\star_X, \Lambda_{\mathrm{loc}}) = \mathcal{B}(\Lambda_{\mathrm{loc}} \to \mathbb{Z}) \times \mathcal{A}^{\mathrm{gp}}(\star_X, \Lambda_{\mathrm{loc}}).$$

Definition 3.21 Two pairs $(M, s_m) \in \mathrm{Herm}(\Lambda_{\mathrm{loc}} \to \mathbb{Z}) \times \mathcal{A}^{\mathrm{gp}}(\star_X, \Lambda_{\mathrm{loc}})$ $m = 1, 2$ are related by a *wrapping move* at the jth gluing site (for some j such that $1 \le j \le g$) if there exists an $s \in \mathcal{A}^{\mathrm{gp}}(\star_{X\cup\{\partial h\}}, \Lambda_{\mathrm{loc}})$ such that:

$$s_m \;=\; \mathrm{con}_{\{h\}}\left(\varphi_m(s) \sqcup \left(\exp_{\sqcup}\left(-\frac{1}{2}\chi_h(M^{-1}\varphi_m M)\right)\right)\right)$$

for $m = 1, 2$ where φ_1 is the identity and φ_2 is the substitution $\varphi_{t_j \to e^{-h} t_j e^h}$, and

$$\chi_h(A) = \circlearrowleft\!\bullet\ \mathrm{tr}\log(A) \in \mathcal{A}(\star_h, \Lambda_{\mathrm{loc}}). \tag{10}$$

The wrapping move generates an equivalence relation (the so-called *group-like wrapping relation*) on $\mathcal{A}^{\mathrm{gp},0}(\star_X, \Lambda_{\mathrm{loc}})$.

Our first task is to make sure that the above definition makes sense. Using the h-graded ring with involution $\widehat{\mathcal{L}_{\mathrm{loc}}}$ from Section 3.4, it follows that $\varphi_m(s_{12}) \in \mathcal{A}^{\mathrm{gp}}(\star_X, \widehat{\mathcal{L}_{\mathrm{loc}}})$. Furthermore, $\chi_h(M^{-1}\varphi_m M)$ can be thought of either as a circle with a bead in $\widehat{\mathcal{L}_{\mathrm{loc}}}/(\mathrm{cyclic})$ that has vanishing h-degree 0 part, or as a sum of wheels with h-colored legs and beads in Λ_{loc}. Thus, the contraction $\mathrm{con}_{\{h\}}$ in the above definition results in an element of $\mathcal{A}^{\mathrm{gp}}(\star_X, \Lambda_{\mathrm{loc}})$. It is easy to see that χ_h satisfies the following properties (compare with [25, Proposition 2.3]

Proposition 3.22 [25] (a) *For Hermitian matrices A, B over Λ_{loc}, nonsingular over $\mathbb{Z}$, we have in $\hat{\Lambda}/(\mathrm{cyclic})$ that*

$$\chi_h(AB) = \chi_h(A) + \chi_h(B) \quad \textit{and} \quad \chi_h(A \oplus B) = \chi_h(A) + \chi_h(B).$$

(b) *χ_h descends to a $\hat{\Lambda}/(\mathrm{cyclic})$-valued invariant of the set $\mathcal{B}(\Lambda_{\mathrm{loc}} \to \mathbb{Z})$.*
(c) *For A as above, $\chi_h(A) = \chi_h(A\,\epsilon A^{-1})$ where $\epsilon : \Lambda \to \mathbb{Z}$.*

From this, it follows that χ_h depends only on the image of M in $\mathcal{B}(\Lambda_{\mathrm{loc}} \to \mathbb{Z})$ and the above definition makes sense in $\mathcal{A}^{\mathrm{gp},0}(\star_X, \Lambda_{\mathrm{loc}})$.

Remark 3.23 In the above definition 2.9 of the wrapping relation, we could have chosen equivalently that φ_1 and φ_2 denote the substitutions $\varphi_{t_j \to t_j e^h}$ and $\varphi_{t_j \to e^h t_j}$.

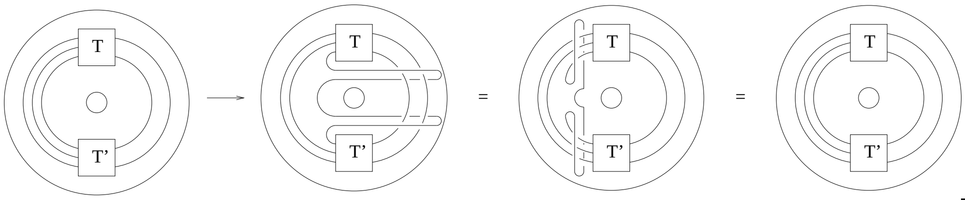
Remark 3.24 In the above definition, the ∂h colored legs are allowed to have beads in Λ_{loc}. Also, we allow the case of the empty matrix M, with the understanding that the χ_h term is absent.

Remark 3.25 An important special case of the group-like wrapping relation is when M is the empty matrix, and the beads lie in Λ. Without loss of generality, we may assume that the beads are monomials in t_i. In that case, if $s_i \in \mathcal{A}^{\mathrm{gp}}(\star_X, \Lambda)$ for $i = 1, 2$, and e^h appears before (resp. after) the beads t_i, then s_1 is wrapping move equivalent to s_2. For example, if all occurrences of t_j are shown below, then we have:

$$\begin{array}{cccc} \uparrow e^h & \downarrow e^{-h} & \downarrow e^{-h} & \uparrow e^h \\ \uparrow t_j & \downarrow t_j^{-1} & \downarrow t_j^{-1} & \uparrow t_j \end{array} \overset{\omega_j^{\mathrm{gp}}}{=} \begin{array}{cccc} \uparrow t_j & \downarrow t_j^{-1} & \downarrow t_j^{-1} & \uparrow t_j \\ \uparrow e^h & \downarrow e^{-h} & \downarrow e^{-h} & \uparrow e^h \end{array}.$$

Lemma 3.26 *For $g = 1$, the wrapping relation ω^{gp} on $\mathcal{A}^{\mathrm{gp},0}(\star_X, \Lambda_{\mathrm{loc}})$ is trivial.*

Proof It follows from a swapping argument analogous to the isotopy of the following figure



□

We end this section with a comparison between $\mathcal{A}^0$ and $\mathcal{A}^{\mathrm{gp},0}$. There is an obvious inclusion

$$\mathcal{A}^{\mathrm{gp},0}(\star_X, \Lambda_{\mathrm{loc}}) \longrightarrow \mathcal{A}^0(\star_X, \Lambda_{\mathrm{loc}}).$$

The next proposition states that

Proposition 3.27 *The inclusion* $\mathcal{A}^{\mathrm{gp},0}(\star_X, \Lambda_{\mathrm{loc}}) \longrightarrow \mathcal{A}^0(\star_X, \Lambda_{\mathrm{loc}})$ *maps the* ω^{gp}*-relations to* ω*-relations, and thus induces a map*

$$\mathcal{A}^{\mathrm{gp},0}(\star_X, \Lambda_{\mathrm{loc}})/\langle\omega^{\mathrm{gp}}\rangle \longrightarrow \mathcal{A}^0(\star_X, \Lambda_{\mathrm{loc}})/\langle\omega\rangle.$$

Proof See Appendix C. □

4 An invariant of links in the complement of an unlink

The goal of this section is to define an invariant

$$\check{Z}^{\mathrm{rat}} : \mathcal{N}_X(\mathcal{O}) \longrightarrow \mathcal{A}^{\mathrm{gp}}(\circledast_X, \Lambda)$$

as stated in Fact 7 of Section 1.5.

Given a finite set X, an X*-indexed sliced crossed link on* D_g is a sliced crossed link on D_g equipped with bijection of the set of components of the presented link with X. All the definitions above have obvious adaptations for X-indexed objects, indicated by an appropriate X subscript when required.

Theorem 2 *There exists an invariant*

$$\check{Z}^{\mathrm{rat}} : \mathcal{D}_X^{\times}(D_g)/\langle r\rangle \longrightarrow \mathcal{A}^{\mathrm{gp}}(\uparrow_X, \Lambda)$$

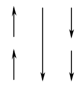
that maps the topological basing relations to group-like basing relations and the topological wrapping relations to wrapping relations. Combined with Proposition 2.10, and symmetrizing using Lemma 3.10, it induces a map

$$\sigma_X \circ \check{Z}^{\mathrm{rat}} : \mathcal{N}_X(\mathcal{O}) \longrightarrow \mathcal{A}^{\mathrm{gp}}(\circledast_X, \Lambda).$$

Proof Our first task is to define $\check{Z}^{\mathrm{rat}}$. In order to do so, we need to introduce some more notation. Given a word w in the letters $\uparrow$ and $\downarrow$, let

- $\uparrow_w$ denote the corresponding skeleton.
- $I_w \in \mathcal{A}(\uparrow_w, \Lambda)$ denote the identity element.
- $G_{i,w}$ denote the gluing word at the ith gluing site, that is, I_w with a label of t_i on each component, in the sense described by the following example: if $w = \uparrow\downarrow\downarrow$, then $G_{2,\uparrow\downarrow\downarrow} = \uparrow t_2 \downarrow t_2^{-1} \downarrow t_2^{-1}$.

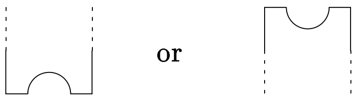
Given a word w in the symbols $\uparrow, \downarrow, \not\uparrow$ and $\not\downarrow$, let w' denote w with the crosses forgotten. Let $I_w^{\times}$ be $I_{w'}$ with every crossed strand broken. For example,

$$I^{\times}_{\not\uparrow\downarrow\not\downarrow} =$$ 

Definition 4.1 Take an X-indexed element $D = \{T_1, \ldots, T_k\} \in \mathcal{D}^{\times}_X(D_g)$.

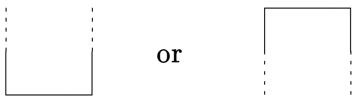
(1) Select q-tangle lifts of each tangle in the sequence, $\{T'_1, \ldots, T'_k\}$, such that

  (a) the strands of each q-tangle which lie on boundary lines of the following form

  are bracketed $(w_1)(w_2)$, where w_1 (resp. w_2) corresponds to the strands going to the left (resp. right) of the removed disc, and these bracketings match up,

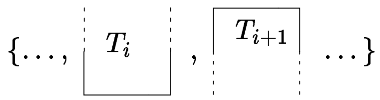
  (b) boundary words at boundaries of the form

  are given the canonical left bracketing.

(2) Compose the (usual) Kontsevich invariants of these q-tangles in the following way:

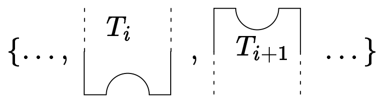
  (a) if two consecutive tangles are as follows,

with a word w in the symbols $\uparrow, \downarrow, \not\uparrow, \not\downarrow$ describing their matching boundary word, then they are to be composed

$$\cdots \circ Z(T_i') \circ I_w^{\times} \circ Z(T_{i+1}') \circ \ldots$$

(b) if, on the other hand, two consecutive tangles meet as follows,



with their matching boundary word at the jth gluing site factored as $(w_1)(w_2)$, then they are to be composed

$$\cdots \circ Z(T_i') \circ (I_{w_1} \otimes G_{j,w_2}) \circ Z(T_{i+1}') \circ \ldots \ .$$

This completes the definition of $Z^{\mathrm{rat}}(D)$. A normalized version $\check{Z}^{\mathrm{rat}}(D)$ (useful for invariance under Kirby moves) is defined by connect-summing a copy of ν into each component of the crossed link associated to D. Observe that the result does not depend on the choice of a place to connect-sum to.

The proof of Theorem 2 follows from Lemmas 4.3 and 4.4 below. $\square$

The next lemma, observed by the second author in [33], follows from a "sweeping" argument.

Lemma 4.2 *Let s denote an element of $\mathcal{A}(\uparrow_X,\mathbb{Q}) \subset \mathcal{A}(\uparrow_X,\Lambda)$, (ie, some diagram with labels 1 at each edge), where the top boundary word of $\uparrow_X$ is w_1, and the bottom boundary word of $\uparrow_X$ is w_2. Then*

$$G_{w_1} \circ s = s \circ G_{w_2}.$$

Lemma 4.3 *The map*

$$\check{Z}^{\mathrm{rat}} : \mathcal{D}_X^{\times}(D_g) \longrightarrow \mathcal{A}^{\mathrm{gp}}(\uparrow_X,\Lambda) \tag{11}$$

is well defined and preserves the isotopy relation $\overset{r}{\sim}$ of Definition 2.5.

Proof For the first part we only need to show that the function does not depend on the choice of bracketings. This follows because of the following, where ψ_1, ψ_2 are some expressions built from the associator.

$$\begin{aligned}
&\cdots \circ Z(T_i') \circ (\psi_1 \otimes \psi_2) \circ (I_{w_1} \otimes G_{j,w_2}) \circ (\psi_1^{-1} \otimes \psi_2^{-1}) \circ Z(T_{i+1}') \circ \ldots, \\
= \ &\cdots \circ Z(T_i') \circ (I_{w_1} \otimes G_{j,w_2}) \circ (\psi_1 \otimes \psi_2) \circ (\psi_1^{-1} \otimes \psi_2^{-1}) \circ Z(T_{i+1}') \circ \ldots, \\
= \ &\cdots \circ Z(T_i') \circ (I_{w_1} \otimes G_{j,w_2}) \circ Z(T_{i+1}') \circ \ldots .
\end{aligned}$$

where the first step follows from Lemma 4.2. The invariance of $\check{Z}^{\mathrm{rat}}$ under the $\overset{r}{\sim}$ equivalence relation follows the same way. $\square$

Lemma 4.4 *The map $\sigma_X \circ \check{Z}^{\mathrm{rat}}$ respects basing relations.*

Proof It suffices to show that $s_X \circ Z^{\mathrm{rat}}$ respects basing relations. In other words, it suffices to show that $D_1 \overset{\beta_i}{\sim} D_2$ implies that

$$\sigma_X(Z^{\mathrm{rat}}(D_1)) \overset{\beta_1^{\mathrm{gp}}}{\sim} \sigma_X(Z^{\mathrm{rat}}(D_2)),$$

for $i = 1, 2$.

In the first case, consider two diagrams $D_1, D_2 \in \mathcal{D}_X^{\times}(D_g)$ such that $D_1 \overset{\beta_1}{\sim} D_2$, and let D_{12} denote the diagram in $\mathcal{D}^{\times}(D_g)'$ such that D_1 is obtained by forgetting some cross, and D_2 is obtained by forgetting the other cross on that component.

Enlarge $\mathcal{D}_X^{\times}(D_g)$ to $\mathcal{D}_X^{\times}(D_g)'$, the set of X-indexed sliced crossed diagrams on D_g whose presented links in $D_g \times I$ have *at least* one cross on every component, thus $\mathcal{D}^{\times}(D_g) \subset \mathcal{D}^{\times}(D_g)'$. The function Z^{rat} immediately extends to $\mathcal{D}^{\times}(D_g)'$. The key point is the fact that the product of labels around each component of the skeleton equals to 1; this is true since the locations of the two crosses cut the component into two arcs, both of which have trivial intersection word with the gluing sites.

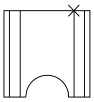
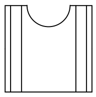
In other words, we have: $Z^{\mathrm{rat}}(D_{12}) \in \mathcal{A}^{\mathrm{gp}}(\uparrow_{\{x\}}\uparrow_{\{x'\}}\uparrow_{X-\{x\}}, \Lambda)$. It follows by construction that

$$\begin{aligned} Z^{\mathrm{rat}}(D_1) &= \vec{m}_x^{x,x'}(Z^{\mathrm{rat}}(D_{12})), \\ Z^{\mathrm{rat}}(D_2) &= \vec{m}_x^{x',x}(Z^{\mathrm{rat}}(D_{12})), \\ Z^{\mathrm{rat}}(D_{12}) &\in \mathcal{A}^{\mathrm{gp}}(\uparrow_{\{x\}}\uparrow_{\{x'\}}\uparrow_{X-\{x\}}, \Lambda_{\mathrm{loc}}). \end{aligned}$$

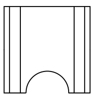
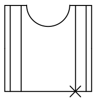
Thus, $\sigma_X(Z^{\mathrm{rat}}(D_1)) \overset{\beta_1^{\mathrm{gp}}}{\sim} \sigma_X(Z^{\mathrm{rat}}(D_2))$.

In the second case, consider $D_1 \overset{\beta_2}{\sim} D_2$. Then, for some appropriate expression

ψ built from the associator, we have that:

$\sigma_X(Z^{\mathrm{rat}}(\{\dots,$  ,  $\dots\}))$

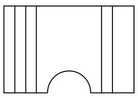
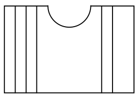
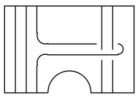
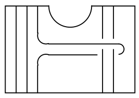
$$\begin{aligned}
&= \sigma_X(\cdots\circ I^{\times}_{w}\circ\psi\circ(I_{w_1}\otimes G_{i,w_2})\circ\psi^{-1}\circ\dots)\\
&\overset{\beta_1^{\mathrm{gp}}}{\sim} \sigma_X(\cdots\circ\psi\circ I^{\times}_{w}\circ(I_{w_1}\otimes G_{i,w_2})\circ\psi^{-1}\circ\dots)\\
&\overset{\beta_2^{\mathrm{gp}}}{\sim} \sigma_X(\cdots\circ\psi\circ(I_{w_1}\otimes G_{i,w_2})\circ I^{\times}_{w}\circ\psi^{-1}\circ\dots)\\
&\overset{\beta_1^{\mathrm{gp}}}{\sim} \sigma_X(\cdots\circ\psi\circ(I_{w_1}\otimes G_{i,w_2})\circ\psi^{-1}\circ I^{\times}_{w}\circ\dots)
\end{aligned}$$

$= \sigma_X(Z^{\mathrm{rat}}(\{\dots,$  ,  $,\dots\}))$.

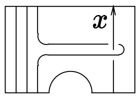
Thus, $\sigma_X(Z^{\mathrm{rat}}(D_1))\overset{\beta_1^{\mathrm{gp}}}{\sim}\sigma_X(Z^{\mathrm{rat}}(D_2))$. □

Lemma 4.5 *The map $\sigma_X\circ\check{Z}^{\mathrm{rat}}$ preserves the wrapping relations.*

Proof Consider two sliced crossed links $D_1, D_2\in\mathcal{D}^{\times}(D_g)$ related by a topological wrapping move at the jth gluing site as in Section 2.2. In other words, D_1 (shown on the left) and D_2 (shown on the right) are given by:

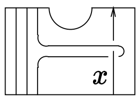
$\{\dots,$  ,  $,\dots\}\longleftrightarrow\{\dots,$  ,  $,\dots\}$

Let

$z_1 = \check{Z}^{\mathrm{rat}}\Big($  $\Big)$,

$z_1' =$  $e^h\cup(\sigma_x(z_1))_{x\to\partial h}$,

x'

$z_2 = \check{Z}^{\mathrm{rat}}\Big($  $\Big)$

Now we can compute, for some element ψ built from an associator:

$$\begin{aligned}
&(\sigma_X\circ\check{Z}^{\mathrm{rat}})(D_2)\\
&\quad= \sigma_X\circ\left(\mathrm{con}_{\{h\}}\left(\cdots\circ\psi\circ\Delta^{w_2}_{x'}(z_1')\circ(I_{w_1}\otimes G_{j,w_2})\circ\Delta^{w_2}_{x}(z_2)\circ\psi^{-1}\circ\dots\right)\right)\\
&\quad\overset{\omega^{\mathrm{gp}}}{\sim} \sigma_X\circ\left(\mathrm{con}_{\{h\}}\left(\cdots\circ\psi\circ(I_{w_1}\otimes G_{j,w_2})\circ\Delta^{w_2}_{x'}(z_1')\circ\Delta^{w_2}_{x}(z_2)\circ\psi^{-1}\circ\dots\right)\right),\\
&\quad= (\sigma_X\circ\check{Z}^{\mathrm{rat}})(D_1).
\end{aligned}$$

In the above, $\Delta_{x'}^{w_2}$ means the comultiplication of the diagram x with pattern given by the gluing word w_2. For example,

$$\Delta^{\uparrow\downarrow\downarrow\uparrow}\left(\uparrow e^{h} \right) = \uparrow e^{h} \downarrow e^{-h} \downarrow e^{-h} \uparrow e^{h}. \qquad \square$$

Theorem 2 now follows, using the identification of $\mathcal{N}_X(\mathcal{O})$ with the set $\mathcal{D}_X^{\times}(D_g)/\langle r,\beta,\omega\rangle$ given by Proposition 2.12.

Remark 4.6 An application of the Z^{rat} invariant is a formula for the LMO invariant of cyclic branched covers of knots, [24]. This application requires a *renormalized version* $Z^{\mathrm{rat}}[\alpha]$ where $\alpha = (\alpha_1, \dots, \alpha_g)$ and $\alpha_i \in \mathcal{A}^{\mathrm{gp}}(\star_k)$. $Z^{\mathrm{rat}}[\alpha]$ is defined using Definition 4.1 and replacing the formula in 2(b) by

$$\cdots \circ Z(T_i') \circ (I_{w_1} \otimes G_{j,w_2}\Delta(\alpha_j)) \circ Z(T_{i+1}') \circ \dots .$$

where $\Delta(\alpha_j)$ places a copy of α_j in each of the skeleton segments of the jth gluing site. The renormalized invariant $Z^{\mathrm{rat}}[\alpha]$ still satisfies the properties of Theorem 2.

5 A rational version of the Aarhus integral

5.1 What is Formal Gaussian Integration?

In this section, we develop a notion of Rational Formal Gaussian Integration $\int^{\mathrm{rat}}$, which is invariant under basing, wrapping and Kirby move relations.

Our theory is a cousin of the Formal Gaussian Integration of [5], and requires good book-keeping but essentially no new ideas.

Before we get involved in details, let us repeat a main idea from [5]: Formal Gaussian Integration is a theory of contraction of legs of diagrams. As such, it does not care about the internal structure of diagrams (such as their internal valency) or the decoration of the internal edges of diagrams.

