ISSN 1364-0380 (on line) 1465-3060 (printed)

Geometry & Topology
Volume 8 (2004) 295–310
Published: 14 February 2004

Witten's conjecture and Property P

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Abstract

Let K be a non-trivial knot in the 3–sphere and let Y be the 3–manifold obtained by surgery on K with surgery-coefficient 1. Using tools from gauge theory and symplectic topology, it is shown that the fundamental group of Y admits a non-trivial homomorphism to the group $SO(3)$. In particular, Y cannot be a homotopy-sphere.

AMS Classification numbers Primary: 57M25, 57R57

Secondary: 57R17

Keywords: 3–manifold, knot, surgery, homotopy sphere, gauge theory

Proposed: Robion Kirby
Seconded: John Morgan, Ronald Stern

Received: 7 December 2003
Revised: 9 December 2003

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1 Introduction

Let K be a knot in S^3 and let Y_1 be the oriented 3–manifold obtained by $+1$–surgery on K. The following is one formulation of the "Property P" conjecture for knots:

Conjecture 1 *If K is a non-trivial knot, then Y_1 is not a homotopy 3–sphere.*

The purpose of this note is to prove the conjecture. The ingredients of the argument are: (a) Taubes' theorem [21] on the non-vanishing of the Seiberg–Witten invariants for symplectic 4–manifolds; (b) the theorem of Gabai [16] on the existence of taut foliations on 3–manifolds with non-zero Betti number; (c) the construction of Eliashberg and Thurston [9], which produces a contact structure from a foliation; (d) Floer's exact triangle [13, 4] for instanton Floer homology; (e) a recent result of Eliashberg [8] on concave filling of contact 3–manifolds[1]; and (f) Witten's conjecture relating the Seiberg–Witten and Donaldson invariants of smooth 4–manifolds. Although the full version of Witten's conjecture remains open, a weaker version that is still strong enough to serve our purposes has recently been established by Feehan and Leness [12], following a program proposed by Pidstrigatch and Tyurin. With these ingredients, we shall prove:

Theorem 2 *Let Y_1 be obtained by $+1$–surgery on a non-trivial knot K in S^3. Then there is a non-trivial homomorphism $\rho\colon \pi_1(Y_1) \to SO(3)$.*

It is known [18] that surgery on a non-trivial knot can never yield S^3, so the Property P conjecture would follow from the Poincaré conjecture. Theorem 2 is a slightly sharper statement which implies Conjecture 1. The same techniques yield a closely-related theorem:

Theorem 3 *Let Y be an irreducible, closed, orientable 3–manifold (not $S^1 \times S^2$), and let v be an element of $H^2(Y;\mathbb{Z}/2)$. Then there is a homomorphism $\rho\colon \pi_1(Y) \to SO(3)$ having $w_2(\rho) = v$.*

Remarks The question whether surgery on a knot could produce a counterexample to the Poincaré conjecture was asked explicitly by Bing in [2], and the question was formalized with the definition of "Property P" by Bing and

[1]The authors have learned that this result was also known to Etnyre, who shows in [10] that it is a straightforward extension of the earlier results of [11].

Martin in [3]. To verify that a knot K has Property P in their sense, it is sufficient to verify that the 3–manifolds Y obtained by non-trivial Dehn surgeries on K all have non-trivial fundamental group. In [6], it was shown that $\pi_1(Y)$ is non-trivial if K is non-trivial and the surgery-coefficient is not ± 1. This is why Conjecture 1 is now equivalent to the original version. The problem appears on Kirby's problem list [19, Problem 1.15], where there is also a summary of some of the contributions that have been made.

It follows from Casson's work (see [1]) that if Y is obtained by Dehn surgery on a knot K whose symmetrized Alexander polynomial Δ_K satisifes $\Delta''_K(1) \neq 0$, then $\pi_1(Y)$ admits a non-trivial homomorphisms to $SO(3)$. Such knots therefore have Property P. The argument used by Casson is closely related to what is done here. The quantity $\Delta''_K(1)$ is equal to the Euler characteristic of a Floer homology group $HF(Y_0)$, associated to the manifold Y_0 obtained by 0–framed surgery on K. We shall show that the Floer homology group $HF(Y_0)$ itself is always non-trivial if K is not the unknot, even though the Euler characteristic may vanish.

