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Formal groups and stable homotopy of commutative rings

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Abstract

We explain a new relationship between formal group laws and ring spectra in stable homotopy theory. We study a ring spectrum denoted DB which depends on a commutative ring B and is closely related to the topological André–Quillen homology of B. We present an explicit construction which to every 1–dimensional and commutative formal group law F over B associates a morphism of ring spectra $F_*\colon H\mathbb{Z} \longrightarrow DB$ from the Eilenberg–MacLane ring spectrum of the integers. We show that formal group laws account for all such ring spectrum maps, and we identify the space of ring spectrum maps between $H\mathbb{Z}$ and DB. That description involves formal group law data and the homotopy units of the ring spectrum DB.

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1 Introduction

In this paper we explain a new relationship between formal group laws and ring spectra in stable homotopy theory. We use formal group laws to construct maps of ring spectra and describe spaces of ring spectrum maps in terms of formal group data.

Our main object of study is a ring spectrum denoted DB which functorially depends on a commutative ring B. The ring spectrum DB controls the stable homotopy theory of augmented commutative B–algebras. Its homotopy groups π_*DB are the Cartan–Bousfield–Dwyer algebra of stable homotopy operations of commutative simplicial B–algebras [10, 7, 12]. The homotopy groups π_*DB are also isomorphic to the Γ*–homology* [42], relative to B, of the polynomial algebra $B[x]$ on one generator, and to the *topological André–Quillen homology* [1] of the associated Eilenberg–MacLane spectra. Both Γ–homology and topological André–Quillen homology arise in obstruction theory for E_∞ ring spectrum structures [41]. We elaborate more on the precise relationship in Remark 3.3 below, and we also refer to the survey paper [3] by Basterra and Richter.

We present an explicit construction which to every 1–dimensional and commutative formal group law F over B associates a homomorphism of ring spectra

$$F_* : \; H\mathbb{Z} \longrightarrow DB$$

from the Eilenberg–MacLane ring spectrum of the integers to DB. We prove that in this way formal group laws account for all ring spectrum maps, ie, we show:

Theorem *The construction which sends a 1–dimensional, commutative formal group law F to the ring spectrum map F_* induces a natural bijection between the set of strict isomorphism classes of formal group laws over B and the set of homotopy classes of ring spectrum maps from $H\mathbb{Z}$ to DB.*

This theorem is a corollary of the identification of the space of all ring spectrum maps between the Eilenberg–MacLane ring spectrum $H\mathbb{Z}$ and DB. That description involves the *homotopy units* $(DB)^\times$ of the ring spectrum DB. The group-like simplicial monoid $(DB)^\times$ of homotopy units acts by conjugation on the space $\mathrm{Ring}(H\mathbb{Z}, DB)$ of ring spectrum maps. The construction F_* of the first theorem extends to a natural weak map from the classifying space of the groupoid of formal group laws and strict isomorphisms $\mathcal{FGL}^{\mathrm{str}}(B)$ to the homotopy orbit space $\mathrm{Ring}(H\mathbb{Z}, DB)/\mathrm{conj.}$ of the conjugation action of the identity component $(DB)_1^\times$ of the units on the space of ring spectrum maps. In Theorem 5.2 we show

Theorem *The weak map*

$$\mathcal{FGL}^{\mathrm{str}}(B) \longrightarrow \mathrm{Ring}(H\mathbb{Z}, DB)/\mathrm{conj}.$$

from the classifying space of the groupoid of formal group laws and strict isomorphisms to the homotopy orbit space of the conjugation action is a weak homotopy equivalence.

Another corollary of this main theorem is that the ring spectrum DB is not equivalent to the Eilenberg–MacLane ring spectrum of any differential graded algebra (or, equivalently, of any simplicial ring); see Corollary 13.2 for the precise statement. This should be compared to the fact that the spectrum underlying DB (ie, ignoring the multiplication) is stably equivalent to the smash product $H\mathbb{Z} \wedge HB$ (see Theorem 3.2 (b)), in particular it is equivalent to a product of Eilenberg–MacLane spectra. Corollary 13.2 says that the multiplicative structure of DB is considerably more complicated.

Prerequisites We freely use the language and standard results from the theory of model categories; the original source for this material is [39], a more modern introduction can be found in [13], and the ultimate reference is currently [23]. Our notion of ring spectrum is that of a *Gamma-ring* (see [29, 2.13] or [44, Def. 2.1]). Gamma-rings are based on a symmetric monoidal smash product for Γ–spaces with good homotopical properties [48, 8, 29]. The foundational material about the homotopy theory of Gamma-rings and their modules can be found in [44]; a summary is also given in Section 2. The results of this paper can be translated into other frameworks for ring spectra by the general comparison procedures described in [32, 45]. We also need a few basic facts from the theory of formal group laws, which in this paper (with the exception of Section 13) are always 1–dimensional and commutative; all we need is contained in [28] or in Chapter III, §1 of [19].

Outline of the paper

Section 2 We review some general facts about Γ–spaces and Gamma-rings. We recall the assembly map (2.5) from the smash product to the composition product of Γ–spaces which is used several times in this paper.

Section 3 We review the Gamma-ring DB and summarize some of its properties in Theorem 3.2. Construction 3.5 associates a homomorphism of ring spectra $F_*\colon H\mathbb{Z} \longrightarrow DB$ to every formal group law F. We state in Theorem 3.8 how this accounts for all homotopy classes of Gamma-ring maps.

Section 4 For every Gamma-ring R we construct a natural conjugation action of the simplicial monoid of homotopy units $R^\times$ (3.9) on R through Gamma-ring homomorphisms.

Section 5 We construct a weak map

$$\mathcal{FGL}^{\mathrm{str}}(B) \xleftarrow{\sim} \widetilde{\mathcal{FGL}}^{\mathrm{str}}(B) \xrightarrow{\kappa} \mathrm{Ring}(H\mathbb{Z}, DB)/\mathrm{conj}.$$

from the classifying space of the groupoid of formal group laws and strict isomorphisms to the homotopy orbit space of the conjugation action of Section 4. The left map is a weak equivalence by construction. Theorem 5.2, which is the main theorem of this paper, says that the right map κ is a weak equivalence.

Section 6 We use a filtration of the Gamma-ring DB, coming from powers of the augmentation ideal, to reduce the proof of Theorem 5.2 to showing that a truncated version

$$\kappa_k :\ \widetilde{\mathcal{B}ud}^k(B) \ \longrightarrow\ \mathrm{Ring}(H\mathbb{Z}, D_kB)/\mathrm{conj}.$$

of the map κ of Section 5 is a weak equivalence for all $k \geq 1$ (see Theorem 6.4). Here $\widetilde{\mathcal{B}ud}^k(B)$ is (weakly equivalent to) the classifying space of the groupoid of k–buds (or k–jets) of formal group laws and D_kB is the "quotient" Gamma-ring of DB by the "ideal" coming from $(k+1)$st powers of the augmentation ideal.

Section 7 We exploit that two successive stages in the filtration of DB form a "singular extension" of ring spectra

$$(B \otimes \mathcal{S}^k)^! \ \longrightarrow\ D_kB \ \longrightarrow\ D_{k-1}B$$

where $(B\otimes\mathcal{S}^k)^!$ is a "square zero ideal" coming from the k-th symmetric power functor. This allows us to reduce the problem to showing that a certain map

$$\kappa_{B\otimes\mathcal{S}^k} :\ \widetilde{\mathcal{Z}}(B \otimes \mathcal{S}^k) \ \longrightarrow\ \mathrm{der}(H\mathbb{Z}, B \otimes \mathcal{S}^k)/\mathrm{conj}.$$

to the homotopy orbit space of the derivations of $H\mathbb{Z}$ with coefficients in the symmetric power functor is a weak equivalence. Here $\widetilde{\mathcal{Z}}(B \otimes \mathcal{S}^k)$ is (weakly equivalent to) the classifying space of the groupoid of symmetric 2–cocycles of degree k over B.

Section 8 For later use we define a map

$$\lambda_G : \mathcal{Z}(G) \longrightarrow \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^{\mathrm{c}}, H\mathbb{Z} \times G^!_{st})_{hG_{st}(\mathbb{Z})}$$

for every functor G from the category of finitely generated free abelian groups to the category of all abelian groups; the important case is when G is a symmetric power functor $B \otimes \mathcal{S}^k$. Here $\mathcal{Z}(G)$ is the classifying space of the groupoid of symmetric 2–cocycles of the functor G (8.1), $\mathrm{map}_{\mathcal{GR}}$ denotes the simplicial mapping space of Gamma-rings, G_{st} is the Dold–Puppe stabilization of G (8.9) and the Gamma-ring $H\mathbb{Z} \times G^!_{st}$ is the split singular extension of $H\mathbb{Z}$ by the bimodule $G^!_{st}$ (7.4).

Section 9 We compare the map $\kappa_{B\otimes\mathcal{S}^k}$ of Section 7 with the map $\lambda_{B\otimes\mathcal{S}^k}$ of Section 8 by means of a commutative square

$$\begin{tikzcd}
\widetilde{\mathcal{Z}}(B \otimes \mathcal{S}^k) \arrow[r, "\kappa_{B\otimes\mathcal{S}^k}"] \arrow[d, "\sim"'] & \mathrm{der}(H\mathbb{Z}, B \otimes \mathcal{S}^k)/\mathrm{conj}. \arrow[d, "\sim"] \\
\mathcal{Z}(B \otimes \mathcal{S}^k) \arrow[r, "\lambda_{B\otimes\mathcal{S}^k}"'] & \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^{\mathrm{c}}, H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!_{st})_{h(B\otimes\mathcal{S}^k)_{st}(\mathbb{Z})}
\end{tikzcd} \tag{9.1}$$

in which the vertical maps are weak equivalences. Hence instead of showing that $\kappa_{B\otimes\mathcal{S}^k}$ is a weak equivalence we may show that $\lambda_{B\otimes\mathcal{S}^k}$ is.

Section 10 By the results of the previous sections, the proof of the main theorem is reduced to an identification of the space of Gamma-ring maps

$$\mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^{\mathrm{c}}, H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!_{st})$$

(or more precisely a certain homotopy orbit space thereof) with the classifying space of symmetric 2–cocycles. In this section we reinterpret the above mapping space in terms of the category $s\mathcal{F}$ of simplicial functors from the category of finitely generated free abelian groups to the category of abelian groups. We note that the construction which sends $G \in s\mathcal{F}$ to the split extension $H\mathbb{Z} \times G^!$ (7.4) has a left adjoint

$$\mathcal{L} : \mathcal{G}R/H\mathbb{Z} \longrightarrow s\mathcal{F}$$

from the category of Gamma-rings over $H\mathbb{Z}$ to the category of simplicial functors. Moreover, the two functors form a Quillen adjoint pair between model categories. In order to identify the above mapping space, we evaluate the left adjoint $\mathcal{L}$ on the Gamma-ring $H\mathbb{Z}^{\mathrm{c}}$, the cofibrant replacement of $H\mathbb{Z}$. We denote by J the functor which supports the universal symmetric 2–cocycle (8.3). The main result of this section, Theorem 10.2, states that the map

$$\mathcal{L}(H\mathbb{Z}^{\mathrm{c}}) \longrightarrow J$$

which is adjoint to the "universal derivation" (8.6)

$$(1, d_u): \ H\mathbb{Z}^{\mathrm{c}} \longrightarrow H\mathbb{Z} \times J^!$$

is a stable equivalence of simplicial functors. This implies that for any reduced functor G the homotopy groups of the space

$$\mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^{\mathrm{c}}, H\mathbb{Z} \times G^!_{st}) \cong \mathrm{map}_{s\mathcal{F}}(\mathcal{L}(H\mathbb{Z}^{\mathrm{c}}), G_{st})$$

are isomorphic to the hyper-cohomology groups $\mathbb{E}\mathrm{xt}^*_{\mathcal{F}}(J, QG)$, for $* \leq 0$, in the abelian category $\mathcal{F}$ of reduced functors from the category of finitely generated abelian groups to the category of abelian groups. The chain complex QG is MacLane's cubical construction for the functor G (8.9).

Section 11 We prove a homological criterion, Theorem 11.1, in terms of the functor G for when the map λ_G defined in Section 8 is a weak equivalence. Loosely speaking, the criterion requires that "MacLane cohomology equals topological Hochschild cohomology" for the functor G, compare Remark 11.2. The precise meaning of this is that for all integers $m \leq 2$ the map

$$\mathrm{Ext}^m_{\mathcal{F}}(I, G) \longrightarrow \mathbb{E}\mathrm{xt}^m_{\mathcal{F}}(I, QG)$$

be an isomorphism. The map $G \longrightarrow QG$ is a model for Dold–Puppe stabilization and it is initial, in the derived category of $\mathcal{F}$, among maps from G to complexes whose homology functors are additive. $\mathbb{E}\mathrm{xt}^m_{\mathcal{F}}(I, -)$ denotes hyper-Ext groups of the functor I with coefficients in a chain complex of functors. In Example 11.3 we also give a functor for which the criterion fails.

Section 12 In Theorem 12.1 we verify the homological criterion of Theorem 11.1 for the symmetric power functors $G = B \otimes \mathcal{S}^k$. This finishes the proof of our main theorem: Theorem 11.1 shows that the map $\lambda_{B \otimes \mathcal{S}^k}$ is a weak equivalence, hence by the commutative square (9.1) the map $\kappa_{B \otimes \mathcal{S}^k}$ is a weak equivalence. The commutative diagram (7.7) of fibre sequences shows inductively that the maps $\kappa_k \colon \widetilde{\mathcal{B}ud}^k(B) \longrightarrow \mathrm{Ring}(H\mathbb{Z}, N_k)/\mathrm{conj.}$ of Theorem 6.4 are weak equivalences. Hence the (weak) map $\kappa \colon \mathcal{FGL}^{\mathrm{str}}(B) \longrightarrow \mathrm{Ring}(H\mathbb{Z}, DB)/\mathrm{conj.}$ of Theorem 5.2 is a weak equivalence.

Section 13 In the last section we give an application of the main theorem as well as an outlook towards possible generalizations and future directions. Variations of the main construction are possible, and some of them are described in this final section. For example, formal groups can be replaced by formal A–modules where A is any ring (not necessarily commutative). Such formal A–modules give rise to Gamma-ring maps from the Eilenberg–MacLane Gamma-ring of A into DB. Furthermore when considering higher dimensional formal group laws, the natural target of the construction is a matrix Gamma-ring over DB of the corresponding dimension.

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2 Review of Γ–spaces and Gamma-rings

In this section we review some general facts about Γ–spaces and Gamma-rings, and we fix notation and terminology. None of this material is new, but we present it in a form which is convenient for this paper. We also prove certain properties of the assembly map (2.5) from the smash product to the composition product of Γ–spaces which are used several times in this paper.

2.1 Γ–spaces The category of Γ–spaces was introduced by Segal [48], who showed that it has a homotopy category equivalent to the stable homotopy category of connective spectra. Bousfield and Friedlander [8] considered a bigger category of Γ–spaces in which the ones introduced by Segal appeared as the *special* Γ–spaces (2.3). Their category admits a closed simplicial model category structure with a notion of stable weak equivalences giving rise again to the homotopy category of connective spectra. Then Lydakis [29] introduced internal function objects and a symmetric monoidal smash product with good homotopical properties.

The category Γ^{op} has one object $n^+ = \{0, 1, \ldots, n\}$ for every non-negative integer n, and morphisms are the maps of sets which send 0 to 0. Γ^{op} is equivalent to the opposite of Segal's category Γ [48]. A Γ–space is a covariant functor from Γ^{op} to the category of simplicial sets taking 0^+ to a one point simplicial set. A morphism of Γ–spaces is a natural transformation of functors. We denote by $\mathbb{S}$ the Γ–space which takes n^+ to n^+, considered as a constant simplicial set. If X is a Γ–space and K a pointed simplicial set, a new Γ–space $X \wedge K$ is defined by setting $(X \wedge K)(n^+) = X(n^+) \wedge K$.

A Γ–space X can be prolonged, by direct limit, to a functor from the category of finite pointed sets to the category of (not necessarily finite) pointed sets. By degreewise evaluation and formation of the diagonal of the resulting bisimplicial sets, it can furthermore be promoted to a functor from the category of pointed simplicial sets to itself [8, §4]. The extended functor preserves weak equivalences of simplicial sets [8, Prop. 4.9] and is automatically simplicial, ie, it comes with coherent natural maps $K \wedge X(L) \longrightarrow X(K \wedge L)$. We will not distinguish notationally between the prolonged functor and the original Γ–space.

The homotopy groups of a Γ–space X are defined as

$$\pi_n X \;=\; \operatorname{colim}_i \pi_{n+i}|X(S^i)| \;,$$

where the colimit is formed using the maps

$$S^1 \wedge X(S^n) \;\longrightarrow\; X(S^1 \wedge S^n) \;.$$

A map of Γ–spaces is a *stable equivalence* if it induces isomorphisms on homotopy groups. Since the functor given by a prolonged Γ–space preserves connectivity [8, 4.10], the homotopy groups of a Γ–space are always trivial in negative dimensions.

2.2 Smash products In [29, Thm. 2.2], Lydakis defines a smash product for Γ–spaces which is characterized by the universal property that Γ–space maps $X \wedge Y \longrightarrow Z$ are in bijective correspondence with maps

$$X(k^+) \wedge Y(l^+) \quad\longrightarrow\quad Z(k^+ \wedge l^+)$$

which are natural in both variables. By [29, Thm. 2.18], the smash product of Γ–spaces is associative and commutative with unit $\mathbb{S}$, up to coherent natural isomorphism. There are also internal homomorphism Γ–spaces [29, 2.1], adjoint to the smash-product, so that the category of Γ–spaces forms a symmetric monoidal closed category.

2.3 Special Γ–spaces A Γ–space X is called *special* if the map $X(k^+ \vee l^+) \longrightarrow X(k^+) \times X(l^+)$ induced by the projections from $k^+ \vee l^+$ to k^+ and l^+ is a weak equivalence for all k and l. In this case, the weak map

$$X(1^+) \times X(1^+) \xleftarrow{\;\sim\;} X(2^+) \xrightarrow{X(\nabla)} X(1^+)$$

induces an abelian monoid structure on $\pi_0(X(1^+))$. Here $\nabla\colon 2^+ \longrightarrow 1^+$ is the fold map defined by $\nabla(1) = 1 = \nabla(2)$. X is called *very special* if it is special and the monoid $\pi_0(X(1^+))$ is a group. By Segal's theorem ([48, Prop. 1.4], see also [8, Thm. 4.2]), the spectrum associated to a very special Γ–space X is an Ω–spectrum in the sense that the maps $|X(S^n)| \longrightarrow \Omega|X(S^{n+1})|$ adjoint to the spectrum structure maps are homotopy equivalences. In particular, the homotopy groups of a very special Γ–space X are naturally isomorphic to the homotopy groups of the simplicial set $X(1^+)$.

2.4 Model structures Bousfield and Friedlander introduce two model category structures for Γ–spaces called the *strict* and the *stable* model categories [8, 3.5, 5.2]. It will be more convenient for our purposes to work with slightly different model category structures, which we called the Quillen- or Q–model category structures in [44] and [46]. The strict and stable Q–structures have the same weak equivalences, hence the same homotopy categories, as the corresponding Bousfield–Friedlander model category structures. In this paper we never consider the Bousfield–Friedlander model structures, so we drop the decoration 'Q' for the other model structure.

We call a map of Γ–spaces a *strict fibration* (resp. a *strict equivalence*) if it is a Kan fibration (resp. weak equivalence) of simplicial sets when evaluated at every $n^+ \in \Gamma^{\mathrm{op}}$. *Cofibrations* are defined as the maps having the left lifting property with respect to all strict acyclic fibrations. The cofibrations can be characterized as the injective maps with projective cokernel, see [44, Lemma A3 (b)] for the precise statement. By [39, II.4 Thm. 4], the strict equivalences, strict fibrations and cofibrations make the category of Γ–spaces into a closed simplicial model category.

More important is the *stable* model category structure. This one is obtained by localizing the strict model structure with respect to the stable equivalences. We call a map of Γ–spaces a *stable fibration* if it has the right lifting property with respect to the cofibrations which are also stable equivalences. By [44, Thm. 1.5], the stable equivalences, stable fibrations and cofibrations make the category of Γ–spaces into a closed simplicial model category. A Γ–space X is stably fibrant if and only if it is very special and $X(n^+)$ is fibrant as a simplicial set for all $n^+ \in \Gamma^{\mathrm{op}}$.

A Γ–space X defines a spectrum $X(S)$ (in the sense of [8, Def. 2.1]) whose n-th term is the value of the prolonged Γ–space at S^n. For example, the Γ–space $\mathbb{S}$ becomes isomorphic to the identity functor of the category of pointed simplicial sets after prolongation. So the associated spectrum is the sphere spectrum. The functor that sends a Γ–space X to the spectrum $X(S)$ has a right adjoint [8, Lemma 4.6], and these two functors form a Quillen pair. One of the main theorems of [8] says that this Quillen pair induces an equivalence between the homotopy category of Γ–spaces, taken with respect to the stable equivalences, and the stable homotopy category of connective spectra (see [8, Thm. 5.8]). We do not use this result here, but it is the main motivation for the study of Γ–spaces.

2.5 The assembly map Given two Γ–spaces X and Y, there is a natural map $X \wedge Y \longrightarrow X \circ Y$ from the smash product to the composition product [29, 2.12], [46, 1.8]. The formal and homotopical properties of this *assembly map* are of importance to this paper. Since Γ–spaces prolong to functors defined on the category of pointed simplicial sets, they can be composed. Explicitly, for Γ–spaces X and Y, we set $(X \circ Y)(n^+) = X(Y(n^+))$. This composition $\circ$ is a monoidal (though not symmetric monoidal) product on the category of Γ–spaces. The unit is the same as for the smash product, it is the Γ–space $\mathbb{S}$ which as a functor is the inclusion of Γ^{op} into all pointed simplicial sets.