If diagrams are thought as tensors (as is common in the world of perturbative Quantum Field Theory) which represent differential operators with polynomial coefficients, then contractions of legs corresponds to differentiation.

The reader is encouraged to read [5, part II] for a lengthy introduction to the need and use of Formal Gaussian Integration.

5.2 The definition of $\int^{\mathrm{rat}}$

Recall that $\mathrm{Herm}(\Lambda_{\mathrm{loc}} \to \mathbb{Z})$ denotes the set of Hermitian matrices over Λ_{loc}, invertible over $\mathbb{Z}$. Let X denote a finite set and let $X' = \{x_i\}$ denote an arbitrary subset of X. A diagram $D \in \mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$ is called *X'-substantial* if there are no *strut* components (ie, components comprising of a single edge) both of whose univalent vertices are labeled from the subset X'. An element of $\mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$ is X'-substantial if it is a series of X'-substantial diagrams.

Definition 5.1 (a) An element $s \in \mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$ is called *integrable with respect to X'* if there exists an Hermitian matrix $M \in \mathrm{Herm}(\Lambda_{\mathrm{loc}} \to \mathbb{Z})$ such that

$$s = \exp\left(\frac{1}{2}\sum_{i,j} \overset{x_j}{\underset{x_i}{\updownarrow}} M_{ij}\right) \sqcup R, \tag{12}$$

with R an X'-substantial element. Notice that M, the *covariance matrix* $\mathrm{cov}(s)$ of s, and R, the *X'-substantial part* of s, are uniquely determined by s.
(b) Let $\mathrm{Int}_{X'}\mathcal{A}(\star_X, \Lambda_{\mathrm{loc}}) \subset \mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$ denote the subset of elements that are *integrable with respect* to X'.
(c) Observe that a basing move preserves integrability. Thus, we can define

$$\mathrm{Int}_{X'}\mathcal{A}^{\mathrm{gp}}(\circledast_X, \Lambda_{\mathrm{loc}}) = \left(\mathcal{A}^{\mathrm{gp}}(\star_X, \Lambda_{\mathrm{loc}}) \cap \mathrm{Int}_{X'}\mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})\right) / \langle \beta_1^{\mathrm{gp}}, \beta_2^{\mathrm{gp}} \rangle.$$

For the next definition, recall that every matrix $M \in \mathrm{Herm}(\Lambda_{\mathrm{loc}} \to \mathbb{Z})$ is invertible over Λ_{loc}.

Definition 5.2 Define a map (*Rational Formal Gaussian Integration*)

$$\int^{\mathrm{rat}} dX' : \mathrm{Int}_{X'}\mathcal{A}(\star_X, \Lambda_{\mathrm{loc}}) \to \mathcal{A}(\star_{X-X'}, \Lambda_{\mathrm{loc}})$$

as follows. If $s \in \mathrm{Int}_{X'}\mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$ has associated decomposition given in Equation (12), then

$$\int^{\mathrm{rat}} dX'(s) = \left\langle \exp_{\sqcup}\left(-\frac{1}{2}\sum_{i,j} \overset{\partial x_j}{\underset{\partial x_i}{\updownarrow}} M_{ij}^{-1}\right), R \right\rangle_{X'}.$$

The gluing of the legs of diagrams occurs according to Remark 3.17.

Remark 5.3 The definition of $\int^{\mathrm{rat}}$ for elements s whose covariance matrix M is defined over Λ, and invertible over $\mathbb{Z}$, requires beads labeled by the inverse matrix M^{-1}. This necessitates the need to replace Λ by an appropriate ring Λ_{loc} that makes all such matrices M invertible. It follows by definition that the universal choice of Λ_{loc} is the *Cohn localization* of Λ.

5.3 $\int^{\mathrm{rat}}$ respects the basing relations

The next theorem shows that $\int^{\mathrm{rat}}$ is compatible with the basing relations.

Theorem 3 $\int^{\mathrm{rat}}$ *descends to a map:*

$$\int^{\mathrm{rat}} dX' : \mathrm{Int}_{X'}\mathcal{A}^{\mathrm{gp}}(\star_X, \Lambda_{\mathrm{loc}})/\langle \beta_X^{\mathrm{gp}} \rangle \to \mathcal{A}^{\mathrm{gp}}(\star_{X-X'}, \Lambda_{\mathrm{loc}})/\langle \beta_{X-X'}^{\mathrm{gp}} \rangle$$

Proof Consider $s_1, s_2 \in \mathrm{Int}_{X'}\mathcal{A}(\star_X, \Lambda_{\mathrm{loc}}) \cap \mathcal{A}^{\mathrm{gp}}(\star_X, \Lambda_{\mathrm{loc}})$ such that $s_1 \overset{\beta_1^{\mathrm{gp}'}}{\sim} s_2$ or $s_1 \overset{\beta_2^{\mathrm{gp}}}{\sim} s_2$ (based on some element $x_k \in X'$). Using Lemma 3.19, we need to show that

$$\int^{\mathrm{rat}} dX'\,(s_1) = \int^{\mathrm{rat}} dX'\,(s_2)\,.$$

The case of β_2^{gp}. Consider the decomposition of s_1

$$s_1 = \exp\left(\frac{1}{2}\sum \overset{x_j}{\underset{x_i}{\updownarrow}} M_{ij}\right) \sqcup R.$$

Suppose that an element $f \in F_g$ is pushed onto the legs labeled by $x_k \in X'$, to form s_2. Now, let $D_{f,k}$ be $\mathrm{diag}(1,1,\ldots,f,\ldots,1)$, where f is in the kth entry, and let R' denote R with an f pushed onto every leg marked x_k. Then,

$$\begin{aligned}
\int^{\mathrm{rat}} dX'\,(s_2) &= \int^{\mathrm{rat}} dX' \left(\exp\left(\frac{1}{2}\sum \overset{x_j}{\underset{x_i}{\updownarrow}} (D_{f,k} M D_{f^{-1},k})_{ij}\right) \sqcup R'\right) \\
&= \left\langle \exp\left(-\frac{1}{2}\sum \overset{x_j}{\underset{x_i}{\updownarrow}} (D_{f,k} M^{-1} D_{f^{-1},k})_{ij}\right), R' \right\rangle \\
&= \left\langle \exp\left(-\frac{1}{2}\sum \overset{x_j}{\underset{x_i}{\updownarrow}} (M^{-1})_{ij}\right), R \right\rangle \\
&= \int^{\mathrm{rat}} dX'\,(s_1)
\end{aligned}$$

because all the beads labeled f and f^{-1} match up and cancel, after we pairwise glue the X'-colored legs.

The case of $\beta_1^{\mathrm{gp}'}$. Consider $s \in \mathcal{A}^{\mathrm{gp}}(\star_{X \cup \{\partial h\}}, \Lambda_{\mathrm{loc}})$ such that

$$s_1 = \mathrm{con}_{\{h\}}(s) \quad \text{and} \quad s_2 = \mathrm{con}_{\{h\}}(s|_{x_k \to x_k e^h}).$$

Notice that if s_1, s_2 are X'-integrable, so is s. Furthermore, $\mathrm{con}_{\{h\}}$ commutes with $\int^{\mathrm{rat}}$. The result follows from Lemma 5.4 below. $\square$

The next lemma says that pushing e^h on X'-colored legs commutes with X'-integration. Of course, after X'-integration there is no X'-colored leg, thus X'-integration is invariant under pushing e^h on X'-colored legs.

Lemma 5.4 *Let $s \in \mathrm{Int}_{X'}\mathcal{A}^{\mathrm{gp}}(\star_{X\cup\{\partial h\}}, \Lambda_{\mathrm{loc}})$, and let $x \in X' \subset X$.*

$$\int^{\mathrm{rat}} dX' \,(s|_{x\to xe^h}) = \int^{\mathrm{rat}} dX' \,(s) \in \mathcal{A}^{\mathrm{gp}}(\star_{X\cup\{\partial h\}}, \Lambda_{\mathrm{loc}}).$$

Proof The idea of this lemma is the same as "in the case β_2^{gp}", but instead of pushing an element of the free group onto legs labeled by some element, we push an exponential of hair.

The substitution $x \to xe^h$ defines a map

$$\mathcal{A}(\star_{X\cup\{\partial h\}}, \Lambda_{\mathrm{loc}}) \longrightarrow \mathcal{A}(\star_{X\cup\{h,\partial h\}}, \Lambda_{\mathrm{loc}})$$

with the property that $s - s|_{x\to xe^h}$ is X'-substantial for all $s \in \mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$.

Assume that the canonical decomposition of s is given by Equation (12). Let us define

$$D_{e^h,k} = \mathrm{diag}(1, 1, \ldots, e^h, \ldots, 1),$$

where e^h is in the kth entry. Then, Theorem 18 from Appendix E implies that

$$\begin{aligned}
&\int^{\mathrm{rat}} dX' \,(s|_{x\to xe^h}) \\
&= \exp\left(-\frac{1}{2}\chi_h\left(MD_{e^h,k}M^{-1}D_{e^{-h},k}\right)\right) \sqcup \left\langle \exp\left(-\frac{1}{2}\sum \underset{x_i}{\overset{x_j}{\updownarrow}} M_{ij}^{-1}\right), R\right\rangle \\
&= \left\langle \exp\left(-\frac{1}{2}\sum \underset{x_i}{\overset{x_j}{\updownarrow}} M_{ij}^{-1}\right), R\right\rangle
\end{aligned}$$

where the last equation follows from Proposition 3.22. □

Remark 5.5 An alternative proof of Theorem 3 can be obtained by adapting the proof of [5, Part II, Proposition 5.6] to the rational integration $\int^{\mathrm{rat}}$.

5.4 $\int^{\mathrm{rat}}$ respects the Kirby moves

Recall from Section 2.2 the κ handleslide move on the set $\mathcal{D}^{\times}(D_g)$. In a similar fashion, we define an X'-*handleslide move* on $\mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$, as follows: For two elements $s_1, s_2 \in \mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$, say that $s_1 \overset{\kappa_{X'}}{\sim} s_2$, if for some $x_i \in X$, and $x_j \in X'$, $i \neq j$,

$$s_2 = \left(\sigma_{\{x_i\}} \circ \vec{m}^{y,x_i}_{x_i} \circ \chi_{\{y,x_i\}}\right)\left(s_1\left(x_1, \ldots, x_{j-1}, x_j + y, x_{j+1}, \ldots\right)\right).$$

Notice that if $s_1, s_2 \in \mathcal{A}^{\mathrm{gp}}(\star_X, \Lambda_{\mathrm{loc}})$ and $s_1 \overset{\kappa_{X'}}{\sim} s_2$, then if one of s_i lies in $\mathrm{Int}_{X'}\mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$, so does the other. This is because the covariance matrix of s_2 is obtained from the covariance matrix of s_1 by a unimodular congruence.

Theorem 4 *The map $\sigma_X \circ \check{Z}^{\mathrm{rat}}$ preserves the Kirby relations.*

Proof The argument of [38, Proposition 1], adapts to this setting without difficulty. There are two possible subtleties that must be observed.

(a) The proof in [38] is based on taking a parallel of the parts of a component (here, the component x_j) away from the site of the cross. In our setting, the paralleling operation of the involved components of tangles behaves as usual (sums over lifts of legs). At a gluing word, one simply labels both of the two components arising from the parallel operation with a bead labeled by the appropriate letter. Then, when $\sigma_{\{x_i,y\}}$ is applied, these two identical beads are pushed onto the legs sitting on those components side-by-side. This may alternatively be done by first pushing the original bead up onto the legs, *and then* taking the parallel, as is required in this situation.

(b) We have defined the bracketing at boundary lines with crosses to be the standard left-bracketing, while the move in [38] is given for a presentation where the two involved strands are bracketed together. This is treated by preceding and following the $\kappa_{X'}$ with a β_1 move, to account for the change in bracketing, exactly as in the proof of the β_2-case of Lemma 4.4. □

Theorem 5 *The map $\int^{\mathrm{rat}} dX'$ descends to a map:*

$$\int^{\mathrm{rat}} dX' : \mathrm{Int}_{X'}\mathcal{A}^{\mathrm{gp}}(\star_X, \Lambda_{\mathrm{loc}})/\langle\kappa_{X'}\rangle \to \mathcal{A}^{\mathrm{gp}}(\star_{X-X'}, \Lambda_{\mathrm{loc}}).$$

This is proved by the following lemma.

Lemma 5.6 *Let $s_1 \in \mathrm{Int}_{X'}\mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$ and let*

$$s_2 = \left(\sigma_{\{x_i\}} \circ \vec{m}^{y,x_i}_{x_i} \circ \chi_{\{y,x_i\}}\right)\left(s_1\left(x_1, \ldots, x_{j-1}, x_j + y, x_{j+1}, \ldots\right)\right).$$

If $x_j \in X'$ then

$$\int^{\mathrm{rat}} dX'(s_1) = \int^{\mathrm{rat}} dX'(s_2).$$

Proof We prove this in two steps, which usually go by the names of the "lucky" and "unlucky" cases.

The lucky case: $s_1, s_2 \in \mathrm{Int}_{\{x_j\}}\mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$. In this case, we can use the Iterated Integration Lemma A.2.

$$\begin{aligned}
&\int^{\mathrm{rat}} dX'(s_2)\\
&= \int^{\mathrm{rat}} d(X'-\{x_j\})\left(\int^{\mathrm{rat}} d\{x_j\}(s_2)\right),\\
&= \int^{\mathrm{rat}} d(X'-\{x_j\})\\
&\quad\left(\int^{\mathrm{rat}} d\{x_j\}\left(\sigma_{\{x_i\}}\circ \vec{m}^{y,x_i}_{x_i}\circ\chi_{\{y,x_i\}}\right)(s_1(x_1,\dots,x_{j-1},x_j+y,x_{j+1},\dots))\right),\\
&= \int^{\mathrm{rat}} d(X'-\{x_j\})\\
&\quad\left(\left(\sigma_{\{x_i\}}\circ \vec{m}^{y,x_i}_{x_i}\circ\chi_{\{y,x_i\}}\right)\left(\int^{\mathrm{rat}} d\{x_j\}(s_1(x_1,\dots,x_{j-1},x_j+y,x_{j+1},\dots))\right)\right),\\
&= \int^{\mathrm{rat}} d(X'-\{x_j\})\left(\left(\sigma_{\{x_i\}}\circ \vec{m}^{y,x_i}_{x_i}\circ\chi_{\{y,x_i\}}\right)\right.\\
&\quad\left.\left(\int^{\mathrm{rat}} d\{x_j\}\left(\exp\begin{pmatrix} y\\ \updownarrow 1\\ \partial x_j\end{pmatrix}\flat_{\{x_j\}}s_1(x_1,\dots,x_{j-1},x_j,x_{j+1},\dots)\right)\right)\right)
\end{aligned}$$

Now, using the Integration by Parts Lemma A.4, it follows that

$$\begin{aligned}
&\int^{\mathrm{rat}} d\{x_j\}\left(\exp\begin{pmatrix} y\\ \updownarrow 1\\ \partial x_j\end{pmatrix}\flat_{\{x_j\}}s_1(x_1\dots)\right)\\
&\quad= \int^{\mathrm{rat}} d\{x_j\}\left(\mathrm{div}_{\{x_j\}}\left(\exp\begin{pmatrix} y\\ \updownarrow 1\\ \partial x_j\end{pmatrix}\right)\flat_{\{x_j\}}s_1(x_1,\dots)\right)\\
&\quad= \int^{\mathrm{rat}} d\{x_j\}(s_1(x_1,\dots)),
\end{aligned}$$

because $\mathrm{div}_{\{x_j\}}\left(\exp\begin{pmatrix} y\\ \updownarrow 1\\ \partial x_j\end{pmatrix}\right)=1$.

The unlucky case: $s_1, s_2 \notin \mathrm{Int}_{\{x_j\}}\mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$. This pathology is treated by deformation. Namely, we "deform" s_1 by multiplying it by $\exp\begin{pmatrix} x_j\\ \epsilon \updownarrow 1\\ x_j\end{pmatrix}$, for

ϵ a "small real," commuting with Λ_{loc}. Formally, we let ϵ be a variable and consider an integration theory based on $\mathbb{Q}(\epsilon)[F_g]$ and its Cohn localization $\Lambda^{\epsilon}_{\text{loc}}$. We denote by $\mathbb{Q}^0(\epsilon) \subset \mathbb{Q}(\epsilon)$, the subring of rational functions in ϵ non-singular at zero, and we denote by $\Lambda^{\epsilon,0}_{\text{loc}}$ the corresponding noncommutative localization of $\mathbb{Q}^0(\epsilon)$.

Letting

$$\begin{aligned} s_1^\epsilon &= \exp\left(\epsilon \overset{x_j}{\underset{x_j}{\updownarrow}} 1\right) \sqcup s_1 \in \mathcal{A}(\star_X, \Lambda^\epsilon_{\text{loc}}), \\ s_2^\epsilon &= \left(\sigma_{\{x_i\}} \circ \vec{m}^{y,x_i}_{x_i} \circ \chi_{\{y,x_i\}}\right)\left(s_1^\epsilon\left(x_1, \ldots, x_{j-1}, x_j + y, x_{j+1}, \ldots\right)\right), \end{aligned}$$

it follows from the argument of the "lucky case" that

$$\int^{\text{rat}} dX'\,(s_2^\epsilon) = \int^{\text{rat}} dX'\,(s_1^\epsilon) \in \mathcal{A}(\star_X, \Lambda^{\epsilon,0}_{\text{loc}}).$$

There is a ring homomorphism $\phi : \Lambda^{\epsilon,0}_{\text{loc}} \to \Lambda_{\text{loc}}$, defined by "setting ϵ to zero". Since

$$\phi\left(\int^{\text{rat}} dX'\,(s_i^\epsilon)\right) = \int^{\text{rat}} dX'\,(s_i),$$

the result follows. □

Remark 5.7 These last lines are best understood from the realization of Λ_{loc} and $\Lambda^\epsilon_{\text{loc}}$ as spaces of formal power series. The elements $\int^{\text{rat}} dX'\,(s_i^\epsilon)$ lie in $\mathcal{A}(\star_X, \Lambda^{\epsilon,0}_{\text{loc}})$ because the determinant of the augmentation of the covariance matrix of s_i is invertible in $\mathbb{Q}^0(\epsilon)$.

5.5 $\int^{\text{rat}}$ respects the wrapping relations

In this section we show that $\int^{\text{rat}}$-integration respects the group-like wrapping relations ω^{gp} that were introduced in Definition 2.9. Since the wrapping relations involve pairs of matrices and graphs, we begin by extending the definition of $\int^{\text{rat}}$ to pairs (M, s) by setting

$$\int^{\text{rat}} dX'(M, s) = (M \oplus \text{cov}(s), \int^{\text{rat}} dX'(s)),$$

where $\text{cov}(s)$ is the covariance matrix of s. The motivation for this comes from the next theorem which states that $\int^{\text{rat}}$ preserves the wrapping relations.

Theorem 6 *The map $\int^{\mathrm{rat}} dX'$ descends to a map*

$$\int^{\mathrm{rat}} dX' : \mathrm{Int}_{X'}\mathcal{A}^{\mathrm{gp},0}(\star_X, \Lambda_{\mathrm{loc}})/\langle\omega^{\mathrm{gp}}\rangle \to \mathcal{A}^{\mathrm{gp},0}(\star_{X-X'}, \Lambda_{\mathrm{loc}})/\langle\omega^{\mathrm{gp}}\rangle$$

Proof Consider two pairs $(M, s_1) \overset{\omega^{\mathrm{gp}}}{\sim} (M, s_2)$ related by a wrapping move, as in Definition 3.21. Observe that a wrapping move preserves the covariance matrix.

Using Theorem 18 in Appendix E, it follows that if W is the common covariance matrix of s_1, s_2 and s, then

$$\begin{aligned}
&\int^{\mathrm{rat}} dX'\,(M, s_m)\\
&= \left(M \oplus W, \mathrm{con}_{\{h\}}\left(\left(\int^{\mathrm{rat}} dX'\,(\varphi_m(s)) \sqcup \left(\exp\left(-\frac{1}{2}\chi_h(M^{-1}\varphi_m M)\right)\right)\right)\right)\right)\\
&= \left(M \oplus W, \mathrm{con}_{\{h\}}\left(\varphi_m\left(\left(\int^{\mathrm{rat}} dX'(s)\right) \sqcup \left(\exp\left(-\frac{1}{2}\chi_h(W^{-1}\varphi_m W)\right)\right)\right.\right.\right.\\
&\qquad \left.\left.\left.\sqcup \left(\exp\left(-\frac{1}{2}\chi_h(M^{-1}\varphi_m M)\right)\right)\right)\right)\right)\\
&= \left(M \oplus W, \mathrm{con}_{\{h\}}\left(\varphi_m\left(\left(\int^{\mathrm{rat}} dX'(s)\right)\right.\right.\right.\\
&\qquad \left.\left.\left.\sqcup \left(\exp\left(-\frac{1}{2}\chi_h((W \oplus M)^{-1}\varphi_m(W \oplus M))\right)\right)\right)\right)\right)\\
&\overset{\omega^{\mathrm{gp}}}{\sim} \left(M, \int^{\mathrm{rat}} dX'\,(M, s)\right).
\end{aligned}$$

∎

Remark 5.8 The group-like wrapping relation is the minimal relation that we need to impose on diagrams so that $\int^{\mathrm{rat}} \circ \sigma \circ Z^{\mathrm{rat}}$ is invariant under the topological wrapping relation, as follows from the proof of Theorem 6.

5.6 The definition of the rational invariant Z^{rat} of F-links

We have all the ingredients to define our promised invariant of F-links.