The authors were aware some time ago that Property P could be deduced from Witten's conjecture and other known results, if one only had a suitably general "concave filling" result for symplectic 4–manifolds with contact boundary, as explained later in this paper. At the time (around 1996), no concave filling results were known. The first general result on concave filling of contact 3–manifolds is given in [11], using results on open-book decompositions from [17]. More recently, Eliashberg has shown [8] that one can construct a concave filling compatible with a given symplectic form on a collar of the contact 3–manifold, provided only that the symplectic form is positive on the contact planes. It is this stronger result from [8] that we need here.

Acknowledgements The first author was supported by NSF grant DMS-0100771. The second author was supported by NSF grants DMS-0206485, DMS-0111298 and FRG-0244663. Both authors would like to thank Yasha Eliashberg for generously sharing his expertise.

2 Donaldson and Seiberg–Witten invariants

2.1 Donaldson invariants and simple type

Let X be a smooth, closed, oriented 4–manifold, with $b^+(X)$ odd and greater than 1, and $b_1(X) = 0$. Fix a homology orientation for X. For each $w \in$

$H^2(X;\mathbb{Z})$, the Donaldson invariants of X (constructed using $U(2)$ bundles with first Chern class w) constitute a linear map

$$D_X^w\colon\ \mathbb{A}(X) \to \mathbb{Z},$$

where $\mathbb{A}(X)$ is the symmetric algebra on $H_2(X;\mathbb{Z}) \oplus H_0(X;\mathbb{Z})$. Our notation here follows [20], and we write x for the element of $\mathbb{A}(X)$ corresponding to the positive generator of $H_0(X;\mathbb{Z})$. We make $\mathbb{A}(X)$ a graded algebra, by putting the generators from $H_2(X;\mathbb{Z})$ in degree 2 and the generator x in degree 4. With this grading, the restriction

$$D_X^w\colon\ \mathbb{A}_{2d}(X) \to \mathbb{Z}$$

is non-zero only when

$$d \equiv -w^2 - \frac{3}{2}(b^+(X)+1) \pmod 4. \tag{1}$$

The manifold X is said to have *simple type* if the invariant satisfies

$$D_X^w(x^2 z) = 4 D_X^w(z)$$

for all z in $\mathbb{A}(X)$. This notion was introduced in [20], where it was shown that X has simple type if it contains a *tight surface*: a smoothly embedded oriented surface Σ whose genus g satisfies $2g-2 = [\Sigma]\cdot[\Sigma] > 0$. For manifolds of simple type, it is natural to introduce

$$\bar{D}_X^w\colon\ \mathbb{A}(X) \to \mathbb{Z},$$

defined by

$$\bar{D}_X^w(z) = D_X^w(z) + D_X^w(zx/2).$$

If z is homogeneous of degree $2d$, then only one of the terms on the right can be non-zero because of the congruence (1); and both terms are zero unless

$$d \equiv -w^2 - \frac{3}{2}(b^+(X)+1) \pmod 2. \tag{2}$$

We combine the Donaldson invariants to form a series

$$\begin{aligned} \mathcal{D}_X^w(h) &= \bar{D}_X^w(e^h) \\ &= \sum D_X^w(h^d)/d! \ + \ \tfrac{1}{2}\sum D_X^w(xh^d)/d! \end{aligned}$$

We regard this as a formal power series for $h \in H_2(X;\mathbb{R})$. The main result of [20] contains the following:

Theorem 4 [20] *Let X be a 4–manifold of simple type with $b_1 = 0$. Then the Donaldson series converges for all w and there exist finitely many cohomology* *classes $K_1, \ldots, K_s \in H^2(X;\mathbb{Z})$ and non-zero rational numbers $\beta_1, \ldots, \beta_s$ (both independent of w) such that*

$$\mathcal{D}^w_X = \exp\left(\frac{Q}{2}\right) \sum_{r=1}^{s} (-1)^{(w^2+K_r\cdot w)/2} \beta_r e^{K_r}$$

as analytic functions on $H_2(X;\mathbb{R})$. Here Q is the intersection form, regarded as a quadratic function. Each of the classes K_r is an integral lift of $w_2(X)$.

Remarks The classes K_r are called the *basic classes* of X. The theorem is supposed to include the case that the Donaldson invariants are identically zero. This is the case $s = 0$. The Donaldson series is always either an even or an odd function of h, so the non-zero basic classes come in pairs differing by sign.