The assembly map is obtained as follows. Prolonged Γ–spaces are naturally simplicial functors [8, §3], which means that there are natural coherent maps $X(K) \wedge L \longrightarrow X(K \wedge L)$. This simplicial structure gives maps

$$X(n^+) \wedge Y(m^+) \longrightarrow X(n^+ \wedge Y(m^+)) \longrightarrow X(Y(n^+ \wedge m^+))$$

natural in both variables. From this the assembly map $X \wedge Y \longrightarrow X \circ Y$ is obtained by the universal property of the smash product of Γ–spaces. The assembly map is associative and unital, $\mathbb{S}$ being the unit for both $\wedge$ and $\circ$. In technical terms: the identity functor on the category of Γ–spaces becomes a lax monoidal functor from $(\mathcal{G}S, \wedge)$ to $(\mathcal{G}S, \circ)$. The homotopical properties of smash and composition product and of the assembly map are summarized in the following theorem, which is due to Lydakis [29].

Theorem 2.6 (a) *The composition product of* Γ*–spaces preserves stable equivalences in each of its variables.*

(b) *The smash product with a cofibrant* Γ*–space preserves stable equivalences.*

(c) *Let* X *and* Y *be* Γ*–spaces, one of which is cofibrant. Then the assembly map*

$$X \wedge Y \longrightarrow X \circ Y$$

is a stable equivalence.

Proof Parts (b) and (c) are [29, Prop. 5.12] and [29, Prop. 5.23] respectively.

For any Γ–space F the structure map $S^1 \wedge F(S^n) \longrightarrow F(S^1 \wedge S^n)$ is $(2n+1)$–connected [29, Prop. 5.21]. So the map $\pi_* \Omega^n |F(S^n)| \longrightarrow \pi_* F$ is an isomorphism for $* < n$. Hence if $F \longrightarrow F'$ is a stable equivalence of Γ–spaces, then the map $F(S^n) \longrightarrow F'(S^n)$ is $2n$–connected. If X is any Γ–space, then the prolonged functor preserves connectivity [29, Prop. 5.20], so the map $X(F(S^n)) \longrightarrow X(F'(S^n))$ is also $2n$–connected. Thus the map $X \circ F \longrightarrow X \circ F'$ is a stable equivalence.

It remains to show that the map $F \circ X \longrightarrow F' \circ X$ is also a stable equivalence. By the previous paragraph we may assume that X is cofibrant, and then the claim follows from parts (b) and (c). □

2.7 Gamma-rings and their modules Our notion of ring spectrum is that of a *Gamma-ring*. Gamma-rings are the monoids in the symmetric monoidal category of Γ–spaces with respect to the smash product and they are special cases of 'Functors with Smash Product' (FSPs, cf. [5, 1.1], [37, 2.2]). One can describe Gamma-rings as 'FSPs defined on finite sets'. From a Gamma-ring one obtains an FSP or *symmetric ring spectrum* ([24], [49, 2.1.11]) by prolongation and, in the second case, evaluation on spheres. A more detailed discussion of the homotopy theory of Gamma-rings can be found in [44]. Homotopy theoretic results about the Gamma-rings can be translated into other frameworks for ring spectra by the general comparison procedures of [32].

Explicitly, a Gamma-ring is a Γ–space R equipped with maps

$$\mathbb{S} \longrightarrow R \quad \text{and} \quad R \wedge R \longrightarrow R \,,$$

called the unit and multiplication map, which satisfy certain associativity and unit conditions (see [31, VII.3]). A morphism of Gamma-rings is a map of Γ–spaces commuting with the multiplication and unit maps. If R is a Gamma-ring, a *left R–module* is a Γ–space N together with an action map $R \wedge N \longrightarrow N$ satisfying associativity and unit conditions (see again [31, VII.4]). A morphism of left R–modules is a map of Γ–spaces commuting with the action of R. We denote the category of left R–modules by R–mod.

One similarly defines right modules and bimodules. Because of the universal property of the smash product of Γ–spaces (2.2), Gamma-rings are in bijective correspondence with lax monoidal functors from the category Γ^{op} to the category of pointed simplicial sets (both under smash product).

2.8 Examples The unit $\mathbb{S}$ of the smash product is a Gamma-ring in a unique way. The category of $\mathbb{S}$–modules is isomorphic to the category of Γ–spaces. Other standard examples of Gamma-rings are monoid rings over the sphere Gamma-ring $\mathbb{S}$ and Eilenberg–MacLane models of classical rings. If M is a simplicial monoid, we define a Γ–space $\mathbb{S}[M]$ by

$$\mathbb{S}[M]\,(n^+) \;=\; M^+ \wedge n^+$$

(so $\mathbb{S}[M]$ is isomorphic, as a Γ–space, to $\mathbb{S} \wedge M^+$, see (2.1)). There is a unit map $\mathbb{S} \longrightarrow \mathbb{S}[M]$ induced by the unit of M and a multiplication map $\mathbb{S}[M] \wedge \mathbb{S}[M] \longrightarrow \mathbb{S}[M]$ induced by the multiplication of M which turn $\mathbb{S}[M]$

into a Gamma-ring. This construction of the monoid ring over $\mathbb{S}$ is left adjoint to the functor which takes a Gamma-ring R to the simplicial monoid $R(1^+)$.

If A is an ordinary ring, then the Eilenberg–MacLane Γ–space HA is given by the functor which takes a pointed set K to the reduced free A–module $\widetilde{A}[K]$ generated by K. The unit map

$$\eta:\ K \longrightarrow \widetilde{A}[K]$$

is the inclusion of generators. The multiplication map

$$\mu:\ \widetilde{A}[K] \wedge \widetilde{A}[L] \longrightarrow \widetilde{A}[K \wedge L]$$

takes a smash product $(\sum_{k\in K} a_k \cdot k) \wedge (\sum_{l\in L} b_l \cdot l)$ to the element $\sum a_k b_l \cdot (k \wedge l)$. For later reference we note that the multiplication $\mu\colon HA \wedge HA \longrightarrow HA$ factors as the composition

$$HA \wedge HA \xrightarrow{\text{assembly}} HA \circ HA \xrightarrow{\text{eval.}} HA \qquad (2.9)$$

of the assembly map (2.5) and the evaluation map.

More examples of Gamma-rings arise from algebraic theories and as endomorphism Gamma-rings, see [46, 4.5, 4.6]. The Gamma-ring DB (3.1) is such an example.

The modules over a fixed Gamma-ring and the category of all Gamma-rings form simplicial model categories, created by the forgetful functor to Γ–spaces [44, Thm. 2.2 and Thm. 2.5]. In these model structures a map is a weak equivalence (resp. fibration) if it is a stable equivalence (resp. stable fibration) as a map of Γ–spaces. For a ring A, the Eilenberg–MacLane functor H is the right adjoint of a Quillen equivalence between the model category of HA–modules and the model category of simplicial A–modules [44, Thm. 4.4].

3 The Gamma-ring DB and formal group laws

In this section we recall the definition and some basic properties of our main object of study, the Gamma-ring DB, for B a fixed commutative ring. This Gamma-ring parameterizes the stable homotopy theory of augmented commutative simplicial B–algebras. In Construction 3.5 we explain how a formal group law F over B gives rise to a homomorphism of ring spectra $F_*\colon H\mathbb{Z} \longrightarrow DB$. The rest of this paper is then devoted to studying the homotopical significance of that construction.

By "parameterizing the stable homotopy theory" we mean that there is a Quillen-equivalence between the model category of DB–modules and the model category of spectra of simplicial commutative B–algebras, see Theorem 3.2 (d). Commutative simplicial algebras have been the object of much study [38, 12, 20, 21, 43, 51]. The homology theory arising as the derived functor of abelianization in this category is known as André–Quillen homology for commutative rings. The stable homotopy category of simplicial commutative B–algebras is equivalent to the homotopy category of DB–modules. The homotopy groups of DB are the coefficients of the universal homology theory for commutative algebras.

3.1 The Gamma-ring DB The Γ–space underlying the Gamma-ring DB takes a pointed set K to the augmentation ideal of the power series ring generated by K, considered as a constant simplicial set:

$$DB(K) \;=\; \mathrm{kernel}\,(\, \widetilde{B}[\![K]\!] \longrightarrow \widetilde{B}[\![*]\!] = B\,)\ .$$

The tilde over $\widetilde{B}[\![K]\!]$ indicates that the power series generator corresponding to the basepoint of K has been set equal to 0; thus $\widetilde{B}[\![*]\!]$ reduces to the coefficient ring.

The product which makes DB into a Gamma-ring comes from substitution of power series. To define the multiplication map $\mu\colon DB \wedge DB \longrightarrow DB$ we need to describe a natural associative map

$$\mu:\; DB(K) \wedge DB(L) \;\longrightarrow\; DB(K \wedge L)$$

for pointed sets K and L. An element of $DB(K)$ is represented by a power series f in variables K without constant term. Similarly, an element of $DB(L)$ is represented by a power series g in variables L. The multiplication map μ takes the smash product

$$f \wedge g \;\in\; DB(K) \wedge DB(L)$$

to the power series $\mu(f \wedge g)$ in the variables $K \wedge L$ defined by

$$\begin{aligned}\mu(f(k_1,\ldots,k_m) \wedge g(l_1,\ldots,l_n)) \quad &= \\ & f(\,g(k_1 \wedge l_1,\ldots,k_1 \wedge l_n),\ \ldots\ ,g(k_m \wedge l_1,\ldots,k_m \wedge l_n)\,)\ .\end{aligned}$$

The unit map $\eta\colon \mathbb{S} \longrightarrow DB$, ie, natural transformation

$$\eta:\; K \longrightarrow DB(K)\ ,$$

sends an element of K to the generator it represents. The multiplication and unit transformations μ and η are associative and unital because substitution of power series is, so DB is in fact a Gamma-ring.

Some important properties of DB are summarized in the following theorem. Most of these results are compiled from [46]. In [46, 7.9], we use the notation DB for a slightly different Gamma-ring, namely the Gamma-ring associated to the algebraic theory of augmented commutative B–algebras. Let DB^{pol} denote the sub-Γ–space of DB whose value at a pointed set K consists of all *polynomials* in K, ie, the power series in $DB(K)$ with only finitely many non-zero coefficients. The sub-Γ–space DB^{pol} is closed under the multiplication of DB, and the unit map $\eta\colon \mathbb{S} \longrightarrow DB$ factors through DB^{pol}. Hence DB^{pol} is a Gamma-ring and the inclusion $DB^{\mathrm{pol}} \longrightarrow DB$ is a homomorphism. DB^{pol} is exactly the Gamma-ring which is denoted DB in [46, 7.9]; there is no homotopical difference between the two Gamma-rings since the inclusion $DB^{\mathrm{pol}} \longrightarrow DB$ is a stable equivalence, see Theorem 3.2 (a) below. However, in this paper it is more convenient to work with the power series model, so we give it the simpler name.

Theorem 3.2 (a) *The inclusion $DB^{\mathrm{pol}} \longrightarrow DB$ is a stable equivalence of Gamma-rings.*

(b) *As a Γ–space, DB is stably equivalent to the derived smash product of the Eilenberg–MacLane Γ–spaces $H\mathbb{Z}$ and HB; in particular, the stable homotopy groups of DB are additively isomorphic to the spectrum homology of the Eilenberg–MacLane spectrum of B.*

(c) *The graded ring of homotopy groups of DB is isomorphic to the ring of stable homotopy operations of commutative augmented simplicial B–algebras.*

(d) *There is a Quillen adjoint functor pair between the model category of DB–modules and the model category $\mathcal{S}p(B\text{-alg})$ of spectra of commutative augmented simplicial B–algebras. The adjoint pair passes to an equivalence between the homotopy category of DB–modules and the homotopy category of connective spectra of commutative augmented simplicial B–algebras.*

Proof (a) Let $(B\otimes\mathcal{S}^k)^!$ denote the Γ–space defined by

$$(B\otimes\mathcal{S}^k)^!(K) \;=\; B\otimes(\widetilde{\mathbb{Z}}[K]^{\otimes k}/\Sigma_k)\ ,$$

the tensor product of B with the k-th symmetric power of the reduced free abelian group on K. An isomorphic description of $(B\otimes\mathcal{S}^k)^!(K)$ is as the free reduced B–module on the k-th symmetric power of K,

$$(B\otimes\mathcal{S}^k)^!(K) \;\cong\; \widetilde{B}[K^{\wedge k}/\Sigma_k]\ .$$

The underlying Γ–space of DB is the infinite product of the symmetric power functors $(B \otimes \mathcal{S}^k)^!$ for all $k \geq 1$. The polynomial model DB^{pol} is the *weak* product of these symmetric power functors. Since the stable homotopy groups of the Γ–space $(B \otimes \mathcal{S}^k)^!$ are trivial up to dimension $2k-3$ [11, 12.3], the weak product and the product are stably equivalent.

(b) We let SP denote the Γ–space which takes a pointed set K to its infinite symmetric product, ie, the free abelian monoid generated by K with basepoint as identity element. By the Dold–Thom theorem, the group completion map $SP \longrightarrow H\mathbb{Z}$ is a stable equivalence of Γ–spaces. We choose a cofibrant replacement HB^{c} of HB as a Γ–space and obtain a chain of homomorphisms of Γ–spaces

$$\begin{aligned} HB^{\mathrm{c}} \wedge H\mathbb{Z} \;\xleftarrow{\sim}\; & HB^{\mathrm{c}} \wedge SP \xrightarrow{\sim} HB^{\mathrm{c}} \circ SP \\ \xrightarrow{\sim}\; & HB \circ SP \cong DB^{\mathrm{pol}} \xrightarrow{\sim} DB \;. \end{aligned}$$

The first map is a stable equivalence since smashing with a cofibrant Γ–space preserves stable equivalences [29, 5.12]. The second map is the assembly map (2.5), which is a stable equivalence by [29, 5.23]. The third map is a stable equivalence since the composition product of Γ–spaces preserves stable equivalences in both variables (Theorem 2.6 (a)). The reduced free B–module generated by $SP(K)$ is isomorphic to the polynomials without constant term generated by K, subject to the basepoint relation. This gives as isomorphism of Γ–spaces between $HB \circ SP$ and DB^{pol}. The last map is a stable equivalence by part (a).

Part (c) and (d) are special cases of [46, 4.11]. and [46, 4.4] respectively; see also [46, 7.9]. □

The ring of stable homotopy operations of commutative augmented simplicial B–algebras — ie, the graded ring of homotopy groups of DB — is sometimes called the stable Cartan–Bousfield–Dwyer-algebra since these authors calculated the unstable operations for $B = \mathbb{F}_p$, see [10, 7, 12]. An explicit calculation of $\pi_* D\mathbb{F}_p$ can be found as Theorems 12.3 (for $p = 2$) and 12.6 (for p odd) in Bousfield's unpublished paper [7]. The fact that the ring of stable homotopy operations is generally not commutative shows that DB is *not* stably equivalent to the derived smash product $HB \wedge^L H\mathbb{Z}$ as a Gamma-ring (unless B is a $\mathbb{Q}$–algebra).

3.3 Relation to topological André-Quillen homology and Γ-homology

There are isomorphisms of graded abelian groups

$$\pi_* DB \;\cong\; H\Gamma_*(B[x]|B;B) \;\cong\; TAQ_*(HB[x]|HB;HB) \;. \tag{3.4}$$

Here $H\Gamma_*(B[x]|B;B)$ is the Γ*–homology* in the sense of Robinson and Whitehouse [42], of the polynomial algebra $B[x]$, considered as an augmented B–algebra; moreover, $TAQ_*(HB[x]|HB;HB)$ denotes the *topological André–Quillen homology* [1] of the Eilenberg–MacLane ring spectrum $HB[x]$ relative to HB. Both Γ–homology and topological André–Quillen homology groups are studied because they carry obstructions to the existence of E_∞ ring spectrum structures [41] (it is a coincidence that the symbol Γ occurs both in Γ–homology and as the category Γ^{op}).

The first isomorphism in (3.4) comes about as follows. By 3.2 (a) above, the Gamma-ring DB has a stably equivalent polynomial model DB^{pol}; as a Γ–space, DB^{pol} is isomorphic to the functor which assigns to the object n^+ of Γ^{op} the B–module $B[x]^{\otimes_B n}$. By a theorem of Pirashvili and Richter [36, Thm. 1], the homotopy groups $\pi_* DB$ are thus isomorphic to the Γ–homology $H\Gamma_*(B[x]|B;B)$ of the polynomial algebra $B[x]$ relative to B. The second isomorphism in (3.4) is due to Basterra and McCarthy [2], who show that for Eilenberg–MacLane spectra of classical rings, topological André–Quillen homology coincides with Γ–homology. The survey article [3] by Basterra and Richter discusses all these identifications in more detail.

The isomorphisms (3.4) do not mention the multiplicative structure of $\pi_* DB$. In Sections 5.1 and 7.9 of [46] we associate to any augmented commutative B–algebra A a DB–module $\Sigma^\infty_B A$ which models the suspension spectrum of A as an augmented commutative B–algebra. The underlying Γ–space of $\Sigma^\infty_B A$ sends $n^+ \in \Gamma^{\mathrm{op}}$ to the B–module $A^{\otimes_B n}$. So [36, Thm. 1] yields an isomorphism of graded abelian groups

$$\pi_*(\Sigma^\infty_B A) \;\cong\; H\Gamma_*(A|B;B)$$

(at least if A is flat as a B–module). The DB–action on $\Sigma^\infty_B A$ gives more structure to Γ–homology, since the left hand side above has a natural action of the graded ring $\pi_* DB \cong H\Gamma_*(B[x]|B;B)$. We see moreover that the homotopical object underlying Γ–homology is not just a chain complex, but a DB–module spectrum.

The main objective of this paper is the study of the homotopy type of DB as a ring spectrum and of a close relationship to formal group theory. More precisely we will describe the space of Gamma-ring maps from $H\mathbb{Z}$, the Eilenberg–MacLane Gamma-ring of the integers (2.8), to DB in terms of formal group law data. Unless stated otherwise, formal group laws will always be *1–dimensional and commutative.* To see how non-commutative and higher-dimensional formal group laws fit into our context see Remarks 13.4 and 13.5.

Construction 3.5 Suppose that F is a 1–dimensional and commutative formal group law over the commutative ring B. In other words, F is a power series in two variables x and y with coefficients in B which satisfies

$$\begin{aligned} F(x,0) &= x = F(0,x) , \\ F(x,y) &= F(y,x) \quad \text{and} \\ F(F(x,y),z) &= F(x,F(y,z)) . \end{aligned}$$

We define a map $F_*\colon H\mathbb{Z} \longrightarrow DB$ of Gamma-rings. For every pointed set K we have to specify a map

$$F_*(K) : \ \widetilde{\mathbb{Z}}[K] = H\mathbb{Z}(K) \ \longrightarrow \ DB(K) \ \subseteq \ \widetilde{B}[\![K]\!]$$

which is natural in K and respects the multiplication and unit maps. The map $F_*(K)$ simply takes a sum $\sum_{k\in K} a_k \cdot k$ of generators of the free abelian group on K to the formal sum

$$\sum_{k\in K}^{F} [a_k]_F(k)$$

with respect to F, of the same elements viewed as generators of the power series ring. Here $[n]_F$ denotes the n–series of the formal group law F for every integer $n \in \mathbb{Z}$. We omit the verification that the map F_* indeed commutes with the multiplication and unit map. For example, on the level of underlying monoids this means that the map

$$[-]_F : \ \mathbb{Z} \cong H\mathbb{Z}(1^+) \ \longrightarrow \ DB(1^+) \ = \ x \cdot B[\![x]\!]$$

is a homomorphism from the multiplicative monoid of the integers to the monoid of power series without constant term under substitution, ie, it boils down to the relation

$$[n]_F([m]_F(x)) \ = \ [n \cdot m]_F(x)$$

for $n, m \in \mathbb{Z}$.

Remark 3.6 We offer two additional ways of looking at the above construction of the Gamma-ring map F_*; the three definitions correspond to looking at a commutative 1–dimensional formal group law as either

(a) a power series $F(x,y)$ in two variables with certain properties,

(b) an abelian cogroup structure, in the category of complete, augmented commutative B–algebras, on the power series ring $B[\![x]\!]$,

(c) or a morphism of algebraic theories from the theory of abelian groups to the theory of complete, augmented, commutative B–algebras.

The first point of view leads to the explicit formula for the Gamma-ring map F_* that was just given in Construction 3.5.

The interpretation (b) of a formal group law exhibits the Gamma-ring map F_* as a special case of a more general construction associated to an abelian cogroup object. Indeed, to every object X in a pointed category with coproducts $\mathcal{C}$ one can associate an *endomorphism Gamma-ring* $\mathrm{End}_{\mathcal{C}}(X)$ (see [46, 4.6] or 13.3). Then every abelian cogroup structure on the object X leads to a map of Gamma-rings $H\mathbb{Z} \longrightarrow \mathrm{End}_{\mathcal{C}}(X)$; we refer to 13.3 for more details.

Perspective (c) leads to a compact description in the language of algebraic theories [6, 3.3.1]. Specifying a 1–dimensional, commutative formal group law over B is the same as specifying a morphism of algebraic theories [6, 3.7.1] from the theory of abelian groups to the theory of complete, augmented, commutative B–algebras. The construction [44, 4.5] which associates to a pointed algebraic theory T its stable Gamma-ring T^s is functorial for morphisms of algebraic theories. Now $H\mathbb{Z}$ is the Gamma-ring associated to the theory of abelian groups and DB is the Gamma-ring associated to the theory of complete, augmented, commutative B–algebras. Hence a formal group law F defines a morphism of algebraic theories, thus a morphisms of associated Gamma-rings.

Example 3.7 The *additive formal group law* is given by

$$F^a(x,y) \;=\; x+y \;.$$

The associated Gamma-ring map F^a_* is the composite

$$H\mathbb{Z} \longrightarrow HB \xrightarrow{\text{incl.}} DB$$

of the unique Gamma-ring map $H\mathbb{Z} \longrightarrow HB$ with the 'inclusion' of HB into DB as the linear power series. Conversely, this is the only case in which the Gamma-ring map F_* factors over the inclusion $HB \longrightarrow DB$ on the point-set level: the power series $F(x,y)$ can be recovered from the Gamma-ring map F_* as the image of $x+y \in \mathbb{Z}[x,y]$ under

$$F_* : \; \mathbb{Z}[x,y] \;\cong\; H\mathbb{Z}(\{x,y\}^+) \;\longrightarrow\; DB(\{x,y\}^+) \;\subseteq B[\![x,y]\!] \;;$$

so if F_* factors over HB, then $F(x,y)$ has only linear terms, so that necessarily $F(x,y) = x+y$.