Theorems 2, 3, 5, and 6 imply that:

Definition 5.9 There is a map

$$Z^{\mathrm{rat,gp}} : F\text{-links} \longrightarrow \mathcal{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}})/\langle\omega^{\mathrm{gp}}\rangle$$

defined as follows: for a F-link (M, L) of g components, choose a sliced crossed link C in $\mathcal{D}^\times(D_g)$ whose components are in 1-1 correspondence with a set X, and define

$$Z^{\mathrm{rat,gp}}(M, L) = \frac{\int^{\mathrm{rat}} dX \left(\sigma \circ \check{Z}^{\mathrm{rat}}(C)\right)}{c_+^{\sigma_+(B)} c_-^{\sigma_-(B)}} \in \mathcal{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}})/\langle \omega^{\mathrm{gp}} \rangle$$

where $c_\pm = \int dU \check{Z}(S^3, U_\pm)$ are some universal constants of the unit-framed unknot $U_\pm$ and where B is the linking matrix of C and $\sigma_\pm(B)$ is the number of positive (resp. negative) eigenvalues of the symmetric matrix B.

The slight renormalization given by multiplication by $c_\pm$ is needed to deal with the stabilization Kirby move on C, and is justified as follows: recall that $\mathcal{A}^{\mathrm{gp}}(\phi)$ is a group, which acts on the set $\mathcal{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}})$ by $a \cdot (M, s) = (M, a \sqcup s)$. Since the beads of the graphs in $\mathcal{A}^{\mathrm{gp}}(\phi)$ are rational numbers independent of the t_i, it follows easily that the above action of $\mathcal{A}^{\mathrm{gp}}(\phi)$ on $\mathcal{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}})$ induces an action of $\mathcal{A}^{\mathrm{gp}}(\phi)$ on the quotient set $\mathcal{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}})/\langle \omega \rangle$. This is exactly what appears in the above definition of Z^{rat}.

6 The String action and invariants of boundary links

In this section we show how the invariant of F-links of Definition 5.9 can be modified in a natural way to give an invariant of boundary links.

The key geometric observation is that the set of boundary links can be defined with the set of orbits of an action

$$\mathrm{String} \times F\text{-links} \to F\text{-links}$$

for a suitable group String. Starting from this observation, we refine the results of Sections 2-5 to take into account the action of the group String.

6.1 A surgery description of boundary links

We begin by giving a surgery description of boundary links. Recall that the set of ∂-links can be identified as a quotient of the set of F-links modulo an action of a group of automorphisms String_g of the free group, [32, 11], where

$$\mathrm{String}_g = \{f \in \mathrm{Aut}(F) | f(t_i) = \alpha_i^{-1} t_i \alpha_i, \quad i = 1, \dots, g\}.$$

In particular, String_g contains (and for $g = 1, 2$ coincides with) the group of inner automorphisms of F_g. It is easy to see that String_g is generated by the special automorphisms α_{ij} (for $1 \leq i \neq i \leq g$) acting on the generators of F_g by

$$\alpha_{ij}(t_k) = \begin{cases} t_k & \text{if} \quad k \neq i \\ t_j^{-1} t_i t_j & \text{if} \quad k = i; \end{cases}$$

see [32, Lemma 2.4]. The group String_g acts on the set of F-links by changing the basing of the F-link, in other words by sending (M, L_b, ϕ) to $\alpha \circ \phi$ for $\alpha \in \mathrm{String}_g$. An alternative and more geometric way to describe this action is as follows: identify the set of F-links with the set of Seifert surfaces modulo isotopy and *tube equivalence*, see [20]. There is an action of String_g on the set of Seifert surfaces, described on generators α_{ij} by *cocooning* the ith surface to the jth surface using suitable arcs to a base point, well-defined modulo isotopy and tube equivalence.

Since the set $\mathcal{N}(\mathcal{O})/\langle\kappa\rangle$ is in 1-1 correspondence with the set of F-links and since String acts on the latter set, it follows that it acts on the former set, too. We now describe this action of String_g (or rather, of its generating set $\{\alpha_{ij}\}$) on the set $\mathcal{N}(\mathcal{O})/\langle\kappa\rangle$.

Consider a link $L \in \mathcal{N}(\mathcal{O})$. The action of α_{ij} on such a link is the following: "the ith unlink component is isotoped through the jth unlink component, and then put back where it started".

Let us examine what happens when we try to implement such a principle. To do this, we take a parallel copy of the ith unlink component and drag it so that, at the end, the initial link (resp. the link that results from the move) is recovered by forgetting the parallel copy (resp. forgetting the original copy).

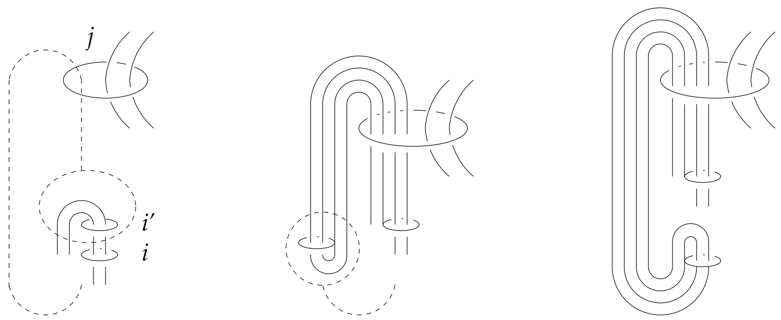


Figure 5: Dragging the ith unlink component through the jth unlink component, and then put back where it started.

It is easy to see that the α_{ij} move does not depend on the choice of arc, modulo Kirby moves, and that the above discussion together with Theorem 1 imply that:

Theorem 7 *There exists an action*

$$\mathrm{String} \times (\mathcal{N}(\mathcal{O})/\langle\kappa\rangle) \longrightarrow \mathcal{N}(\mathcal{O})/\langle\kappa\rangle$$

which induces a 1-1 onto correspondence

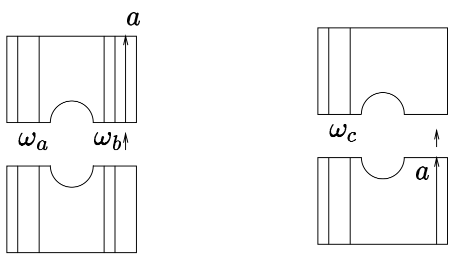
$$\mathcal{N}(\mathcal{O})/\langle\kappa, \mathrm{String}\rangle \longleftrightarrow \partial\text{-links.}$$

Remark 6.1 It is instructive to compare the above theorem with a result of Roberts [46], which states that two 3-manifolds N_L and $N_{L'}$ are diffeomorphic relative boundary iff L and L' are related by a sequence of handle-slides (κ_1), stabilization (κ_2), and insertion/deletion of an arbitrary knot in N together with a 0-framed meridian of it (κ_3). In case the knot participating in the κ_3 move is nullhomotopic, the move is equivalent to a composition of κ_1 and κ_2 moves, as was observed by Kirby and Fenn-Rourke. In general, the set $\mathcal{N}(\mathcal{O})$ is not closed under the moves κ_3, however we can consider the set $\widehat{\mathcal{N}(\mathcal{O})}$ that consists of all links in $S^3 - \mathcal{O}$ that are obtained from nullhomotopic links by $\kappa_{1,2,3}$ moves. Roberts' theorem implies that there is a 1-1 onto correspondence

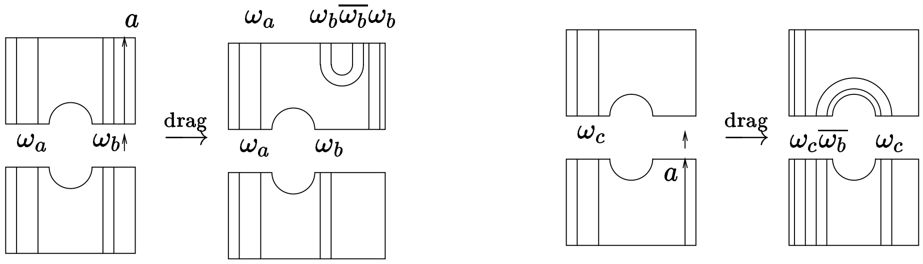
$$\widehat{\mathcal{N}(\mathcal{O})}/\langle\kappa_1, \kappa_2, \kappa_3\rangle \longleftrightarrow \partial\text{-links.}$$

We leave it as an exercise to show that the links of Figure 5 are related by a sequence of $\kappa_{1,2,3}$ moves. □

In order to give a tangle description of the set of ∂-links, we need to introduce a geometric action of the set String_g on the set $\mathcal{D}^\times(D_g)$. Recalling the action of String_g on the set $\mathcal{N}(\mathcal{O})$, we see that this action can be specified by a framed arc, and that as this proceeds, then all the strands going between the two copies are parallel to each other (that is, dragging the copy away from the original "lines-up" all the strands in between); see also Figure 5. This will be the idea of our action of String_g on $\mathcal{D}^\times(D_g)$. To formalize this, we (briefly) enlarge our notion of sliced crossed links, to allow for such an arc. That is, we will consider the set $\mathcal{D}^{\times,\mathrm{arc}}(D_g)$ of *arc-decorated sliced crossed diagrams*, represented by sequences of tangles such that, in addition to all closed components, there is an arc which starts and finishes at some meridional disc:



For an element of this set, we will say that the result of a *drag* along the arc is the sliced crossed link that arises by making the following replacements, accompanied by taking a parallel of the arc a according to the appropriate word (in the example below, according to $w_b\overline{w_b}$).



Given two sliced crossed links $L_1, L_2 \in \mathcal{D}^{\times}(D_g)$, we say that L_2 is obtained from L_1 by an α_{ij} move if there exists an arc-decorated sliced crossed link $L_{12} \in \mathcal{D}^{\times,\mathrm{arc}}(D_{g+1})$ with the following properties:

(1) there are no intersections of link components with the $(i+1)$st meridianal disc,

(2) gluing back in the $(i+1)$st removed disc (and forgetting the arc) gives a diagram which presents L_1,

(3) the arc travels from the ith meridianal disc, intersects only the jth meridianal disc, and only once, and then ends at the ith meridianal disc.

(4) Dragging L_{12} along the arc and then gluing back the ith removed unlink component recovers L_2.

Remark 6.2 The above definition implements the procedure described in Figure 5. Actually, this α_{ij} move does not depend on the choice of arc (this is an instructive exercise). In addition, this move translated to the set of F-links coincides with the action of String_g by cocooning.

Proposition 2.12 and Theorem 7 imply that:

Proposition 6.3 *There is an action*

$$\mathrm{String} \times \mathcal{D}^{\times}(D_g)/\langle r, \beta, \omega, \kappa\rangle \longleftrightarrow \mathcal{D}^{\times}(D_g)/\langle r, \beta, \omega, \kappa\rangle$$

which induces a 1-1 correspondence

$$\mathcal{D}^{\times}(D_g)/\langle r, \beta, \omega, \kappa, \mathrm{String}\rangle \longleftrightarrow \partial\text{-links.}$$

6.2 The action of String on diagrams

We now define an algebraic action of String_g on diagrams. Recall the group String_g that acts on the free group F_g (see Section 6.1), thus also on Λ. The action of String_g on Λ extends to the localization Λ_{loc}, since for $\alpha \in \mathrm{String}_g$, the induced ring homomorphism $\Lambda \to \Lambda \subset \Lambda_{\mathrm{loc}}$ is Σ-inverting, and thus extends to a ring homomorphism $\Lambda_{\mathrm{loc}} \to \Lambda_{\mathrm{loc}}$. This induces an action

$$\mathrm{String}_g \times \mathcal{A}^0(\star_X, \Lambda_{\mathrm{loc}}) \to \mathcal{A}^0(\star_X, \Lambda_{\mathrm{loc}})$$

given by acting on the beads of diagrams and the entries of matrices by special automorphisms of the free group. Following Remark 3.9, we denote by $\mathcal{A}^0(\star_X, \Lambda_{\mathrm{loc}})/\langle \mathrm{String}\rangle$ the quotient space.

Similarly, there is an action

$$\mathrm{String}_g \times \mathcal{A}^{\mathrm{gp},0}(\star_X, \Lambda_{\mathrm{loc}}) \to \mathcal{A}^{\mathrm{gp},0}(\star_X, \Lambda_{\mathrm{loc}}),$$

which results in a quotient set $\mathcal{A}^{\mathrm{gp},0}(\star_X, \Lambda_{\mathrm{loc}})/\langle \mathrm{String}\rangle$, too.

The next theorem shows that the action of String_g on the set $\mathcal{A}^{\mathrm{gp},0}(\star_X, \Lambda_{\mathrm{loc}})$ descends an action on the quotient set $\mathcal{A}^{\mathrm{gp},0}(\star_X, \Lambda_{\mathrm{loc}})/\langle \beta^{\mathrm{gp}}, \omega^{\mathrm{gp}}\rangle$.

Theorem 8 *The* String_g *action on* $\mathcal{A}^{\mathrm{gp},0}(\star_X, \Lambda_{\mathrm{loc}})$ *descends to an action on* $\mathcal{A}^{\mathrm{gp},0}(\star_X, \Lambda_{\mathrm{loc}})/\langle \beta^{\mathrm{gp}}, \omega^{\mathrm{gp}}\rangle$.

Proof It is an exercise in the definitions to see that if $(M, s_1), (M, s_2) \in \mathcal{A}^{\mathrm{gp},0}(\star_X, \Lambda_{\mathrm{loc}})$ and and $(M, s_1) \overset{\beta^{\mathrm{gp}}}{\sim} (M, s_2)$, then $\alpha_{ij}.(M, s_1) \overset{\beta^{\mathrm{gp}}}{\sim} \alpha_{ij}.(M, s_2)$. However, this is not so clear for the case of the wrapping move ω^{gp}. Consider pairs $(M, s_1) \overset{\omega_k^{\mathrm{gp}}}{\sim} (M, s_2)$ by a wrapping move at the kth site. We need to show that $\alpha_{ij}.(M, s_1) \overset{\beta^{\mathrm{gp}}}{\sim} \alpha_{ij}.(M, s_2)$.

By definition, there exists an $s \in \mathcal{A}^{\mathrm{gp}}(\star_{X \cup \{\partial h\}}, \Lambda_{\mathrm{loc}})$ such that

$$\begin{aligned} s_1 &= \mathrm{con}_{\{h\}}(s) \\ s_2 &= \mathrm{con}_{\{h\}}\left(\phi_{t_k \to e^{-h} t_k e^h}(s) \sqcup \exp_{\sqcup}\left(-\frac{1}{2}\chi_h(M^{-1}\phi_{t_k \to e^{-h} t_k e^h} M)\right)\right). \end{aligned}$$

Then, we have that

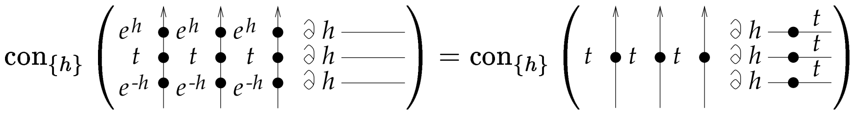
$$\begin{aligned}\alpha_{ij}s_1 &= \mathrm{con}_{\{h\}}\left(\alpha_{ij}(s)\right)\\ \alpha_{ij}s_2 &= \mathrm{con}_{\{h\}}\left(\alpha_{ij}\phi_{t_k\to e^{-h}t_k e^h}(s)\sqcup\left(\exp_{\sqcup}\left(-\frac{1}{2}\chi_h(M^{-1}\phi_{t_k\to e^{-h}t_k e^h}M)\right)\right)\right).\end{aligned}$$

If $k\neq i,j$ then α_{ij} commutes with $\phi_{t_k\mapsto e^{-h}t_k e^h}$, thus the result follows. It remains to consider the cases of $k=i$ or $k=j$.

The case of $k=i$. Observe that α_{ij} equals to the substitution $\phi_{t_i\mapsto t_j^{-1}t_i t_j}$, thus

$$\alpha_{ij}\circ\phi_{t_i\mapsto e^{-h}t_i e^h}=\phi_{t_i\mapsto e^{-h}t_j^{-1}t_i t_j e^h}$$

We will first compare $\phi_{t_i\mapsto e^{-h}t_j^{-1}t_i t_j e^h}$ and $\phi_{t_i\mapsto t_j^{-1}e^{-h}t_i e^h t_j}$ using a Vertex Invariance (in short, VI) Relation, and then $\phi_{t_i\mapsto t_j^{-1}e^{-h}t_i e^h t_j}$ with the identity using the wrapping move to reach the desired conclusion. Explicitly, the Vertex Invariance Relation of Figure 4 implies that



if all appearances of t and all legs ∂h are drawn. Let $s(\partial h)$ denote s with the special leg colored explicitly by ∂h, and use the following abbreviation

$$\Psi(N)=\left(\exp_{\sqcup}\left(-\frac{1}{2}\chi_h(N)\right)\right).$$

The above discussion implies that:

$$\begin{aligned}&\alpha_{ij}(M,s_2)\\ &=\left(\alpha_{ij}M,\mathrm{con}_{\{h\}}\left(\phi_{t_i\mapsto e^{-h}t_j^{-1}t_i t_j e^h}(s(\partial h))\sqcup\Psi(\alpha_{ij}M^{-1}.\phi_{t_i\mapsto e^{-h}t_j^{-1}t_i t_j e^h}(M))\right)\right)\\ &\overset{\mathrm{VI}}{=}\left(\alpha_{ij}M,\mathrm{con}_{\{h\}}\left(\phi_{t_i\mapsto t_j^{-1}e^{-h}t_i e^h t_j}s(t_j\partial h)\sqcup\Psi\left(\alpha_{ij}M^{-1}.\phi_{t_i\mapsto t_j^{-1}e^{-h}t_i e^h t_j}M\right)\right)\right)\\ &\overset{\omega_i^{\mathrm{gp}}}{\sim}\left(\alpha_{ij}M,\mathrm{con}_{\{h\}}\left(\phi_{t_i\mapsto t_j^{-1}t_i t_j}s(t_j\partial h)\sqcup\Psi\left(\alpha_{ij}M^{-1}.\phi_{t_i\mapsto t_j^{-1}t_i t_j}M\right)\right)\right)\\ &=(\alpha_{ij}M,(\alpha_{ij}s(t_j\partial h)|_{\partial h\mapsto 0}))\\ &=\left(\alpha_{ij}M,\mathrm{con}_{\{h\}}(\alpha_{ij}s)\right)\\ &=\alpha_{ij}(M,s_1).\end{aligned}$$

The case of $k=j$. Observe that

$$\alpha_{ij}\circ\phi_{t_j\mapsto e^{-h}t_j e^h}=\phi_{t_j\mapsto e^{-h}t_j e^h,t_i\mapsto t_j^{-1}t_i t_j}$$

First, we will compare the substitutions $\phi^{(0)}$ with $\phi^{(4)}$ (defined below) using a wrapping move ω_j^{gp}, the following Ψ-identity

$$\Psi(A^{-1}B) \sqcup \Psi(B^{-1}C) = \Psi(A^{-1}C) \tag{13}$$

(which is a consequence of Proposition 3.22) and Equation (25) of Lemma A.1. Then, we will compare $\phi^{(4)}$ with $\phi^{(8)}$ using the wrapping move ω_i^{gp}. Finally, we will compare $\phi^{(8)}$ with the identity using the case of $k=i$ above. Here, $\phi^{(a)}$ denote the substitutions:

$$\begin{aligned}
\phi^{(0)} &= & t_j &\mapsto e^{-h}t_je^h, & t_i &\mapsto t_j^{-1}t_it_j,\\
\phi^{(2)} &= & t_j &\mapsto e^{-h}e^{h'}t_je^{-h'}e^h, & t_i &\mapsto e^{h'}t_j^{-1}e^{-h'}t_ie^{h'}t_je^{-h'},\\
\phi^{(3)} &= & t_j &\mapsto e^{h'}t_je^{-h'}, & &\\
\phi^{(4)} &= & t_j &\mapsto t_j, & t_i &\mapsto e^ht_j^{-1}e^{-h}t_ie^ht_je^{-h},\\
\phi^{(6)} &= & t_j &\mapsto t_j, & t_i &\mapsto e^ht_j^{-1}e^{-h}e^{h'}t_ie^{-h'}e^ht_je^{-h},\\
\phi^{(7)} &= & t_i &\mapsto e^{h'}t_ie^{-h'}, & &\\
\phi^{(8)} &= & t_j &\mapsto t_j, & t_i &\mapsto e^ht_j^{-1}t_it_je^{-h}.
\end{aligned}$$

Then, we have that

$$\begin{aligned}
&\alpha_{ij}(M,s_2)\\
&= \Big(\alpha_{ij}M, \mathrm{con}_{\{h\}}\Big(\phi^{(0)}s(\partial h)\sqcup\Psi(\alpha_{ij}M^{-1}\phi^{(0)}M)\Big)\Big)\\
&\overset{\omega_j^{\mathrm{gp}}}{\sim} \Big(\alpha_{ij}M, \mathrm{con}_{\{h,h'\}}\Big(\phi^{(2)}s(\partial h+\partial h')\sqcup\Psi(\phi^{(3)}\alpha_{ij}M^{-1}\phi^{(2)}M)\\
&\qquad \sqcup\Psi(\alpha_{ij}M^{-1}\phi^{(3)}\alpha_{ij}M)\Big)\Big)\\
&\overset{(13)}{=} \Big(\alpha_{ij}M, \mathrm{con}_{\{h,h'\}}\Big(\phi^{(2)}s(\partial h+\partial h')\sqcup\Psi(\alpha_{ij}M^{-1}\phi^{(2)}M)\Big)\Big)\\
&\overset{(25)}{=} \Big(\alpha_{ij}M, \mathrm{con}_{\{h\}}\Big(\phi^{(4)}s(\partial h)\sqcup\Psi(\alpha_{ij}M^{-1}\phi^{(4)}M)\Big)\Big)\\
&\overset{\omega_i^{\mathrm{gp}}}{\sim} \Big(\alpha_{ij}M, \mathrm{con}_{\{h,h'\}}\Big(\phi^{(6)}s(\partial h+\partial h')\sqcup\Psi(\phi^{(7)}\alpha_{ij}M^{-1}\phi^{(6)}M)\\
&\qquad \sqcup\Psi(\alpha_{ij}M^{-1}\phi^{(7)}\alpha_{ij}M)\Big)\Big)\\
&\overset{(13)}{=} \Big(\alpha_{ij}M, \mathrm{con}_{\{h,h'\}}\Big(\phi^{(6)}s(\partial h+\partial h')\sqcup\Psi(\alpha_{ij}M^{-1}\phi^{(6)}M)\Big)\Big)\\
&\overset{(25)}{=} \Big(\alpha_{ij}M, \mathrm{con}_{\{h\}}\Big(\phi^{(8)}s(\partial h)\sqcup\Psi(\alpha_{ij}M^{-1}\phi^{(8)}M)\Big)\Big)\\
&\overset{k=i}{\sim} \left(\alpha_{ij}M, \mathrm{con}_{\{h\}}\left(\alpha_{ij}s(\partial h)\right)\right)\\
&= \alpha_{ij}(M,s_1).
\end{aligned}$$

□

6.3 Z^{rat} commutes with the String action

Recall the invariant Z^{rat} of sliced crossed links of Section 4.