It is more common today to use the terms "simple type" and "basic classes" to refer to properties defined not by the Donaldson invariants but by the Seiberg–Witten invariants, as explained below. We will therefore refer to these as *D–simple type* and *D–basic classes* henceforth, to avoid ambiguity.

2.2 Seiberg–Witten invariants and Witten's conjecture

The Seiberg–Witten invariants of a 4–manifold X such as the one we are considering (with b^+ odd and greater than 1 and $b_1 = 0$) are a function on the set of Spin^c structures on X. For each Spin^c structure $\mathfrak{s}$, they define an integer $SW(\mathfrak{s}) \in \mathbb{Z}$. To simplify our notation, we shall assume that X has no 2–torsion in its second cohomology: in this case, $\mathfrak{s}$ is determined by the first Chern class K of the corresponding half-spin bundle S^+, and we can regard SW as a function of K:

$$SW\colon H^2(X;\mathbb{Z}) \to \mathbb{Z}.$$

The manifold X is said to have *SW–simple type* if $SW(K) = 0$ whenever K^2 is not equal to $2\chi+3\sigma$. The *SW–basic classes* are the classes $K \in H^2(X;\mathbb{Z})$ with $SW(K) \neq 0$. The following is a stripped-down version of Witten's conjecture from [22].

Conjecture 5 *Let X be a 4–manifold with b^+ odd and greater than 1, with $b^1(X) = 0$ and with no 2–torsion in $H^2(X;\mathbb{Z})$. Suppose X has SW–simple type. Then X has D–simple type, the D–basic classes are the SW–basic classes, and for each basic class K_r, the corresponding rational number β_r in the statement of Theorem 4 is given by*

$$\beta_r = c(X) SW(K_r),$$

where $c(X)$ is a non-zero rational number depending on X.

An important corollary of this conjecture is the assertion that the Donaldson invariants are non-zero if the Seiberg–Witten invariants are non-zero and X has SW–simple type. Witten's conjecture also gives the value of $c(X)$ as

$$c(X) = 2^{2+\frac{1}{4}(7\chi+11\sigma)},$$

but we will not need this statement.

2.3 The theorem of Feehan and Leness

A weaker version of Witten's conjecture is proved by Feehan and Leness in [12]. We rephrase Theorem 1.1 of [12] here, specializing to the case that X has SW–simple type, and simplifying the statement to suit our needs, as follows. The theorem involves a choice of auxiliary class $\Lambda \in H^2(X;\mathbb{Z})$ with $\Lambda - w = w_2(X)$ mod 2. In the version we state here, we take Λ to be the class dual to a tight surface in X. This ensures that $\Lambda \cdot K$ is zero, for all SW–basic classes K. The presence of a tight surface ensures that X has D–simple type. We choose Λ to be divisible by 2 and w to be an integer lift of $w_2(X)$. Set

$$N = \Lambda^2 \in \mathbb{Z}.$$

We may replace Λ by any multiple of Λ, to make N as large as we might need.

Theorem 6 [12] *Let X be a 4–manifold with $b_1 = 0$ and b^+ odd and greater than 1. Suppose that X has no 2–torsion in its second cohomology and has SW–simple type. Suppose in addition that X contains a tight surface with positive self-intersection number. Let Λ and N be as above, and let d be an integer in the range*

$$0 \le d < N - \tfrac{1}{4}(\chi + \sigma) - 2$$

satisfying the congruence (2). Then for any class h in $H_2(X;\mathbb{R})$ with $\Lambda \cdot h = 0$, we have

$$\bar{D}^w_X(h^d) = \sum_K (-1)^{(w^2+K\cdot w)/2} SW(K)\, p_d(K\cdot h, Q(h)).$$

Here p_d is a weighted-homogeneous polynomial,

$$p_d(s,t) = \sum_{a+2b=d} C_{a,b} s^a t^b,$$

whose coefficients $C_{a,b} \in \mathbb{Q}$ are universal functions of $\chi(X)$, $\sigma(X)$ and N.

From this result, it is straightforward to deduce:

Corollary 7 *Witten's conjecture, in the form of Conjecture 5, holds for X as long as X satisfies the following three additional conditions:*

(1) *X contains a tight surface Σ with positive self-intersection;*

(2) *X has the same Euler number and signature as some smooth hypersurface in $\mathbb{CP}^3$ whose degree is even and at least 6;*

(3) *X contains a sphere of self-intersection -1.*

Remark The second condition is much more restrictive than necessary, but suffices for our application.