Another formal group law which exists over any ring B is the *multiplicative* one given by

$$F^m(x,y) \;=\; x+y+xy \;.$$

The multiplicative formal group law can be used to express the Gamma-ring DB additively as the smash product of the Eilenberg–MacLane Gamma-rings for B and $\mathbb{Z}$; indeed the composite map

$$HB \wedge^L H\mathbb{Z} \xrightarrow{\text{incl.} \wedge^L F^m_*} DB \wedge^L DB \xrightarrow{\mu} DB$$

is in the same homotopy class as the stable equivalence of Theorem 3.2 (b).

The homotopical significance of the point-set level construction of 3.5 is summarized in the following

Theorem 3.8 *Construction 3.5 which to a 1–dimensional, commutative formal group law F associates the Gamma-ring map $F_*\colon H\mathbb{Z} \longrightarrow DB$ induces a natural bijection*

$$\mathrm{FGL}(B)/\text{strict isomorphism} \xrightarrow{\cong} [H\mathbb{Z}, DB]_{\text{Gamma-rings}}$$

between the strict isomorphism classes of formal group laws over B and the set of maps from $H\mathbb{Z}$ to DB in the homotopy category of Gamma-rings.

Theorem 3.8 is the π_0–part of a space level statement relating formal groups to Gamma-ring maps between $H\mathbb{Z}$ and DB in Theorem 5.2. In the simplicial model category of Gamma-rings every pair of objects has a homomorphism space (ie, simplicial set). As usual with model categories, in order to give the morphism space a homotopy invariant meaning, the source object has to be replaced by a weakly equivalent cofibrant one, and the target object has to be replaced by a weakly equivalent fibrant one. Then the components of the (derived) space of Gamma-ring maps are the morphisms in the homotopy category of Gamma-rings. Theorem 5.2 below identifies the derived homomorphism space of Gamma-ring maps from $H\mathbb{Z}$ to DB. The answer is given in terms of formal group law data and the simplicial monoid of *homotopy units* of DB.

3.9 Homotopy units Let R be a Gamma-ring and R^{f} a stably fibrant replacement of R in the model category structure of [44, 2.5]. As for any Gamma-ring, the underlying space $R^{\mathrm{f}}(1^+)$ is a simplicial monoid with product induced by the multiplication map $R^{\mathrm{f}} \wedge R^{\mathrm{f}} \longrightarrow R^{\mathrm{f}}$. Moreover, $R^{\mathrm{f}}(1^+)$ is a model for the infinite loop space of the spectrum represented by R. We define the *homotopy units* $R^\times$ as the union of the invertible components of the simplicial monoid $R^{\mathrm{f}}(1^+)$. So $R^\times$ is a group-like simplicial monoid which is independent up to weak equivalence of the choice of fibrant replacement. Moreover there are natural isomorphisms of homotopy groups

$$\pi_0 R^\times \cong \text{units}(\pi_0 R) \quad \text{and} \quad \pi_i R^\times \cong \pi_i R \quad \text{for } i \geq 1.$$

For any classical ring A, the units of A act by conjugation on A and hence on the set of ring homomorphisms from any other ring to A. In Section 4 we make sense of the analogous conjugation action for Gamma-rings. For any Gamma-ring R we construct, after change of models, a strict action of the homotopy units $R^\times$ on R by Gamma-ring homomorphisms. More precisely we introduce a simplicial group $UR^\times$, weakly equivalent to $R^\times$, and a fibrant Gamma-ring, stably equivalent to R, and on which $UR^\times$ acts by conjugation. Below we only use the action of the simplicial subgroup $UR^\times_1$, the connected component of the unit element. We can consider the homotopy orbit space of the conjugation action of $UR^\times_1$ on the simplicial set $\mathrm{Ring}(H\mathbb{Z}, R) = \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^{\mathrm{c}}, R^{\mathrm{f}})$. We denote that homotopy orbit space by $\mathrm{Ring}(H\mathbb{Z}, R)/\mathrm{conj}$.

Construction 3.5 associates to every formal group law over B a Gamma-ring map from the Eilenberg–MacLane Gamma-ring $H\mathbb{Z}$ to DB. This map gives rise to a point in the space of Gamma-ring maps $\mathrm{Ring}(H\mathbb{Z}, DB)$. In Section 5 we extend this to a natural weak map from the classifying space $\mathcal{FGL}^{\mathrm{str}}(B)$ of the groupoid of formal group laws and strict isomorphisms to the homotopy orbits space of the conjugation action, ie, we construct a diagram of simplicial sets

$$\mathcal{FGL}^{\mathrm{str}}(B) \xleftarrow{\sim} \widetilde{\mathcal{FGL}}^{\mathrm{str}}(B) \xrightarrow{\kappa} \mathrm{Ring}(H\mathbb{Z}, DB)/\mathrm{conj}.$$

in which the first map is a weak equivalence. The main result of this paper, Theorem 5.2, says that the map $\kappa\colon \widetilde{\mathcal{FGL}}^{\mathrm{str}}(B) \longrightarrow \mathrm{Ring}(H\mathbb{Z}, DB)/\mathrm{conj}.$ is a weak equivalence. Theorem 3.8 is the bijection induced on path components by the weak equivalence κ.

4 The conjugation action

In this section we construct the conjugation action of the homotopy units of a Gamma-ring on the Gamma-ring. In view of the application to DB we need a construction relative to a group which maps to the multiplicative monoid of the Gamma-ring. In the example of DB that group is the group of invertible power series on one generator over B.

Construction 4.1 We consider a Gamma-ring R together with a simplicial group G and a homomorphism of simplicial monoids $\psi\colon G \longrightarrow R(1^+)$. The conjugation action of G on R is described by a map of simplicial monoids

$$c : \; G \;\longrightarrow\; \mathrm{map}_{\mathcal{GR}}(R, R) \; .$$

Here for $g \in G$ the conjugation map $c(g)\colon R \longrightarrow R$ is defined at a pointed set K as the composite

$$R(K) \xrightarrow{g \wedge \mathrm{id} \wedge g^{-1}} G_+ \wedge R(K) \wedge G_+ \xrightarrow{\psi \wedge \mathrm{id} \wedge \psi} R(1^+) \wedge R(K) \wedge R(1^+)$$
$$\xrightarrow{\text{mult.}} R(1^+ \wedge K \wedge 1^+) \cong R(K) .$$

We omit the formal verification that the conjugation map is in fact a homomorphism of Gamma-rings, that the definition extends to higher dimensional simplices of G, and that the formula $c(g \cdot g') = c(g) \circ c(g')$ holds. The monoid map c can now be used to let the group G act on the space of Gamma-ring maps from any Gamma-ring S to R via

$$G \times \mathrm{map}_{\mathcal{GR}}(S,R) \longrightarrow \mathrm{map}_{\mathcal{GR}}(S,R) , \quad (g,f) \mapsto c(g) \circ f .$$

The goal of this section is to extend the conjugation action from the given group G to the homotopy units (3.9) of R. The problem is that Construction 4.1 makes use of strict inverses, whereas the homotopy units $R^\times$ are only a group-like simplicial monoid. One way to solve this would be to find a stable equivalence of Gamma-rings from R to some stably fibrant Gamma-ring R^{f} which has the property that every element in an invertible component of $R^{\mathrm{f}}(1^+)$ has a point-set inverse. But it seems unlikely that this can be done in general, and we use a different approach.

Construction 4.2 As above we consider a Gamma-ring R together with a simplicial group G and a homomorphism of simplicial monoids $\psi\colon G \longrightarrow R(1^+)$. If $R \longrightarrow R^{\mathrm{f}}$ is a stably fibrant replacement of R in the model category of Gamma-rings then $R^\times$ was defined in 3.9 as the simplicial monoid of invertible components in $R^{\mathrm{f}}(1^+)$. The image of the simplicial group G under the map $R(1^+) \longrightarrow R^{\mathrm{f}}(1^+)$ is contained in the invertible components, which provides a morphism of simplicial monoids $G \longrightarrow R^\times$. We explained in 4.1 how the group G acts on spaces of Gamma-ring maps into R, and we now want to extend this to a conjugation action of the homotopy units $R^\times$.

We start by factoring the homomorphism $G \longrightarrow R^\times$ in the model category of simplicial monoids (a special case of [39, II.4 Thm. 4]) as a cofibration followed by an acyclic fibration

$$G \rightarrowtail cR^\times \xrightarrow{\sim} R^\times .$$

We denote by $UR^\times$ the algebraic group completion of the simplicial monoid $cR^\times$. So $UR^\times$ is obtained from $cR^\times$ by formally adjoining inverses in every simplicial dimension. Since the map $G \longrightarrow cR^\times$ is a cofibration of simplicial monoids, the monoid $cR^\times$ is a retract of a simplicial monoid which is dimensionwise a free product of a group and a free monoid [39, II p. 4.11 Rem. 4]. By the following Lemma the group completion map $cR^\times \longrightarrow UR^\times$ is thus a weak equivalence.

Lemma 4.3 *Let M be a simplicial monoid which in every dimension is a free product of a group and a free monoid, and such that $\pi_0 M$ is a group. Then the group completion map $M \longrightarrow UM$ is a weak equivalence.*

Proof We call a simplicial monoid N *good* if the group completion map $N \longrightarrow UN$ induces a weak equivalence $BN \longrightarrow BUN$ of classifying spaces. For any simplicial monoid N whose components form a group, BN is a delooping of N, ie, the map $|N| \longrightarrow \Omega|BN|$ is a weak equivalence; this follows for example by applying [8, B.4] to the sequence of bisimplicial sets $N \longrightarrow E_\bullet N \longrightarrow B_\bullet N$. So it suffices to show that any monoid as in the statement of the lemma is good.

Suppose M and N are good, discrete monoids. We claim that then the free product $M * N$ is also good. By [33, Lemma 4], the canonical map of simplicial sets $BM \vee BN \longrightarrow B(M * N)$ is a weak equivalence. Since $U(M * N) \cong UM * UN$ (the coproduct in the category of groups coincides with the free product of underlying monoids), the map $BUM \vee BUN \longrightarrow BU(M * N)$ is also a weak equivalence and the claim follows.

Every group, viewed as a constant simplicial monoid, is good. The classifying spaces of the free monoid on one generator and of the free group on one generator are both weakly equivalent to a circle. So a free monoid on one generator is good. By the above and by direct limit, a free product of a group with a free monoid is good.

The classifying space BN of a simplicial monoid N is the diagonal of the bisimplicial set given by the classifying spaces BN_m of the individual monoids N_m in the various simplicial dimensions. Hence if N is a simplicial monoid such that the discrete monoid N_m is good for all $m \geq 0$, then N itself is good. This proves the lemma. □

We take the adjoint $\mathbb{S}[cR^\times] \longrightarrow R^{\mathrm{f}}$ of the monoid map $cR^\times \longrightarrow R^\times \longrightarrow R^{\mathrm{f}}(1^+)$ (where $\mathbb{S}[cR^\times]$ is the monoid Gamma-ring (2.8)) and factor it in the model category of Gamma-rings as a cofibration followed by an acyclic fibration

$$\mathbb{S}[cR^\times] \rightarrowtail R_1 \xrightarrow{\sim} \!\!\!\!\!\rightarrow R^{\mathrm{f}} \ .$$

We then define another Gamma-ring R_2 as the pushout, in the category of Gamma-rings, of the diagram

$$\begin{tikzcd} \mathbb{S}[cR^\times] \arrow[r, tail] \arrow[d, "\sim"'] & R_1 \arrow[d] \\ \mathbb{S}[UR^\times] \arrow[r] & R_2 . \end{tikzcd}$$

For every pointed simplicial set K the Γ–space $\mathbb{S} \wedge K$ is cofibrant; in particular the underlying Γ–spaces of $\mathbb{S}[cR^\times]$ and $\mathbb{S}[UR^\times]$ are cofibrant. By [47, 4.1 (3)] the underlying Γ–space of R_1 is also cofibrant, so by the following lemma the map $R_1 \longrightarrow R_2$ is a stable equivalence of Gamma-rings.

Lemma 4.4 *Consider a diagram of Gamma-rings*

$$X \xleftarrow{\sim} Y \longrightarrow Z$$

in which the left map is a stable equivalence, the right map is a cofibration and all three Gamma-rings are cofibrant as Γ–spaces. Then the map from Z to the pushout of the diagram is also a stable equivalence.

Proof We denote the pushout of the diagram by P. We first consider the situation where the map $Y \longrightarrow Z$ is obtained by cobase change from a generating cofibration. In other words we assume that there exists a cofibration $K \longrightarrow L$ of Γ–spaces such that Z is the pushout of the diagram of Gamma-rings

$$Y \longleftarrow T(K) \longrightarrow T(L)$$

where T denotes the tensor algebra functor. This special kind of pushout in the category of Gamma-rings is analyzed in the proof of [47, Lemma 6.2]. The pushout Z is then the colimit of a sequence of cofibrations of Γ–spaces

$$Y = Z_0 \longrightarrow Z_1 \longrightarrow \cdots \longrightarrow Z_n \longrightarrow \cdots$$

for which the subquotient Z_n/Z_{n-1} is isomorphic to $(L/K)^{\wedge n} \wedge Y^{\wedge(n+1)}$. In the same way the composite pushout P is the colimit of a sequence of cofibrations of Γ–spaces with subquotients isomorphic to $(L/K)^{\wedge n} \wedge X^{\wedge(n+1)}$. The smash product of Γ–spaces preserves stable equivalences between cofibrant objects [29, Thm. 5.12], so the induced maps on the subquotients of the filtrations for Z and P are all stable equivalences. So the maps induced on all finite stages, and finally the map $Z \longrightarrow P$ on colimits are also stable equivalences.

By induction the lemma thus holds whenever the cofibration $Y \longrightarrow Z$ is the composite of finitely many maps obtained by cobase changes from generating cofibrations. Since homotopy groups of Γ–spaces commute with transfinite compositions over cofibrations, the lemma holds whenever $Y \longrightarrow Z$ is such a transfinite composition of cobase changes of generating cofibrations. Finally, if the lemma holds for a cofibration $Y \longrightarrow Z$, then it also holds for any retract. But all cofibrations of Gamma-rings are obtained by a sequence of these constructions by the small object argument. $\square$

We finally let R_3 be a stably fibrant replacement of the Gamma-ring R_2. We can display all the relevant objects thus constructed in a commutative diagram of Gamma-rings

$$\begin{tikzcd}
\mathbb{S}[G] \arrow[rr] \arrow[d] & & R \arrow[d, tail, "\sim"] \\
\mathbb{S}[cR^\times] \arrow[r, tail] \arrow[d, "\sim"'] & R_1 \arrow[r, two heads, "\sim"] \arrow[d, "\sim"] & R^{\mathrm{f}} \\
\mathbb{S}[UR^\times] \arrow[r] & R_3 &
\end{tikzcd}$$

In this diagram the Gamma-rings R^{f}, R_1 and R_3 are fibrant, and the maps between them are stable equivalences. Furthermore, the induced map from the simplicial group $UR^\times$ to the invertible components of the underlying monoid of R_3 is a weak equivalence. As described in 4.1, the simplicial group $UR^\times$ acts by conjugation on R_3 via homomorphisms of Gamma-rings, and this action extends the action of G.

Remark 4.5 Construction 4.2 can be made functorial in the triple $(R, G, \psi : G \longrightarrow R(1^+))$ since the factorizations in the model categories of simplicial monoids and of Gamma-rings can be made functorial.

5 The comparison map

We now use the conjugation action of the previous section in the case of the Gamma-ring DB. We obtain a model for the homotopy invariant space of Gamma-ring maps from $H\mathbb{Z}$ to DB on which the homotopy units of DB act by conjugation. So we can form the homotopy orbit space with respect to the conjugation action. We then construct a weak map from the classifying space of the groupoid of formal group laws and strict isomorphisms to the homotopy orbit space of the conjugation action. In the remaining sections we show that that map is a weak equivalence.

In the later sections we need a version of the weak map from the classifying space $\mathcal{FGL}^{\mathrm{str}}(B)$ to the homotopy orbit space for other Gamma-rings. So we set up the construction of the (weak) map in a slightly more general context.

Construction 5.1 Again we consider a Gamma-ring R, a simplicial group G and a homomorphism of simplicial monoids $\psi\colon G \longrightarrow R(1^+)$. We now make the additional assumption that *the image of G lands in the unit component of R*, ie, the composite map

$$G \xrightarrow{\psi} R(1^+) \longrightarrow \pi_0 R$$

is constant with value $1 \in \pi_0 R$. This assumption is not very important, but it slightly simplifies certain arguments later. Moreover we are given a simplicial set X with an action of the simplicial group G and suppose we are are also given a G–equivariant map $X \longrightarrow \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}, R)$, where G acts on the mapping space by conjugation (4.1). In our main example, R will be the Gamma-ring DB and G will be the discrete group of power series in one variable with leading term x and with multiplication give by substitution. Furthermore, X will be the set of formal group laws over B, and the map $\mathrm{FGL}(B) \longrightarrow \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}, DB)$ takes F to F_*.

In Construction 4.2 we produced a simplicial group $UR^\times$, a homomorphism of simplicial groups $G \longrightarrow UR^\times$ and a commutative diagram of Gamma-rings

$$\begin{tikzcd} \mathbb{S}[G] \arrow[r, equal] \arrow[d] & \mathbb{S}[G] \arrow[r] \arrow[d] & \mathbb{S}[UR^\times] \arrow[d] \\ R^{\mathrm{f}} & R_1 \arrow[l, two heads, "\sim"'] \arrow[r, "\sim"'] & R_3 \end{tikzcd}$$

in which the lower horizontal maps are stable equivalences between stably fibrant models of R. Furthermore, the induced map from the simplicial group $UR^\times$ to the invertible components of the underlying monoid of R_3 is a weak equivalence. The simplicial group G acts by conjugation on R, R_1 and R_3, hence on the spaces of Gamma-ring maps from any other Gamma-ring into R, R_1 and R_3. If S is a cofibrant Gamma-ring, then the lower horizontal maps induce weak equivalences

$$\mathrm{map}_{\mathcal{G}R}(S, R^{\mathrm{f}}) \xleftarrow{\sim} \mathrm{map}_{\mathcal{G}R}(S, R_1) \xrightarrow{\sim} \mathrm{map}_{\mathcal{G}R}(S, R_3)$$

which are G–equivariant. Furthermore the action of G on $\mathrm{map}_{\mathcal{G}R}(S, R_3)$ extends to an action of the group $UR^\times$. So we have extended, up to weak equivalence, the action of the group G to the action of a simplicial group weakly equivalent to the homotopy units of R.

If we now choose a cofibrant replacement $H\mathbb{Z}^c \longrightarrow H\mathbb{Z}$ in the model category of Gamma-rings, then the space $\mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^c, R_3)$ is a model for the homotopy invariant space of Gamma-ring maps. Furthermore, the simplicial group $UR^\times$ acts on this space by conjugation, hence so does its subgroup $UR_1^\times$, the connected component of the identity element. We abbreviate the space $\mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^c, R_3)$ to $\mathrm{Ring}(H\mathbb{Z}, R)$ and write $\mathrm{Ring}(H\mathbb{Z}, R)/\mathrm{conj.}$ for the homotopy orbit space of the conjugation action of the connected simplicial group $UR_1^\times$,

$$\mathrm{Ring}(H\mathbb{Z}, R)/\mathrm{conj.} \;=\; \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^c, R_3)_{hUR_1^\times} \;.$$

We now use the G–space X and the equivariant map $X \longrightarrow \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}, R)$ to construct a weak map from the homotopy orbit space X_{hG} to the homotopy orbit space $\mathrm{Ring}(H\mathbb{Z}, R)/\mathrm{conj.}$. We let $\widetilde{X}$ denote the pullback of the diagram

$$\begin{tikzcd} & \mathrm{map}_{\mathcal{G}R}(H\mathbb{Z}^c, R_1) \arrow[d, "\sim"] \\ X \arrow[r] & \mathrm{map}_{\mathcal{G}R}(H\mathbb{Z}^c, R^f) \end{tikzcd}$$

Since all spaces in the diagram have an action by the group G and all maps are equivariant, the group G acts on the space $\widetilde{X}$. Since the map $R_1 \longrightarrow R^f$ is an acyclic fibration of Gamma-rings, the induced map on homomorphism spaces is an acyclic fibration, hence so is the map $\widetilde{X} \longrightarrow X$. The stable equivalence of Gamma-rings $R_1 \longrightarrow R_3$ induces a G–equivariant map of homomorphism spaces $\mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^c, R_1) \longrightarrow \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^c, R_3)$.