Theorem 9 *The map*

$$\sigma_X \circ Z^{\text{rat}} : \mathcal{D}^{\times}_X(D_g) \to \mathcal{A}^{\text{gp}}(\star_X, \Lambda)/\langle \omega^{\text{gp}} \rangle$$

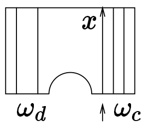
is String_g*-equivariant, and induces a well-defined map*

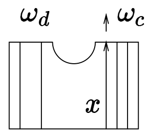
$$\sigma_X \circ Z^{\text{rat}} : \mathcal{D}^{\times}_X(D_g)/\langle \text{String}_g \rangle \to \mathcal{A}^{\text{gp}}(\star_X, \Lambda)/\langle \omega^{\text{gp}}, \text{String}_g \rangle.$$

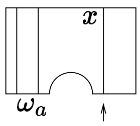
Proof We will show that for $D \in \mathcal{D}^{\times}_X(D_g)$, we have

$$\alpha_{ij}.Z^{\text{rat}}(D) = Z^{\text{rat}}(\alpha_{ij}.D).$$

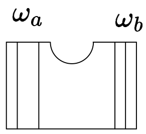
Given sliced crossed links L_1, L_2 such that $L_2 = \alpha_{ij}.L_1$, consider an accompanying link L_{12} as in the previous section. L_{12} may be presented so that all the strands going through the ith meridianal disc proceed, parallel to each other, through the jth meridianal disc, and then onto the $(i+1)$st meridianal disc. That is, we can find a sequence of tangles, containing some arc 'x' (not exactly the arc 'a' of Section 2.2), such that the link L_{12} is presented by taking a parallel the arc 'x' according to some word (in the example at hand, w_b). Let us define $T_p, T_q, T_r, T_s, T_t, T_u$ by

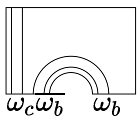
$T_p =$ 

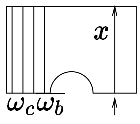
$T_q =$ 

T_r 

and

$T_s =$ 

$T_t =$ 

$T_u =$ 

Continuing to work with the example of the above paragraph, then the sequence will look as follows (where the jth, the ith and then the $(i+1)$st meridianal discs are displayed):

$$\{\ldots, \Delta^{w_b}_x(T_p), \Delta^{w_b}_x(T_q), \ldots, \Delta^{w_b}_x(T_r), T_s, \ldots, T_t, \Delta^{w_b}_x(T_u), \ldots\}$$

Now, the link L_1 (resp. L_2) is presented by the diagrams recovered by gluing back in the $(i+1)$st (resp. ith) removed disc.

So, on the one hand we have:

$$\begin{aligned}
&Z^{\mathrm{rat}}(L_1)\\
=\quad &\cdots\circ\Delta_x^{w_b}(Z(T_p))\circ(I_{w_d}\otimes G_{j,w_bw_e})\circ\Delta_x^{w_b}(Z(T_q))\circ\dots\\
&\cdots\circ\Delta_x^{w_b}(Z(T_r))\circ(I_{w_a}\otimes G_{i,w_b})\circ Z(T_s)\circ\cdots\circ Z(T_t)\circ\Delta_x^{w_b}(Z(T_u))\circ\dots\\
\overset{\omega^{\mathrm{gp}}}{\sim}\quad &\cdots\circ\Delta_x^{w_b}(Z(T_p))\circ(I_{w_d}\otimes G_{j,w_bw_e})\circ(I_{w_d}\otimes G_{i,w_b}\otimes I_{w_e})\circ\Delta_x^{w_b}(Z(T_q))\circ\\
&\cdots\circ\Delta_x^{w_b}(Z(T_r))\circ Z(T_s)\circ\cdots\circ Z(T_t)\circ\Delta_x^{w_b}(Z(T_u))\circ\dots
\end{aligned}$$

A ω^{gp}-move has been used here to move the gluing word $G_{i,w}$ up to (just underneath) the gluing word arising from the jth meridional slice. This is legitimate because the meaning of the ω^{gp}-move is that such an element (if it involves *every* appearance of t_i) can be slid along a *paralleled* arc (here arising from the component x).

On the other hand, we have:

$$\begin{aligned}
&Z^{\mathrm{rat}}(L_2)\\
&=\cdots\circ\Delta_x^{w_b}(Z(T_p))\circ(I_{w_d}\otimes G_{j,w_bw_e})\circ\Delta_x^{w_b}(Z(T_q))\circ\dots\\
&\quad\cdots\circ\Delta_x^{w_b}(Z(T_r))\circ Z(T_s)\circ\cdots\circ Z(T_t)\circ(I_{w_c\overline{w_b}}\otimes G_{i,w_b})\circ\Delta_x^{w_b}(Z(T_u))\circ\dots\\
&\overset{\omega^{\mathrm{gp}}}{\sim}\cdots\circ\Delta_x^{w_b}(Z(T_p))\circ(I_{w_d}\otimes G_{i,w_b}\otimes I_{w_e})\circ(I_{w_d}\otimes G_{j,w_bw_e})\circ\Delta_x^{w_b}(Z(T_q))\circ\\
&\quad\cdots\circ\Delta_x^{w_b}(Z(T_r))\circ Z(T_s)\circ\cdots\circ Z(T_t)\circ\Delta_x^{w_b}(Z(T_u))\circ\dots
\end{aligned}$$

Observe that this line results from the line ending the previous paragraph of equations by the algebraic action of α_{ij}, as is required for the proof. □

6.4 $\int^{\mathrm{rat}}$ commutes with the String_g action

The next theorem says that the action of String_g on diagrams commutes with $\int^{\mathrm{rat}}$-integration.

Theorem 10 *The map*

$$\int^{\mathrm{rat}} dX' : \mathrm{Int}_{X'}\mathcal{A}^{\mathrm{gp},0}(\star_X,\Lambda_{\mathrm{loc}})\to\mathcal{A}^{\mathrm{gp},0}(\star_{X-X'},\Lambda_{\mathrm{loc}})$$

is String_g *equivariant and induces a well-defined map:*

$$\int^{\mathrm{rat}} dX' : \mathrm{Int}_{X'}\mathcal{A}^{\mathrm{gp},0}(\star_X,\Lambda_{\mathrm{loc}})/\langle\mathrm{String}\rangle\to\mathcal{A}^{\mathrm{gp},0}(\star_{X-X'},\Lambda_{\mathrm{loc}})/\langle\mathrm{String}\rangle.$$

Proof Let the canonical decomposition of $s \in \mathrm{Int}_{X'}\mathcal{A}^{\mathrm{gp}}(\star_X, \Lambda_{\mathrm{loc}})$ with respect to X' be

$$s = \exp\left(\frac{1}{2}\sum_{kl} \underset{x_l}{\overset{x_k}{\updownarrow}} M_{kl}\right) \sqcup R,$$

for some matrix $M \in \mathrm{Herm}(\Lambda_{\mathrm{loc}} \to \mathbb{Z})$. Then, the canonical decomposition of $\alpha_{ij}.s$ with respect to X' is

$$\alpha_{ij}.s = \exp\left(\frac{1}{2}\sum_{kl} \underset{x_l}{\overset{x_k}{\updownarrow}} \alpha_{ij}.M_{kl}\right) \sqcup \alpha_{ij}.R.$$

Observe that $(\alpha_{ij}.M)^{-1} = \alpha_{ij}.M^{-1} \in \mathrm{Herm}(\Lambda_{\mathrm{loc}} \to \mathbb{Z})$ which implies that $\alpha_{ij}.s$ is X'-integrable. Furthermore, we have that

$$\begin{aligned}
\int^{\mathrm{rat}} dX'\,(\alpha_{ij}.s) &= \int^{\mathrm{rat}} dX' \left(\exp\left(\frac{1}{2}\sum_{kl} \underset{x_l}{\overset{x_k}{\updownarrow}} \alpha_{ij}.M_{kl}\right) \sqcup (\alpha_{ij}.R)\right) \\
&= \left\langle \exp\left(-\frac{1}{2}\sum_{kl} \underset{x_l}{\overset{x_k}{\updownarrow}} \alpha_{ij}.(M^{-1}_{kl})\right), \alpha_{ij}.R \right\rangle \\
&= \alpha_{ij}.\int^{\mathrm{rat}} dX'\,(s)\,.
\end{aligned}$$

Note that we could have used Theorem 18 to deduce the above equality. □

We end this section with the following

Lemma 6.4 (a) *For all g, the identity map $\mathcal{A}(\star_X, \Lambda_{\mathrm{loc}}) \to \mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$ maps the Inn_g-relation to the basing relation.*

(b) *For $g = 1, 2$, the identity map $\mathcal{A}(\star_X, \Lambda_{\mathrm{loc}}) \to \mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$ maps the String_g-relation to the basing relation.*

Proof Recall that Inn_g is the subgroup of String_g that consists of inner automorphisms of the free group F.

For the first part, an inner automorphism by an element $g \in F$ maps a bead $b \in \Lambda_{\mathrm{loc}}$ to $g^{-1}bg \in \Lambda_{\mathrm{loc}}$. The relations of Figure 3 cancel the g at every trivalent vertex of a diagram. The remaining vertices change a diagram by a basing move β_2^{gp}.

The second part follows from the fact that $\mathrm{Inn}_g = \mathrm{String}_g$ for $g = 1, 2$, see [11, 32]. □

6.5 The definition of the rational invariant Z^{rat} of ∂-links

We can now define an invariant of boundary links as was promised in Fact 10 of Section 1.5.

Definition 5.9, together with Theorems 7, 6.3, 9 and 10 imply that:

Definition 6.5 There is a map

$$Z^{\mathrm{rat,gp}} : \partial\text{-links} \to \mathcal{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}})/\langle \omega^{\mathrm{gp}}, \mathrm{String}\rangle$$

defined as follows: for a ∂-link (M, L) of g components, choose a sliced crossed link C in $\mathcal{D}^{\times}(D_g)$ whose components are in 1-1 correspondence with a set X, and define

$$Z^{\mathrm{rat,gp}}(M, L) = \frac{\int^{\mathrm{rat}} dX \left(\sigma \circ \check{Z}^{\mathrm{rat}}(C)\right)}{c_+^{\sigma_+(B)} c_-^{\sigma_-(B)}} \in \mathcal{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}})/\langle \omega^{\mathrm{gp}}, \mathrm{String}\rangle$$

where $c_\pm = \int dU \check{Z}(S^3, U_\pm)$ are some universal constants of the unit-framed unknot $U_\pm$ and where B is the linking matrix of C and $\sigma_\pm(B)$ is the number of positive (resp. negative) eigenvalues of B.

The above definition makes sense and uses the fact that as in Section 5.6, the set $\mathcal{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}})/\langle \omega^{\mathrm{gp}}\rangle$ has a well-defined action of the group $\mathcal{A}^{\mathrm{gp}}(\phi)$, which commutes with the String action (where String acts trivially on $\mathcal{A}^{\mathrm{gp}}(\phi)$).

6.6 Some more appearances of the String group

Let us finish this section with some more information on the String group, which we will not need in our paper. We wish to thank K. Vogtmann for bringing to our attention the references in this section.

The group String_g (and its quotient $\mathrm{String}_g/\mathrm{Inn}_g$) appears in three different contexts:

- $\mathrm{String}_g/\mathrm{Inn}_g$ is the group that acts on the set of F-links whose set of orbits is identified with the set of boundary links.

- String_g is the *group of motions* of a standard unlink in $\mathbb{R}^3$, as explained by Goldsmith following unpublished results of Dahm, [27, Theorem 5.4]. Let us explain a bit more. Using terminology from [27], recall that a given a submanifold N in a noncompact manifold M, a *motion* of N in M is a one-parameter family f_t of diffeomorphisms of M with compact support, for $0 \leq t \leq 1$, such that f_0 and f_1 pointwise fix N. A motion is *stationary* if it is homotopic to a motion that pointwise fixes N at all times t. The set of equivalence classes of motions (modulo stationary ones) is a group. In the case of interest, N is a standard oriented unlink of ordered components in $\mathbb{R}^3$ (resp. S^3), and the group of motions is identified with the group String_g (resp. $\mathrm{String}_g/\mathrm{Inn}_g$). That is, String_g is the group of motions of a unknotted unlinked *string* in 3-space. In this context, see also Figure 5.

- String_g is the subgroup of the automorphism group of the free group that sends every generator t_i to a conjugate of itself. Geometric group theory tells us quite a bit about this algebraic group. McCool has given a presentation in terms of the generators α_{ij}, for $1 \leq i \neq j \leq g$

$$\mathrm{String}_g = \langle\ \alpha_{ij}\ |[\alpha_{ij}, \alpha_{kl}] = 1,\ [\alpha_{ik}, \alpha_{jk}] = 1,\ [\alpha_{ij}, \alpha_{ik}\alpha_{jl}] = 1\,\rangle$$

where the indices occurring in each relation are assumed to be distinct. Bounds for the cohomological dimension of String_g are known, as well as normal forms for the associated language of the elements in String_g, [29].

We will not use explicitly the presentation of String_g, however the reader may keep it in mind in the proof of 8. The reader should consult [8, 27, 29, 40] for further information on the String group.

7 A comparison between Z and Z^{rat}

Fact 10 of Section 1.5 asks for an invariant Z^{rat} which is a rational form of the Kontsevich integral. The goal of this section is to show that the invariant of Definition 6.5 is indeed a rational form of the Kontsevich integral of a boundary link.

This will be obtained by comparing the construction of the Aarhus integral $\int$ with that of $\int^{\mathrm{rat}}$.

7.1 The Hair map

First, we need to define a Hair map which replaces beads by the exponential of hair.

Given generators $\{t_1, \dots, t_g\}$ of the free group, let $\{h_1, \dots, h_g\}$ be noncommuting variables such that $t_i = e^{h_i}$.

Definition 7.1 Let

$$\mathrm{Hair} : \mathcal{A}(\star_X, \Lambda_{\mathrm{loc}}) \to \mathcal{A}(\star_{X \cup H})$$

be the map that replaces each variable t_i by an exponential of h_i-colored hair, as in Figure 6.

Observe that the image of the Hair map lies in the span of all diagrams that do not contain a H-labeled tree as a connected component, and that it maps group-like elements to group-like elements.

We will extend the Hair map on $\mathcal{A}^{\mathrm{gp},0}(\star_X, \Lambda_{\mathrm{loc}})$ (see Definition 3.20) by sending $(M, D) \in \mathcal{B}(\Lambda \to \mathbb{Z}) \times \mathcal{A}^{\mathrm{gp}}(\star_X, \Lambda_{\mathrm{loc}})$ to

$$\mathrm{Hair}^{\nu}(M, D) = \exp_{\sqcup}\left(-\frac{1}{2}\, \chi(M)\right) \sqcup \mathrm{Hair}(D) \sqcup \nu(h_1) \cdots \sqcup \nu(h_g),$$

where $\nu(h) = Z(S^3, U)$ whose legs are colored by h and

$$\chi(M) = \mathrm{tr}\log'(M) := \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\, \mathrm{tr}\left((M(\epsilon M)^{-1} - I)^n\right)$$

where $\epsilon : \Lambda \to \mathbb{Z}$. Note that the $\exp_{\sqcup}$ term in the above expression is a disjoint union of H-colored *wheels*.

Finally, let Hair^{ν} be a slightly renormalized version Hair^{ν} of the Hair map defined by

$$\mathrm{Hair}^{\nu}(M, D) = \mathrm{Hair}(M, D) \sqcup \nu(h_1) \sqcup \cdots \sqcup \nu(h_g).$$

$$t_i \to \sum_{n=0}^{\infty} \frac{1}{n!} \quad n\ h_i\text{-labeled legs}$$

Figure 6: From beads to legs

Lemma 7.2 *There exist maps that fit in a commutative diagram*

$$\begin{tikzcd}
\mathcal{A}^{\mathrm{gp},0}(\circledast_X, \Lambda_{\mathrm{loc}})/\langle \omega^{\mathrm{gp}} \rangle \arrow[rr] \arrow[d] & & \mathcal{A}^{0}(\circledast_X, \Lambda_{\mathrm{loc}})/\langle \omega \rangle \arrow[d] \\
\mathcal{A}^{\mathrm{gp},0}(\circledast_X, \Lambda_{\mathrm{loc}})/\langle \omega^{\mathrm{gp}}, \mathrm{String}_g \rangle \arrow[rr] \arrow[dr, "\mathrm{Hair}"] & & \mathcal{A}^{0}(\circledast_X, \Lambda_{\mathrm{loc}})/\langle \omega, \mathrm{String}_g \rangle \arrow[dl, "\mathrm{Hair}"'] \\
& \mathcal{A}(\circledast_{X \cup H}) &
\end{tikzcd}$$

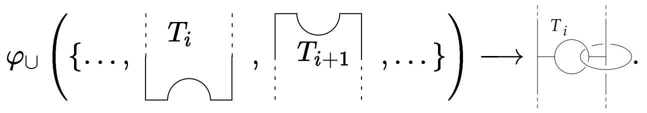
Proof Proposition D.1 implies that the horizontal maps are well-defined. It is easy to show that the X-flavored group-like and infinitesimal basing relations (defined in Section D) are mapped by the Hair-map to X-flavored infinitesimal basing relations. It is also easy to see that the ω^{gp} and String_g relations are mapped to H-flavored basing relations. $\square$

7.2 A comparison between Z^{rat} and Z

Consider the set $\mathcal{L}_X(\mathcal{O})$ of links $L \cup \mathcal{O}$ in S^3 where $\mathcal{O}$ is an unlink whose components are in 1-1 correspondence with the set $H = \{h_1, \dots, h_g\}$ and L is a link whose components are in 1-1 correspondence with a set X. Define a map

$$\varphi_\cup : \mathcal{D}^\times_X(\mathcal{O}) \to \mathcal{L}_X(\mathcal{O})$$

that maps the associated link L of a sliced crossed diagram in D_g to the link $L \cup \mathcal{O} \subset S^3$ according to a standard inclusion of $D_g \times I$ in S^3, by adding a g component unlink $\mathcal{O}$, one component for each gluing site of D_g as follows:



Consider the Kontsevich integral

$$\sigma_{X\cup H} \circ Z : \mathcal{L}_X(\mathcal{O}) \to \mathcal{A}(\circledast_{X\cup H})$$

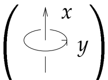
which takes values in the completed space of unitrivalent graphs whose legs are colored by $X \cup H$, modulo the AS, IHX relations and the *infinitesimal basing relations* discussed in Appendix C. The purpose of this section is to show that

Theorem 11 *The following diagram commutes*

$$\begin{tikzcd}
\mathcal{N}_X(\mathcal{O}) \arrow[r, "\sigma\circ Z^{\mathrm{rat}}"] \arrow[d, "\varphi_\cup"'] & \mathcal{A}^{\mathrm{gp}}(\circledast_X, \Lambda)/\langle \omega^{\mathrm{gp}} \rangle \arrow[d, "\mathrm{Hair}^\nu"] \\
\mathcal{L}_X(\mathcal{O}) \arrow[r, "\sigma\circ Z"] & \mathcal{A}(\circledast_{X\cup H})
\end{tikzcd}$$

The proof will utilize the following *magic formula* for the Kontsevich integral of the Long Hopf Link, conjectured in [4] (in conjunction with the so-called Wheels and Wheeling conjectures) and proven in [7].

Theorem 12 [7]

$$\left(\sigma_{\{x,y\}} \circ Z\right)\left(\right.$$


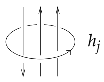
$$\left.\right) = \overset{x}{\updownarrow}\, e^{h} \sqcup \nu(h) \in \mathcal{A}(\uparrow_{\{x\}} \circledast_{\{y\}}).$$

Proof of Theorem 11 We will show first that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}^{\times}_X(\mathcal{O}) \arrow[r, "\sigma\circ Z^{\mathrm{rat}}"] \arrow[d, "\varphi\cup"'] & \mathcal{A}^{\mathrm{gp}}(\circledast_X, \Lambda) \arrow[d, "\mathrm{Hair}^{\nu}"] \\
\mathcal{L}_X(\mathcal{O}) \arrow[r, "\sigma\circ Z"] & \mathcal{A}(\circledast_{X\cup H})
\end{tikzcd} \tag{14}$$

Given a word w in the symbols $\uparrow$ and $\downarrow$, let

- $\mathrm{LongHopf}_{j,w}$ denote the long Hopf link cabled according to the word w, with the closed component labeled h_j. For example:

$$\mathrm{LongHopf}_{j,\downarrow\uparrow\uparrow} =$$


- $G^h_{j,w}$ denote the jth hairy gluing word corresponding to w. This is definition by example:

$$G^h_{j,\downarrow\uparrow\uparrow} = \updownarrow e^{-h_j} \updownarrow e^{h_j} \updownarrow e^{h_j}$$

We use the following corollary to Theorem 12.