Proof of the corollary The assertion of Conjecture 5 is an equality

$$\mathcal{D}_X^w = c(X)\exp(Q/2)\sum_K (-1)^{(w^2+K\cdot w)/2} SW(K)e^K \tag{3}$$

of analytic functions on $H^2(X;\mathbb{R})$, where $c(X)$ is a non-zero rational number. We are assuming that X contains a tight surface, so X has D–simple type. If we change w to w', then we know how $\mathcal{D}_X^w$ changes, from Theorem 4, and we know also how the right-hand side changes. It is therefore enough to verify the conjecture for one particular w. We take w to be an integral lift of $w_2(X)$.

Let $\Lambda \in H^2(X;\mathbb{Z})$ be some large even multiple of the class dual to the tight surface. All the SW–basic classes and all the D–basic classes are orthogonal to Λ by the adjunction inequality. If we write $h = h_1 + h_2$, where $\Lambda \cdot h_1 = 0$ and h_2 is in the span of the dual of Λ, then

$$\mathcal{D}_X^w(h_1 + h_2) = \mathcal{D}_X^w(h_1)\exp(Q(h_2)/2).$$

The same holds for the function defined by the right-hand side of (3). So it is enough to verify that the conjecture holds for the restriction of the Donaldson series to the kernel of Λ.

Let X_* be a hypersurface in $\mathbb{CP}^3$ with the same Euler number and signature as X. We take Λ_* in $H^2(X_*;\mathbb{Z})$ to be a class orthogonal to the canonical class K_* of X_*, represented by a tight surface. By replacing Λ_* and Λ by suitable multiples, we can arrange that they have the same square N. When the degree of X_* is even, the congruence (2) asserts that d is even. The Donaldson invariants of X and X_* are even functions on the second homology in this case, and the Seiberg–Witten invariants satisfy $SW(K) = SW(-K)$ in both cases.

We apply Theorem 6 to X_*, with $w = 0$. The SW–basic classes are $\pm K_*$, and it is known that $SW(\pm K_*) = 1$. We learn that

$$\bar{D}^0_{X_*}(h^d) = 2 \sum_{a+2b=d} C_{a,b}(K_* \cdot h)^a Q(h)^b$$

for all h orthogonal to Λ_*. This formula determines the coefficients $C_{a,b}$ entirely, in terms of the Donaldson invariants of X_*, because the linear function K_* and the quadratic form Q are algebraically independent as functions on this vector space.

In particular, we see that $C_{a,b}$ is independent of N. We can therefore sum over all d, and write

$$\mathcal{D}^0_{X_*}(h) = 2 \sum_{d \text{ even}} (1/d!) \sum_{a+2b=d} C_{a,b}(K_* \cdot h)^a Q(h)^b.$$

On the other hand, we know from Theorem 4 that $\mathcal{D}^0_{X_*}$ has the special form given there; and we also know that the Donaldson invariants of this complex surface are not identically zero. Thus

$$2 \sum_{d \text{ even}} (1/d!) \sum_{a+2b=d} C_{a,b}(K_* \cdot h)^a Q(h)^b = \exp(Q(h)/2) f(K_* \cdot h)$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a non-zero even function of the form

$$f(t) = \sum_{r=1}^{m} \alpha_r \cosh(\lambda_r t)$$

for some rational numbers α_r and $\lambda_r \geq 0$. The rational numbers λ_r are such that the D–basic classes of X are $\pm\lambda_r K_*$. The basic classes are supposed to be integer classes, and this constrains the denominator of λ_r. The adjunction inequality also implies that $\lambda_r \leq 1$.

With this information about $C_{a,b}$, we can now apply Theorem 6 to our original X, to deduce that

$$\mathcal{D}^w_X = \exp(Q/2) \sum_K (-1)^{(w^2 + K\cdot w)/2} SW(K) f(K)$$

as functions on the orthogonal complement of Λ. If any of the λ_r are not integral, then this formula is inconsistent with Theorem 4, because the SW–basic classes K for X are primitive, because X contains a sphere of square -1. The D–basic classes are also all non-zero for X, for the same reason, and this means that no λ_r can be zero. So λ_r can only be ± 1, and it follows that $f(K)$ is simply a multiple of $\cosh(K)$. This establishes the result. □

3 Proofs of the theorems

3.1 Concave filling

Let Y be a closed oriented 3–manifold (not necessarily connected), and ξ an oriented contact structure compatible with the orientation of Y. This means that ξ is the 2–plane field defined as the kernel of a 1–form α on Y, and $\alpha \wedge d\alpha$ is a positive 3–form. If Y is the oriented boundary of an oriented 4–manifold of W, then a symplectic form ω on W is said to be weakly compatible with ξ if the restriction $\omega|_Y$ is positive on the 2–plane field ξ; or equivalently, if $\alpha \wedge \omega|_Y > 0$. The following is proved in [8].