The conjugation action of G on the target space extends to an action of the simplicial group $UR^\times$. By our assumption on the homomorphism $\psi\colon G \longrightarrow R(1^+)$ the image of G lands in the identity component $UR_1^\times$. We denote by κ the map induced on homotopy orbit spaces

$$\begin{aligned} \kappa:\ \widetilde{X}_{hG} &\longrightarrow \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^c, R_1)_{hG} \\ &\longrightarrow \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^c, R_3)_{hUR_1^\times} \;=\; \mathrm{Ring}(H\mathbb{Z}, R)/\mathrm{conj.} \end{aligned}$$

So altogether we have obtained a weak map of homotopy orbit spaces

$$X_{hG} \xleftarrow{\;\sim\;} \widetilde{X}_{hG} \xrightarrow{\;\kappa\;} \mathrm{Ring}(H\mathbb{Z}, R)/\mathrm{conj.}$$

Now we return to the main example and apply Construction 5.1 to the Gamma-ring DB. In this case G is the discrete group $\Phi(B)$ of power series $\varphi(x)$ in one variable over B with leading term x, with composition (substitution) of power series as the group structure. This group acts on the set of formal group laws over B via

$$F^\varphi(x,y) \;=\; \varphi(F(\varphi^{-1}(x), \varphi^{-1}(y))) \;.$$

In fact, F^φ is defined so that $\varphi\colon F \longrightarrow F^\varphi$ is a strict isomorphism of formal group laws. The homomorphism $\Phi(B) \longrightarrow DB(1^+) \cong x \cdot B[\![x]\!]$ is the inclusion. Because of the equality

$$(F^\varphi)_* \;=\; \varphi \cdot F_* \cdot \varphi^{-1}$$

as maps of Gamma-rings $H\mathbb{Z} \longrightarrow DB$, the assignment $F \mapsto F_*$ from the set $\mathrm{FGL}(B)$ of formal group laws to the set of Gamma-ring maps from $H\mathbb{Z}$ to DB is $\Phi(B)$–equivariant. The homotopy orbit space $\mathrm{FGL}(B)_{h\Phi(B)}$ is isomorphic to the classifying space of the groupoid of formal group laws and strict isomorphisms, which we denote by $\mathcal{FGL}^{\mathrm{str}}(B)$. So Construction 5.1 yields maps

$$\begin{tikzcd} \mathcal{FGL}^{\mathrm{str}}(B) = \mathrm{FGL}^{\mathrm{str}}(B)_{h\Phi(B)} & \widetilde{\mathrm{FGL}^{\mathrm{str}}}(B)_{h\Phi(B)} \arrow[l, "\sim"'] \arrow[d, "\kappa"] \\ & \mathrm{Ring}(H\mathbb{Z}, DB)/\mathrm{conj.} \end{tikzcd}$$

where the upper map is a weak equivalence. We use the notation $\widetilde{\mathcal{FGL}}^{\mathrm{str}}(B)$ for the homotopy orbit space $\widetilde{\mathrm{FGL}^{\mathrm{str}}}(B)_{h\Phi(B)}$. The following theorem is our main result:

Theorem 5.2 *The map*

$$\kappa : \widetilde{\mathcal{FGL}}^{\mathrm{str}}(B) \;\longrightarrow\; \mathrm{Ring}(H\mathbb{Z}, DB)/\mathrm{conj.}$$

is a weak equivalence.

The proof of Theorem 5.2 occupies the rest of this paper. Since the homotopy orbit space construction defining $\mathrm{Ring}(H\mathbb{Z}, DB)/\mathrm{conj.}$ involves a connected simplicial group, the quotient map

$$\mathrm{Ring}(H\mathbb{Z}, DB) \;\longrightarrow\; \mathrm{Ring}(H\mathbb{Z}, DB)/\mathrm{conj.}$$

induces a bijection of path components. The set $[H\mathbb{Z}, DB]_{\text{Gamma-rings}}$ is canonically isomorphic to the components of the mapping space $\mathrm{Ring}(H\mathbb{Z}, DB)$, so Theorem 3.8 is just the bijection of path components induced by the weak equivalence κ of Theorem 5.2.

In addition to the homotopy classes of Gamma-ring maps, Theorem 5.2 allows us to identify the higher homotopy groups of the space $\mathrm{Ring}(H\mathbb{Z}, DB)$ of Gamma-ring maps. Since the group $\pi_1(DB)^\times \cong \pi_1 DB$ is trivial by 3.2 (b), the simplicial group $(DB)_1^\times$ is 1–connected, so the quotient map $\mathrm{Ring}(H\mathbb{Z}, DB) \longrightarrow \mathrm{Ring}(H\mathbb{Z}, DB)/\mathrm{conj.}$ induces an equivalence of fundamental groupoids. Together with Theorem 5.2 this implies that the fundamental groupoid of the space $\mathrm{Ring}(H\mathbb{Z}, DB)$ is equivalent to the groupoid $\mathcal{FGL}^{\mathrm{str}}(B)$. In particular this yields isomorphisms

$$\pi_1 \mathrm{Ring}(H\mathbb{Z}, DB; F_*) \;\cong\; \mathrm{Aut}^{\mathrm{strict}}(F)$$

between the fundamental group at the basepoint $F_* \in \mathrm{Ring}(H\mathbb{Z}, DB)$ and the strict automorphism group of F. Since the homotopy orbit space of the conjugation action is weakly equivalent to the classifying space of a groupoid, its homotopy groups are trivial above dimension 1; so for every F the action map

$$U(DB)_1^\times \longrightarrow \mathrm{Ring}(H\mathbb{Z}, DB)\ , \quad u \longmapsto u \cdot F_* \cdot u^{-1}$$

induces isomorphisms of homotopy groups $\pi_n DB \cong \pi_n \mathrm{Ring}(H\mathbb{Z}, DB; F_*)$ for $n \geq 2$.

Example 5.3 For algebras over the rational numbers, the map κ of Theorem 5.2 is trivially an equivalence. Indeed, if B is a $\mathbb{Q}$–algebra, and F a 1–dimensional and commutative formal group law over B, then there is a strict isomorphism (called the *logarithm* of F) between F and the additive formal group law [19, III.1 Cor. 1]. Moreover, F has no non-trivial strict automorphisms, so the classifying space of the groupoid $\mathcal{FGL}^{\mathrm{str}}(B)$ is weakly contractible.

On the other hand, the Gamma-ring DB is now stably equivalent to the Eilenberg–MacLane Gamma-ring HB by Theorem 3.2 (b). So both the space $\mathrm{Ring}(H\mathbb{Z}, DB)$ and the unit component of $(DB)^\times$ are weakly contractible, hence so is the homotopy orbit space $\mathrm{Ring}(H\mathbb{Z}, DB)/\mathrm{conj}$.

Remark 5.4 Instead of taking homotopy orbits with respect to the connected simplicial group $U(DB)_1^\times$ one can divide out the conjugation action of the entire homotopy units $U(DB)^\times$ on the space $\mathrm{Ring}(H\mathbb{Z}, DB)$. The resulting orbit space receives a (weak) map from the groupoid of formal group laws and all (ie, not necessarily strict) isomorphisms. The same proof as for Theorem 5.2 shows that that map

$$\widetilde{\mathcal{FGL}}(B) \longrightarrow \mathrm{Ring}(H\mathbb{Z}, DB)_{hU(DB)^\times}$$

is a weak equivalence.

6 A filtration of DB

The Gamma-ring DB has a natural filtration arising from powers of the augmentation ideal of the power series rings. There are truncated versions $D_k B$ of the Gamma-ring DB and analogues of the map κ of Theorem 5.2 for every k. In this section we reduce the proof of Theorem 5.2 to the analogous statement about the stages of the filtration, see Theorem 6.4.

For $k \geq 1$ we denote by D_kB the truncated version of the Gamma-ring DB obtained by dividing out all power series in the $(k+1)$-st power of the augmentation ideal. So as a Γ–space,

$$D_kB(K) \;=\; \mathrm{kernel}\,(\,\widetilde{B}[\![K]\!]/I^{k+1} \longrightarrow \widetilde{B}[\![*]\!] = B\,)$$

where $I = K \cdot \widetilde{B}[\![K]\!]$ is the ideal of $\widetilde{B}[\![K]\!]$ consisting of power series without constant term. The unit map again comes from the inclusion of generators and the multiplication is induced by substitution of (truncated) power series, similar to the definition for DB. In other words: D_kB has a unique Gamma-ring structure for which the natural projection map $DB \longrightarrow D_kB$ is a homomorphism of Gamma-rings. Note that D_1B is isomorphic to the Eilenberg–MacLane Gamma-ring HB. There are maps of Gamma-rings $DB \longrightarrow D_kB$ and $D_kB \longrightarrow D_{k-1}B$ induced by truncation of power series.

Now we apply Construction 5.1 to the Gamma-ring D_kB. The group we work relative to is $\Phi_k(B)$, the quotient of the group $\Phi(B)$ of power series over B with leading term x by the normal subgroup of power series which are congruent to the power series x modulo x^{k+1}. The group $\Phi_k(B)$ injects into the monoid $D_kB(1^+)$, so Construction 4.2 provides a diagram of Gamma-rings

$$\begin{tikzcd}
\mathbb{S}[\Phi_k(B)] \arrow[r, equal] \arrow[d] & \mathbb{S}[\Phi_k(B)] \arrow[r] \arrow[d] & \mathbb{S}[U(D_kB)^{\times}] \arrow[d] \\
(D_kB)^{\mathrm{f}} & (D_kB)_1 \arrow[l, two heads, "\sim"'] \arrow[r, "\sim"'] & (D_kB)_3
\end{tikzcd}$$

in which the lower horizontal maps are stable equivalences between fibrant Gamma-rings. Furthermore the induced map from the simplicial group $U(D_kB)^{\times}$ to the invertible components of the underlying monoid of $(D_kB)_3$ is a weak equivalence. Hence the simplicial group $U(D_kB)^{\times}$ acts by conjugation on the space

$$\mathrm{Ring}(H\mathbb{Z}, D_kB) \;=\; \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^{\mathrm{c}}, (D_kB)_3)$$

extending the action of the group $\Phi_k(B)$. As in (5.1) we denote by

$$\mathrm{Ring}(H\mathbb{Z}, D_kB)/\mathrm{conj}.$$

the homotopy orbit space of $\mathrm{Ring}(H\mathbb{Z}, D_kB)$ by the conjugation action of $U(D_kB)^{\times}_1$, the identity component of $U(D_kB)^{\times}$.

Since the Constructions 4.2 and 5.1 can be made functorial, truncations induce compatible maps

$$\begin{array}{rrcl} & \mathrm{Ring}(H\mathbb{Z}, DB) & \longrightarrow & \mathrm{Ring}(H\mathbb{Z}, D_k B) \\ \text{and} & \mathrm{Ring}(H\mathbb{Z}, D_k B) & \longrightarrow & \mathrm{Ring}(H\mathbb{Z}, D_{k-1} B) \end{array}$$

and similarly for the orbit spaces by the conjugation actions.

Lemma 6.1 *The maps*

$$\mathrm{Ring}(H\mathbb{Z}, DB) \longrightarrow \mathrm{holim}_k\ \mathrm{Ring}(H\mathbb{Z}, D_k B)$$

and

$$\mathrm{Ring}(H\mathbb{Z}, DB)/\mathrm{conj.} \longrightarrow \mathrm{holim}_k\ (\mathrm{Ring}(H\mathbb{Z}, D_k B)/\mathrm{conj.})$$

induced by truncation are weak equivalences.

Proof We apply the homotopy limit construction [9] objectwise to Gamma-rings to obtain a construction of homotopy limits for Gamma-rings. As we explained in the proof of Theorem 3.2 (a), the homotopy fibre of the projection $DB \longrightarrow D_k B$ is the product of certain Γ–spaces $(B \otimes \mathcal{S}^m)^!$ for $m > k$, each of which is $(2k-1)$–connected. Hence the map

$$(DB)_3 \longrightarrow \mathrm{holim}_k\ (D_k B)_3$$

is a stable equivalence of Gamma-rings. So after taking homomorphism spaces from $H\mathbb{Z}^{\mathrm{c}}$ we obtain a weak equivalence of spaces

$$\begin{array}{rcl} \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^{\mathrm{c}}, (DB)_3) & \xrightarrow{\sim} & \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^{\mathrm{c}}, \mathrm{holim}_k\ (D_k B)_3) \\ & & \cong\ \mathrm{holim}_k\ \mathrm{Ring}(H\mathbb{Z}, D_k B) \end{array}$$

which proves the first statement. Similarly, the induced map of homotopy units

$$(DB)^{\times} \longrightarrow \mathrm{holim}_k\ (D_k B)^{\times}$$

is a weak equivalence, and so also the homotopy orbit space $\mathrm{Ring}(H\mathbb{Z}, DB)/\mathrm{conj.}$ is the homotopy inverse limit of the truncated versions. □

On the formal group law side, the classifying space $\mathcal{FGL}^{\mathrm{str}}(B)$ can also be expressed as a homotopy limit of suitable truncated versions. We denote by $\mathcal{B}ud_B^k$ the classifying space of the groupoid of k–buds [28, Def. 2.1] (also called k–jets) of formal group laws over B and k–buds of strict isomorphisms. Truncation induces maps

$$\mathcal{FGL}^{\mathrm{str}}(B) \longrightarrow \mathcal{B}ud_B^k \qquad \text{and} \qquad \mathcal{B}ud_B^k \longrightarrow \mathcal{B}ud_B^{k-1}\ .$$

The classifying spaces $\mathcal{FGL}^{\mathrm{str}}(B)$ and $\mathcal{B}ud^k_B$ are isomorphic to the homotopy orbit spaces $\mathrm{FGL}(B)_{h\Phi(B)}$ and $(\mathrm{Bud}^k_B)_{h\Phi_k(B)}$ respectively. Since the group $\Phi(B)$ is the inverse limit of the groups $\Phi_k(B)$ and the set of formal group laws is the inverse limit of the sets of k–buds, we have

Lemma 6.2 *The map*

$$\mathcal{FGL}^{str}(B) \longrightarrow \mathrm{holim}_k\ \mathcal{B}ud^k_B$$

induced by truncation on classifying spaces is a weak equivalence.

Proof This an instance of a general fact about homotopy orbits of groups acting on sets, alias translation categories. Suppose $\{G_k \longrightarrow G_{k-1}\}_{k\geq 1}$ is a sequence of surjective group homomorphisms, $\{X_k \longrightarrow X_{k-1}\}_{k\geq 1}$ a tower of sets, and suppose that G_k acts on X_k in such a way that the map $X_k \longrightarrow X_{k-1}$ is G_k–equivariant. Then the inverse limit $G = \lim_k G_k$ of groups acts on the inverse limit $X = \lim_k X_k$ of sets and the canonical map

$$X_{hG} \longrightarrow \mathrm{holim}_k\ (X_k)_{hG_k}$$

is a weak equivalence. This follows from the homotopy fibre sequences

$$X_k \longrightarrow (X_k)_{hG_k} \longrightarrow BG_k$$

by passage to homotopy inverse limit together with the fact that the natural maps

$$X = \lim_k X_k \longrightarrow \mathrm{holim}_k\ X_k \qquad \text{and} \qquad BG \longrightarrow \mathrm{holim}_k\ BG_k$$

are weak equivalences (we note that the path components of the homotopy inverse limit of the classifying spaces BG_k are in bijective correspondence with the set $\lim^1_k G_k$; since we consider surjective group homomorphisms, this $\lim^1$–term is trivial and the map from BG to the homotopy inverse limit is indeed a weak equivalence). In our example X_k is the set of k–buds of formal group laws and G_k is the group $\Phi_k(B)$ of k–buds of power series conjugating the k–buds of formal group laws. □

Every k–bud of formal group law F gives rise to a map of Gamma-rings $H\mathbb{Z} \longrightarrow D_kB$ in the same fashion as genuine formal group laws give maps to DB (3.5). The group $\Phi_k(B)$ of truncated invertible power series conjugates the set of k–buds and the map $\mathrm{Bud}^k_B \longrightarrow \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}, D_kB)$ is $\Phi_k(B)$–equivariant. If we carry out Construction 5.1 with $R = D_kB$ and with the group $\Phi_k(B)$

conjugating the k–buds of formal group laws, we obtain a map of homotopy orbit spaces

$$\begin{aligned}\kappa_k : \widetilde{\mathcal{B}ud}^k_B = (\widetilde{\mathrm{Bud}^k_B})_{h\Phi_k(B)} &\longrightarrow \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^c, (D_kB)_3)_{hU(D_kB)_1^\times} \\ &= \mathrm{Ring}(H\mathbb{Z}, D_kB)/\mathrm{conj}.\end{aligned}$$

The constructions are natural, so we end up with a commutative diagram

$$\begin{tikzcd} \widetilde{\mathcal{FGL}}^{\mathrm{str}}(B) \arrow[rr, "\kappa"] \arrow[d, "\sim"'] & & \mathrm{Ring}(H\mathbb{Z}, DB)/\mathrm{conj}. \arrow[d, "\sim"] \\ \mathrm{holim}_k\ \widetilde{\mathcal{B}ud}^k_B \arrow[rr, "\kappa_k"'] & & \mathrm{holim}_k\ \mathrm{Ring}(H\mathbb{Z}, D_kB)/\mathrm{conj}. \end{tikzcd} \tag{6.3}$$

in which the vertical maps are weak equivalences by Lemmas 6.1 and 6.2. So Theorem 5.2 follows once we have shown

Theorem 6.4 *For all $k \geq 1$ the map*

$$\kappa_k : \widetilde{\mathcal{B}ud}^k_B \longrightarrow \mathrm{Ring}(H\mathbb{Z}, D_kB)/\mathrm{conj}.$$

is a weak equivalence.

Remark 6.5 If k is not a prime power, then the functor $B\otimes\mathcal{S}^k$ is a retract of a diagonalizable functor (Lemma 12.5), and so the Γ–space underlying $(B\otimes\mathcal{S}^k)^!$ is stably contractible (8.9 (e)). Hence the reduction map $D_kB \longrightarrow D_{k-1}B$ is a stable equivalence of Gamma-rings and the induced map on homotopy units $(D_kB)^\times \longrightarrow (D_{k-1}B)^\times$ is a weak equivalence of simplicial monoids. Thus the map of homotopy orbit spaces

$$\mathrm{Ring}(H\mathbb{Z}, D_kB)/\mathrm{conj}. \xrightarrow{\text{red.}} \mathrm{Ring}(H\mathbb{Z}, D_{k-1}B)/\mathrm{conj}.$$

is a weak equivalence. By Remark 7.3 below, the reduction functor $\mathcal{B}ud^k_B \longrightarrow \mathcal{B}ud^{k-1}_B$ is an equivalence of categories. Hence in the inductive step nothing happens unless $k = p^h$ is a prime power. However it is convenient to make this case distinction only at the very end (see Step 1 in the proof of Theorem 12.3).

7 Some singular extensions of Gamma-rings

In this section we start the inductive proof of Theorem 6.4. We exploit that the truncation maps $D_kB \longrightarrow D_{k-1}B$ are "singular extensions" of Gamma-rings. This lets us reduce the problem to a comparison of the derivation space of the

Gamma-ring $H\mathbb{Z}$ with coefficients in the "kernel" $(B \otimes \mathcal{S}^k)^!$ of the extension to the groupoid of symmetric 2–cocycles.

The case $k = 1$ of Theorem 6.4 is straightforward. There is only one 1–bud of formal group law, and the only 1–bud of strict automorphism is the identity. On the other hand, $D_1 B$ is isomorphic to the Eilenberg–MacLane Gamma-ring HB, so both the space $\mathrm{Ring}(H\mathbb{Z}, D_1 B)$ and the unit component $(D_1 B)_1^\times$ of the homotopy units are weakly contractible. Hence source and target of the map

$$\kappa_1 : \widetilde{\mathcal{B}ud}_B^1 \longrightarrow \mathrm{Ring}(H\mathbb{Z}, D_1 B)/\mathrm{conj.}$$

are weakly contractible and κ_1 is a weak equivalence.

7.1 Symmetric 2–cocycles For the inductive step we recall how the difference between k–buds and $(k-1)$–buds of formal group laws is controlled by symmetric 2–cocycles. A *symmetric 2-cocycle* of degree k with values in B is a homogeneous polynomial $c(x, y) \in B[x, y]$ of degree k which satisfies the relations

$$c(x, y) = c(y, x) \quad \text{and} \quad c(x, y) + c(x + y, z) = c(x, y + z) + c(y, z) .$$

If F is any k–bud of formal group law over B and c is a k–homogenous 2–cocycle, then the truncated power series $F(x, y) + c(x, y)$ is another k–bud of formal group law with the same $(k-1)$–bud as F. Conversely, if F and F' are two k–buds with the same reduction modulo the k–th powers of the augmentation ideal, then $c = F - F'$ is a k–homogenous 2–cocycle. The proof of this is straightforward, compare [28, Sec. II] or [19, III.1 Lemma 2].

We define $\mathcal{Z}(B \otimes \mathcal{S}^k)$, the *groupoid of symmetric 2-cocycles* of degree k, as the category whose objects are the symmetric 2–cocycles of degree k over B. The set of morphisms from a cocycle c to a cocycle c' consists of those $b \in B$ satisfying $c' = c + b \cdot \left[x^k + y^k - (x + y)^k\right]$; composition is given by addition in B.

Suppose F is a k–bud of formal group law. Then we can define a functor

$$F + - : \mathcal{Z}(B \otimes \mathcal{S}^k) \longrightarrow \mathcal{B}ud_B^k$$

on objects by $(F + -)(c) = F + c$ and on morphisms by $(F + -)(b) = x + b \cdot x^k$. The relation

$$F^{x + bx^k}(y, \bar{y}) \equiv F(y, \bar{y}) + b \cdot \left[y^k + \bar{y}^k - (y + \bar{y})^k\right] \quad \mathrm{mod}\ (y, \bar{y})^{k+1}$$

shows that if $b \colon c \longrightarrow c'$ is a morphism of cocycles, then the power series $x + b \cdot x^k$ is indeed an isomorphism from $F + c$ to $F + c'$.

Lemma 7.2 *For every k–bud of formal group law F, the functor*

$$F + - : \ \mathcal{Z}(B \otimes \mathcal{S}^k) \ \longrightarrow \ \mathcal{B}ud^k_B$$

induces a weak equivalence from the classifying space of the groupoid $\mathcal{Z}(B \otimes \mathcal{S}^k)$ to the homotopy fibre of the truncation map $\mathcal{B}ud^k_B \longrightarrow \mathcal{B}ud^{k-1}_B$ over the basepoint F.

Proof This again is an instance of a general fact about homotopy orbits of groups acting on sets, alias translation categories. Suppose $G \longrightarrow \bar{G}$ is an epimorphism of groups with kernel K. Moreover, let X be a G–set, $\bar{X}$ a $\bar{G}$–set and $\pi\colon X \longrightarrow \bar{X}$ a G–equivariant map. Then for every point $x \in \bar{X}$ the kernel K acts on the preimage $\pi^{-1}(x)$. In this situation the sequence of homotopy orbit spaces

$$\pi^{-1}(x)_{hK} \ \longrightarrow \ X_{hG} \ \longrightarrow \ \bar{X}_{h\bar{G}}$$

is a homotopy fibre sequence over the point x.