Corollary 7.3

$$\left(\sigma_{\{h_j\}} \circ Z\right)(\mathrm{LongHopf}_{j,w}) = G^h_{j,w} \sqcup \nu(h_j) \in \mathcal{A}(\uparrow_w \circledast_{\{h_j\}}).$$

The commutativity of diagram (14) now follows by the definition of Z^{rat}. Theorem 11 follows from the commutativity of diagram (14), after considering the quotient $\mathcal{N}_X(\mathcal{O})$ of $\mathcal{D}^{\times}_X(\mathcal{O})$ (given by Proposition 2.12) and symmetrizing (ie, composing with $\sigma : \mathcal{A}^{\mathrm{gp}}(\uparrow_X, \Lambda) \to \mathcal{A}^{\mathrm{gp}}(\circledast_X, \Lambda)$) together with the fact that Hair^{ν} commutes with symmetrization. □

7.3 A comparison between $\int^{\mathrm{rat}}$ and $\int$

Recall that the Aarhus integral respects the group-like basing relations as shown in [5, Part II, Proposition 5.6] and also the infinitesimal basing relations as was observed by D. Thurston, see [6, Proposition 2.2]. In other words, there is a well-defined map

$$\int dX' : \mathrm{Int}_{X'}\mathcal{A}(\circledast_{X\cup H}) \to \mathcal{A}(\circledast_{(X-X')\cup H}),$$

where $H = \{h_1, \dots, h_g\}$. The purpose of this section is to show that

Theorem 13 *The following diagram commutes*

$$\begin{array}{ccc}
\mathrm{Int}_{X'}\mathcal{A}^{\mathrm{gp},0}(\circledast_X, \Lambda_{\mathrm{loc}}) & \xrightarrow{\int^{\mathrm{rat}} dX'} & \mathcal{A}^{\mathrm{gp},0}(\circledast_{X-X'}, \Lambda_{\mathrm{loc}}) \\
\Big\downarrow{\scriptstyle \mathrm{Hair}^{\nu}} & & \Big\downarrow{\scriptstyle \mathrm{Hair}^{\nu}} \\
\mathrm{Int}_{X'}\mathcal{A}(\circledast_{X\cup H}) & \xrightarrow{\int dX'} & \mathcal{A}(\circledast_{(X-X')\cup H})
\end{array}$$

Proof It follows from Remark E.1 of Appendix E. □

7.4 Z^{rat} is a rational form of the Kontsevich integral of a boundary link

Let us now formulate a main result of the paper:

Theorem 14 *There exists an invariant of F-links*

$$Z^{\mathrm{rat}} : F\text{-links} \longrightarrow \mathcal{A}^0(\Lambda_{\mathrm{loc}})/\langle\omega\rangle$$

which

(a) *descends to an invariant of boundary links*

$$Z^{\mathrm{rat}} : \partial - \text{links} \longrightarrow \mathcal{A}^0(\Lambda_{\mathrm{loc}})/\langle\omega, \mathrm{String}\rangle$$

(b) *determines the Kontsevich integral Z of a boundary link by*

$$Z = \mathrm{Hair}^{\nu} \circ Z^{\mathrm{rat}}.$$

(c) *determines the Blanchfield pairing of an F-link.*

Proof The invariant Z^{rat} that satisfies (a) was defined in Section 6.5.

Let us show (b). We begin by observing that the following diagram commutes, by the definition of the invariant $Z^{\mathrm{rat,gp}}$, using the work of Section 6:

$$\begin{tikzcd}
F\text{-links} \arrow[r, "Z^{\mathrm{rat,gp}}"] \arrow[d] & \mathcal{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}})/\langle \omega^{\mathrm{gp}} \rangle \arrow[d] \\
\partial\text{-links} \arrow[r, "Z^{\mathrm{rat,gp}}"] & \mathcal{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}})/\langle \omega^{\mathrm{gp}}, \mathrm{String} \rangle
\end{tikzcd}$$

We now claim that the following diagram commutes:

$$\begin{tikzcd}
\partial\text{-links} \arrow[r, "Z^{\mathrm{rat,gp}}"] \arrow[d, equal] & \mathcal{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}})/\langle \omega^{\mathrm{gp}}, \mathrm{String} \rangle \arrow[d, "\mathrm{Hair}^{\nu}"] \\
\partial\text{-links} \arrow[r, "Z"] & \mathcal{A}(\circledast_H)
\end{tikzcd} \tag{15}$$

The above two diagrams, together with Lemma 7.2 Theorem 14.In order to show the commutativity of (15), we will assemble several commutative diagrams in one with two layers. The top layer consists of the rational invariants Z^{rat} and its integration theory $\int^{\mathrm{rat}}$; the bottom layer consists of the LMO-Aarhus-Kontsevich integral Z and its integration theory $\int$.

$$\begin{tikzcd}
& \mathcal{N}_X(\mathcal{O}) \arrow[rr, two heads] \arrow[dl] \arrow[dd, dashed] & & \mathcal{N}_X(\mathcal{O})/\langle \kappa_X, \mathrm{String} \rangle \arrow[dl, "Z^{\mathrm{rat,gp}}"'] \arrow[dd, dashed] \\
\mathcal{A}^{\mathrm{gp}}(\circledast_X, \Lambda)/\langle \omega^{\mathrm{gp}} \rangle \arrow[rr, "\int^{\mathrm{rat}} dX"] \arrow[dd, dashed] & & \mathcal{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}})/\langle \omega^{\mathrm{gp}}, \mathrm{String} \rangle \arrow[dd, dashed] & \\
& \mathcal{L}_X(\mathcal{O}) \arrow[rr] \arrow[dl] & & \partial\text{-links} \arrow[dl, "Z"] \\
\mathcal{A}(\circledast_{X \cup H}) \arrow[rr, "\int dX"] & & \mathcal{A}(\circledast_H) &
\end{tikzcd}$$

Theorems 2, 11 and 13 imply that all faces of the above cube except the side right one commute. Since the map $\mathcal{N}_X(\mathcal{O}) \to \mathcal{L}_X(\mathcal{O})/\langle \kappa, \mathrm{String} \rangle$ is onto, it follows that the remaining face of the cube commutes.

Using the identification $\mathcal{A}(\circlearrowleft_H) \cong \mathcal{A}(\circledast_H)$ (see Equation (3)) it follows that diagram (15) commutes. This concludes the proof (b) of Theorem 14. (c) follows from [25]; see also the discussion of Section 2.1. $\square$

7.5 Comments on Theorem 14

Let us make some complimentary remarks on Theorem 14.

Remark 7.4 First a remark about the empty set. For the unlink $\mathcal{O}$ of g components, we have that $Z^{\mathrm{rat}}(\mathcal{O}) = (\text{empty matrix}, 1) \in \mathcal{A}^0(\Lambda_{\mathrm{loc}})/\langle \omega, \mathrm{String}\rangle$. On the other hand, the Kontsevich intregal of $\mathcal{O}$ is given by $Z(\mathcal{O}) = \nu(h_1) \sqcup \cdots \sqcup \nu(h_g)$ (where $\nu(h_i)$ is the Kontsevich integral of the unknot, whose diagrams are colored by h_i), and

$$\mathrm{Hair}^{\nu}(\text{empty matrix}, 1) = \nu(h_1) \sqcup \cdots \sqcup \nu(h_g).$$

This is how $Z^{\mathrm{rat}}(\mathcal{O})$ determines $Z(\mathcal{O})$, as stated in the above theorem.

Remark 7.5 Erasing a component L_i of a boundary link corresponds to substituting $t_i = 1$. In other words,

$$Z^{\mathrm{rat}}(M, L - L_i) = Z^{\mathrm{rat}}(M, L)|_{t_i=1}.$$

Remark 7.6 The Z^{rat} invariant of F-links takes values in a quotient of a set of pairs of matrices and vectors $\mathcal{A}^0(\Lambda_{\mathrm{loc}})$ by an equivalence relation. If we restrict $Z^{\mathrm{rat}}|_{\mathcal{S}}$ to a class $\mathcal{S}$ of F-links with the same Blanchfield pairing (or the same S-equivalence class), then $Z^{\mathrm{rat}}|_{\mathcal{S}}$ is equivalent to an invariant

$$\mathcal{S} \longrightarrow \mathcal{A}(\Lambda_{\mathrm{loc}})/(\omega)$$

that takes values in a *graded vector space*, where the degree of a graph is the number of trivalent vertices. In addition, the edges of the trivalent graphs are decorated by elements of a subring of Λ_{loc} that depends on $\mathcal{S}$. In particular, for acyclic F-links (ie, for links whose free cover is acyclic, or equivalently, whose Blanchfield pairing is trivial) Z^{rat} is equivalent to an invariant with values in the more manageable graded vector space $\mathcal{A}(\Lambda)/(\omega, \mathrm{String})$; see Section 5.6.

Proof Recall that the matrix part W of Z^{rat} is an element of $\mathcal{B}(\Lambda_{\mathrm{loc}} \to \mathbb{Z})$ and determines the S-equivalence class of the F-link, as follows from [25]. Recall also that the wrapping relations ω^{gp} and ω depend only on W. Thus, it follows that S-equivalent F-links have equal matrix-part, and that the matrix-free part of Z^{rat} makes sense as an element of the graded vector space $\mathcal{A}(\Lambda_{\mathrm{loc}})/(\omega)$.

[25, Theorem 3] implies that (M, L) is a good boundary link, (ie, $H_1(X^{\omega}, \mathbb{Z}) = 0$) iff its associated matrix W is invertible over Λ. In that case, the construction of the Z^{rat} invariant implies that $Z^{\mathrm{rat}}(M, L) \in \mathcal{A}^0(\Lambda)/\langle \omega, \mathrm{String}\rangle$. □

Remark 7.7 For $g = 1$, the sets of 1-component F-links and ∂-links coincide with the set of knots in integral homology 3-spheres, and the relations ω and String_1 are trivial, see Lemmas 3.26 and 6.4. In this case, Z^{rat} takes values in $\mathcal{B}(\Lambda_{\text{loc}} \to \mathbb{Z}) \times \mathcal{A}(\Lambda_{\text{loc}})$, and the matrix-free part of Z^{rat} takes values in the *graded algebra* $\mathcal{A}(\Lambda_{\text{loc}})$, where the multiplication in $\mathcal{A}(\Lambda_{\text{loc}})$ is given by the disjoint union of graphs. It turns out that the matrix-free part of Z^{rat} is a group-like element in $\mathcal{A}(\Lambda_{\text{loc}})$, ie, it equals to the exponential of a series of *connected* graphs.

For $g = 2$ (ie, for two component ∂-links) the relation String_g is trivial, see Lemma 6.4 and the set of 2-component F-links coincides with the one of ∂-links.

Remark 7.8 In a sense, Z is a graph-valued $\hat{\Lambda}$-invariant of ∂-links, where $\hat{\Lambda}$ is the completion of $\Lambda \otimes \mathbb{Q}$ with respect to the augmentation ideal whereas Z^{rat} is a graph-valued Λ_{loc}-invariant. It often happens that $\hat{\Lambda}$-invariants can be lifted to Λ_{loc}-invariants, see for example the discussion of Farber-Ranicki, [16], on a problem of Novikov about circle-valued Morse theory. It would be nice if an extension of this circle-valued Morse theory to infinite dimensions (using the circle-valued Chern-Simons action on the space of gauge equivalence classes of connections on a principal bundle over a 3-manifold) would lead to an independent construction of the Z^{rat}-invariant.

Remark 7.9 As will become apparent from the construction of the Z^{rat} invariant, it is closely related to equivariant linking matrices of nullhomotopic links in the complement of an F-link, see Proposition 8.1. It is an interesting question to ask whether the Z^{rat} invariant of an F-link can be defined in terms of an equivariant construction on the free cover of the F-link.

8 A universal property of Z^{rat}

8.1 Statement of the result

In this section we will show that the rational invariant Z^{rat}, evaluated on acyclic F-links, is characterized by a universal property. Namely, it is the universal finite type invariant of acyclic F-links with respect to the null filtration, [23].

Using the terminology of [23], a *null-move* on an F-link (M, L, ϕ) can be described by surgery on a clover G in $M - L$ whose leaves lie in the kerner of ϕ, [23]. The result $(M, L, \phi)_G$ of surgery is another F-link. As usual, one can define a notion of finite type invariants on the set of F-links using null moves; we will call the corresponding filtration the *Euler filtration*. The number of invariants of degree n can be identified with a quotient vector space $\mathcal{G}_n^{\mathrm{null}}$. For simplicity, we will confine ourselves to the case of *acyclic F-links*, that is F-links whose free cover X^ω satisfies $\bar{H}_\star(X^\omega, \mathbb{Z}) = 0$. In [23], it was shown that there is an onto map

$$\mathcal{A}(\Lambda)/\langle\omega\rangle \longrightarrow \mathcal{G}^{\mathrm{null}}$$

which preserves degrees. The next theorem constructs an inverse to this map and characterizes Z^{rat} by a universal property.

Theorem 15 *Z^{rat} is a universal $\mathcal{A}(\Lambda)/\langle\omega\rangle$-valued finite type invariant of acyclic F-links with respect to the Euler filtration. In particular, we have that $\mathcal{A}(\Lambda) \cong \mathcal{G}^{\mathrm{null}}$, over $\mathbb{Q}$.*

The arguments in this section are minor modifications of arguments given in [18] for the case of knots. Since the work of [18] predated the existence of the rational form Z^{rat}, and since we prefer to be self-sufficient, we will repeat the arguments here.

8.2 A relative version of the Z^{rat} invariant

We already discussed in the introduction (before the statement of Theorem 15) the notion of finite type invariants of F-links, based on the null move. The null move is described in terms of surgery on a clasper whose leaves lie in the kernel of the basing map ϕ of an F-link.

Our task here is to give a *relative version* of the Z^{rat} invariant for null links in the complement of an F-link. Given an F-link (M, L, ϕ), where $\phi : \pi_1(M \smallsetminus L) \to F$, we will call a link $C \in M \smallsetminus L$ *null relative to* (M, L, ϕ) if each component of C is mapped to 1 under ϕ. Notice that surgery under a null link preserves the underlying F-structure. In what follows, we will omit ϕ from the notation, for typographical reasons.

Given a null link C relative to an F-link (M, L), choose a sliced crossed link $L' \cup C' \in \mathcal{D}^\times(D_g)$ such that after surgery on $L' \subset S^3 \smallsetminus \mathcal{O}$ it gives rise to the F-link (M, L) with $C' \subset S^3 \smallsetminus \mathcal{O}$ transformed to a null link C relative to (M, L). We define

$$\begin{aligned} Z^{\mathrm{rat,gp}}((M,L),C) &= \frac{\int^{\mathrm{rat}} dX(\sigma \circ \check{Z}^{\mathrm{rat}}(L' \cup C'))}{(\text{normalization})} \in \mathcal{A}^{\mathrm{gp},0}(\circledast_Y, \Lambda_{\mathrm{loc}})/\langle\omega^{\mathrm{gp}}\rangle \\ Z^{\mathrm{rat,gp}}((M,L)_C) &= \frac{\int^{\mathrm{rat}} dY(Z^{\mathrm{rat}}((M,L),C))}{(\text{normalization})} \in \mathcal{A}^{\mathrm{gp},0}(\circledast, \Lambda_{\mathrm{loc}})/\langle\omega^{\mathrm{gp}}\rangle \end{aligned}$$

where X (resp. Y) is in 1-1 correspondence with the components of L (resp. C), and where (normalization) refers to factors inserted to deal with the Kirby move of adding a distant union of unit-framed unlinks, as in Section 5.6. Theorem 14 has the following relative version (as is revealed by its proof in Section 5.6):

Theorem 16 *There exists an invariant*

$$Z^{\mathrm{rat,gp}} : F\text{-links} \cup \mathrm{null}\ C \to \mathcal{A}^{\mathrm{gp},0}(\circledast_C, \Lambda_{\mathrm{loc}})/\langle\omega\rangle$$

which is local (ie, compatible with respect to considering sublinks of C) and fits in a commutative diagram

$$\begin{array}{ccc} F\text{-links} \cup \mathrm{null}\ C & \xrightarrow{Z^{\mathrm{rat,gp}}} & \mathcal{A}^{\mathrm{gp},0}(\circledast_Y, \Lambda_{\mathrm{loc}})/\langle\omega^{\mathrm{gp}}\rangle \\ {\scriptstyle C-\mathrm{surgery}}\Big\downarrow & & \Big\downarrow{\scriptstyle \int^{\mathrm{rat}} dY} \\ F\text{-links} & \xrightarrow{Z^{\mathrm{rat,gp}}} & \mathcal{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}})/\langle\omega^{\mathrm{gp}}\rangle \end{array}$$

As in Section 5.6, we denote by Z^{rat} the composition of $Z^{\mathrm{rat,gp}}$ with the map

$$\mathcal{A}^{\mathrm{gp},0}(\circledast_Y, \Lambda_{\mathrm{loc}})/\langle\omega^{\mathrm{gp}}\rangle \to \mathcal{A}^{0}(\circledast_Y, \Lambda_{\mathrm{loc}})/\langle\omega\rangle.$$

Given a null link C relative to (M,L), one can define an *equivariant linking matrix* $\mathrm{lk}_{(M,L)}(C)$, by considering the linking numbers of the components of the lift $\widetilde{C}$ of C in the free cover of $M-L$. $\mathrm{lk}_{(M,L)}(C)$ is a Hermitian matrix over Λ_{loc}, well defined up to conjugation by a diagonal matrix with elements in F. The details of the definition, together with the following proposition will appear in a subsequent publication, [23]:

Proposition 8.1 *For C and (M,L) as above, the covariance matrix* cov *of Z^{rat} satisfies*

$$\mathrm{cov}(Z^{\mathrm{rat}}((M,L),C)) = \mathrm{lk}_{(M,L)}(C).$$

The discussion in this section simplifies considerably when the F-links in question are acyclic. In this case, there is no need to localize, and the wrapping relation becomes trivial. In other words, we get an invariant

$$Z^{\mathrm{rat}} : \text{Acyclic } F\text{-links} \cup \text{null links} \longrightarrow \mathcal{A}(\circledast_Y, \Lambda)/\langle\omega\rangle.$$

8.3 Surgery on claspers

Let us review in brief some elementary facts about surgery on claspers. For a more detailed discussion, we refer the reader to [22] and also [18, Section 3].

Consider a null clasper G of degree 1 in $S^3 - \mathcal{O}$ (a so-called Y-graph). Surgery on G can be described by surgery on a six component link $E \cup L$ associated to G, where E (resp. L) is the three component link that consists of the edges (resp. leaves) of G.

Lemma 8.2 *The equivariant linking matrix of $E \cup L$ and its negative inverse are given as follows:*

$$\begin{pmatrix} 0 & I \\ I & \mathrm{lk}_{(S^3,\mathcal{O})}(L_i, L_j) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathrm{lk}_{(S^3,\mathcal{O})}(L_i, L_j) & -I \\ -I & 0 \end{pmatrix}. \tag{16}$$

Proof The 0 block follows from the fact that $\{E_i, E_j, \mathcal{O}\}$ is an unlink. The I block follows from the fact that E_i is a meridian of L_i, using the formula for the Kontsevich integral of the Long Hopf Link, together with the fact that the linking number between L_i and $\mathcal{O}$ is zero. □

The six component link $E \cup L$ is partitioned in three blocks of two component links $A_i = \{E_i, L_i\}$ each for $i = 1, 2, 3$, the *arms* of G. A key feature of surgery on G is the fact that surgery on any proper subset of the set of arms does not alter M. In other words, alternating $(S^3, \mathcal{O})$ with respect to surgery on G equals to alternating $(S^3, \mathcal{O})$ with respect to surgery on all nine subsets of the set of arms $A = \{A_1, A_2, A_3\}$. That is,

$$Z^{\mathrm{rat}}([(S^3, \mathcal{O}), G]) = Z^{\mathrm{rat}}([(S^3, \mathcal{O}), A]). \tag{17}$$

Due to the locality property of the Z^{rat} invariant (ie, Theorem 16) the nontrivial contributions to the right hand side of Equation (17) come from the *strutless part* $Z^{\mathrm{rat},t}((S^3, \mathcal{O}), A_1 \cup A_2 \cup A_3)$ of $Z^{\mathrm{rat},t}((S^3, \mathcal{O}), A_1 \cup A_2 \cup A_3)$ that consists of graphs with legs *touch* (ie, are colored by) all three arms of G. The above discussion generalizes to the case of an arbitrary disjoint union of claspers.

8.4 Counting above the critical degree

Proposition 8.3 *The Euler degree n part of Z^{rat} is a type n invariant of acyclic F-links with values in $\mathcal{A}_n(\Lambda)$.*

Proof Since $\mathcal{A}_{\text{odd}}(\Lambda) = 0$, it suffices to consider the case of even n. Suppose that $G = \{G_1, \dots, G_m\}$ (for $m \geq 2n+1$) is a collection of null claspers in $S^3 - \mathcal{O}$ each of degree 1, and let A denote the set of arms of G. Equation (17) and its following discussion implies that

$$Z^{\text{rat}}([(S^3, \mathcal{O}), G]) = Z^{\text{rat}}([(S^3, \mathcal{O}), A])$$

and that the nonzero contribution to the right hand side come from diagrams in $Z^{\text{rat},t}((S^3, \mathcal{O}), A)$ that touch all arms. Thus, contributing diagrams have at least $3(2n+1)+1 = 6n+4$ A-colored legs, to be glued pairwise.

Notice that the diagrams in $Z^{\text{rat},t}((S^3, \mathcal{O}), A)$ contain no struts. Thus, at most three A-colored legs meet at a vertex, and after gluing the A-colored legs we obtain trivalent graphs with at least $(6n+4)/3 = 2n + 4/3$ trivalent vertices, in other words of Euler degree at least $2n+2$. Thus, $Z^{\text{rat}}_{2n}([(S^3, \mathcal{O}), G]) = 0 \in \mathcal{A}_{2n}(\Lambda)$, which implies that Z^{rat}_{2n} is a invariant of acyclic F-links of type $2n$ with values in $\mathcal{A}_{2n}(\Lambda)$. □

Sometimes the above vanishing statement is called *counting above the critical degree.*

8.5 Counting on the critical degree

Our next statement can be considered as *counting on the critical degree.* We need a preliminary definition.

Definition 8.4 Consider a null clasper G in $S^3 - \mathcal{O}$ of degree $2n$, and let $G^{break} = \{G_1, \dots, G_{2n}\}$ denote the collection of degree 1 claspers G_i which are obtained by inserting a Hopf link in the edges of G. Let $G^{nl} = \{G_1^{nl}, \dots, G_{2n}^{nl}\}$ denote the collection of abstract unitrivalent graph obtained by removing the leaves of the G_i (and leaving one leg, or univalent vertex, for each leave behind). Choose arcs from a fixed base point to the trivalent vertex of each G_i^{nl}, which allows us to define the equivariant linking numbers of the leaves of G^{break}. Then the *complete contraction* $\langle G \rangle \in \mathcal{A}(\Lambda)$ of G is defined to be the sum over all ways of gluing pairwise the legs of G^{nl}, and placing as a bead the equivariant linking number of the corresponding leaves.