Theorem 8 [8] *Let Y be the oriented boundary of a 4–manifold W and let ω be a symplectic form on W. Suppose there is a contact structure ξ on Y that is compatible with the orientation of Y and weakly compatible with ω. Then we can embed W in a closed symplectic 4–manifold (X, Ω) in such a way that $\Omega|_W = \omega$.*

Because we will need to construct an (X, Ω) satisfying some additional mild restrictions, we summarize how X is constructed in [8] as a smooth manifold (without concern for the symplectic form). If the components of Y are $Y_1, \dots, Y_n$, then the first step is to choose an open-book decomposition of each Y_i with binding B_i. These open-book decompositions are required to be compatible with the contact structures $\xi|_{Y_i}$ in the sense of [17]. We can take each binding B_i to be connected. Let W' be obtained from W by attaching a 2–handle along each knot B_i with zero framing. The boundary $Y' = \partial W'$ is the union of 3–manifolds Y_i', obtained from Y_i by zero surgery: each Y_i' fibers over the circle with typical fiber Σ_i. The genus of Σ_i is the genus of the leaves of the open-book decomposition of Y_i. For each i, one then constructs a symplectic Lefschetz fibration

$$p_i\colon\ Z_i \to B_i \tag{4}$$

over a 2–manifold-with-boundary B_i, with $\partial B_i = S^1$. One constructs Z_i to have the same fiber Σ_i, and $\partial Z_i = -Y_i'$. The 4–manifold X is obtained as the union of W' and the Z_i, joined along their common boundaries Y_i'.

There is considerable freedom in this construction. We exploit this freedom in a sequence of lemmas, each of which states that we can choose Z_i so as to fulfill a particular additional property.

Lemma 9 *We can choose the Lefschetz fibration $p_i\colon\ Z_i \to B_i$ so that the base B_i is a disk D^2.*

Proof The constructions in [8] already establish this. We present a slight variation of the argument.

As a component of $\partial W'$, the 3–manifold Y_i' carries a 2–form $\eta' \in \Omega^2(Y_i')$, the restriction of the symplectic form ω' from W'. This form is positive on the fibers of the fibration $p'\colon\, Y_i' \to S^1$, and its kernel is a line-field on Y_i' transverse to the fibers. There is a unique vector field V' on Y_i' contained in the line-field, with $p'_*(V') = \partial/\partial S^1$ on the circle S^1. The flow generated by V' preserves η'; and at time 2π the flow determines a holonomy automorphism $\mathrm{Hol}(\eta')$ of the fiber over $1 \in S^1$, which is an area-preserving map of the surface.

Since positive Dehn twists generate the mapping class group, we can construct a Lefschetz fibration $p_i^0\colon\, Z_i^0 \to D^2$ whose boundary is topologically $-Y_i'$, as a surface bundle over S^1. This Lefschetz fibration can be made symplectic; and we write η'' for the restriction of the symplectic form from Z_i^0 to Y_i'. We can assume that η' and η'' have the same integral on the fiber Σ_i.

If we can choose Z_i^0 so that

$$\mathrm{Hol}(\eta') = \mathrm{Hol}(\eta'') \tag{5}$$

as area-preserving maps of the fiber over 1, then there is a fiber-preserving diffeomorphism ψ of Y_i' with $\psi^{-1}(\eta') = \eta''$. We can then use ψ to attach Z_i^0 to W' along Y_i' (see [8]) and our task will be complete. At this point however, we only know that the map $\phi = \mathrm{Hol}(\eta') \circ \mathrm{Hol}(\eta'')^{-1}$ is isotopic to the identity in $\mathrm{Diff}(\Sigma_i)$.