In the situation at hand the epimorphism is the truncation $\Phi_k(B) \longrightarrow \Phi_{k-1}(B)$, whose kernel is isomorphic to the additive group of B via $b \longmapsto x + b \cdot x^k$. The groups conjugate the k–buds respectively $(k-1)$–buds of formal group laws. For every choice of k–bud of formal group law F, the map $F + -$ is an isomorphism, equivariant for $B \cong$ kernel: $\Phi_k(B) \longrightarrow \Phi_{k-1}(B)$, from the symmetric 2–cocycles to the k–buds which have the same $(k-1)$–bud as F. Hence the lemma follows. □

Remark 7.3 The symmetric 2–cocycles have been identified by Lazard [28, II Lemme 3], see also [19, III.1 Thm. 1a]. There is a universal integral symmetric 2–cocycle $c_k \in \mathbb{Z}[x,y]$ of degree k given by

$$c_k(x,y) \ = \ \frac{1}{d_k}\left[x^k + y^k - (x+y)^k\right]$$

and the degree k symmetric 2–cocycles over B are precisely the multiples $b \cdot c_k$ for $b \in B$. Here d_k is the greatest common divisor of the binomial coefficients $\binom{k}{i}$ for $1 \leq i \leq k-1$, which evaluates to

$$d_k \ = \ \begin{cases} p & \text{if } k = p^h \text{ for a prime } p \text{ and } h \geq 1 \\ 1 & \text{else.} \end{cases}$$

Hence the classifying space of $\mathcal{Z}(B \otimes \mathcal{S}^k)$ can be identified as follows: if k is not a prime power, then the classifying space is weakly contractible and the reduction functor $\mathcal{B}ud^k_B \longrightarrow \mathcal{B}ud^{k-1}_B$ is an equivalence of categories. If $k = p^h$

for some prime p, then the groupoid $\mathcal{Z}(B \otimes \mathcal{S}^{p^h})$ is isomorphic to the translation category of the action of B on itself given by

$$(b, x) \longmapsto p \cdot b + x \ .$$

Hence the components of the classifying space of $\mathcal{Z}(B \otimes \mathcal{S}^{p^h})$ are in bijective correspondence with the set B/pB and the fundamental group at each basepoint is isomorphic to the group of those $b \in B$ such that $pb = 0$. However, in the rest of this paper we will not use this explicit knowledge about the symmetric 2–cocycles.

Now we identify the difference between the spaces of Gamma-ring maps from $H\mathbb{Z}$ to $D_k B$ and to $D_{k-1}B$, ie, we study the homotopy fibers of the reduction map

$$\mathrm{Ring}(H\mathbb{Z}, D_k B)/\mathrm{conj.} \longrightarrow \mathrm{Ring}(H\mathbb{Z}, D_{k-1}B)/\mathrm{conj.}$$

For this purpose we consider the Gamma-ring $H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!$. We present the construction in a more general context, since we need it later.

7.4 Split singular extensions Let G be a functor from the category of finitely generated free abelian groups to the category of all abelian groups, and suppose that G is *reduced* in the sense that $G(0) \cong 0$. We define a Gamma-ring $H\mathbb{Z} \times G^!$, the *split singular extension* of $H\mathbb{Z}$ by the bimodule $G^!$.

First there is an $H\mathbb{Z}$–bimodule $G^!$ associated to the functor G; the notation is taken from [37, Ex. 2.6], where the construction first appeared. As a Γ–space, $G^!$ is the composite

$$\Gamma^{\mathrm{op}} \xrightarrow{\tilde{\mathbb{Z}}} (\text{f. g. free ab. groups}) \xrightarrow{G} \mathcal{A}b \xrightarrow{\Phi} (\text{pt. sets})$$

of the reduced free functor $\tilde{\mathbb{Z}}$, followed by G and the forgetful functor Φ from abelian groups to pointed sets. The Γ–space $G^!$ has the structure of an $H\mathbb{Z}$–bimodule via the composite

$$H\mathbb{Z} \wedge G^! \wedge H\mathbb{Z} \xrightarrow{\text{assembly}} H\mathbb{Z} \circ G^! \circ H\mathbb{Z} \longrightarrow G^! \ .$$

The first map is an instance of the assembly map (2.5) from the smash product to the composition product of Γ–spaces; the second map is induced by evaluation maps (this uses that the original functor G was defined for and takes values in abelian groups).

The product $H\mathbb{Z} \times G^!$ becomes a Gamma-ring as follows. The unit map of $H\mathbb{Z} \times G^!$ is the composition of the unit $\eta\colon \mathbb{S} \longrightarrow H\mathbb{Z}$ with the inclusion $H\mathbb{Z} \longrightarrow$

$H\mathbb{Z} \times G^!$. The multiplication map is the composite

$$\begin{array}{rcl}(H\mathbb{Z} \times G^!) \wedge (H\mathbb{Z} \times G^!) & \longrightarrow & (H\mathbb{Z} \wedge H\mathbb{Z}) \times (H\mathbb{Z} \wedge G^!) \times (G^! \wedge H\mathbb{Z}) \\ & \xrightarrow{(\mu_{H\mathbb{Z}},\ l+r)} & H\mathbb{Z} \times G^!\end{array}$$

where l and r denote the left respectively right action of $H\mathbb{Z}$ on $G^!$.

In particular we can apply Construction 7.4 to the functor $B \otimes \mathcal{S}^k$ which takes a finitely generated free abelian group A to the tensor product of B with the k-th symmetric power of A. If F is any k–bud of formal group law, then $F_*\colon H\mathbb{Z} \longrightarrow D_kB$ is a morphism of Gamma-rings. Homogenous polynomials of degree k naturally inject into the quotient of power series by terms of degree $k+1$, which gives a map of Γ–spaces $\mathrm{Incl}\colon (B \otimes \mathcal{S}^k)^! \longrightarrow D_kB$. Moreover, their pointwise sum in D_kB

$$F_* + \mathrm{Incl.}:\ H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^! \ \longrightarrow\ D_kB$$

is again a map of Gamma-rings.

Lemma 7.5 *The commutative square*

$$\begin{tikzcd} H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^! \arrow[r, "F_* + \mathrm{Incl.}"] \arrow[d, "\mathrm{proj.}"'] & D_kB \arrow[d] \\ H\mathbb{Z} \arrow[r, "F_*"'] & D_{k-1}B \end{tikzcd}$$

is a homotopy fibre square of Gamma-rings.

Proof It suffices to show that the underlying square of Γ–spaces is homotopy cartesian. As a Γ–space DB splits as a product

$$D_kB \ \cong\ D_{k-1}B \ \times\ (B \otimes \mathcal{S}^k)^!$$

and under this isomorphism the map $F_* + \mathrm{Incl.}$ becomes the map

$$F_* \times \mathrm{Id}:\ H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^! \ \longrightarrow\ D_{k-1}B \ \times\ (B \otimes \mathcal{S}^k)^!\ ,$$

so the claim follows. □

The additive group of the ring B includes into the underlying monoid of the Gamma-ring $H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!$ via the map which sends $b \in B$ to the polynomial $x + b \cdot x^k$, considered as an element of $(H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!)(1^+) \cong \mathbb{Z}[x] \times B[x^k]$ (where x is an indeterminate corresponding to the non-basepoint element of 1^+). We can now apply Construction 4.2 to the Gamma-ring $H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!$

relative to the additive group of B. The construction produces a commutative diagram of Gamma-rings

$$\begin{tikzcd}
\mathbb{S}[B] \arrow[r, equal] \arrow[d] & \mathbb{S}[B] \arrow[r] \arrow[d] & \mathbb{S}[U(H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!)^\times] \arrow[d] \\
(H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!)^{\mathrm{f}} & (H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!)_1 \arrow[l, "\sim"'] \arrow[r, "\sim"'] & (H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!)_3
\end{tikzcd}$$

in which the lower horizontal maps are stable equivalences between fibrant Gamma-rings. The induced map from the simplicial group $U(H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!)^\times$ to the invertible components of the underlying monoid of $(H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!)_3$ is a weak equivalence.

We denote by

$$\mathrm{der}(H\mathbb{Z}, B \otimes \mathcal{S}^k) \;=\; \mathrm{map}_{\mathcal{G}R}(H\mathbb{Z}^{\mathrm{c}}, (H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!)_3)$$

the space of Gamma-ring maps from $H\mathbb{Z}^{\mathrm{c}}$ to $(H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!)_3$ and refer to this space as the space of *derivations* of $H\mathbb{Z}$ with coefficients in $B \otimes \mathcal{S}^k$. The group $U(H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!)^\times$ acts by conjugation on the space of derivations and we denote by $\mathrm{der}(H\mathbb{Z}, B \otimes \mathcal{S}^k)/\mathrm{conj.}$ the homotopy orbit space of the conjugation action of the identity component $U(H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!)_1^\times$. Since the square of monoids

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "b \,\mapsto\, x + b\cdot x^k"'] & (H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!)(1^+) \arrow[d, "F_* + \mathrm{Incl.}"] \\
\Phi_k(B) \arrow[r] & D_k B(1^+)
\end{tikzcd}$$

commutes and the constructions of Sections 4 are functorial in Gamma-rings equipped with a map from a group to the underlying monoid, the map

$$F_* + \mathrm{Incl.}:\; H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^! \;\longrightarrow\; D_k B$$

induces maps between the respective derived spaces of Gamma-ring maps from $H\mathbb{Z}$ and their homotopy orbit spaces.

Lemma 7.6 *The two maps*

$$\mathrm{der}(H\mathbb{Z}, B \otimes \mathcal{S}^k) \xrightarrow{F_* + \mathrm{Incl.}} \mathrm{Ring}(H\mathbb{Z}, D_k B) \qquad \textit{and}$$
$$\mathrm{der}(H\mathbb{Z}, B \otimes \mathcal{S}^k)/\mathrm{conj.} \xrightarrow{F_* + \mathrm{Incl.}} \mathrm{Ring}(H\mathbb{Z}, D_k B)/\mathrm{conj.}$$

induce weak equivalences between the derivation space, respectively its homotopy orbit space, and the respective homotopy fibres of the truncation maps

$$\begin{array}{rcll} \mathrm{Ring}(H\mathbb{Z}, D_k B) & \longrightarrow & \mathrm{Ring}(H\mathbb{Z}, D_{k-1}B) & \textit{and} \\ \mathrm{Ring}(H\mathbb{Z}, D_k B)/\mathrm{conj.} & \longrightarrow & \mathrm{Ring}(H\mathbb{Z}, D_{k-1}B)/\mathrm{conj.} & \end{array}$$

over the basepoint F_*.

Proof The first statement is a direct consequence of the fact that the square of Lemma 7.5 is homotopy cartesian. The second follows from the first since the sequence of simplicial groups

$$U(H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!)_1^\times \longrightarrow U(D_k B)_1^\times \longrightarrow U(D_{k-1}B)_1^\times$$

is also a homotopy fibre sequence, again because of Lemma 7.5. □

We denote by $\widetilde{\mathrm{Z}}^2_s(B \otimes \mathcal{S}^k)$ the pullback of the diagram

$$\begin{tikzcd} & \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^c, (H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!)_1) \arrow[d, two heads, "\sim"] \\ \mathrm{Z}^2_s(B \otimes \mathcal{S}^k) \arrow[r] & \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^c, (H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!)^{\mathrm{f}}) \end{tikzcd}$$

All maps are equivariant with respect to the action of the additive group of B, so this group acts on $\widetilde{\mathrm{Z}}^2_s(B \otimes \mathcal{S}^k)$. Furthermore the weak equivalence

$$\mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^c, (H\mathbb{Z} \times (B \otimes \mathcal{S}^k))_1) \xrightarrow{\sim} \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^c, (H\mathbb{Z} \times (B \otimes \mathcal{S}^k))_3)$$

is B–equivariant, and on the target the action extends to an action by the simplicial group $U(H\mathbb{Z} \times (B \otimes \mathcal{S}^k))_1^\times$. So we get an induced map on homotopy orbits

$$\begin{aligned} \kappa_{B \otimes \mathcal{S}^k} : \widetilde{\mathrm{Z}}^2_s(B \otimes \mathcal{S}^k)_{hB} &\longrightarrow \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^c, (H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!)_1)_{hB} \longrightarrow \\ & \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^c, (H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!)_3)_{hU(H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!)_1^\times} \\ &= \mathrm{der}(H\mathbb{Z}, B \otimes \mathcal{S}^k)/\mathrm{conj.} \end{aligned}$$

Note that the homotopy orbit space of the action of B on the set $\mathrm{Z}^2_s(B \otimes \mathcal{S}^k)$ of symmetric 2–cocycles is isomorphic to the classifying space of the groupoid $\mathcal{Z}(B \otimes \mathcal{S}^k)$ (7.1); hence we use the notation $\widetilde{\mathcal{Z}}(B \otimes \mathcal{S}^k)$ for the weakly equivalent homotopy orbit space $\widetilde{\mathrm{Z}}^2_s(B \otimes \mathcal{S}^k)_{hB}$.

Now we can reduce the inductive step of Theorem 6.4 to a statement about the map $\kappa_{B \otimes \mathcal{S}^k}$. We assume inductively that the map

$$\kappa_{k-1} : \widetilde{\mathcal{B}ud}^{k-1}_B \longrightarrow \mathrm{Ring}(H\mathbb{Z}, D_{k-1}B)/\mathrm{conj.}$$

is a weak equivalence. This guarantees in particular that the $(k-1)$–buds of formal group laws account for all components of the target space. Since every $(k-1)$–bud of formal group law extends to a k–bud, the reduction map

$$\mathrm{Ring}(H\mathbb{Z}, D_k B)/\mathrm{conj.} \longrightarrow \mathrm{Ring}(H\mathbb{Z}, D_{k-1} B)/\mathrm{conj.}$$

is surjective on components. If we fix a k–bud of F of formal group law, then the diagram

$$\begin{tikzcd} \widetilde{\mathcal{Z}}(B \otimes \mathcal{S}^k) \arrow[r, "F+-"] \arrow[d, "\kappa_{B\otimes\mathcal{S}^k}"'] & \widetilde{\mathcal{B}ud^k_B} \arrow[d, "\kappa_k"] \\ \mathrm{der}(H\mathbb{Z}, B \otimes \mathcal{S}^k)/\mathrm{conj.} \arrow[r, "F_*+\mathrm{Incl.}"'] & \mathrm{Ring}(H\mathbb{Z}, D_k B)/\mathrm{conj.} \end{tikzcd} \tag{7.7}$$

is commutative. By Lemmas 7.2 and 7.6, the horizontal maps identify the respective homotopy fibres of the truncation maps over the basepoints F and F_*. Since formal group laws account for all components of the target space, we have thus reduced the inductive step of the proof of Theorem 6.4, and hence of the main theorem, to showing

Theorem 7.8 *The map*

$$\kappa_{B\otimes\mathcal{S}^k} : \widetilde{\mathcal{Z}}(B \otimes \mathcal{S}^k) \longrightarrow \mathrm{der}(H\mathbb{Z}, B \otimes \mathcal{S}^k)/\mathrm{conj.}$$

is a weak equivalence for all commutative rings B and all $k \geq 1$.

The remaining sections are spent verifying that the map $\kappa_{B\otimes\mathcal{S}^k}$ is indeed a weak equivalence. If k is not a prime power, then source and target of the map $\kappa_{B\otimes\mathcal{S}^k}$ are weakly contractible, compare Remark 6.5. However there is no need to make this case distinction until the very end (see Step 1 in the proof of Theorem 12.3).

8 Symmetric 2–cocycles and derivations

In this section we provide some general constructions which will be needed in the sequel. We consider functors G from finitely generated free abelian groups to all abelian groups which are *reduced* in the sense that $G(0) \cong 0$. For such functors we discuss symmetric 2–cocycles and show how these lead to Gamma-ring maps into the split extension $H\mathbb{Z} \times G^!$ (7.4). We also recall the Dold–Puppe stabilization G_{st} of the functor G (8.9). The work of this section is summarized in a certain map (8.10)

$$\lambda_G : \mathcal{Z}(G) \longrightarrow \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^c, H\mathbb{Z} \times G^!_{st})_{hG_{st}(\mathbb{Z})}$$

with source the groupoid $\mathcal{Z}(G)$ of symmetric 2–cocycles of G. In the case where G is the symmetric power functor $B \otimes \mathcal{S}^k$, we identify the map $\lambda_{B\otimes\mathcal{S}^k}$ (up to weak equivalence) with the map $\kappa_{B\otimes\mathcal{S}^k}$ of Theorem 7.8 in Section 9.

8.1 Cocycles of a functor We let $\mathcal{F}$ denote the abelian category of reduced functors from finitely generated free abelian groups to all abelian groups. Suppose $G \in \mathcal{F}$ is a such functor. Then a *symmetric 2–cocycle* with values in G is an element $c \in G(\mathbb{Z}\oplus\mathbb{Z})$ which is

(a) fixed by the involution of $G(\mathbb{Z}\oplus\mathbb{Z})$ induced by the interchange of summands and

(b) in the kernel of the map

$$G\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - G\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} + G\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - G\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} :$$
$$G(\mathbb{Z}\oplus\mathbb{Z}) \longrightarrow G(\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}) .$$

We denote by $\mathrm{Z}^2_s(G)$ the group of symmetric 2–cocycles in G.

A group homomorphism

$$\theta : \ G(\mathbb{Z}) \longrightarrow \mathrm{Z}^2_s(G) \subset G(\mathbb{Z}\oplus\mathbb{Z})$$

is defined by

$$\theta \ = \ G(1,0) - G(1,1) + G(0,1) \ .$$

The cocycle $\theta(a)$ associated to $a \in G(\mathbb{Z})$ is sometimes referred to as the *principal* cocycle of a. We denote by $\mathcal{Z}(G)$ the translation category of the action of $G(\mathbb{Z})$ on the set $\mathrm{Z}^2_s(G)$ of symmetric 2–cocycles given by $(a, c) \longmapsto \theta(a) + c$. More precisely, $\mathcal{Z}(G)$ is the groupoid whose objects are the symmetric 2–cocycles $\mathrm{Z}^2_s(G)$ and where the set of morphisms from a cocycle c to a cocycle c' consists of those elements $a \in G(\mathbb{Z})$ such that $c' = \theta(a) + c$. Composition in $\mathcal{Z}(G)$ is given by addition in the group $G(\mathbb{Z})$.

Example 8.2 If $G = B \otimes \mathcal{S}^k$ is the symmetric power functor, then $G(\mathbb{Z}\oplus\mathbb{Z})$ is the group of homogenous polynomials of degree k in two variables over the ring B. For $c \in (B\otimes\mathcal{S}^k)(\mathbb{Z}\oplus\mathbb{Z})$, the cocycle condition (a) translates into $c(x, y) = c(y, x)$ and condition (b) translates into the equation

$$c(x, y) + c(x + y, z) \ = \ c(x, y + z) + c(y, z) \ .$$

Hence the symmetric 2–cocycles with values in $B\otimes\mathcal{S}^k$ coincide with the homogenous 2–cocycles of degree k as defined in 7.1. Moreover, the group $(B\otimes\mathcal{S}^k)(\mathbb{Z})$ is isomorphic to the additive group of B and the map

$$\theta : \ B \cong (B\otimes\mathcal{S}^k)(\mathbb{Z}) \longrightarrow \mathrm{Z}^2_s(B\otimes\mathcal{S}^k)$$

sends $b \in B$ to the cocycle $b \cdot \left[x^k + y^k - (x+y)^k\right]$. Hence the cocycle groupoid for the functor $B \otimes \mathcal{S}^k$ as defined in 8.1 coincides with $\mathcal{Z}(B \otimes \mathcal{S}^k)$ as defined in 7.1.

8.3 The universal 2–cocycle We let $I \in \mathcal{F}$ denote the inclusion functor and $P \in \mathcal{F}$ the functor which takes an abelian group A to the reduced free abelian group generated by the underlying pointed set of A. The Yoneda isomorphism $\mathrm{Hom}_{\mathcal{F}}(P, G) \cong G(\mathbb{Z})$ shows that P is a projective object of $\mathcal{F}$. Evaluation gives a natural epimorphism

$$P(A) \;=\; \widetilde{\mathbb{Z}}[A] \;\longrightarrow\; A = I(A) \;,$$

ie, an epimorphism $\epsilon\colon P \longrightarrow I$ in the category $\mathcal{F}$. We let J denote the kernel; the relevance for us is that J represents the symmetric 2–cocycle functor. The element

$$c_u \;=\; [1,0] - [1,1] + [0,1] \;\in\; \widetilde{\mathbb{Z}}[\mathbb{Z} \oplus \mathbb{Z}] = P(\mathbb{Z} \oplus \mathbb{Z}) \tag{8.4}$$

is in the kernel of the evaluation map, so it is an element of $J(\mathbb{Z} \oplus \mathbb{Z})$. As an element of $P(\mathbb{Z} \oplus \mathbb{Z})$, c_u is the principal cocycle associated to $[1] \in P(\mathbb{Z})$. Hence c_u is a symmetric 2–cocycle of J (but it is not principal for J because the element $[1] \in P(\mathbb{Z})$ does not belong to $J(\mathbb{Z})$).

Lemma 8.5 *The symmetric 2–cocycle c_u with values in the functor J is universal in the sense that the map*

$$\mathrm{Hom}_{\mathcal{F}}(J, G) \;\longrightarrow\; \mathrm{Z}^2_s(G) \;, \quad f \longmapsto f(c_u)$$

is an isomorphism for all functors $G \in \mathcal{F}$.

Proof Both $\mathrm{Hom}_{\mathcal{F}}(J, G)$ and $\mathrm{Z}^2_s(G)$ are additive and left exact in the functor G. Hence it suffices to check the claim for a set of injective cogenerators of the category $\mathcal{F}$. If A is a finitely generated free abelian group we define a functor $\mathcal{I}_A$ by the formula

$$\mathcal{I}_A(M) \;=\; \mathrm{map}_*(\mathrm{Hom}(M, A), \mathbb{Q}/\mathbb{Z}) \;.$$

Here $\mathrm{Hom}(M, A)$ denotes the set of group homomorphisms from M to A and 'map_*' refers to the group of set-theoretic maps from $\mathrm{Hom}(M, A)$ to $\mathbb{Q}/\mathbb{Z}$ preserving 0, with group structure by pointwise addition. The Yoneda isomorphism

$$\mathrm{Hom}_{\mathcal{F}}(G, \mathcal{I}_A) \;\cong\; \mathrm{Hom}(G(A), \mathbb{Q}/\mathbb{Z})$$

implies that $\mathcal{I}_A$ is injective and that the functors $\mathcal{I}_A$ form a collection of injective cogenerators as A varies.