The result of a complete contraction of a null clasper G is a well-defined element of $\mathcal{A}(\Lambda)$. Changing the arcs is taken care by the Vertex Invariance relations in $\mathcal{A}(\Lambda)$. The next proposition computes the *symbol* of Z^{rat}:

Proposition 8.5 *If G is a null clasper of degree $2n$ in $S^3 \smallsetminus \mathcal{O}$, then*

$$Z^{\mathrm{rat}}_{2n}([(S^3,\mathcal{O}),G]) = \langle G\rangle \in \mathcal{A}_{2n}(\Lambda)/\langle\omega\rangle$$

Proof It suffices to consider a collection $G=\{G_1,\dots,G_{2n}\}$ of claspers in S^3 each of degree 1. Let A denote the set of arms of G. The counting argument of the above Proposition shows that the contributions to $Z^{\mathrm{rat}}_{2n}([(S^3,\mathcal{O}),G]) = Z^{\mathrm{rat}}_{2n}([(S^3,\mathcal{O}),A])$ come from complete contractions of a disjoint union $D = Y_1\cup\dots\cup Y_{2n}$ of $2n$ *vortices.* A vortex is the diagram Y, the next simplest unitrivalent graph after the strut. Furthermore, the $6n$ legs of D should touch all $6n$ arms of G. In other words, there is a 1-1 correspondence between the legs of such D and the arms of G.

Consider a leg l of D that touches an arm $A_l=\{E_l,L_l\}$ of G. If l touches L_l, then due to the restriction of the negative inverse linking matrix of G (see Lemma 8.2), it needs to be contracted to another leg of D that touches E_l. But this is impossible, since the legs of D are in 1-1 correspondence with the arms of G.

Thus, each leg of D touches presicely one edge of G. In particular, each leg of D is colored by three edges of G.

Consider a vortex colored by three edges of G as part of the Z^{rat} invariant. Since the three edges of G is an unlink in a ball disjoint from $\mathcal{O}$, it follows that the beads on the edges of the vortex are 1, if the three edges belong to the same G_i, and 0 otherwise.

Thus, the diagrams D that contribute are a disjoint union of $2n$ vortices $Y=\{Y_1,\dots,Y_{2n}\}$ and these votrices are in 1-1 correspondence with the set of claspers $\{G_1,\dots,G_{2n}\}$, in such a way that the legs of each vortex Y_i are colored by the edges of a unique clasper G_j.

After we glue the legs of such Y using the negative inverse linking matrix of G, the result follows. □

Let us mention that the discussion of Propositions 8.3 and 8.5 really applied to the unnormalized rational Aarhus integral; however since we are counting above the critical degree, we need only use the degree 0 part of the normalization which equals to 1; in other words we can forget about the normalization.

The above proposition is useful in realization properties of the Z_{2n} invariant, but also in proving the following *Universal Property*:

Proposition 8.6 *For all n, the composite map of [23] with that of Proposition 8.5*

$$\mathcal{A}_{2n}(\Lambda)/\langle\omega\rangle \longrightarrow \mathcal{G}_{2n}^{\mathrm{null}} \xrightarrow{Z_{2n}^{\mathrm{rat}}} \mathcal{A}_{2n}(\Lambda)/\langle\omega\rangle$$

is the identity. Since the map on the left is onto, it follows that the above maps are isomorphism.

Remark 8.7 In the above propositions 8.3-8.6, S^3 can be replaced by any integral homology 3-sphere (or even a rational homology 3-spheres) M. □

Clearly, Propositions 8.3-8.6 imply Theorem 15.

9 Relations of the Z^{rat} invariant with Homology Surgery

In our paper, we constructed an invariant Z^{rat} of F-links. The construction of this invariant leads naturally to the noncommutative localization of $\Lambda = \mathbb{Z}[F]$; see Fact 8 in Section 1.5. Thus, our paper leads to a natural relation (from the point of view of Qunatum Topology) between F-links and noncommutative localization.

Two previous relations between F-links and noncommutative localization are known through the work of Cappell-Shaneson and Farber-Vogel; see [10, 11, 49, 14]. Let us sidcuss those briefly, pointing out that this subject, well-known among senior topologists, is not as well-known to quantum topologists.

Cappell-Shaneson reduced the problems of *Homology Surgery* to the computation of Ga-groups; [10, 11]. A typical problem of Homology Surgery the following: given a knot K in S^3, consider the manifold $M = S^3_{K,0}$ obtained by 0-surgery on K. Of course it is true that $H_\star(M,\mathbb{Z}) = H_\star(S^2\times S^1,\mathbb{Z})$, but the problem is to decide when is the case that the equivariant homology of M and $S^2\times S^1$ coincides: $H_\star(M,\Lambda) = H_\star(S^2\times S^1,\Lambda)$ for $\Lambda = \mathbb{Z}[Z]$.

The Γ groups $\Gamma(\Lambda\to\mathbb{Z})$ of Cappell-Shaneson (interpreted in terms of matrices, a la Levine) ask for a classification of cobordism of Hermitian matrices over Λ which are invertible over $\mathbb{Z}$.

These Γ groups are hard to compute, but Cappell-Shaneson reduced the problem of classification of F-links modulo cobordism (in high enough dimensions) to the computation of their Γ-groups.

Vogel identified these Γ-groups in terms of Wall surgery L-groups of a suitable localization of Λ, [49]. Later on, Vogel and Farber identified this localization with the ring of rational functions in noncommuting variables, [14].

This explains the two previously known relations among F-links and the noncommutative localization.

Later on, Farber constructed an invariant of F-links with values in Λ_{loc}, [15]. This invariant was reinterpreted in terms Seifert surfaces (and the surgery view of F-links) in [20].

This reinterpretation of Farber's invariant, together with the results of [25] imply that the "matrix part" of our Z^{rat}-invariant of F-links determines the Blanchfield pairing of them.

One last comment: if one tries to develop a theory of finite type invariants for degree 1 maps, then one is naturally lead to the notion of beads, which lie in $\mathbb{Z}[\pi]$ for the fundamental group π of the target manifold. This was discovered in joint work of Levine and the first author; see [19]. Presumably, one could try to construct an invariant for degree 1 maps in the spirit of the Z^{rat} invariant, which would take values in a completed space of trivalent graphs with beads in the noncommutative localization of $\mathbb{Z}[\pi]$. We will develop this in a later publication.

Appendices

A A brief review of diagrammatic calculus

A.1 Some useful identities

In this section we establish some notations and collect some lemmas. Let X denote some finite set. Given a sum of diagrams s with legs colored by a set X, and a subset $X' = \{x_1, \dots, x_r\}$ of X, we will often denote s by $s(x_1, \dots, x_r)$ or simply by $s(x)$. This notation allows us to extend linearly to diagrams whose legs are labeled by formal linear combinations of elements of X. For example, for $a, b \in \mathbb{Q}$, $v, w \in X$, $s(\dots, x_{i-1}, av + bw, x_{i+1}, \dots)$ denotes the element obtained from s by replacing each diagram appearing in s by the sum of all diagrams obtained by relabeling each leg labeled x_i by either v (with a multiplicative factor of a) or w (with a multiplicative factor of b). Given diagrams $A(x), B(x)$, we let

$$\langle A(\partial x), B(x) \rangle_X \quad \text{and} \quad A(\partial x) \flat_X B(x)$$

denote respectively the sum of all ways of pairing all X-colored legs of A with all (resp. some) X-colored legs of B.

Furthermore, for some $q \in \Lambda_{\mathrm{loc}}$, $s(\ldots, x_{i-1}, x_i q, x_{i+1}, \ldots)$ denotes the element obtained by pushing a bead labeled by q onto each leg labeled x_i in the sense specified by the following diagram:

$$\underset{x_i}{\uparrow}\, 1 \to \underset{x_i}{\uparrow}\, q$$

Taken together, these conventions give meaning to such expressions as $s(xM)$, for $M \in \mathrm{Herm}(\Lambda_{\mathrm{loc}} \to \mathbb{Z})$.

The following lemma collects a number of useful identities, which the reader would be well-advised to understand.

Lemma A.1 (a) *Let $A \in \mathcal{A}(\star_X, \Lambda_{\mathrm{loc}}), B \in \mathcal{A}(\star_{X\cup Y}, \Lambda_{\mathrm{loc}})$ (for Y a bijective copy of X), and let X' and X'' denote bijective copies of X.*

$$\langle A(\partial x), B(x,y)|_{x=y}\rangle_X = \langle A(\partial x' + \partial x''), B(x', x'')\rangle_{X'\cup X''} \tag{18}$$

(b) *Let $A_1, A_2, B \in \mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$.*

$$\langle A_1(\partial x) A_2(\partial x), B(x)\rangle_X = \langle A_2(\partial X), A_1(\partial x) \flat_X B(x)\rangle_X. \tag{19}$$

(c) *Let $A, B \in \mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$, and let Y be a bijective copy of X.*

$$A(\partial x) \flat_X B(x) = \langle A(\partial y), B(x+y)\rangle_Y \tag{20}$$

(d) *Let $A = \exp(a) \in \mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$ and $B = \exp(b) \in \mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$. Then*

$$\exp(a) \flat_{X'} \exp(b) = \exp(c), \tag{21}$$

where c is the sum of all connected *diagrams that arise by pairing* all *legs labeled from X' of some diagram (not necessarily connected) appearing in A* to some *legs of some diagram (not necessarily connected) appearing in B.*

(e) *Let $A \in \mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$, let Y be a bijective copy of X, and let $M \in \mathrm{Herm}(\Lambda_{\mathrm{loc}} \to \mathbb{Z})$.*

$$\left(\exp\left(\overset{\partial x_j}{\underset{y_i}{\uparrow}} M_{ij}\right)\right) \flat_X A(x) = A(yM). \tag{22}$$

(f) *In particular,*

$$\left\langle \exp\begin{pmatrix}\partial x\\ |\\ y\end{pmatrix}, B(x)\right\rangle_x = B(y) \tag{23}$$

$$\exp\begin{pmatrix}\partial x\\ |\\ y\end{pmatrix} \flat_x \exp(B(x)) = \exp\left(\sum \begin{array}{l}\text{all ways of replacing some of}\\ \text{the } x\text{-legs of B by } y\text{-legs}\end{array}\right) \tag{24}$$

(g) *In addition, if $s(\partial z) \in \mathcal{A}(\star_{X\cup\{h',h'',\partial z\}}, \Lambda_{\mathrm{loc}})$ then*

$$\mathrm{con}_{\{h',h''\}}(s(\partial h' + \partial h'')) = \mathrm{con}_{\{h\}}(s(\partial h)|_{h',h''\mapsto h}). \tag{25}$$

A.2 Iterated Integration

This section proves an Iterated Integration property of the $\int^{\mathrm{rat}}$ integral. Fix labeling sets $X'' \subset X' \subset X$.

Lemma A.2 (Iterated integration) *Given* $s \in \mathrm{Int}_{X'}\mathcal{A}(\star_X, \Lambda_{\mathrm{loc}}) \cap \mathrm{Int}_{X''}\mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$, *then,*

$$\begin{aligned} \int^{\mathrm{rat}} dX''(s) &\in \mathrm{Int}_{X'-X''}\mathcal{A}(\star_{X-X''}, \Lambda_{\mathrm{loc}}) \\ \int^{\mathrm{rat}} d(X'-X'')\left(\int^{\mathrm{rat}} dX''(s)\right) &= \int^{\mathrm{rat}} dX'(s). \end{aligned}$$

Proof Denote the components of X' and X'' by a_i and b_j respectively. Consider the covariance matrix of s with respect to X', presented with respect to this basis:

$$M = \begin{bmatrix} A & C \\ C^\star & B \end{bmatrix},$$

where $A = A^\star$ and $B = B^\star$, where here and below $A^\star$ denotes the transpose of C, followed by the conjugation $t_i \to t_i^{-1}$. If the canonical decomposition of s with respect to X' is then,

$$\exp\left(\frac{1}{2} \overset{a_j}{\underset{a_i}{\updownarrow}} A_{ij} + \frac{1}{2} \overset{b_j}{\underset{b_i}{\updownarrow}} B_{ij} + \overset{b_j}{\underset{a_i}{\updownarrow}} C_{ij}\right) \sqcup R,$$

(suppressing the summations) then the canonical decomposition of s with respect to X'' is

$$\exp\left(\frac{1}{2} \overset{a_j}{\underset{a_i}{\updownarrow}} A_{ij}\right) \sqcup \left(\exp\left(\frac{1}{2} \overset{b_j}{\underset{b_i}{\updownarrow}} B_{ij} + \overset{b_j}{\underset{a_i}{\updownarrow}} C_{ij}\right) \sqcup R\right).$$

Using the identities of Lemma A.1, we have

$$\begin{aligned} &\int^{\mathrm{rat}} dX''(s) \\ &= \left\langle \exp\left(-\frac{1}{2} \overset{\partial a_j}{\underset{\partial a_i}{\updownarrow}} A^{-1}_{ij}\right), \exp\left(\frac{1}{2} \overset{b_j}{\underset{b_i}{\updownarrow}} B_{ij} + \overset{b_j}{\underset{a_i}{\updownarrow}} C_{ij}\right) \sqcup R \right\rangle_{X''} \\ &= \left\langle \left(\exp\left(\overset{b_j}{\underset{\partial a_i}{\updownarrow}} C_{ij}\right) \flat_{X'} \exp\left(-\frac{1}{2} \overset{a_j}{\underset{a_i}{\updownarrow}} A^{-1}_{ij}\right)\right)\Bigg|_{a_i \to \partial a_i}, \right. \\ &\qquad \left. \exp\left(\frac{1}{2} \overset{b_j}{\underset{b_i}{\updownarrow}} B_{ij}\right) \sqcup R \right\rangle_{X''} \\ &= \exp\left(\frac{1}{2} \overset{b_j}{\underset{b_i}{\updownarrow}} (B - C^\star A^{-1} C)_{ij}\right) \sqcup \\ &\qquad \left(\left\langle \exp\left(-\frac{1}{2} \overset{\partial a_j}{\underset{\partial a_i}{\updownarrow}} A^{-1}_{ij} - \overset{b_j}{\underset{\partial a_i}{\updownarrow}} (A^{-1}C)_{ij}\right), R \right\rangle_{X''}\right) \end{aligned}$$

It is easy to see that if M and A are invertible over $\mathbb{Z}$, then so is $B - C^\star A^{-1}C$ over $\mathbb{Z}$, see [5, Part II, Proposition 2.13]. Thus, the first part of the theorem follows. Continuing,

$$
\begin{aligned}
&\int^{\mathrm{rat}} d(X'' - X')\left(\int^{\mathrm{rat}} dX'(s)\right) \\
&= \left\langle \exp\left(-\frac{1}{2} \overset{\partial b_j}{\underset{\partial b_i}{\uparrow}} (B - C^\star A^{-1}C)^{-1}_{ij}\right), \right. \\
&\qquad \left. \left\langle \exp\left(-\frac{1}{2} \overset{\partial a_j}{\underset{\partial a_i}{\uparrow}} A^{-1}_{ij} \;-\; \overset{b_j}{\underset{\partial a_i}{\uparrow}} (A^{-1}C)_{ij}\right), R\right\rangle_{X''} \right\rangle_{X'-X''} \\
&= \left\langle \left(\exp\left(- \overset{\partial b_j}{\underset{\partial a_i}{\uparrow}} (A^{-1}C)_{ij}\right) \flat_{X'-X''} \exp\left(-\frac{1}{2} \overset{b_j}{\underset{b_i}{\uparrow}} (B - C^\star A^{-1}C)^{-1}_{ij}\right)\right)\Bigg|_{b_i \to \partial b_i} \right. \\
&\qquad \left. \exp\left(-\frac{1}{2} \overset{\partial a_j}{\underset{\partial a_i}{\uparrow}} A^{-1}_{ij}\right), R\right\rangle_{X'} \\
&= \left\langle \exp\left(-\frac{1}{2} \overset{\partial a_j}{\underset{\partial a_i}{\uparrow}} (A^{-1} + A^{-1}C(B - C^\star A^{-1}C)^{-1}C^\star A^{-1})_{ij} \right.\right. \\
&\qquad \left.\left. + \overset{\partial b_j}{\underset{\partial a_i}{\uparrow}} (A^{-1}C(B - C^\star A^{-1}C)^{-1})_{ij} - \frac{1}{2} \overset{\partial b_j}{\underset{\partial b_i}{\uparrow}} (B - C^\star A^{-1}C)^{-1}_{ij}\right), R\right\rangle_{X'}.
\end{aligned}
$$

The identification of the term on the left of the pairing with $\exp\left(-\frac{1}{2} \overset{\partial x_j}{\underset{\partial x_i}{\uparrow}} M^{-1}_{ij}\right)$ completes the proof. □

Remark A.3 With the notation of the previous lemma, the matrices $A \oplus (B - C^\star A^{-1}C)$ and M are congruent by a product of elementary matrices, as follows from the following identity

$$
\begin{bmatrix} A & 0 \\ 0 & B - C^\star A^{-1}C \end{bmatrix} = \begin{bmatrix} I & -AC^\star \\ 0 & I \end{bmatrix}^\star \begin{bmatrix} A & C \\ C^\star & B \end{bmatrix} \begin{bmatrix} I & -AC^\star \\ 0 & I \end{bmatrix}.
$$

and from the fact that the conjugating matrix is a product of elementary matrices. It follows from this, that the Iterated Integration Lemma A.2 is valid on $\mathcal{A}^{\mathrm{gp},0}(\star_X, \Lambda_{\mathrm{loc}})$.

A.3 Integration by parts

Fix labeling sets X and $X' \subset X$. Recall from Section 5.2 the notion of X'-substantial diagrams, that is diagrams that do not contain a strut labeled from $X' \cup \partial X'$. The *divergence* of an X'-substantial diagram (extended by linearity to sums of diagrams) with respect to $X' \subset X$ is defined to be the sum of diagrams obtained by pairing *all* legs labeled from $\partial_{X'}$ with *some* legs labeled from X'.

Lemma A.4 (Integration by Parts) *Let s be an X'-substantial diagram and $t \in \mathrm{Int}_{X'}\mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$. Then*

$$\begin{aligned} s \flat_X t &\in \mathrm{Int}_{X'}\mathcal{A}(\star_X, \Lambda_{\mathrm{loc}}) \\ \int^{\mathrm{rat}} dX'(s \flat_X t) &= \int^{\mathrm{rat}} dX'(\mathrm{div}_{X'}(s) \flat_{X-X'} t). \end{aligned}$$

Proof For the first statement, assume, for convenience, that $X' = \{f_i\}$ and $X - X' = \{g_j\}$, and rename X' and $X - X'$ by F and G respectively. Applying Lemma A.1.(20), introducing a labeling set $\tilde{F}$ (resp. $\tilde{G}$), bijective with F (resp. G) (and associated sets $\partial_{\tilde{F}}$ and $\partial_{\tilde{G}}$), gives the expression in question as:

$$s(f, g, \partial f, \partial g) \flat_{F\cup G} t(f, g) = \left\langle s(f, g, \partial \tilde{f}, \partial \tilde{g}), t(f + \tilde{f}, g + \tilde{g}) \right\rangle_{\tilde{F}\cup\tilde{G}}$$

If the canonical decomposition of t with respect to F is

$$t(f, g) = \exp\left(\frac{1}{2} \overset{f_j}{\underset{f_i}{\uparrow}} M_{ij}\right) \sqcup T(f, g),$$

then the previous expression is equal to

$$\exp\left(\frac{1}{2} \overset{f_j}{\underset{f_i}{\uparrow}} M_{ij}\right) \sqcup \left\langle s(f, g, \partial \tilde{f}, \partial \tilde{g}), \exp\left(\overset{\tilde{f}_j}{\underset{f_i}{\uparrow}} M_{ij} + \frac{1}{2} \overset{\tilde{f}_j}{\underset{\tilde{f}_i}{\uparrow}} M_{ij}\right) \sqcup T(f + \tilde{f}, g + \tilde{g}) \right\rangle_{\tilde{F}\cup\tilde{G}}.$$

Now, because s is an X'-substantial operator, this gives the required decomposition of $s \flat_{F\cup G} t$ with respect to F.

For the second statement, we compute

$$\begin{aligned} &\int^{\mathrm{rat}} dF(s \flat_{F\cup G} t) \\ &= \Bigg\langle \exp\left(-\frac{1}{2} \overset{\partial f_j}{\underset{\partial f_i}{\uparrow}} M^{-1}_{ij}\right), \\ &\qquad \left\langle s(f, g, \partial \tilde{f}, \partial \tilde{g}), \exp\left(\overset{\tilde{f}_j}{\underset{f_i}{\uparrow}} M_{ij} + \frac{1}{2} \overset{\tilde{f}_j}{\underset{\tilde{f}_i}{\uparrow}} M_{ij}\right) \sqcup T(f + \tilde{f}, g + \tilde{g}) \right\rangle_{\tilde{F}\cup\tilde{G}} \Bigg\rangle_F. \end{aligned}$$

To proceed, we must adjust this calculation so that it can distinguish struts, appearing in the left of the pairing, according to which factors, appearing in the right of the pairing, they are glued to. This adjustment is made with Lemma A.1.(18), and leads to the following (introducing sets F^a, F^b and F^c, bijective copies of F):

$$\Bigg\langle \exp\Bigg(-\frac{1}{2}\overset{\partial f^a_j}{\underset{\partial f^a_i}{\uparrow}} M^{-1}_{ij} \;\; -\frac{1}{2}\overset{\partial f^b_j}{\underset{\partial f^b_i}{\uparrow}} M^{-1}_{ij} \;\; -\frac{1}{2}\overset{\partial f^c_j}{\underset{\partial f^c_i}{\uparrow}} M^{-1}_{ij}$$
$$-\overset{\partial f^a_j}{\underset{\partial f^b_i}{\uparrow}} M^{-1}_{ij} \;\; -\overset{\partial f^a_j}{\underset{\partial f^c_i}{\uparrow}} M^{-1}_{ij} \;\; -\overset{\partial f^b_j}{\underset{\partial f^c_i}{\uparrow}} M^{-1}_{ij} \Bigg),$$
$$\Bigg\langle s(f^a, g, \partial\tilde{f}, \partial\tilde{g}), \exp\Bigg(\overset{\tilde{f}_j}{\underset{f^b_i}{\uparrow}} M_{ij} + \frac{1}{2}\overset{\tilde{f}_j}{\underset{\tilde{f}_i}{\uparrow}} M_{ij} \Bigg)$$
$$\sqcup T(f^c + \tilde{f}, g + \tilde{g}) \Bigg\rangle_{\tilde{F}\cup\tilde{G}} \Bigg\rangle_{F^a\cup F^b\cup F^c}.$$

The calculation proceeds by removing factors appearing in left hand side of the pairing, at the expense of operating with them on the right hand side, follow Lemma A.1.(19).