To complete the proof of the lemma, it will be enough to construct a symplectic Lefschetz fibration

$$p\colon\, (V, \omega_V) \to D^2$$

whose boundary is the topologically trivial surface bundle over S^1 and whose holonomy is given by $\mathrm{Hol}(\eta) = \phi$, where $\eta = \omega_V|_{\partial V}$. We can then form Z_i as the union of Z_i^0 and V, attached along a neighborhood of a fiber in their boundaries. That such a V exists is the content of the next lemma, which is a variant of [8, Lemma 3.4]. □

Lemma 10 *Let Σ be a closed symplectic surface of area 1 and genus 2 or more. Let $\phi\colon\, \Sigma \to \Sigma$ be an area-preserving map that is isotopic to the identity through diffeomorphisms. Then there is a symplectic Lefschetz fibration $p\colon\, (V, \omega) \to D^2$ with $p^{-1}(1) = \Sigma$ and $\mathrm{Hol}(\omega|_V) = \phi$.*

Proof As explained in [8], it will be enough if we can find a (V, ω) such that $\mathrm{Hol}(\omega_V)$ has the same *flux* as ϕ. In this context, the flux has the following interpretation. Because the identity component of the diffeomorphism group is contractible, we can identify ∂V with $S^1 \times \Sigma$ canonically up to fiber-preserving isotopy; so we have a canonical map

$$H_1(\Sigma) \to H_2(\partial V)$$

given by $[\gamma] \mapsto [S^1 \times \gamma]$. The flux is the element of $H^1(\Sigma;\mathbb{R})$ corresponding to the homomorphism

$$\begin{aligned} f\colon\ H_1(\Sigma) &\to \mathbb{R} \\ [\gamma] &\mapsto \int_{S^1\times\gamma} \omega|_{\partial V}. \end{aligned}$$

So the assertion of the lemma is that we can choose $p\colon (V,\omega) \to D^2$ so that the cohomology class of $\omega|_{\partial V}$ is any given class in $H^2(S^1 \times \Sigma;\mathbb{R})$, subject only to the constraint that the area of Σ is 1.

To see that this is possible, we observe that we can find first an example $p_0\colon (V_0,\omega_0) \to D^2$ whose flux f is *zero* and such that the map $H_2(\partial V_0;\mathbb{R}) \to H_2(V_0;\mathbb{R})$ induced by the inclusion $\partial V_0 \hookrightarrow V_0$ is injective. Such an example is obtained by removing a neighborhood of a fiber in a closed Lefschetz fibration $\bar{p}_0\colon (\bar{V}_0,\bar{\omega}_0) \to S^2$; the condition on the second homology is achieved if $H_1(\bar{V}_0)$ is zero.

Next, because non-degeneracy is an open condition on 2–forms, there exists a neighborhood $\mathcal{U}$ of $0 \in H^1(\Sigma;\mathbb{R})$ such that, for all $f \in \mathcal{U}$, there exists a form ω_f on V_0 such that

$$p_0\colon\ (V_0,\omega_f) \to D^2$$

is a symplectic Lefschetz fibration whose holonomy on the boundary has flux f. Finally, given a general f, we can find an integer N such that f/N belongs to $\mathcal{U}$. We then construct (V,ω) by attaching N copies of $(V_0,\omega_{f/N})$ along neighborhoods of fibers in their boundaries. □

From now on, we may assume that the base of the fibration Z_i is a disk. We can now arrange that $H_1(Z_i;\mathbb{Z})$ is zero. Indeed, $H_1(Z_i;\mathbb{Z})$ is generated by a collection of 1–cycles on the fiber Σ_i, and we can arrange that these are vanishing cycles in the Lefschetz fibration. Thus we can state:

Lemma 11 *If the map $H_1(Y;\mathbb{Z}) \to H_1(W;\mathbb{Z})$ is surjective, then we can choose X in Theorem 8 so that $H_1(X;\mathbb{Z})$ is zero.*

Proof The hypothesis implies that $H_1(Y';\mathbb{Z}) \to H_1(W';\mathbb{Z})$ is surjective also. Choose the Z_i to have trivial first homology, as explained above, and the lemma then follows from the Mayer–Vietoris sequence. □

In a similar vein, we have:

Lemma 12 *We can choose X so the restriction map $H^2(X;\mathbb{Z}) \to H^2(W;\mathbb{Z})$ is surjective.*