It remains to verify that for all finitely generated free abelian groups A evaluation at the cocycle $c_u \in \mathrm{Z}^2_s(J)$ is an isomorphism from $\mathrm{Hom}_{\mathcal{F}}(J, \mathcal{I}_A)$ to $\mathrm{Z}^2_s(\mathcal{I}_A)$. We claim that the group $\mathrm{Z}^2_s(\mathcal{I}_A)$ can be identified with the quotient of the group $\mathrm{map}_*(A, \mathbb{Q}/\mathbb{Z})$ of pointed set maps from A to $\mathbb{Q}/\mathbb{Z}$ by the subgroup $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ of additive maps. We use the natural basis of $\mathbb{Z} \oplus \mathbb{Z}$ to identify $\mathcal{I}_A(\mathbb{Z} \oplus \mathbb{Z}) = \mathrm{map}_*(\mathrm{Hom}(\mathbb{Z} \oplus \mathbb{Z}, A), \mathbb{Q}/\mathbb{Z})$ with the group of set-theoretic maps $g\colon A \oplus A \longrightarrow \mathbb{Q}/\mathbb{Z}$ satisfying $g(0,0) = 0$. Under this identification the cocycle conditions for elements of $\mathcal{I}_A(\mathbb{Z} \oplus \mathbb{Z})$ translate into the conditions

$$g(x,y) \;=\; g(y,x) \quad \text{and} \quad g(x,y) \;+\; g(x+y,z) \;=\; g(x,y+z) \;+\; g(y,z)$$

for all $x, y, z \in A$ on the function $g \in \mathrm{map}_*(A \oplus A, \mathbb{Q}/\mathbb{Z})$. In other words: g is a factor set of an abelian group extension of A by $\mathbb{Q}/\mathbb{Z}$. Since A is free abelian, every such extension splits, so g is principal, ie, there exists a function $h\colon A \longrightarrow \mathbb{Q}/\mathbb{Z}$ satisfying $g(x,y) \;=\; h(x) - h(x+y) + h(y)$ and $h(0) = 0$. Moreover, h is uniquely determined by this up to an additive function.

All this means that the bottom row in the commutative diagram of abelian groups

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{F}}(I, \mathcal{I}_A) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathcal{F}}(P, \mathcal{I}_A) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathcal{F}}(J, \mathcal{I}_A) \arrow[r] \arrow[d, "f \mapsto f(c_u)"] & 0 \\
0 \arrow[r] & \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \arrow[r] & \mathrm{map}_*(A, \mathbb{Q}/\mathbb{Z}) \arrow[r, "\vartheta"'] & \mathrm{Z}^2_s(\mathcal{I}_A) \arrow[r] & 0
\end{tikzcd}$$

is exact where ϑ is defined by $\vartheta(h)(x,y) \;=\; h(x) \;-\; h(x\,+\,y) + h(y)$. The upper row is exact since $\mathcal{I}_A$ is an injective object and J is the kernel of the epimorphism $\epsilon\colon P \longrightarrow I$. The left and middle vertical maps are special cases of the Yoneda isomorphism $\mathrm{Hom}_{\mathcal{F}}(G, \mathcal{I}_A) \cong \mathrm{Hom}(G(A), \mathbb{Q}/\mathbb{Z})$, so the right vertical map is also an isomorphism and $c_u \in \mathrm{Z}^2_s(J)$ is indeed a universal symmetric 2–cocycle. □

8.6 The universal derivation We saw in Lemma 8.5 that the functor $J = \mathrm{kernel}(\epsilon\colon P \longrightarrow I)$ supports a universal symmetric 2–cocycle. Now we construct a universal derivation, ie, a certain homomorphism of Gamma-rings

$$H\mathbb{Z} \;\longrightarrow\; H\mathbb{Z} \times J^!$$

into the split extension (7.4) of $H\mathbb{Z}$ by $J^!$. The first component of this map is the identity map of $H\mathbb{Z}$. To describe the second component we consider the map of Γ–spaces

$$\eta \circ 1 \;-\; 1 \circ \eta\colon\; H\mathbb{Z} \;\longrightarrow\; H\mathbb{Z} \circ H\mathbb{Z}$$

where $\eta\colon \mathbb{S} \longrightarrow H\mathbb{Z}$ is the unit map of $H\mathbb{Z}$ given by inclusion of generators into the free abelian groups, and where we use the identifications $\mathbb{S} \circ H\mathbb{Z} \cong H\mathbb{Z} \cong$

$H\mathbb{Z} \circ \mathbb{S}$. We note that $H\mathbb{Z} \circ H\mathbb{Z} = P^!$ and that the composite of $\eta \circ 1 - 1 \circ \eta$ with the evaluation map $\epsilon^!\colon H\mathbb{Z} \circ H\mathbb{Z} = P^! \longrightarrow I^! = H\mathbb{Z}$ is trivial. So we can define $d_u\colon H\mathbb{Z} \longrightarrow J^!$ as the unique morphism of Γ–spaces whose composite with the inclusion $J^! \subseteq P^!$ is the difference $\eta \circ 1 - 1 \circ \eta$. We refer to d_u as the *universal derivation.* Now we claim that the map

$$(1, d_u)\colon\ H\mathbb{Z} \ \longrightarrow\ H\mathbb{Z} \times J^!$$

is a homomorphism of Gamma-rings. The only thing to verify is the multiplicativity, and it suffices to do this after composition with the injective Gamma-ring map $H\mathbb{Z} \times J^! \longrightarrow H\mathbb{Z} \times P^!$ induced by the inclusion $J \longrightarrow P$. By the definition of the product of $H\mathbb{Z} \times P^! = H\mathbb{Z} \times (H\mathbb{Z} \circ H\mathbb{Z})$ in 7.4 this boils down to verifying the commutativity of the diagram

$$\begin{tikzcd}
H\mathbb{Z} \circ H\mathbb{Z} \arrow[rr, "{(1\circ(\eta\circ 1-1\circ\eta)\ ,\ (\eta\circ 1-1\circ\eta)\circ 1)}"] \arrow[d, "\mu"'] & & H\mathbb{Z}^{\circ 3} \times H\mathbb{Z}^{\circ 3} \arrow[d, "(\mu\circ 1)\times(1\circ\mu)"] \\
H\mathbb{Z} \arrow[dr, "\eta\circ 1-1\circ\eta"'] & & (H\mathbb{Z} \circ H\mathbb{Z}) \times (H\mathbb{Z} \circ H\mathbb{Z}) \arrow[dl, "+"] \\
& H\mathbb{Z} \circ H\mathbb{Z} &
\end{tikzcd}$$

where $\mu\colon H\mathbb{Z} \circ H\mathbb{Z} \longrightarrow H\mathbb{Z}$ is the "multiplication" induced by the evaluation map $P \longrightarrow I$. The commutativity of the diagram in turn follows from the identities

$$(\mu \circ 1)(1 \circ \eta \circ 1) \ = \ 1_{H\mathbb{Z}\circ H\mathbb{Z}} \qquad \text{and} \qquad (\mu \circ 1)(1 \circ 1 \circ \eta) \ = \ (1 \circ \eta)\mu$$

(here juxtaposition means composition of Γ–space maps) and their analogues for $\mu \circ 1$ replaced by $1 \circ \mu$.

Hence for a functor $G \in \mathcal{F}$ we can define a map

$$\mathrm{Z}^2_s(G) \ \cong\ \mathrm{Hom}_{\mathcal{F}}(J, G) \ \longrightarrow\ \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}, H\mathbb{Z} \times G^!) \tag{8.7}$$

by sending a morphism $f\colon\ J \longrightarrow G$ to the composite Gamma-ring map

$$H\mathbb{Z} \ \xrightarrow{\ d_u\ }\ H\mathbb{Z} \times J^! \ \xrightarrow{\ 1\times f^!\ }\ H\mathbb{Z} \times G^! \ .$$

If $c \in \mathrm{Z}^2_s(G)$ is a cocycle represented by $f_c\colon\ J \longrightarrow G$, then we refer to the above Gamma-ring map as the *derivation* associated to the 2–cocycle c.

The group $G(\mathbb{Z})$ maps to the underlying monoid $(H\mathbb{Z} \times G^!)(1^+) \cong \mathbb{Z} \times G(\mathbb{Z})$ of the split extension via $a \longmapsto (1, a)$, hence it acts on the Gamma-ring $H\mathbb{Z} \times G^!$ by conjugation (4.1). The group $G(\mathbb{Z})$ also acts on the symmetric 2–cocycles via the homomorphism $\theta\colon\ G(\mathbb{Z}) \longrightarrow \mathrm{Z}^2_s(G)$ (8.1).

Lemma 8.8 *The map*

$$\mathrm{Z}^2_s(G) \;\cong\; \mathrm{Hom}_{\mathcal{F}}(J,G) \;\longrightarrow\; \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}, H\mathbb{Z}\times G^!)$$

which sends a morphism $f:\ J \longrightarrow G$ to the Gamma-ring map $(1, f^! \circ d_u):$ $H\mathbb{Z} \longrightarrow H\mathbb{Z}\times G^!$ is equivariant for the action of $G(\mathbb{Z})$.

Proof Let $a \in G(\mathbb{Z})$ be an element and $f\colon\ J \longrightarrow G$ a morphism of functors. Under the Yoneda isomorphism $G(\mathbb{Z}) \cong \mathrm{Hom}_{\mathcal{F}}(P,G)$ and the isomorphism $\mathrm{Z}^2_s(G) \cong \mathrm{Hom}_{\mathcal{F}}(J,G)$ of Lemma 8.5, the map $\theta\colon\ G(\mathbb{Z}) \longrightarrow \mathrm{Z}^2_s(G)$ corresponds to the map $\mathrm{Hom}_{\mathcal{F}}(P,G) \longrightarrow \mathrm{Hom}_{\mathcal{F}}(J,G)$ induced by the inclusion $\iota\colon\ J \longrightarrow P$.

So we have to verify the equality

$$(1, d_{f+a\circ\iota}) \;=\; (1,a)\cdot(1,d_f)\cdot(1,a)^{-1}$$

as maps $H\mathbb{Z} \longrightarrow H\mathbb{Z}\times G^!$. Only the second component matters. If we substitute definitions 4.1 of the conjugation action and 7.4 of the multiplication in $H\mathbb{Z}\times G^!$ we see that the second component of right hand side is a sum

$$\mathrm{proj}_{G^!}\circ\left[(1,a)\cdot(1,d_f)\cdot(1,a)^{-1}\right] \;=\; r_{\text{-}a} \;+\; d_f \;+\; l_a\ .$$

Here $l_a\colon\ H\mathbb{Z} \longrightarrow G^!$ is left multiplication with $a \in G^!(1^+) \cong G(\mathbb{Z})$, ie, the composition

$$H\mathbb{Z} \cong \mathbb{S}\wedge H\mathbb{Z} \xrightarrow{\ a\wedge 1\ } G^!\wedge H\mathbb{Z} \xrightarrow{\ \text{action}\ } G^!\ ,$$

and $r_{\text{-}a}$ is right multiplication with $-a \in G^!(1^+)$.

Since $d_{f+a\circ\iota} = d_f + d_{a\circ\iota}$ it suffices to show that

$$d_{a\circ\iota} \;=\; r_{\text{-}a} \;+\; l_a$$

as maps $H\mathbb{Z} \longrightarrow G^!$. By naturality it is enough to check the universal example, ie, to take $G = P$ and $a = [1] \in \widetilde{\mathbb{Z}}[\mathbb{Z}] = P(\mathbb{Z})$, which corresponds to the identity of P under the Yoneda isomorphism. By definition

$$d_\iota \;=\; \eta\circ 1 - 1\circ\eta:\ H\mathbb{Z} \;\longrightarrow\; P^! = H\mathbb{Z}\circ H\mathbb{Z}\ .$$

So the claim follows since $l_{[1]} = \eta\circ 1$ and $r_{\text{-}[1]} = -1\circ\eta$. □

8.9 Dold–Puppe stabilization and MacLane's Q–construction We recall the Dold–Puppe stabilization of a functor $G \in \mathcal{F}$ (compare [11, 8.3]). We work with a specific model for the Dold–Puppe stabilization which uses MacLane's *cubical construction* [14, Sec. 12]. In the original paper of Eilenberg and MacLane the cubical construction was defined for the free functor $P = \widetilde{\mathbb{Z}}$, but the definition extends to arbitrary reduced functors in $\mathcal{F}$, see [35, Sec.

4] or [27, 6.2] for the definition. A convenient reference for the relationship between Dold–Puppe stabilization and MacLane's cubical construction is [27]. The cubical construction QG of a functor $G \in \mathcal{F}$ is a chain complex of functors, concentrated in non-negative dimensions, with the following properties:

(a) QG is *homotopy-additive* in the sense that for every pair of finitely generated free abelian groups A and A' the canonical map
$$QG(A) \oplus QG(A') \longrightarrow QG(A \oplus A')$$
is a quasi-isomorphism.

(b) there is a natural isomorphism
$$H_* QG(\mathbb{Z}) \cong L_*^{\mathrm{st}} G \cong \pi_* G^! ,$$
ie, the homology groups of the complex $QG(\mathbb{Z})$ are isomorphic to the Dold–Puppe stable derived functors of G and to the stable homotopy groups of the Γ–space $G^!$.

(c) In dimension zero, $(QG)_0 = G$ and in positive dimensions QG is a finite sum of higher order cross-effects (see [15, Sec.9] or [27, Sec. 7]) of G.

(d) As a functor of G, the assignment $G \longmapsto QG$ is additive, exact, and commutes with limits and colimits.

(e) Suppose the functor G is *diagonalizable*, ie, there exists a functor
$$\bar{G} : \text{ (f.g. free ab. groups)} \times \text{(f.g. free ab. groups)} \longrightarrow \mathcal{A}b$$
of two variables satisfying $\bar{G}(A,0) \cong 0 \cong \bar{G}(0,A)$ and a natural isomorphism $G(A) \cong \bar{G}(A,A)$. Then the complex QG is acyclic.

Property (a) is proved in [35, 4.2] and [27, 6.3]. The first isomorphism of part (b) follows from [35, 4.1] or [27, 7.5]; essentially by definition the stable homotopy groups of the Γ–space $G^!$ are the Dold–Puppe stable derived functors of G. Part (c) is proved in [27, 6.3]. Property (d) follows from (c) since taking cross-effects is exact and commutes with limits and colimits. If G is diagonalizable, then [11, 5.20] shows that the stable derived functors of G are trivial. So by part (b) the complex $QG(\mathbb{Z})$ is acyclic and by part (a) QG is acyclic as a complex of functors.

Properties (a) and (b) already characterize QG up to a chain of natural quasi-isomorphisms; this is because on the level of homotopy categories, $G \longmapsto QG$ is left adjoint to the inclusion of the subcategory of homotopy additive complexes of functors.

We denote by G_{st} the simplicial functor which corresponds to QG under the Dold–Kan equivalence between simplicial objects and non-negative chain complexes in the abelian category $\mathcal{F}$. So G_{st} is defined by the property that its normalized chain complex is isomorphic to QG. By property (c), the functor of zero-simplices of G_{st} is G, which gives a map $G \longrightarrow G_{st}$ which induces isomorphism of stable homotopy groups upon passage to associated Γ–spaces $G^! \longrightarrow G^!_{st}$.

Construction 7.4 which associates the Gamma-ring $H\mathbb{Z} \times G^!$ to a functor $G \in \mathcal{F}$ makes perfect sense for *simplicial functors*, ie, simplicial objects in the abelian category $\mathcal{F}$. Moreover, the stabilization map $G \longrightarrow G_{st}$ induces a stable equivalence of Gamma-rings $H\mathbb{Z} \times G^! \longrightarrow H\mathbb{Z} \times G^!_{st}$. Combining the map (8.7) with this Gamma-ring homomorphism and the approximation map $H\mathbb{Z}^{\mathrm{c}} \longrightarrow H\mathbb{Z}$ gives a map

$$\begin{aligned} \mathrm{Z}^2_s(G) \;\cong\; \mathrm{Hom}_{\mathcal{F}}(J,G) \;&\xrightarrow{f \,\mapsto\, (1,\; f^! \circ d_u)}\; \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}, H\mathbb{Z} \times G^!) \\ &\xrightarrow{\qquad\qquad}\; \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^{\mathrm{c}}, H\mathbb{Z} \times G^!_{st}) \;. \end{aligned}$$

By Lemma 8.8 this map is equivariant for the action of $G(\mathbb{Z})$ by translation and conjugation respectively. Hence passing to homotopy orbits yields a map

$$\lambda_G : \; \mathcal{Z}(G) \;\cong\; \mathrm{Z}^2_s(G)_{hG(\mathbb{Z})} \;\longrightarrow\; \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^{\mathrm{c}}, H\mathbb{Z} \times G^!_{st})_{hG_{st}(\mathbb{Z})} \;. \tag{8.10}$$

Here we identified the classifying space of the groupoid $\mathcal{Z}(G)$ of symmetric 2–cocycles (8.1) with the homotopy orbit space of the action of $G(\mathbb{Z})$ on the set $\mathrm{Z}^2_s(G)$.

9 A change of models

So far we have reduced the proof of our main theorem, Theorem 5.2, to the verification that the map

$$\kappa_{B\otimes\mathcal{S}^k} : \; \widetilde{\mathcal{Z}}(B \otimes \mathcal{S}^k) \;\longrightarrow\; \mathrm{der}(H\mathbb{Z}, B \otimes \mathcal{S}^k)/\mathrm{conj.}$$

constructed in Section 7 is a weak equivalence for all $k \geq 1$. In this section we bring the map $\kappa_{B\otimes\mathcal{S}^k}$ into a more manageable form by constructing a commutative square

$$\begin{tikzcd} \widetilde{\mathcal{Z}}(B \otimes \mathcal{S}^k) \arrow[r, "\kappa_{B\otimes\mathcal{S}^k}"] \arrow[d, "\sim"'] & \mathrm{der}(H\mathbb{Z}, B \otimes \mathcal{S}^k)/\mathrm{conj.} \arrow[d, "\sim"] \\ \mathcal{Z}(B \otimes \mathcal{S}^k) \arrow[r, "\lambda_{B\otimes\mathcal{S}^k}"'] & \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^{\mathrm{c}}, H\mathbb{Z} \times (B \otimes \mathcal{S}^k)^!_{st})_{h(B\otimes\mathcal{S}^k)_{st}(\mathbb{Z})} \end{tikzcd} \tag{9.1}$$

in which the vertical maps are weak equivalences. The lower horizontal map $\lambda_{B\otimes\mathcal{S}^k}$ is an instance of (8.10). This then leaves us with the task to verify that $\lambda_{B\otimes\mathcal{S}^k}$ is a weak equivalence for all $k \geq 1$.

The construction of the square (9.1) is technical. The idea is that by properties 8.9 (a) and (b) of the cubical construction, $(B\otimes\mathcal{S}^k)^!_{st}$ is a stably fibrant model of the Γ–space $(B\otimes\mathcal{S}^k)^!$. Hence the map $H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^! \longrightarrow H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!_{st}$ is a stable equivalence of Gamma-rings with fibrant target. In particular, the space of Gamma-ring maps $\mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^c, H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!_{st})$ is a model for the homotopy invariant morphism space, ie, it is weakly equivalent to the derivation space $\mathrm{der}(H\mathbb{Z}, B\otimes\mathcal{S}^k)$. In order to identify these two derivation spaces we have to take a little more care because we need to work relative to the classifying space of symmetric 2–cocycles.

We reexamine Construction 4.2, applied to the Gamma-ring $H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!$. This construction yields a commutative diagram of Gamma-rings

$$\begin{tikzcd}
\mathbb{S}[B] \arrow[r, equal] \arrow[d] & \mathbb{S}[B] \arrow[r] \arrow[d] & \mathbb{S}[U(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)^\times] \arrow[d] \\
(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)^{\mathrm{f}} & (H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)_1 \arrow[l, "\sim"'] \arrow[r, "\sim"'] & (H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)_3
\end{tikzcd}$$

in which the lower horizontal maps are stable equivalences between stably fibrant Gamma-rings. Here we work relative to the homomorphism

$$B \longrightarrow H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!(1^+) = \mathbb{Z}[x]\times B[x^k]$$

which sends $b\in B$ to $x+b\cdot x^k$. Furthermore, the induced map from the simplicial group $U(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)^\times$ to the invertible components of the underlying monoid of $(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)_3$ is a weak equivalence.

Since the Gamma-ring $H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!_{st}$ is stably fibrant and the map

$$H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^! \longrightarrow (H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)^{\mathrm{f}}$$

in the functorial fibrant replacement is an acyclic cofibration, we can choose a Gamma-ring map from $(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)^{\mathrm{f}}$ to $H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!_{st}$ under $H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!$. This map will automatically be a stable equivalence. We can perform constructions 4.2 and 5.1 starting from either of these two fibrant replacements. The stable equivalence between them induces a weak equivalence between the two homotopy orbit spaces of the conjugation action. In other words, we can assume that the stably fibrant replacement of the split extension $H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!$ which was chosen in the beginning is *equal* to $H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!_{st}$.

The simplicial group $U(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)^\times$ is defined according to the recipe (4.2) by a factorization in the model category of simplicial monoids

$$B \longrightarrow c(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)^\times \xrightarrow{\ \sim\ } (H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)^\times$$

and then forming the algebraic group completion of $c(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)^\times$. The simplicial monoid $(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)^\times$ is, by definition, the union of the invertible components in $(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!_{st})(1^+)\cong\mathbb{Z}\times(B\otimes\mathcal{S}^k)_{st}(\mathbb{Z})$. So the simplicial monoid

$$(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!_{st})^\times \;\cong\; \{\pm1\}\times(B\otimes\mathcal{S}^k)_{st}(\mathbb{Z})$$

is already a simplicial group, hence there exists a unique extension to a homomorphism of simplicial groups $U(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)^\times\longrightarrow\{\pm1\}\times(B\otimes\mathcal{S}^k)_{st}(\mathbb{Z})$ which is necessarily a weak equivalence by Lemma 4.3.