The first term to be treated in this way will be $\exp\Bigg(-\frac{1}{2}\overset{\partial f^b_j}{\underset{\partial f^b_i}{\uparrow}} M^{-1}_{ij} \Bigg)$, giving (following an application of Lemma A.1.(21)):

$$\exp\Bigg(-\frac{1}{2}\overset{\partial f^b_j}{\underset{\partial f^b_i}{\uparrow}} M^{-1}_{ij} \Bigg) \flat_{F^b} \exp\Bigg(\overset{\tilde{f}_j}{\underset{f^b_i}{\uparrow}} M_{ij} \Bigg) = \exp\Bigg(\overset{\tilde{f}_j}{\underset{f^b_i}{\uparrow}} M_{ij} \Bigg) \sqcup \exp\Bigg(-\frac{1}{2}\overset{\tilde{f}_j}{\underset{\tilde{f}_i}{\uparrow}} M_{ij} \Bigg).$$

The next term to be so treated will be $\exp\Bigg(-\overset{\partial f^c_j}{\underset{\partial f^b_i}{\uparrow}} M^{-1}_{ij} \Bigg)$, which gives:

$$\exp\Bigg(-\overset{\partial f^c_j}{\underset{\partial f^b_i}{\uparrow}} M^{-1}_{ij} \Bigg) \flat_{F^b\cup F^c} \Bigg(\exp\Bigg(\overset{\tilde{f}_j}{\underset{f^b_i}{\uparrow}} M_{ij} \Bigg) \sqcup T(f^c + \tilde{f}, g + \tilde{g}) \Bigg)$$
$$= \exp\Bigg(\overset{\tilde{f}_j}{\underset{f^b_i}{\uparrow}} M_{ij} \Bigg) \sqcup T((f^c - \tilde{f}) + \tilde{f}, g + \tilde{g}).$$

The result of these operations is

$$\Bigg\langle \exp\Bigg(-\frac{1}{2}\overset{\partial f^a_j}{\underset{\partial f^a_i}{\uparrow}} M^{-1}_{ij} \;\; -\frac{1}{2}\overset{\partial f^c_j}{\underset{\partial f^c_i}{\uparrow}} M^{-1}_{ij} \;\; -\overset{\partial f^a_j}{\underset{\partial f^b_i}{\uparrow}} M^{-1}_{ij} \;\; -\overset{\partial f^a_j}{\underset{\partial f^c_i}{\uparrow}} M^{-1}_{ij} \Bigg),$$
$$\Bigg\langle s(f^a, g, \partial\tilde{f}, \partial\tilde{g}), \exp\Bigg(\overset{\tilde{f}_j}{\underset{f^b_i}{\uparrow}} M_{ij} \Bigg) \sqcup T(f^c, g + \tilde{g}) \Bigg\rangle_{\tilde{F}\cup\tilde{G}} \Bigg\rangle_{F^a\cup F^b\cup F^c}.$$

Performing the pairing with respect to the variables $\tilde{F}$, leads to the replacement of the the inner $\langle \cdot, \cdot \rangle$ with

$$\big\langle s(f^a, g, f^b M, \partial\tilde{g}), T(f^c, g+\tilde{g}) \big\rangle_{\tilde{G}}\,.$$

Trading the factor $\exp\left(- \underset{\partial f_i^a}{\overset{\partial f_j^b}{\uparrow}} M_{ij}^{-1} \right)$, gives:

$$\begin{aligned}
\exp\left(- \underset{\partial f_i^a}{\overset{\partial f_j^b}{\uparrow}} M_{ij}^{-1} \right) & \flat_{F^a \cup F^b} s(f^a, g, f^b M, \partial\tilde{g}) \\
&= \quad \mathrm{div}_{F^a} \left(s(f^a, g, f^b M + \partial f^a, \partial g) \right)\big|_{\partial g \to \partial\tilde{g}}\,.
\end{aligned}$$

Performing the pairing with respect to F^b sets f^b to zero, and thus we arrive at the following expression.

$$\begin{aligned}
\Bigg\langle \exp\left(-\frac{1}{2} \underset{\partial f_i^a}{\overset{\partial f_j^a}{\uparrow}} M_{ij}^{-1} \; -\frac{1}{2} \underset{\partial f_i^c}{\overset{\partial f_j^c}{\uparrow}} M_{ij}^{-1} \; - \underset{\partial f_i^c}{\overset{\partial f_j^a}{\uparrow}} M_{ij}^{-1} \right), & \\
\Big\langle \mathrm{div}_F \left(s(f^a, g, \partial f^a, \partial g) \right)\big|_{\partial g \to \partial\tilde{g}}, T(f^c, g+\tilde{g}) \Big\rangle_{\tilde{G}} \Bigg\rangle_{F^a \cup F^c} &.
\end{aligned}$$

The proof follows by inspection. $\square$

B The Wheels Identity

The goal of this section is to show the following *Wheels Identity*:

Theorem 17 *For every $M \in \mathrm{Herm}(\Lambda_{\mathrm{loc}} \to \mathbb{Z})$, we have that*

$$\int dX \left(\exp\left(\frac{1}{2} \sum \underset{x_i}{\overset{x_j}{\uparrow}} M_{ij}(e^{h_1}, \dots, e^{h_g}) \right) \right) = \exp_{\sqcup}\left(-\frac{1}{2} \circlearrowleft\!\chi(M) \right),$$

where we think of $\chi(M)$ as taking values in $\hat{\Lambda}/(\mathrm{cyclic})$, which is identified via the map $t_i \to e^{h_i}$ with the set $\mathbb{Q}\langle\langle h_1, \dots, h_g \rangle\rangle/(\mathrm{cyclic})$ of formal power series in noncommuting variables modulo cyclic permutations.

Proof Choose generators $\{t_1, \dots, t_g\}$ for the free group F and identify $\hat{\Lambda}$ with the ring of noncommutative power series $\{h_1, \dots, h_n\}$. Recall the map $\Lambda \to \hat{\Lambda}$ given by $t_i \to e^{h_i}$.

The Aarhus integral is calculated by splitting the quadratic part from the rest of the integrand as follows:

$$\begin{aligned}
&\int dX \left(\exp\left(\frac{1}{2}\sum \overset{x_j}{\underset{x_i}{\uparrow}} M_{ij}(e^{h_1},\dots,e^{h_g})\right)\right)\\
&= \int dX \left(\exp\left(\frac{1}{2}\sum \overset{x_j}{\underset{x_i}{\uparrow}} M_{ij}(1,\dots,1)\right)\right.\\
&\qquad \left.\sqcup \exp\left(\frac{1}{2}\sum\left(\overset{x_j}{\underset{x_i}{\uparrow}} M_{ij}(e^{h_1},\dots,e^{h_g}) - \overset{x_j}{\underset{x_i}{\uparrow}} M_{ij}(1,\dots,1)\right)\right)\right)\\
&= \left\langle \exp\left(-\frac{1}{2}\sum \overset{\partial x_j}{\underset{\partial x_i}{\uparrow}} M_{ij}(1,\dots,1)^{-1}\right),\right.\\
&\qquad \left.\exp\left(\frac{1}{2}\sum\left(\overset{x_j}{\underset{x_i}{\uparrow}} M_{ij}(e^{h_1},\dots,e^{h_g}) - \overset{x_j}{\underset{x_i}{\uparrow}} M_{ij}(1,\dots,1)\right)\right)\right\rangle_X .
\end{aligned}$$

With the following notation for "Red" and "Green" struts,

$$\begin{aligned}
\uparrow R &= \sum \overset{\partial x_j}{\underset{\partial x_i}{\uparrow}} M_{ij}(1,\dots,1)^{-1},\\
\uparrow G &= \sum \left(\overset{x_j}{\underset{x_i}{\uparrow}} M_{ij}(e^{h_1},\dots,e^{h_g}) - \overset{x_j}{\underset{x_i}{\uparrow}} M_{ij}(1,\dots,1)\right).
\end{aligned}$$

it follows that the desired integral equals to the following formal series (which makes sense, since finitely many terms appear in each degree)

$$\begin{aligned}
&\left\langle \exp\left(-\frac{1}{2}\uparrow R\right), \exp\left(\frac{1}{2}\uparrow G\right)\right\rangle\\
&= \sum_{n=0}^{\infty}(-1)^n \frac{1}{2^n}\frac{1}{2^n}\frac{1}{n!}\frac{1}{n!}\left\langle \left(\uparrow R\right)^n, \left(\uparrow G\right)^n\right\rangle .
\end{aligned}$$

Specifically, the answer is recovered if each "join" is accompanied by a Kronecker delta. For example:

$$\begin{matrix}\uparrow R\\ \uparrow G\end{matrix} = \sum_{i,j,k} \begin{matrix} x_k \\ \uparrow M_{jk}(1,\dots,1)^{-1} \\ \uparrow M_{ij}(e^{h_1},\dots,e^{h_g}) \\ x_i\end{matrix} - \sum_{i,j,k} \begin{matrix} x_k \\ \uparrow M_{jk}(1,\dots,1)^{-1} \\ \uparrow M_{ij}(1,\dots,1) \\ x_i\end{matrix} .$$

We focus on a particular summand. It will help in the management of the combinatorics if we label each of the arcs that appear in the struts:

$$\left\langle \left(\uparrow R\right)^n, \left(\uparrow G\right)^n\right\rangle \to \left\langle \overset{r_{1,1}}{\underset{r_{1,2}}{\uparrow}} \cdots \overset{r_{n,1}}{\underset{r_{n,2}}{\uparrow}}, \overset{g_{1,1}}{\underset{g_{1,2}}{\uparrow}} \cdots \overset{g_{n,1}}{\underset{g_{n,2}}{\uparrow}} \right\rangle$$

Now we introduce some definitions to assist in the enumeration of the polygons that arise from this pairing.

- An *RG-shape of degree* n is a collection of even-sided polygons whose edges are alternately colored red and green. The total number of edges is $2n$.
- A *labeled RG-shape of degree* n is an RG-shape with an extra bivalent vertex in each edge, such that:
  (1) the resulting set of red (resp. green) edges is equipped with a bijection to $\{r_{1,1}, r_{1,2}, \dots, r_{n,1}, r_{n,2}\}$ (resp. $\{g_{1,1}, g_{1,2}, \dots, g_{n,1}, g_{n,2}\}$),
  (2) if an edge is labeled, for example $r_{i,1}$ (resp. $g_{i,1}$), then its neighboring red (resp. green) edge is labeled $r_{i,2}$ (resp. $g_{i,2}$).
- Let the constant C_τ, for τ some RG-shape, denote the number of labeled RG-shapes which recover τ when their labels and extra bivalent vertices are forgotten.

Observe that:

$$\left\langle \left(\uparrow R \right)^n, \left(\uparrow G \right)^n \right\rangle = \sum_{\tau \text{ a labeled RG-shape of deg } n} \tau = \sum_{\tau \text{ an RG-shape of deg } n} C_\tau \tau.$$

To calculate this constant describe an RG-shape of degree n, τ, by a sequence $(i_1, \dots, i_n)$ where i_j counts the number of polygons with $2j$ edges.

Lemma B.1

$$C_\tau = \frac{n!2^n n!2^n}{(2.1)^{i_1} i_1!(2.2)^{i_2} i_2! \dots (2.n)^{i_n} i_n!}.$$

Proof Fix a copy of τ with its set of edges ordered (any old how, so as to distinguish them from each other). There are precisely $2^n n! 2^n n!$ ways of labeling this so as to respect the conditions of the above definition.

But this overcounts by a multiplicative factor of the order of the symmetry group of τ (that is, with the ordering of the edges forgotten). That symmetry group is precisely the following group (that is, swaps of identical polygons, and rotations and flips of individual polygons):

$$(\Sigma_{i_1} \ltimes (D_1 \times \dots D_1)) \times \dots \times (\Sigma_{i_n} \ltimes (D_n \times \dots D_n)),$$

where D_n represents the nth dihedral group.

Thus we conclude:

$$C_\tau = \frac{2^n n! 2^n n!}{\#\{(\Sigma_{i_1} \ltimes (D_1 \times \dots \times D_1)) \times \dots \times (\Sigma_{i_n} \ltimes (D_n \times \dots \times D_n))\}} \qquad \square$$

If τ_i denotes a connected polygon with $2i$ edges, it follows that

$$\begin{aligned}
&\left\langle \exp\left(-\frac{1}{2}\uparrow R\right), \exp\left(\frac{1}{2}\uparrow G\right)\right\rangle \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{1}{2^n}\frac{1}{2^n}\frac{1}{n!}\frac{1}{n!}\left\langle \left(\uparrow R\right)^n, \left(\uparrow G\right)^n\right\rangle, \\
&= \sum_{n}\sum_{i_1,\dots,i_n}\left(\prod_{p=1}^{n}\left(\frac{(-1)^p}{(2p)}\right)^{i_p}\frac{1}{i_p!}\right)\tau_1^{i_1}\sqcup\cdots\sqcup\tau_n^{i_n}, \\
&= \exp\left(\frac{1}{2}\sum_{p=1}^{\infty}\frac{(-1)^p}{p}\tau_p\right).
\end{aligned}$$

Substituting the definitions of the "Red" and "Green" struts into τ_p yields

$$\begin{aligned}
\tau_p &= \operatorname{tr}\left((M(1,\dots,1)^{-1}(M(e^{h_1},\dots,e^{h_g}) - M(1,\dots,1))^p\right) \\
&= \operatorname{tr}\left((M(1,\dots,1)^{-1}M(e^{h_1},\dots,e^{h_g}) - I)^p\right).
\end{aligned}$$

from which the result follows. □

C Infinitesimal versus group-like wrapping relations

In this section we show that the infinitesimal wrapping relations ω imply the group-like wrapping relation ω^{gp}. In order to achieve this, it will be useful to introduce the variant ω_i' of ω_i for $i = 1,\dots,g$ where we allow the diagram $D \in \mathcal{A}(\star_{X\cup\{\partial h\}}, \Lambda_{\mathrm{loc}})$ to have *several* legs labeled by ∂h. In other words, ω_i' is generated by the move $(M, D_1) \sim (M, D_2)$ where

$$D_m = \mathrm{con}_{\{h\}}\left(\varphi_{m,i}(D)\sqcup\left(\exp_{\sqcup}\left(-\frac{1}{2}\chi_h(M^{-1}\varphi_{m,i}M)\right)\right)\right)$$

for $m = 1, 2$ and $\varphi_{1,i}$ is the identity and $\varphi_{2,i}$ (abbreviated by φ_i below) denotes the substitution $\phi_{t_i\to e^h t_i e^{-h}}$ on beads.

It is obvious that the ω_i' relation includes the ω_i relation. Our goal for the rest of the section is to show that they coincide (see Proposition C.7) and that the ω^{gp}-relations are implied by the ω-relations (see Proposition 3.27).

Recall the maps $\eta_i : \Lambda_{\mathrm{loc}} \to \widehat{\mathcal{L}_{\mathrm{loc}}}$ of Section 3.4 that keep track of the h-degree 1 part of the substitutions $\varphi_{t_i\to e^{-h}t_i e^h}(a)$ of $a \in \Lambda_{\mathrm{loc}}$. We will introduce an extension $\widehat{\eta}_i$ of η_i which will allow us to keep track of the h-degree n part of $\varphi_{t_i\to e^{-h}t_i e^h}(a)$, for all n. The following lemma (which the reader should compare with Lemma 3.13) summarizes the properties of $\widehat{\eta}_i$.

Lemma C.1 *For every $i = 1, \dots, g$, there exists a unique*

$$\widehat{\eta}_i : \widehat{\mathcal{L}_{\mathrm{loc}}} \to \widehat{\mathcal{L}_{\mathrm{loc}}}$$

that satisfies the following properties:

$$\begin{aligned} \widehat{\eta}_i(t_j) &= \delta_{ij}[t_i, h] = \delta_{ij}(t_i h - h t_i) \\ \widehat{\eta}_i(h) &= 0 \\ \widehat{\eta}_i(ab) &= \widehat{\eta}_i(a)b + a\widehat{\eta}_i(b) \\ \widehat{\eta}_i(a+b) &= \widehat{\eta}_i(a) + \widehat{\eta}_i(b). \end{aligned}$$

In particular, the restriction of $\widehat{\eta}_i$ to $\Lambda_{\mathrm{loc}} \subset \widehat{\mathcal{L}_{\mathrm{loc}}}$ equals to η_i.

Proof First, we need to show the existence of such a $\widehat{\eta}_i$. A coordinate approach is to define $\widehat{\eta}_i : \mathcal{R} \to \mathcal{R}$ on the ring $\mathcal{R}$ of Section 3.4 which is isomorphic to $\widehat{\mathcal{L}_{\mathrm{loc}}}$. Given the description of $\mathcal{R}$, it is easy to see that $\widehat{\eta}_i$ exists and is uniquely determined by the above properties.

An alternative, coordinate-free definition of $\widehat{\eta}_i$ is the following. Let $\widehat{\mathcal{L}\mathcal{L}_{\mathrm{loc}}}$ denote the Cohn localization of the group-ring $\mathbb{Q}[F \star \mathbb{Z} \star \mathbb{Z}]$, followed by a completion with respect to $t-1$ and $t'-1$ (where t and t' are generators of $\mathbb{Z}\star\mathbb{Z}$). Let $h = \log t$ and $h' = \log t'$. The map $\varphi_{t_i \to e^{-h}t_i e^h} : \mathbb{Q}[F \star \mathbb{Z} \star \mathbb{Z}] \to \mathbb{Q}[F \star \mathbb{Z} \star \mathbb{Z}]$ is Σ-inverting, thus it extends to a map of the localization, and further to a map $\widehat{\mathcal{L}\mathcal{L}_{\mathrm{loc}}} \to \widehat{\mathcal{L}\mathcal{L}_{\mathrm{loc}}}$ of the completion. $\widehat{\mathcal{L}\mathcal{L}_{\mathrm{loc}}}$ is a h'-graded ring, and we can define

$$\widehat{\eta}_i : \widehat{\mathcal{L}_{\mathrm{loc}}} \to \widehat{\mathcal{L}_{\mathrm{loc}}}$$

by

$$\widehat{\eta}_i = \varphi_{h' \to h} \circ \deg^1_{h'} \circ \varphi_{t_i \to e^{-h}t_i e^h}.$$

It is easy to see that this definition of $\widehat{\eta}_i$ satisfies the properties of the lemma, and that the properties characterize $\widehat{\eta}_i$.

Finally, Lemma 3.13 implies that the restriction of $\widehat{\eta}_i$ on $\Lambda_{\mathrm{loc}} \subset \widehat{\mathcal{L}_{\mathrm{loc}}}$ equals to η_i. □

Lemma C.2 *For all n and i we have that*

$$\frac{1}{n!}\widehat{\eta}_i^n = \deg^n_h \circ \varphi_{t_i \to e^{-h}t_i e^h}.$$

considered as maps $\Lambda \to \widehat{\mathcal{L}_{\mathrm{loc}}}$.

Proof In view of Lemma 3.13, it suffices to show that the maps agree on generators t_j of F_g. For $j \neq i$, both maps vanish. For $j = i$, the following identity

$$e^{-h}t_i e^h = \exp(-\mathrm{ad}_h)(t_i) = \sum_{n=0}^{\infty} \frac{1}{n!}[[[t_i, h], h], \dots, h]$$

and Lemma 3.13 show that both maps agree. The result follows. □

Next, we extend the above lemma for matrices.

Lemma C.3 *For all n and sites i we have that*

$$\frac{1}{n!}\widehat{\eta}_i^n = \deg_h^n \circ \varphi_{t_i \to e^{-h} t_i e^h}.$$

considered as maps $\mathrm{Mat}(\Lambda \to \mathbb{Z}) \to \mathrm{Mat}(\widehat{\mathcal{L}_{\mathrm{loc}}} \to \mathbb{Z})$.

Proof Observe the identity

$$\widehat{\eta}_i(AB) = \widehat{\eta}_i(A)B + A\widehat{\eta}_i(B) \tag{26}$$

for matrices A, B (not necessarily square) that can be multiplied. It implies that

$$0 = \frac{1}{n!}\widehat{\eta}_i^n(I) = \frac{1}{n!}\widehat{\eta}_i^n(MM^{-1}) = \sum_{k=0}^{n} \frac{1}{k!}\widehat{\eta}_i^k(M)\widehat{\eta}_i^{n-k}(M^{-1}).$$

The lemma follows by induction on n and the above identity which solves $\eta_i^n(M)$ in terms of quantities known by induction. □

Proposition C.4 *For all n and sites i we have that*

$$\frac{1}{n!}\widehat{\eta}_i^n = \deg_h^n \circ \varphi_{t_i \to e^{-h} t_i e^h}.$$

considered as maps $\Lambda_{\mathrm{loc}} \to \widehat{\mathcal{L}_{\mathrm{loc}}}$.