Proof The restriction map $H^2(W';\mathbb{Z}) \to H^2(W;\mathbb{Z})$ is surjective, so we may replace W by W' in the statement. If we arrange that $H_1(Z_i;\mathbb{Z})$ is zero, then the restriction map $H^2(Z_i;\mathbb{Z}) \to H^2(Y_i';\mathbb{Z})$ is surjective. The surjectivity of the map $H^2(X;\mathbb{Z}) \to H^2(W';\mathbb{Z})$ now follows from the Mayer–Vietoris sequence for cohomology. □

We can also specify the Euler number and signature quite freely subject to some inequalities:

Lemma 13 *We can choose X so that its Euler number and signature are the same as those of X_*, where X_* is a smooth hypersurface in $\mathbb{CP}^3$ whose degree is even and at least 6. At the same time, we can arrange that X contains a sphere with self-intersection -1.*

Proof Our strategy is to arrange that X has the same b^+ as some X_* but has smaller b^-. We then blow up X at enough points to make the value of b^- agree also.

Let $V \to \mathbb{CP}^1$ be a symplectic Lefschetz fibration with $b_1(V) = 0$ and the same fiber genus as Z_i. Replace Z_i by $\tilde{Z}_i$, the Gompf fiber-sum of Z_i and V. The effect on $b^+(X)$ is to add to it the quantity

$$n^+(V) = b^+(V) + 2g - 1,$$

while $b^-(X)$ changes by

$$n^-(V) = b^-(V) + 2g - 1.$$

Here g is the fiber genus. If we use two different Lefschetz fibrations, V and $\tilde{V}$, for which $n^+(V)$ and $n^+(\tilde{V})$ are coprime, then the set of values that we can achieve for $b^+(X)$ includes all sufficiently large integers.

For hypersurfaces X_* in $\mathbb{CP}^3$ of large degree, the ratio $b^-(X_*)/b^+(X_*)$ approaches 2. We can therefore achieve our objective by forming a fiber-sum with many copies of V, provided the ratio $n^-(V)/n^+(V)$ satisfies

$$n^-(V)/n^+(V) < 2.$$

This ratio condition is quite common for Lefschetz fibrations. For example, if S is an algebraic surface with an ample class H satisfying $K_S \cdot H > 0$, then the Lefschetz fibration V constructed from a pencil in the linear system $|dH|$ satisfies this inequality, once d is sufficiently large. A V constructed in this way may not have the same fiber genus as one of the Z_i, but we can always increase the fiber genus of Z_i by any positive integer, by adjusting the original open-book decomposition of Y_i. □

We need one last lemma of this sort.

Lemma 14 *We can choose X so that it contains a tight surface of positive self-intersection number.*

Proof We can choose a Lefschetz fibration $V \to \mathbb{CP}^1$ containing a tight surface disjoint from a fiber. We then replace one Z_i by a Gompf fiber-sum, as in the previous lemma. □

We now combine the conclusions of the last four lemmas with the construction of Eliashberg and Thurston from [9], to prove the next proposition.

Proposition 15 *Let Y be a closed orientable 3–manifold admitting an oriented taut foliation. Suppose Y is not $S^1 \times S^2$. Then Y can be embedded as a separating hypersurface in a closed symplectic 4–manifold (X, Ω). Moreover, we can arrange that X satisfies the following additional conditions.*

(1) *The first homology $H_1(X; \mathbb{Z})$ vanishes.*
(2) *The Euler number and signature of X are the same as those of some smooth hypersurface in $\mathbb{CP}^3$, whose degree is even and not less than 6.*
(3) *The restriction map $H^2(X; \mathbb{Z}) \to H^2(Y; \mathbb{Z})$ is surjective.*
(4) *The manifold X contains a tight surface of positive self-intersection number, and a sphere of self-intersection -1.*
(5) *The two pieces X_1 and X_2 obtained by cutting X along Y both have b^+ positive.*

Proof By the results of [9], the existence of the foliation implies that the product manifold
$$W = [-1, 1] \times Y$$
carries a symplectic form ω, weakly compatible with contact structures ξ_+ and ξ_- on the boundary components $\{1\} \times Y_0$ and $\{-1\} \times Y_0$. By Theorem 8, we may embed (W, ω) in a closed symplectic 4–manifold (X, Ω). We can choose X to satisfy the extra conditions in Lemmas 11, 12, 13 and 14 above. This gives the first of the four conditions on X. The last condition is straightforward. □