Since $k\geq 2$, the stable derived functors of $B\otimes\mathcal{S}^k$ vanish in dimension 0 and 1 [11, 12.3], so the simplicial abelian group $(B\otimes\mathcal{S}^k)_{st}(\mathbb{Z})$ is simply connected, and the unit component of the monoid $(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!_{st})(1^+)$ is equal to $(B\otimes\mathcal{S}^k)_{st}(\mathbb{Z})$. Restriction to the unit components thus gives a weak equivalence of connected simplicial groups $U(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)^\times_1\longrightarrow(B\otimes\mathcal{S}^k)_{st}(\mathbb{Z})$.

The next step in the construction of the conjugation action (4.2) was the formation of the pushout $(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)_2$ of Gamma-rings:

$$\begin{tikzcd}
\mathbb{S}[c(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)^\times] \arrow[r] \arrow[d, "\sim"'] & (H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)_1 \arrow[r] \arrow[d] & H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!_{st} \\
\mathbb{S}[U(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)^\times] \arrow[r] & (H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)_2 \arrow[d, "\sim"'] \arrow[ur, dashed] & \\
 & (H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)_3 \arrow[uur, dashed] &
\end{tikzcd}$$

The simplicial group map $U(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)^\times\longrightarrow\{\pm1\}\times(B\otimes\mathcal{S}^k)_{st}(\mathbb{Z})$ adjoins to a homomorphism of Gamma-rings from the spherical group ring $\mathbb{S}[U(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)^\times]$ to $H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!_{st}$. This in turn induces a map from the pushout $(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)_2$ to $H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!_{st}$, represented as the upper dotted arrow in the previous diagram. Since the approximation map $(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)_2\longrightarrow(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)_3$ is an acyclic cofibration of Gamma-rings, we can finally choose an extension to a stable equivalence from $(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)_3$ to $H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!_{st}$. Since this map was constructed relative to the group ring of the simplicial group $U(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)^\times$, it is equivariant with respect to the conjugation action of $U(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)^\times_1$ on

$(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)_3$ and (through the map $U(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)_1^\times \longrightarrow (B\otimes\mathcal{S}^k)_{st}(\mathbb{Z})$) on $H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!_{st}$. By passage to homotopy orbits we get a weak equivalence

$$\begin{aligned}\mathrm{der}(H\mathbb{Z}, B\otimes\mathcal{S}^k)/\mathrm{conj.} \;&=\; \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^{\mathrm{c}}, (H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)_3)_{hU(H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!)_1^\times} \\ &\xrightarrow{\sim} \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^{\mathrm{c}}, H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!_{st})_{h(B\otimes\mathcal{S}^k)_{st}(\mathbb{Z})}\end{aligned}$$

Moreover, the square (9.1) commutes.

10 A useful adjunction

By the results of the previous two sections, the proof of the main theorem is reduced to an identification of the space of Gamma-ring maps

$$\mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^{\mathrm{c}}, H\mathbb{Z}\times(B\otimes\mathcal{S}^k)^!_{st})$$

(or more precisely a certain homotopy orbit space thereof) with the classifying space of symmetric 2–cocycles. In this section we use an adjunction to reinterpret the above mapping space in terms of the category $s\mathcal{F}$ of simplicial functors from the category of finitely generated free abelian groups to the category of abelian groups.

We note in Lemma 10.4 that the construction (7.4) of the split singular extension has a left adjoint

$$\mathcal{L}:\ \mathcal{G}R/H\mathbb{Z} \longrightarrow s\mathcal{F}$$

from the category of Gamma-rings over $H\mathbb{Z}$ to the category of simplicial functors. Moreover, the two functors form a simplicial Quillen adjoint pair of model categories. So the mapping space we are interested in is isomorphic to the mapping space

$$\mathrm{map}_{s\mathcal{F}}(\mathcal{L}(H\mathbb{Z}^{\mathrm{c}}), (B\otimes\mathcal{S}^k)_{st})$$

in the category of simplicial functors. To identify the mapping space in this adjoint form, we evaluate the left adjoint $\mathcal{L}$ on the cofibrant replacement of the Eilenberg–MacLane Gamma-ring $H\mathbb{Z}$.

As in (8.3), $J\in\mathcal{F}$ denotes the kernel of the evaluation map $\epsilon\colon P\longrightarrow I$. By Lemma 8.5 the functor J supports the universal symmetric 2–cocycle. The Gamma-ring map

$$H\mathbb{Z}^{\mathrm{c}} \longrightarrow H\mathbb{Z}\times J^!$$

which is the composite of the approximation map $H\mathbb{Z}^{\mathrm{c}}\longrightarrow H\mathbb{Z}$ and the universal derivation (8.6) is adjoint to a map of simplicial functors

$$\delta:\ \mathcal{L}(H\mathbb{Z}^{\mathrm{c}}) \longrightarrow J\ . \tag{10.1}$$

The main result of this section is

Theorem 10.2 *The map $\delta\colon \mathcal{L}(H\mathbb{Z}^c) \longrightarrow J$ which is adjoint to the universal derivation is a stable equivalence of simplicial functors.*

Remark 10.3 For any functor $G \in \mathcal{F}$ the Dold–Puppe stabilization G_{st} was defined so that its normalized chain complex is the cubical construction QG (8.9). So Theorem 10.2 and the Dold–Kan correspondence between simplicial objects and chain complexes in the abelian category $\mathcal{F}$ imply that the homotopy groups of the space

$$\mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^c, H\mathbb{Z} \times G^!_{st}) \;\cong\; \mathrm{map}_{s\mathcal{F}}(\mathcal{L}(H\mathbb{Z}^c), G_{st})$$

are isomorphic to the hyper-cohomology groups $\mathbb{E}\mathrm{xt}^*_{\mathcal{F}}(J, QG)$, for $* \leq 0$.

The category $s\mathcal{F}$ of simplicial functors admits a stable model structure, see [46, 6.13]. In this model structure, a map $G \longrightarrow \bar{G}$ is a weak equivalence or fibration if and only if the associated map of Γ–spaces $G^! \longrightarrow \bar{G}^!$ is a stable equivalence or stable fibration respectively. The stably fibrant objects are precisely the homotopy additive simplicial functors.

The split extension functor (7.4) which sends $G \in s\mathcal{F}$ to $H\mathbb{Z} \times G^!$, considered as a Gamma-ring over $H\mathbb{Z}$, commutes with limits. Moreover, the category $s\mathcal{F}$ of simplicial functors is complete, well-powered and has a set of cogenerators. So by Freyd's Adjoint Functor Theorem [31, V.8 Cor.] there is a left adjoint $\mathcal{L}\colon \mathcal{G}R/H\mathbb{Z} \longrightarrow s\mathcal{F}$. The right adjoint $H\mathbb{Z} \times (-)^!$ preserves stable equivalences and stable fibrations since in both categories these are defined on "underlying" Γ–spaces. Hence we obtain

Lemma 10.4 *The functor which sends a simplicial functor $G \in s\mathcal{F}$ to $H\mathbb{Z} \times G^!$, viewed as a Gamma-ring over $H\mathbb{Z}$, is the right adjoint of a Quillen adjoint pair between the category $s\mathcal{F}$ of simplicial functors, endowed with the stable model structure, and the stable model category of Gamma-rings over $H\mathbb{Z}$.*

We prove Theorem 10.2 by constructing a commutative square of $H\mathbb{Z}^c$–bimodules

$$\begin{tikzcd} \widetilde{\mathcal{B}}(H\mathbb{Z}^c) \arrow[r, "\eta_{H\mathbb{Z}^c}"] \arrow[d, "\text{ass.}"'] & (B\mathcal{L}(H\mathbb{Z}^c))^! \arrow[d, "(B\delta)^!"] \\ \widetilde{\mathcal{B}}(P)^! \arrow[r, "\widetilde{u}^!"'] & (BJ)^! \end{tikzcd} \tag{10.5}$$

For a simplicial functor $G \in s\mathcal{F}$ we denote by $BG = \widetilde{\mathbb{Z}}[S^1] \otimes G$ the (additive) bar construction, another simplicial functor. The bimodules $\widetilde{\mathcal{B}}(H\mathbb{Z}^c)$ and $\widetilde{\mathcal{B}}(P)^!$

are "multiplicative" bar constructions defined below. Three of the four objects are actually $H\mathbb{Z}$–bimodules, which we view as $H\mathbb{Z}^c$–bimodules via restriction of scalars. In the square the left vertical and the two horizontal maps are stable equivalences by Theorems 10.7, 10.9 and 10.10 respectively. Hence the map $(B\delta)^!\colon (B\mathcal{L}(H\mathbb{Z}^c))^! \longrightarrow (BJ)^!$ is a stable equivalence. Since the assignment $G \longmapsto (BG)^!$ detects stable equivalences, the map $\delta\colon \mathcal{L}(H\mathbb{Z}^c) \longrightarrow J$ is indeed a stable equivalence of simplicial functors.

10.6 A bar construction The *reduced bar construction* is a functor

$$\widetilde{\mathcal{B}}:\ \mathcal{GR}/H\mathbb{Z}^c \ \longrightarrow\ H\mathbb{Z}^c\text{-mod-}H\mathbb{Z}^c$$

from the category of Gamma-rings over $H\mathbb{Z}^c$ to the category of $H\mathbb{Z}^c$–bimodules. In this construction it is important that we start with Gamma-rings over the cofibrant approximation $H\mathbb{Z}^c$, not just over $H\mathbb{Z}$. If we worked over $H\mathbb{Z}$, the bar construction would have the wrong homotopy type since the point set level smash product of $H\mathbb{Z}$ with itself is not stably equivalent to $H\mathbb{Z}^c \wedge H\mathbb{Z}^c$

If Q is a Gamma-ring over $H\mathbb{Z}^c$, then the (unreduced) bar construction $\mathcal{B}(Q)$ is defined as the realization of a simplicial $H\mathbb{Z}^c$–bimodule which in simplicial dimension n has the form

$$H\mathbb{Z}^c \wedge \underbrace{Q \wedge \ldots \wedge Q}_{n} \wedge H\mathbb{Z}^c\ .$$

The simplicial structure maps are induced by the multiplication and unit map of Q and the structure map $Q \longrightarrow H\mathbb{Z}^c$. The inclusion of the 0-simplices induces a map

$$H\mathbb{Z}^c \wedge H\mathbb{Z}^c \ \longrightarrow\ \mathcal{B}(Q)$$

of $H\mathbb{Z}^c$–bimodules, and the reduced bar construction $\widetilde{\mathcal{B}}(Q)$ is the quotient of $\mathcal{B}(Q)$ by $H\mathbb{Z}^c \wedge H\mathbb{Z}^c$.

For every simplicial functor G there is a map

$$\tau_G:\ \widetilde{\mathcal{B}}(H\mathbb{Z}^c \times G^!) \ \longrightarrow\ (BG)^!$$

defined as the geometric realization of a map of simplicial $H\mathbb{Z}^c$–bimodules

$$n \mapsto \left[\ H\mathbb{Z}^c \wedge \underbrace{(H\mathbb{Z}^c \times G^!) \wedge \ldots \wedge (H\mathbb{Z}^c \times G^!)}_{n} \wedge H\mathbb{Z}^c \ \longrightarrow\ \underbrace{G^! \times \cdots \times G^!}_{n}\ \right]$$

whose i-th projection, for $1 \leq i \leq n$, is given symbolically by

$$x_0 \wedge (x_1, g_1) \wedge \ldots \wedge (x_n, g_n) \wedge x_{n+1} \quad\mapsto\quad x_0 \cdot x_1 \cdots x_{i-1} \cdot g_i \cdot x_{i+1} \cdots x_n \cdot x_{n+1}\ .$$

We define a map $\eta_Q \colon \widetilde{\mathcal{B}}(Q) \longrightarrow (B\mathcal{L}Q)^!$ as the composite

$$\widetilde{\mathcal{B}}(Q) \longrightarrow \widetilde{\mathcal{B}}(H\mathbb{Z}^c \times (\mathcal{L}Q)^!) \xrightarrow{\tau_{\mathcal{L}Q}} (B\mathcal{L}Q)^! \ ;$$

the first map is induced by the Gamma-ring map $Q \longrightarrow H\mathbb{Z}^c \times (\mathcal{L}Q)^!$, which in turn comes from the structure map $Q \longrightarrow H\mathbb{Z}^c$ and the adjunction unit $Q \longrightarrow H\mathbb{Z} \times (\mathcal{L}Q)^!$. For the proof of Theorem 10.2 we are only interested in the special case $Q = H\mathbb{Z}^c$, but to establish that $\eta_{H\mathbb{Z}^c}$ is a weak equivalence we will use a resolution argument which requires the general case of an arbitrary Gamma-ring over $H\mathbb{Z}^c$.

Theorem 10.7 *The map of $H\mathbb{Z}^c$–bimodules*

$$\eta_Q : \ \widetilde{\mathcal{B}}(Q) \longrightarrow (B\mathcal{L}(Q))^!$$

is a stable equivalence for every cofibrant Gamma-ring Q over $H\mathbb{Z}^c$.

Proof We first assume that Q is free, ie, that

$$Q \ = \ TX \ = \ \bigvee_{n\geq 0} X^{\wedge n}$$

is the tensor algebra generated by a cofibrant Γ–space X over $H\mathbb{Z}^c$. Then the $H\mathbb{Z}^c$–bimodule $\widetilde{\mathcal{B}}(TX)$ can be analyzed through a combinatorial filtration as follows. For $p \geq 0$ we define $F_p\mathcal{B}$ as the realization of a simplicial sub-$H\mathbb{Z}^c$–bimodule of the bar construction $\mathcal{B}(TX)$. In simplicial degree n we set

$$\begin{aligned}(F_p\mathcal{B})_n \ &= \ H\mathbb{Z}^c \wedge \Big(\bigvee_{i_1+\cdots+i_n\leq p} X^{\wedge i_1} \wedge \ldots \wedge X^{\wedge i_n}\Big) \wedge H\mathbb{Z}^c \\ &\subseteq \ H\mathbb{Z}^c \wedge (TX)^{\wedge\, n} \wedge H\mathbb{Z}^c \ = \ \mathcal{B}(TX)_n \ .\end{aligned}$$

The 0-th filtration is $H\mathbb{Z}^c \wedge H\mathbb{Z}^c$ and the subquotient $F_1\mathcal{B}/F_0\mathcal{B}$ is isomorphic to the suspension of $H\mathbb{Z}^c \wedge X \wedge H\mathbb{Z}^c$. To identify the subquotients of the filtration we use certain simplicial sets D_p. We define D_p as the quotient of the standard simplicial p–simplex with the union of the first and last $(p-1)$–face collapsed to a basepoint. Then $D_1 = S^1$ is the simplicial circle and D_p is weakly contractible for $p \geq 2$.

We note that the p-th subquotient is the realization of a simplicial object which in dimension n is of the form

$$(F_p\mathcal{B}/F_{p-1}\mathcal{B})_n \ = \ H\mathbb{Z}^c \wedge \Big(\bigvee_{i_1+\cdots+i_n=p} X^{\wedge i_1} \wedge \ldots \wedge X^{\wedge i_n}\Big) \wedge H\mathbb{Z}^c \ .$$

The map

$$H\mathbb{Z}^c \wedge X^{\wedge p} \wedge H\mathbb{Z}^c \longrightarrow (F_p\mathcal{B}/F_{p-1}\mathcal{B})_p$$

indexed by the p–tuple $(1,1,\ldots,1)$ yields a map

$$H\mathbb{Z}^c \wedge X^{\wedge p} \wedge H\mathbb{Z}^c \wedge \Delta^p \longrightarrow F_p\mathcal{B}/F_{p-1}\mathcal{B}$$

which factors over an isomorphism between $H\mathbb{Z}^c \wedge X^{\wedge p} \wedge H\mathbb{Z}^c \wedge D_p$ and $F_p\mathcal{B}/F_{p-1}\mathcal{B}$. Since D_p is weakly contractible for $p \geq 2$, all the filtration subquotients $F_p\mathcal{B}/F_{p-1}\mathcal{B}$ are stably contractible for $p \geq 2$. So the inclusion

$$H\mathbb{Z}^c \wedge \Sigma X \wedge H\mathbb{Z}^c \cong F_1\mathcal{B}/F_0\mathcal{B} \xrightarrow{\ i\ } \mathcal{B}(TX)/F_0\mathcal{B} = \widetilde{\mathcal{B}}(TX)$$

is a stable equivalence of $H\mathbb{Z}^c$–bimodules. To complete the verification that η_{TX} is a stable equivalence it remains to show that the composite

$$\eta_{TX} \circ i : \ H\mathbb{Z}^c \wedge \Sigma X \wedge H\mathbb{Z}^c \longrightarrow (B\mathcal{L}(TX))^!$$

is a stable equivalence.

We can rewrite the target $(B\mathcal{L}(TX))^!$ in a more familiar form. Let Φ denote the forgetful functor from the category of finitely generated free abelian groups to the category of pointed sets, and let $\widetilde{\mathbb{Z}}$ denote the reduced free functor from the category of pointed sets to the category of all abelian groups. By composing the Γ–space X with these two functor we obtain an object $\widetilde{\mathbb{Z}} \circ X \circ \Phi$ of the category $s\mathcal{F}$. The various adjunctions show that $\widetilde{\mathbb{Z}} \circ X \circ \Phi$ and $\mathcal{L}(TX)$ represent the same functor, namely the one which sends an object $G \in s\mathcal{F}$ to the set of Γ–space from X to the underlying Γ–space of $G^!$. Hence $\mathcal{L}(TX)$ is isomorphic to $\widetilde{\mathbb{Z}} \circ X \circ \Phi$ in the category $s\mathcal{F}$. Since the free functor $\widetilde{\mathbb{Z}}$ takes suspension of simplicial sets to bar construction of simplicial abelian groups, $B\mathcal{L}(TX)$ is then isomorphic to $\widetilde{\mathbb{Z}} \circ \Sigma X \circ \Phi$ in the category $s\mathcal{F}$.

Under the isomorphism $B\mathcal{L}(TX) \cong \widetilde{\mathbb{Z}} \circ \Sigma X \circ \Phi$ the map $\eta_{TX} \circ i$ corresponds to the composite

$$\begin{aligned} H\mathbb{Z}^c \wedge \Sigma X \wedge H\mathbb{Z}^c \longrightarrow H\mathbb{Z}^c \circ \Sigma X \circ H\mathbb{Z}^c \longrightarrow H\mathbb{Z} \circ \Sigma X \circ H\mathbb{Z} \\ = (\widetilde{\mathbb{Z}} \circ \Sigma X \circ \Phi)^! \cong (B\mathcal{L}(TX))^! \, . \end{aligned}$$

The left map is the assembly map (2.5), which is a stable equivalence by [29, 5.23] since $H\mathbb{Z}^c$ and ΣX are cofibrant as Γ–spaces. The second map is a weak equivalence since the composition product of Γ–spaces preserves stable equivalences (Theorem 2.6 (a)).

The general case is proved by resolving an arbitrary cofibrant Gamma-ring by free Gamma-rings as follows. If R is a simplicial object in the category of Gamma-rings, we denote by $|R|_{\mathcal{G}R}$ its geometric realization [22, VII 3.1] in the category of Gamma-rings, ie, the coend [31, IX.6]

$$|R|_{\mathcal{G}R} = \int_{n \in \Delta} R_n \times_{\mathcal{G}R} \Delta^n \ ;$$

here $(-) \times_{\mathcal{G}R} \Delta^n$ refers to the enrichment of the category of Gamma-rings over simplicial sets, which has to be distinguished from the objectwise smash product of the underlying Γ–space with Δ^n_+.

Claim Let R be a simplicial object in the category of Gamma-rings over $H\mathbb{Z}^c$ such that for all $n \geq 0$ the map $\eta_{R_n} \colon \widetilde{\mathcal{B}}(R_n) \longrightarrow (B\mathcal{L}(R_n))^!$ is a stable equivalence. Then the map $\eta_{|R|_{\mathcal{G}R}} \colon \widetilde{\mathcal{B}}(|R|_{\mathcal{G}R}) \longrightarrow (B\mathcal{L}(|R|_{\mathcal{G}R}))^!$ is also a stable equivalence.

To prove the claim it suffices to show that the map is a stable equivalence of underlying Γ–spaces. We consider the commutative square

$$\begin{tikzcd} |\widetilde{\mathcal{B}}(R)| \arrow[r, "\sim"] \arrow[d, "|\eta_R|"'] & \widetilde{\mathcal{B}}(|R|_{\mathcal{G}R}) \arrow[d, "\eta_{|R|_{\mathcal{G}R}}"] \\ |(B\mathcal{L}R)^!| \arrow[r, "\cong"'] & (B\mathcal{L}|R|_{\mathcal{G}R})^! \end{tikzcd}$$

On the left the functors $\widetilde{\mathcal{B}}$ and $(B\mathcal{L}-)^!$ are applied dimensionwise to the simplicial Gamma-ring R, and then we form the realization of the underlying simplicial Γ–space. The two horizontal maps are isomorphisms, so we may show that the left vertical map is a stable equivalence. A map of simplicial Γ–spaces which is dimensionwise a stable equivalence becomes a stable equivalence after realization. So the left vertical map in the above square is a stable equivalence of underlying Γ–spaces, which proves the claim.