Proof Consider a matrix presentation of $s \in \Lambda_{\mathrm{loc}}$. Ie,

$$s = (1, 0, \dots, 0)M^{-1}\vec{b}$$

for $M \in \mathrm{Mat}(\Lambda \to \mathbb{Z})$ and $\vec{b}$ a column vector over Λ. Equation (26) implies that

$$\frac{1}{n!}\widehat{\eta}_i^n(s) = \sum_{k=0}^{n} \frac{1}{k!}\widehat{\eta}_i^k(M^{-1})\widehat{\eta}_i^{n-k}(\vec{b}).$$

The proposition now follows from Lemmas C.2 and C.3. □

Proposition C.5 *For all diagrams* $D \in \mathcal{A}(\star_{X,\partial h}, \Lambda_{\mathrm{loc}})$, $M \in \mathrm{Herm}(\Lambda_{\mathrm{loc}} \to \mathbb{Z})$, *and* $i = 1, \dots, g$, *we have that*

$$\frac{1}{n!}\widehat{\eta}_i^n(D) = \deg_h^n \left(\varphi_{t_i \to e^{-h} t_i e^h}(D) \sqcup \exp_{\sqcup}\left(-\frac{1}{2}\chi_h(M^{-1}\varphi_{t_i \to e^{-h} t_i e^h}(M)) \right) \right).$$

Proof The proof follows from a careful combinatorial counting similar to that of the Wheels Identity of Theorem 17. Let us precede the proof with an informal examination of what takes place when we repeatedly apply $\widehat{\eta}_i$ to the empty diagram, $D = \phi$. The first application results in a term:

$$\frac{1}{2}\sum_{k} \left(M^{-1}(-\widehat{\eta}_i(M))\right)_{kk}$$

The second application requires the result of applying $\widehat{\eta}_i$ to M^{-1}. This can be done using the following special case of Equation (26):

$$\widehat{\eta}_i(M^{-1}) = M^{-1}(-\widehat{\eta}_i(M))M^{-1}. \tag{27}$$

So, the second application results in the following terms:

$$\frac{1}{2}\sum_{k,l} \begin{matrix}\left(M^{-1}(-\widehat{\eta}_i(M))\right)_{lk}\\ \left(M^{-1}(-\widehat{\eta}_i(M))\right)_{kl}\end{matrix} + \frac{1}{2}\sum_{k} \left(M^{-1}\frac{-(\widehat{\eta}_i)^2}{2!}(M)\right)_{kk}$$
$$+\frac{(\frac{1}{2})^2}{2!}\sum_{k,l} \left(M^{-1}(-\widehat{\eta}_i(M))\right)_{kk} \sqcup \left(M^{-1}(-\widehat{\eta}_i(M))\right)_{ll}$$

We see that the terms arising from the application of $\widehat{\eta}_i$ fall into three categories.

$$\widehat{\eta}_i(\phi) = \frac{1}{2}\sum_{k} \left(M^{-1}(-\widehat{\eta}_i(M))\right)_{kk} \tag{28}$$

$$(\widehat{\eta}_i(M^{-1}))_{kl} \;=\; \begin{matrix}(M^{-1})_{ml}\\ (M^{-1}(-\widehat{\eta}_i(M)))_{km}\end{matrix} \tag{29}$$

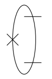
$$(\widehat{\eta}_i((\widehat{\eta}_i)^p(M)))_{kl} \;=\; ((\widehat{\eta}_i)^{p+1}(M))_{kl} \tag{30}$$

Thus, we see that $\widehat{\eta}_i^n(\phi)$ equals to a linear combination of diagrams, each consisting of a number of wheel components, each of which is of the following form (this can be (over-)determined by a sequence $(\kappa_1, \ldots, \kappa_\mu)$ of positive integers):

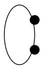
$$\sum_{k_1,\ldots,k_\mu} \begin{matrix}\left(M^{-1}(-(\widehat{\eta}_i)^{\kappa_\mu}(M))\right)_{k_{\mu-1}k_\mu}\\ \cdots\\ \left(M^{-1}(-(\widehat{\eta}_i)^{\kappa_1}(M))\right)_{k_\mu k_1}\end{matrix}$$

The question, then, is to calculate an expression in terms of such diagrams, counting "all ways" that they arise from the application of moves (28)-(30). We will generate a sequence of such moves with the following object, an *edge-list* on a *shape*.

The *shape* determined by a set of positive integers $(\mu_1, \ldots, \mu_\tau)$ is a set of oriented polygons, one with μ_1 edges, and so on, such that each has a distinguished bivalent vertex, its *basepoint*. A *labeled shape* is a shape with its edges labeled by positive integers. The diagram D_κ *determined by some labeled shape* κ is the diagram that arises by replacing each polygon by a series of wheels, in the following manner (to distinguish them from beads, the bivalent vertices of the polygons will be drawn by small horizontal segments, and the base-point by an x).



$$\begin{matrix}\kappa_\mu \\ \cdots \\ \kappa_1\end{matrix} \to \sum_{k_1,\dots,k_\mu}$$



$$\begin{matrix}\left(M^{-1}(-(\widehat{\eta}_i)^{\kappa_\mu}(M))\right)_{k_{\mu-1}k_\mu}\cdots \\ \left(M^{-1}(-(\widehat{\eta}_i)^{\kappa_1}(M))\right)_{k_\mu k_1}\end{matrix}$$

An *edge-list* on a shape is a finite sequence of the edges of the shape such that each edge is represented at least once. (It is convenient to enumerate the edges along the orientation, starting from the basepoint.) The labeled shape associated to some edge-list on that shape is that shape with each edge labeled by the number of times that edge appears in the edge-list.

An edge-list $(\beta_1,\dots,\beta_r)$ on a shape will determine a sequence of moves ((28)-(30)) as follows.

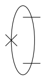
(1) If some β_i is the first appearance in the edge-list of an edge from some polygon with μ edges, then that term determines a move (28), where for the purposes of discussion the introduced wheel is oriented with μ beads, with one such distinguished as the "first", and with the label $M^{-1}(-\widehat{\eta}_i(M))$ sitting on bead number β_i, and with all other beads labeled 1.

(2) If some β_i is the first appearance in the edge-list of some edge of a polygon, some other edge of which has *already* appeared in the edge-list, then that term determines a move (29) applied to the factor of M^{-1} that immediately follows the edge β_i on the (oriented) wheel. (That is, the factor of $M^{-1}(-\widehat{\eta}_i(M))$ that is introduced"slides" back around the polygon from there to the bead number β_i.)

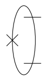
(3) If some edge β_i has already appeared in the path (say, p times) then that step determines the application of a move (30) to the factor $-(\widehat{\eta}_i)^p(M)$ appearing on the bead number β_i.

Remark C.6 Observe that the result of these moves will be the diagram determined by the labeled shape associated to the edge-list.

Observe also that the number of different edge-lists on a shape which give rise to a given associated labeled shape, e.g.



$$\begin{matrix}\kappa_{1,\mu_1} \\ \cdots \\ \kappa_{1,1}\end{matrix} \quad \cdots$$



$$\begin{matrix}\kappa_{\tau,\mu_\tau} \\ \cdots \\ \kappa_{\tau,1}\end{matrix} \quad \text{is} \quad \frac{(\kappa_{1,1}+\cdots+\kappa_{1,\mu_1}+\cdots+\kappa_{\tau,1}+\cdots+\kappa_{\tau,\mu_\tau})!}{(\kappa_{1,1}!\dots\kappa_{1,\mu_1}!)\dots(\kappa_{\tau,1}!\dots\kappa_{\tau,\mu_\tau}!)}.$$

Two edge-lists determine the same sequence of moves if and only if they are related by symmetries of the shape:

(1) rotating individual polygons (ie, shifting the "base-point"),

(2) permuting a subset of polygons which all have the same number of edges.

We now define a function Sym from the set of shapes to the positive integers as follows. If a shape σ is specified by a sequence $(\nu_1, \nu_2, \dots, \nu_k)$, where ν_1 counts the number of polygons with one edge, ν_2 counts the number of polygons with 2 edges, and so on, then

$$\mathrm{Sym}(\sigma) = (\nu_1!\nu_2!\dots\nu_k!)\,(1^{\nu_1}2^{\nu_2}\dots k^{\nu_k}).$$

We are now ready to prove Proposition C.5. First consider the case where the given diagram $D = \emptyset$ is empty.

$$\begin{aligned}
&\sum_n \frac{(\widehat{\eta}_i)^n}{n!}(\emptyset) \\
&= \sum_{\sigma \text{ a shape}} \frac{1}{\mathrm{Sym}(\sigma)} \sum_{\kappa\text{: a labeling of } \sigma} \frac{1}{(\kappa_{1,1}!\dots\kappa_{1,\mu_1}!)\dots(\kappa_{\tau,1}!\dots\kappa_{\tau,\mu_\tau}!)} D_\kappa \\
&= \exp\left(-\frac{1}{2}\mathrm{Tr}\left(-\sum_{m=1}^{\infty}\frac{1}{m}\left(M^{-1}\left(-\widehat{\eta}_i(M) + \frac{-(\widehat{\eta}_i)^2}{2!}(M) + \dots\right)\right)^m\right)\right) \\
&= \exp_\sqcup\left(-\frac{1}{2}\chi_h\left(M^{-1}\phi_{t_i \to e^{-h}t_i e^h}(M)\right)\right),
\end{aligned}$$

as needed. In case D is not empty, assume for convenience that there are k beads on D, and explicitly record their labels $D(q_1, \dots . q_k)$. Now,

$$\sum_n \frac{(\widehat{\eta}_i)^n}{n!}(D) = \sum_{n_1+\dots+n_k+n_{k+1}=n} D\left(\frac{(\widehat{\eta}_i)^{n_1}}{n_1!}(q_1), \dots, \frac{(\widehat{\eta}_i)^{n_k}}{n_k!}(q_k)\right) \sqcup \frac{(\widehat{\eta}_i)^{n_{k+1}}}{n_{k+1}!}(\emptyset),$$

and the Proposition follows from the above case of $D = \emptyset$. ☐

Proposition C.7 *The ω and ω' relations on $\mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$ coincide.*

Proof Consider two pairs (M, s_1) and (M, s_2) related by a wrapping move ω_i' as in definition 2.9. In other words, for some $s \in \mathcal{A}^{\mathrm{gp}}(\star_{X,\partial h}, \Lambda_{\mathrm{loc}})$ we have that

$$s_m \;=\; \mathrm{con}_{\{h\}}\left(\varphi_{m,i}(s) \sqcup \left(\exp_\sqcup\left(-\frac{1}{2}\chi_h(M^{-1}\varphi_{m,i}M)\right)\right)\right)$$

for $m = 1, 2$ where $\varphi_{1,i}$ is the identity and $\varphi_{2,i} = \phi_{t_i \to e^{-h}t_i e^h}$. Consider the case that s has a diagram with n legs labeled by ∂h. Proposition C.5 implies that

$$s_2 = \frac{1}{n!}\mathrm{con}_{\{h\}}(\widehat{\eta}_i^n(s)).$$

Let s' denote the result of gluing all but one of the n ∂h legs of $\widehat{\eta}_i^{n-1}(s)$ and renaming the resulting one $\partial h'$. In other words, $s' = \mathrm{con}_{\{h\}}(\widehat{\eta}_i^{n-1}(s)(\partial h + \partial h'))$. Let $\widehat{\eta}_i'$ denote the action of $\widehat{\eta}_i$ for $\partial h'$-labeled legs. If s has at least one ∂h leg, it follows that

$$s_2 = \mathrm{con}_{\{h'\}}(\widehat{\eta}_i'(s')) = 0 \bmod \omega_i$$

by the infinitesimal wrapping relations, and it follows that $s_1 = 0$ by definition. If s has no ∂h leg, then $s_1 = s_2$. Thus, in all cases, $(M, s_1) \overset{\omega_i}{\sim} (M, s_2)$. The converse is obvious; thus the result follows. $\square$

Proof (of Proposition 3.27) It is obvious that the ω' relation implies the group-like ω^{gp} relation of Section 5.5. Since $\omega' = \omega$, the result follows. $\square$

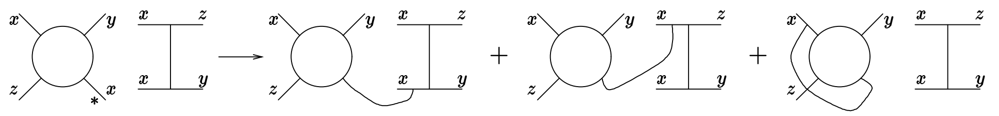
D Infinitesimal basing relations

This section, although it is not used anywhere in the paper, is added for completeness. The reader may have noticed that in Section 3, we mentioned infinitesimal versions of the wrapping and the String relations, but not of the basing relations. The reason is that we did not need them; in addition $\int^{\mathrm{rat}}$ is not known to preserve them. Nevertheless, we introduce them here.

We define

$$\mathcal{A}(\circledast_X, \Lambda_{\mathrm{loc}}) = \mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})/(\beta_1, \beta_2),$$

where $\beta_{1,X}$ is the subspace of X-flavored link relations that appeared in [5, Part II, Section 5.2]:



and $\beta_{2,X}$ denote subspace generated by an action

$$F \times \mathcal{A}(\star_X, \Lambda_{\mathrm{loc}}) \to \mathcal{A}(\star_X, \Lambda_{\mathrm{loc}})$$

described by pushing some element of $f \in F_g$ onto all X-labeled legs of a diagram.

We now compare the infinitesimal and group-like basing relations as follows.

Proposition D.1 (a) *We have an isomorphism*

$$\mathcal{A}(\dots, \circledast_X, \Lambda_{\mathrm{loc}}) = \mathcal{A}(\dots, \star_X, \Lambda_{\mathrm{loc}})/\langle \beta_X \rangle \overset{\cong}{\to} \mathcal{A}(\dots, \circlearrowleft_X, \Lambda_{\mathrm{loc}}).$$

(b) *The inclusion $\mathcal{A}^{\mathrm{gp},0}(\dots, \star_X, \Lambda_{\mathrm{loc}}) \longrightarrow \mathcal{A}^0(\dots, \star_X, \Lambda_{\mathrm{loc}})$ maps the β^{gp}-relations to β-relations, and induces a map*

$$\mathcal{A}^{\mathrm{gp},0}(\dots, \circledast_X, \Lambda_{\mathrm{loc}}) \longrightarrow \mathcal{A}^0(\dots, \star_X, \Lambda_{\mathrm{loc}}).$$

Proof The first part follows from the proof of [5, Part II, Theorem 3] without essential changes. The second part is standard. $\square$

E A useful calculation

In this section, we quote a useful and often appearing computation that uses the identities of Lemma A.1.

Let $\varphi : \mathcal{A}^{\mathrm{gp}}(\star_X, \Lambda_{\mathrm{loc}}) \to \mathcal{A}^{\mathrm{gp}}(\star_{X\cup\{h\}}, \Lambda_{\mathrm{loc}})$ be a map that commutes with respect to gluing X-colored legs.

Theorem 18 *For $X' \subset X$ and $s \in \mathrm{Int}_{X'}\mathcal{A}^{\mathrm{gp}}(\circledast_X, \Lambda_{\mathrm{loc}})$, decomposed as in Equation (12), we have that*

$$\int^{\mathrm{rat}} dX'(\varphi(s)) = \int^{\mathrm{rat}} dX' \left(\exp\left(\frac{1}{2}\sum \underset{x_i}{\overset{x_j}{\uparrow}} \varphi M_{ij}\right)\right) \varphi\left(\int^{\mathrm{rat}} dX'(s)\right).$$

If in addition $\varphi(s) - s$ is X-substantial for all $s \in \mathcal{A}^{\mathrm{gp}}(\star_X, \Lambda_{\mathrm{loc}})$, then

$$\int^{\mathrm{rat}} dX'(\varphi(s)) = \exp\left(-\frac{1}{2}\chi_h(M^{-1}\varphi M)\right) \varphi\left(\int^{\mathrm{rat}} dX'(s)\right).$$

Proof Consider the decomposition of s given by Equation (12). We begin with a "δ-function trick" of Equation (23) of A.1:

$$s = \exp\left(\frac{1}{2}\sum_{i,j} \underset{x_i}{\overset{x_j}{\uparrow}} M_{ij}\right) \sqcup R = \left\langle R(\partial y), \exp\left(\frac{1}{2}\sum \underset{x_i}{\overset{x_j}{\uparrow}} M_{ij} + \underset{x_i}{\overset{y_j}{\uparrow}}\right)\right\rangle_Y.$$

Thus,

$$\int^{\mathrm{rat}} dX'(\varphi(s)) = \left\langle \varphi(R(\partial y)), \int^{\mathrm{rat}} dX' \left(\exp\left(\frac{1}{2}\sum \underset{x_i}{\overset{x_j}{\uparrow}} \varphi M_{ij} + \underset{x_i}{\overset{y_i}{\uparrow}}\right)\right)\right\rangle_Y.$$

Since $\mathrm{div}_{X'} \exp\left(-\sum \underset{y_i}{\overset{\partial x_j}{\uparrow}} \varphi M^{-1}_{ij}\right) = 1$, the integration by parts Lemma A.4 implies that

$$\int^{\mathrm{rat}} dX' \left(\exp\left(\frac{1}{2}\sum \underset{x_i}{\overset{x_j}{\uparrow}} \varphi M_{ij} + \underset{x_i}{\overset{y_j}{\uparrow}}\right)\right) =$$
$$\int^{\mathrm{rat}} dX' \left(\left(\exp\left(-\sum \underset{y_i}{\overset{\partial x_j}{\uparrow}} \varphi M^{-1}_{ij}\right)\right) \flat_{X'} \exp\left(\frac{1}{2}\sum \underset{x_i}{\overset{x_j}{\uparrow}} \varphi M_{ij} + \underset{x_i}{\overset{y_i}{\uparrow}}\right)\right).$$

Completing the square, using Equation (24) of Lemma A.1, implies that the above equals to:

$$\exp\left(-\frac{1}{2}\sum \underset{y_i}{\overset{y_j}{\uparrow}} \varphi M^{-1}_{ij}\right) \int^{\mathrm{rat}} dX' \left(\exp\left(\frac{1}{2}\sum \underset{x_i}{\overset{x_j}{\uparrow}} \varphi M_{ij}\right)\right).$$

Returning to the expression in question:

$$\begin{aligned}\int^{\mathrm{rat}} dX'\,(\varphi(s)) &= \int^{\mathrm{rat}} dX' \left(\exp\left(\frac{1}{2}\sum \overset{x_j}{\underset{x_i}{\uparrow}} \varphi M_{ij}\right)\right) \\ &\qquad \sqcup \left\langle \varphi\,(R(\partial y))\,, \exp\left(-\frac{1}{2}\sum \overset{y_j}{\underset{y_i}{\uparrow}} \varphi M^{-1}_{ij}\right)\right\rangle_Y .\end{aligned}$$

The second factor equals to $\varphi\left(\int^{\mathrm{rat}} dX'(s)\right)$, and concludes the first part of the theorem.

In order to compute the first factor, we need to separate its X'-substantial part. With the added assumption on φ, we can compute the first first factor in a manner analogous to the Wheels Identity of Theorem 17

$$\begin{aligned}&\int^{\mathrm{rat}} dX' \left(\exp\left(\frac{1}{2}\sum \overset{x_j}{\underset{x_i}{\uparrow}} \varphi M_{ij}\right)\right) \\ &= \int^{\mathrm{rat}} dX' \left(\exp\left(\frac{1}{2}\sum\left(\overset{x_j}{\underset{x_i}{\uparrow}} M_{ij} + \overset{x_j}{\underset{x_i}{\uparrow}} \varphi M_{ij} - \overset{x_j}{\underset{x_i}{\uparrow}} M_{ij}\right)\right)\right) \\ &= \left\langle \exp\left(-\frac{1}{2}\sum \overset{\partial x_j}{\underset{\partial x_i}{\uparrow}} M^{-1}_{ij}\right), \exp\left(\frac{1}{2}\, \overset{x_j}{\underset{x_i}{\uparrow}} \varphi M_{ij} - \overset{x_j}{\underset{x_i}{\uparrow}} M_{ij}\right)\right\rangle_{X'} \\ &= \exp\left(-\frac{1}{2}\chi_h(M^{-1}\varphi M)\right).\end{aligned}$$

□

Remark E.1 The proof of Theorem 18 can be applied without changes to the case of the map $\varphi = \mathrm{Hair}_H : \mathcal{A}(\star_X, \Lambda_{\mathrm{loc}}) \to \mathcal{A}(\star_{X\cup H})$ in order to show that for $s \in \mathrm{Int}_{X'}\mathcal{A}^{\mathrm{gp}}(\star_X, \Lambda_{\mathrm{loc}})$, we have that

$$\int dX'(\mathrm{Hair}_H(s)) = \exp\left(-\frac{1}{2}\chi_h(M^{-1}\mathrm{Hair}_H M)\right)\mathrm{Hair}_H\left(\int^{\mathrm{rat}} dX'(s)\right).$$

F The Aarhus integral preserves group-like basing relations

In this appendix we fill a historical gap on the Aarhus integral and how it deals with group-like basing relations.

In [5, Part II, Prop.5.6] it was shown that the Aarhus integral is invariant under infinitesimal basing relations. Later on, in [6, Prop.2.2], it was shown that the Aarhus integral is invariant under group-like basing relations.

Using the results of our paper, one can prove that the Aarhus integral is invariant under group-like basing relations. That is,

Theorem 19 [5] *The Aarhus integral $\int$ descends to a map:*

$$\int^{\mathrm{rat}} dX' : \mathrm{Int}_{X'}\mathcal{A}^{\mathrm{gp}}(\star_X)/\langle\beta_X^{\mathrm{gp}}\rangle \to \mathcal{A}^{\mathrm{gp}}(\star_{X-X'})/\langle\beta_{X-X'}^{\mathrm{gp}}\rangle$$

where $\mathrm{Int}_{X'}\mathcal{A}^{\mathrm{gp}}(\star_X)$ denotes the set of integrable with respect to X' group-like elements and $\beta_{X'}^{\mathrm{gp}}$ denotes the group-like basing relation discussed in Section 3.2.

Proof Use the form of the group-like basing relations given in Definition 3.5 (see Lemma 3.6), and follow the proof of case β_2^{gp} of Theorem 3, using Theorem 18. The result follows. $\square$

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