3.2 Proof of Theorem 2

Let Y_1 be the result of $+1$–surgery on a non-trivial knot K, and let Y_0 be the manifold with $H_1(Y_0) = \mathbb{Z}$ obtained by 0–surgery. According to Floer's exact triangle [13, 4], the instanton Floer homology group $HF(Y_1)$ is isomorphic to the Floer homology group $HF(Y_0)$, where the latter is refers to the group constructed using the $SO(3)$ bundle $P \to Y_0$ with non-zero w_2. We suppose that the knot K contradicts Theorem 2. Then $HF(Y_1)$ is zero, and the exact triangle tells us that $HF(Y_0)$ is zero also. We therefore have:

Proposition 16 *Suppose K is a counterexample to Theorem 2. Let X be a smooth closed 4–manifold containing Y_0 as a separating hypersurface. Suppose that the two pieces X_1, X_2 obtained by cutting X along Y_0 both have b^+ non-zero. Then the Donaldson polynomial invariant D^w_X is identically zero for any class $w \in H^2(X;\mathbb{Z})$ whose restriction to Y_0 is non-zero mod 2.*

Proof When X is decomposed along Y_0 as in the proposition, the value of $D^w_X(x^m h^n)$ can be expressed as a pairing

$$\langle \psi_{X_1}, \psi_{X_2} \rangle,$$

where ψ_{X_1} and ψ_{X_2} are relative invariants of X_1 and X_2 taking values in the Fukaya–Floer homology group $HFF(Y_0, \delta)$ and its dual, where δ is a 1–cycle in Y_0 (see [14, 5]). The vanishing of $HF(Y_0)$ implies the vanishing of $HFF(Y_0, \delta)$ also, which explains the proposition. □

Remark It is possible to avoid the use of the full exact triangle, and to avoid mentioning any type of Floer homology in the proof of this proposition. The hypothesis on K means that the equations for a flat $SO(3)$ connection on Y_0 with w_2 non-zero admit a holonomy-type perturbation (of the sort described in [4]), so that the resulting equations admit no solutions. (In other language, the Chern–Simons functional has a holonomy-type perturbation after which it has no critical points.) The vanishing of the Donaldson invariants for X then follows from a straightforward degeneration argument.

According to [16], the manifold Y_0 has a taut foliation by oriented 2–dimensional leaves and is not the product manifold $S^1 \times S^2$ if K is non-trivial. We may apply Proposition 15 to Y_0, to embed it in (X, Ω) satisfying all the conditions in that proposition. Being symplectic, the manifold X has SW–simple type and non-trivial Seiberg–Witten invariants, by the results of [21]. The conditions imposed in Proposition 15 ensure that Corollary 7 applies, so Witten's conjecture, in the form of Conjecture 5, holds for X. It follows that the Donaldson invariants D^w_X are non-trivial, for all w.

However, the 3–manifold $Y_0 \subset X$ divides X into two pieces X_1 and X_2, both of which have positive b^+. The condition (3) of Proposition 15 allows us to choose a $w \in H^2(X;\mathbb{Z})$ whose restriction to Y_0 is the generator. For this choice of w, Proposition 16 tells us that D^w_X is zero. This is a contradiction. □

3.3 Proof of Theorem 3

Let Y and v be as in the statement of the theorem. If the image of the element v in $\mathrm{Hom}(H_2(Y;\mathbb{Z}),\mathbb{Z}/2)$ is zero, then the result is elementary, for there is an integer lift of v that is a torsion element of $H^2(Y;\mathbb{Z})$, which implies that there is a flat $SO(2)$ bundle on Y with $w_2 = v$. We therefore turn to the interesting case, when v has non-zero pairing with some element of $H_2(Y;\mathbb{Z})$.

Gabai's theorem [15] supplies Y with a taut foliation, so we can embed Y as a separating hypersurface in a symplectic 4–manifold X, as in Proposition 15. Because the restriction map on second cohomology is surjective, there is a class $w \in H^2(X;\mathbb{Z})$ whose restriction to Y becomes v when reduced mod 2.

The hypothesis that v has non-zero pairing with some integer class ensures that there is a well-defined Floer homology group $HF^v(Y)$ constructed from the connections with $w_2 = v$ (see [7]). If there are no such flat connections, then $HF^v(Y)$ is zero, and it follows that $\bar{D}^w_X$ is identically zero, as in Proposition 16. On the other hand, Conjecture 5 holds for X, and we have the same contradiction as before. □

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