We apply the claim to the cotriple resolution [31, VII.6] of a given cofibrant Gamma-ring Q over $H\mathbb{Z}^c$. The tensor algebra functor T from the category of Γ–spaces to the category of Gamma-rings is left adjoint to the forgetful functor. The adjunction gives rise to a cotriple, hence to a simplicial Gamma-ring R which in simplicial dimension n consists of the Gamma-ring $R_n = T^{n+1}Q$. We claim that this simplicial Gamma-ring R is cofibrant in the Reedy model structure ([40, Thm. A], [23, 5.2.5], [22, VII 2.1]). Indeed, the maps from the latching objects to the levels of the simplicial Gamma-ring R are freely generated by a wedge summand inclusion of Γ–spaces whose cokernel is a wedge of smash powers of Q. Since Q is cofibrant as a Gamma-ring, it is cofibrant as a Γ–space [47, 4,1 (3)], hence so are its smash powers. So the maps from the latching objects to the levels of the R are cofibrations of Gamma-rings, ie, R is Reedy cofibrant. In particular, for all $n \geq 0$ the underlying Γ–space of T^nQ is cofibrant, and hence for $R_n = T(T^nQ)$ the map $\eta_{R_n} \colon \widetilde{\mathcal{B}}(R_n) \longrightarrow (B\mathcal{L}(R_n))^!$ is a stable equivalence by the first part of this proof. By the claim, the map $\eta_{|R|_{\mathcal{G}R}} \colon \widetilde{\mathcal{B}}(|R|_{\mathcal{G}R}) \longrightarrow (B\mathcal{L}(|R|_{\mathcal{G}R}))^!$ is also a stable equivalence.

The cotriple resolution comes with an augmentation $|R|_{\mathcal{G}R} \longrightarrow Q$. After forgetting the multiplication, the augmented simplicial Γ–space $|R|_{\mathcal{G}R} \longrightarrow Q$ has an extra degeneracy, so the augmentation map $|R|_{\mathcal{G}R} \longrightarrow Q$ is an objectwise equivalence of Gamma-rings [22, III 5.1]. Since R is Reedy cofibrant, the realization $|R|_{\mathcal{G}R}$ is cofibrant [22, VII 3.6]. Since the functors $\widetilde{\mathcal{B}}$ and $(B\mathcal{L}-)^!$ both preserve stable equivalences between cofibrant Gamma-rings, η_Q is a stable equivalence as claimed. □

10.8 Another bar construction The lower left hand corner of the square (10.5) arises from a simplicial functor $\widetilde{\mathcal{B}}(P) \in s\mathcal{F}$ which is another reduced bar construction. We note that the category $\mathcal{F}$ of reduced functors from finitely generated free abelian groups to abelian abelian groups has a monoidal composition product $\circ$ with unit the inclusion functor I; before composing two functors, one of them has to be left Kan extended [31, X.3] from finitely generated free to all abelian groups.

The functor $P \in \mathcal{F}$ is the composite of the forgetful functor from abelian groups to pointed sets with its adjoint free functor. Hence P has the structure of a cotriple on the category of abelian groups. This cotriple gives rise to a simplicial object $\mathcal{B}(P)$, augmented over the functor I, which in simplicial dimension n is given by

$$\mathcal{B}(P)_n \;=\; \underbrace{P \circ \cdots \circ P}_{n+1},$$

compare [31, VII.6]. The augmentation $\mathcal{B}(P)_0 = P \longrightarrow I$ is given by the evaluation map ϵ.

The $H\mathbb{Z}$–bimodule $\mathcal{B}(P)^!_n = (P^{\circ(n+1)})^!$ is equal to the $(n+2)$–fold composition product of the Eilenberg–MacLane Γ–space $H\mathbb{Z}$. So the assembly map (2.5) induces a map of simplicial $H\mathbb{Z}^c$–bimodules

$$\mathcal{B}(H\mathbb{Z}^c)_n \;=\; (H\mathbb{Z}^c)^{\wedge(n+2)} \xrightarrow{\text{assembly}} H\mathbb{Z}^{\circ(n+2)} \;=\; \mathcal{B}(P)^!_n \,.$$

We denote by $\widetilde{\mathcal{B}}(P)$ the simplicial functor obtained from $\mathcal{B}(P)$ by collapsing the simplicial 0-skeleton. The assembly map passes to a map $\widetilde{\mathcal{B}}(H\mathbb{Z}^c) \longrightarrow \widetilde{\mathcal{B}}(P)^!$ on quotients by the respective simplicial 0-skeleta.

Theorem 10.9 *The assembly map*

$$\widetilde{\mathcal{B}}(H\mathbb{Z}^c) \longrightarrow \widetilde{\mathcal{B}}(P)^!$$

is a stable equivalence of $H\mathbb{Z}^c$–bimodules.

Proof Since $H\mathbb{Z}^c$ is cofibrant as a Γ–space, the assembly map from a smash power of a certain number of copies of $H\mathbb{Z}^c$ to the composition power of the same number of copies of $H\mathbb{Z}$ is a stable equivalence by [29, 5.23] and Theorem 2.6 (a). A map of simplicial Γ–spaces which is dimensionwise a stable equivalence induces a stable equivalence on realizations. So the assembly maps

$$\mathcal{B}(H\mathbb{Z}^c) \longrightarrow \mathcal{B}(P)^! \qquad \text{and} \qquad \widetilde{\mathcal{B}}(H\mathbb{Z}^c) \longrightarrow \widetilde{\mathcal{B}}(P)^!$$

on realizations are stable equivalences of $H\mathbb{Z}^c$–bimodules. $\square$

The lower horizontal map in the square (10.5) arises from an objectwise equivalence

$$u: \mathcal{B}(P) \longrightarrow P \oplus_J EJ$$

of simplicial functors by passage to quotient and application of the $(-)^!$–construction. The target $P \oplus_J EJ$ is the simplicial functor defined as the pushout of the diagram

$$P \xleftarrow{\ \iota\ } J \longrightarrow \widetilde{\mathbb{Z}}[\Delta^1] \otimes J\ .$$

The map $\iota\colon J \longrightarrow P$ is the inclusion and $J \longrightarrow \widetilde{\mathbb{Z}}[\Delta^1] \otimes J$ is induced by the inclusion of the non-basepoint vertex of Δ^1.

We describe the map

$$u_n: \mathcal{B}(P)_n = P^{\circ(n+1)} \longrightarrow (P \oplus_J EJ)_n = P \times J^n$$

in simplicial dimension n by giving the components of the various factors of the target. The projection of u_n to the first factor is the map

$$\epsilon^{\circ n} \circ 1 : P^{\circ(n+1)} \longrightarrow P\ .$$

The projection of u_n to the i-th factor of J, for $1 \le i \le n$ is the map

$$\epsilon^{\circ(n-i)} \circ 1 \circ \epsilon^{\circ i} - \epsilon^{\circ(n-i+1)} \circ 1 \circ \epsilon^{\circ(i-1)} : P^{\circ(n+1)} \longrightarrow J\ ;$$

the target of each of the two summands is really the functor P, but the difference is annihilated by $\epsilon\colon P \longrightarrow I$, so it lands in $J = \mathrm{kernel}(\epsilon)$. Note that P is the functor of 0-simplices in $P \oplus_J EJ$, and the quotient of $P \oplus_J EJ$ by P is isomorphic to the simplicial functor BJ. So the map u passes to quotients and yields a map of simplicial functors $\widetilde{u}\colon \widetilde{\mathcal{B}}(P) \longrightarrow BJ$.

Theorem 10.10 *The map*

$$\widetilde{u}: \widetilde{\mathcal{B}}(P) \longrightarrow BJ$$

is an objectwise weak equivalence of simplicial functors.

Proof The simplicial subfunctor $\widetilde{\mathbb{Z}}[\Delta^1] \otimes J$ of $P \oplus_J EJ$ is objectwise weakly contractible. So the quotient map

$$q: \; P \oplus_J EJ \; \longrightarrow \; (P \oplus_J EJ) \, / \, (\widetilde{\mathbb{Z}}[\Delta^1] \otimes J) \; \cong \; I$$

is an objectwise weak equivalence of simplicial functors.

The composite map $q \circ u\colon \mathcal{B}(P) \longrightarrow I$ is the augmentation of the cotriple resolution. Whether or not it is an objectwise weak equivalence can be checked by looking at the augmented simplicial Γ–space $\mathcal{B}(P)^! \longrightarrow I^!$. However this augmented simplicial Γ–space has an extra degeneracy, so the augmentation $\mathcal{B}(P)^! \longrightarrow I^!$, and hence the map u, is an objectwise weak equivalence [22, III 5.1].

The simplicial 0-skeleta of $\mathcal{B}(P)$ and $P \oplus_J EJ$ are both equal to the functor P. So if we collapse the 0-skeleta, then the induced map of quotients $\widetilde{u}\colon \widetilde{\mathcal{B}}(P) \longrightarrow BJ$ is also an objectwise weak equivalence. $\square$

11 A homological criterion

In this section we give a homological condition, Theorem 11.1 below, for when the map

$$\lambda_G: \; \mathcal{Z}(G) \; \longrightarrow \; \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^{c}, H\mathbb{Z} \times G^!_{st})_{hG_{st}(\mathbb{Z})}$$

defined in (8.10) is a weak equivalence. Here $\mathcal{Z}(G)$ denotes the groupoid of symmetric 2–cocycles of the functor G (8.1). G_{st} is the Dold–Puppe stabilization of G, a simplicial functor which corresponds to the cubical construction QG under the Dold–Kan equivalence between simplicial objects and non-negative complexes in the category $\mathcal{F}$, compare (8.9). In the next section we verify the criterion of Theorem 11.1 in the case of the symmetric power functors $B \otimes \mathcal{S}^k$.

As before, $\mathcal{F}$ denotes the abelian category of reduced functors from finitely generated free abelian groups to all abelian groups and $I \in \mathcal{F}$ is the inclusion functor. The functor category $\mathcal{F}$ is abelian and exactness can be checked objectwise; $\mathcal{F}$ has enough projectives and injectives. The map $G \longrightarrow QG$ is the inclusion as the object in dimension zero. Also, $\mathbb{E}\mathrm{xt}^*_{\mathcal{F}}(I, -)$ denotes hyper-Ext groups of the inclusion functor I with coefficients in a chain complex of functors, ie, the graded abelian group of maps out of I in the derived category $\mathcal{D}^+(\mathcal{F})$ of bounded below complexes of functors. A priori, these hyper-Ext groups can be non-trivial in negative dimensions.

Theorem 11.1 *Let $G \in \mathcal{F}$ be a functor such that for all integers $m \leq 2$ the map*

$$\mathrm{Ext}^m_{\mathcal{F}}(I, G) \longrightarrow \mathbb{E}\mathrm{xt}^m_{\mathcal{F}}(I, QG)$$

is an isomorphism. Then the map

$$\lambda_G : \mathcal{Z}(G) \longrightarrow \mathrm{map}_{\mathcal{GR}}(H\mathbb{Z}^c, H\mathbb{Z} \times G^!_{st})_{hG_{st}(\mathbb{Z})}$$

is a weak equivalence of simplicial sets.

Remark 11.2 By [46, 6.1], the hyper-cohomology groups $\mathbb{E}\mathrm{xt}^m_{\mathcal{F}}(I, QG)$ are isomorphic to the topological Hochschild cohomology groups of $H\mathbb{Z}$ with coefficients in the bimodule $G^!$.

On the other hand, if A is an abelian group, then a theorem of Jibladze and Pirashvili [26, Thm. A] identifies the cohomology groups $\mathrm{Ext}^*_{\mathcal{F}}(I, A \otimes -)$ with the MacLane cohomology groups of A [30]. Because of this, for an arbitrary functor $G \in \mathcal{F}$ the groups $\mathrm{Ext}^*_{\mathcal{F}}(I, G)$ are sometimes referred to as the MacLane cohomology groups of $\mathbb{Z}$ with coefficients in the functor G. So the criterion of Theorem 11.1 ask whether the natural map from the MacLane cohomology to the topological Hochschild cohomology of the functor G is an isomorphism.

A theorem of Pirashvili and Waldhausen [37, 3.2] says that if cohomology is replaced by *homology*, then the MacLane theory coincides with the topological Hochschild theory for arbitrary coefficient functors. By [46, 6.7], the cohomological theories also agree if the coefficient functor is *additive*.

However, as the following example shows, the hypothesis of Theorem 11.1 is not satisfied for an arbitrary functor $G \in \mathcal{F}$. So for general coefficients, MacLane cohomology and topological Hochschild cohomology do *not* coincide.

Example 11.3 We give an example of a functor for which the hypothesis of Theorem 11.1 fails. For a fixed prime p the Frobenius maps

$$\mathbb{Z}/p \otimes \mathcal{S}^{p^{h-1}} \longrightarrow \mathbb{Z}/p \otimes \mathcal{S}^{p^h} \ , \quad a \otimes x \longmapsto a \otimes x^p$$

define a morphism in the category $\mathcal{F}$. We consider the functor

$$G \ = \ \mathrm{colim}_h \ \mathbb{Z}/p \otimes \mathcal{S}^{p^h}$$

defined as the colimit of the sequence of Frobenius maps. Since the Frobenius transformations are injective, the natural map

$$I \xrightarrow{\text{projection}} \mathbb{Z}/p \otimes I = \mathbb{Z}/p \otimes \mathcal{S}^{p^0} \xrightarrow{\text{inclusion}} G$$

is a non-trivial element of $\mathrm{Hom}_{\mathcal{F}}(I, G)$. On the other hand, the stable derived functors of the symmetric power functor $A \otimes \mathcal{S}^k$ are trivial up to dimension $2k-3$

[11, 12.3], so $Q(\mathbb{Z}/p \otimes \mathcal{S}^{p^h})$ has trivial homology up to dimension $(2p^h - 3)$. Since the Q–construction and homology commute with filtered colimits, the complex QG is acyclic, so the hyper-cohomology groups $\mathbb{E}\mathrm{xt}^*_{\mathcal{F}}(I, QG)$ are trivial. In particular the map

$$\mathrm{Ext}^0_{\mathcal{F}}(I, G) \longrightarrow \mathbb{E}\mathrm{xt}^0_{\mathcal{F}}(I, QG)$$

is not injective.

Proof of Theorem 11.1 The map λ_G (8.10) is obtained from the commutative square of simplicial abelian groups

$$\begin{tikzcd}
G(\mathbb{Z}) \arrow[r] \arrow[dd, "\Theta"'] & G_{st}(\mathbb{Z}) \arrow[d, "\Theta"] \\
 & \mathrm{Z}^2_s(G_{st}) \arrow[d] \\
\mathrm{Z}^2_s(G) \arrow[r] & \mathrm{map}_{\mathcal{G}R}(H\mathbb{Z}^{\mathrm{c}}, H\mathbb{Z} \times G^!_{st})
\end{tikzcd}$$

by passage to vertical homotopy cofibres in the category of simplicial abelian groups. So it suffices to show that the square is homotopy cocartesian (in the category of simplicial abelian groups). Evaluation at $\mathbb{Z}$ is represented by the projective functor P, symmetric 2–cocycles are represented by the functor J (Lemma 8.5), and the split extension construction (7.4) has a left adjoint $\mathcal{L}$ (Lemma 10.4). So the square is isomorphic to the square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{F}}(P, G) \arrow[r] \arrow[d, "\mathrm{Hom}(\iota, G)"'] & \mathrm{map}_{s\mathcal{F}}(P, G_{st}) \arrow[d] \\
\mathrm{Hom}_{\mathcal{F}}(J, G) \arrow[r] & \mathrm{map}_{s\mathcal{F}}(\mathcal{L}(H\mathbb{Z}^{\mathrm{c}}), G_{st})
\end{tikzcd}$$

where $\iota\colon J \longrightarrow P$ is the inclusion.

The map $\delta\colon \mathcal{L}(H\mathbb{Z}^{\mathrm{c}}) \longrightarrow J$ (10.1) which is adjoint to the universal derivation is a stable equivalence by Theorem 10.2. We let $\alpha\colon J^{\mathrm{c}} \longrightarrow J$ be a cofibrant approximation of the functor J in the strict model structure of simplicial functors where the weak equivalences are defined objectwise; equivalently, the normalized chain complex of J^{c} is a projective resolution of J. Then $\delta\colon \mathcal{L}(H\mathbb{Z}^{\mathrm{c}}) \longrightarrow J$ lifts to a map of simplicial functors $\bar{\delta}\colon \mathcal{L}(H\mathbb{Z}^{\mathrm{c}}) \longrightarrow J^{\mathrm{c}}$. The lift $\bar{\delta}$ is then a stable equivalence between cofibrant simplicial functors. Since G_{st} is homotopy additive, alias stably fibrant, the map $\bar{\delta}$ induces a weak equivalence of simplicial abelian groups upon application of $\mathrm{map}_{s\mathcal{F}}(-, G_{st})$. Since G is a constant simplicial functor, the map

$$\mathrm{map}_{s\mathcal{F}}(\alpha, G):\ \mathrm{Hom}_{\mathcal{F}}(J,G) = \mathrm{map}_{s\mathcal{F}}(J,G) \longrightarrow \mathrm{map}_{s\mathcal{F}}(J^{\mathrm{c}},G)$$

is an isomorphism. In other words, it suffices to show that the square in the category of simplicial abelian groups

$$\begin{tikzcd} \mathrm{map}_{s\mathcal{F}}(P,G) \arrow[r] \arrow[d, "\mathrm{map}(\iota\circ\alpha,G)"'] & \mathrm{map}_{s\mathcal{F}}(P,G_{st}) \arrow[d, "\mathrm{map}(\iota\circ\alpha,G_{st})"] \\ \mathrm{map}_{s\mathcal{F}}(J^{\mathrm{c}},G) \arrow[r] & \mathrm{map}_{s\mathcal{F}}(J^{\mathrm{c}},G_{st}) \end{tikzcd}$$

is homotopy cocartesian. For this in turn it is enough to show that the map on horizontal homotopy cofibres

$$\mathrm{map}_{s\mathcal{F}}(\iota\circ\alpha, G_{st}/G):\ \mathrm{map}_{s\mathcal{F}}(P, G_{st}/G) \longrightarrow \mathrm{map}_{s\mathcal{F}}(J^{\mathrm{c}}, G_{st}/G)$$

is a weak equivalence where G_{st}/G denotes to cofibre of the stabilization map.

If K and K' are two simplicial functors such that K is cofibrant, then the Dold–Kan theorem provides a natural isomorphisms of groups

$$\pi_n\ \mathrm{map}_{s\mathcal{F}}(K,K') \ \cong\ [NK[n], NK']$$

for $n \geq 0$, where N is the normalized chain complex and $[-,-]$ denotes maps in the derived category $\mathcal{D}^+(\mathcal{F})$ of bounded below chain complexes of functors. The normalized chain complex of J^{c} is quasi-isomorphic to J, and the normalized chain complex of G_{st} is the cubical construction QG. So we need to show that the map

$$[P[n], QG/G] \ \longrightarrow\ [J[n], QG/G]$$

is an isomorphism for $n \geq 0$. The short exact sequence of functors $J \longrightarrow P \longrightarrow I$ yields a long exact sequence after applying $[-, QG/G]$, so it is enough to show that the groups

$$[I[n], QG/G] \ =\ \mathbb{E}\mathrm{xt}_{\mathcal{F}}^{-n}(I, QG/G)$$

vanish for $n \geq -1$. This in turn follows from the assumption that the map

$$\mathrm{Ext}^m_{\mathcal{F}}(I,G) \ \longrightarrow\ \mathbb{E}\mathrm{xt}^m_{\mathcal{F}}(I,QG)$$

is an isomorphism for all integers $m \leq 2$. □

12 Cohomology of symmetric power functors

The purpose of this section is to prove that the symmetric power functors satisfy the homological criterion of Theorem 11.1; this completes the proof of the main theorem.

Theorem 12.1 *For all $m \in \mathbb{Z}$, all $k \geq 1$ and all abelian groups A the map*

$$\mathrm{Ext}^m_{\mathcal{F}}(I, A \otimes \mathcal{S}^k) \;\longrightarrow\; \mathbb{E}\mathrm{xt}^m_{\mathcal{F}}(I, Q(A \otimes \mathcal{S}^k))$$

is an isomorphism.

Remark 12.2 Theorem 12.1 can be interpreted as saying that MacLane cohomology coincides with topological Hochschild cohomology for the symmetric power functors, compare Remark 11.2. The groups $\mathrm{Ext}^m_{\mathcal{F}}(I, A \otimes \mathcal{S}^k)$ have been calculated for $A = \mathbb{Z}/p$ and for $A = \mathbb{Z}$ by Franjou-Lannes-Schwartz and Franjou-Pirashvili, see [17, Thm. 6.6 and Prop. 9.1] and [18, 2.1]. So Theorem 12.1 is a calculation of the topological Hochschild cohomology groups of $H\mathbb{Z}$ with coefficients in the bimodule $(A \otimes \mathcal{S}^k)^!$. In particular, the topological Hochschild cohomology groups of $H\mathbb{Z}$ with coefficients in $(A \otimes \mathcal{S}^k)^!$ are trivial in negative dimensions.

Our proof of Theorem 12.1 is not completely satisfactory because it uses the explicit calculations of the groups $\mathrm{Ext}^m_{\mathcal{F}}(I, \mathcal{S}^k)$; these enter as the sparseness hypothesis (c) of Theorem 12.3 below. It would be desirable to have a direct proof of Theorem 12.1 which would hopefully shed more light on the question for which coefficient functors MacLane cohomology coincides with topological Hochschild cohomology. Example 11.3 shows that some restriction on the functor has to be imposed.

Theorem 12.1 is a special case of the following Theorem 12.3. To apply it, we choose a projective resolution $P_* \longrightarrow I$ of the functor I in the abelian category $\mathcal{F}$. Then we let $T\colon \mathrm{Ch}^+(\mathcal{F}) \longrightarrow \mathrm{coCh}$ be the homomorphism complex out of this resolution,

$$T(X) \;=\; \mathrm{Hom}^\bullet_{\mathcal{F}}(P_*, X) \;.$$

So $T(X)$ is a (usually unbounded) cochain complex of abelian groups and as a functor of X it is additive, exact, and preserves inverse limits and quasi-isomorphisms. The cohomology groups of $\mathrm{Hom}^\bullet_{\mathcal{F}}(P_*, X)$ are the hyper-cohomology groups $\mathbb{E}\mathrm{xt}^*_{\mathcal{F}}(I, X)$. By a theorem of Pirashvili ([34, 2.15], see also [17, 0.4] or the appendix of [4]), the extension groups $\mathrm{Ext}^*_{\mathcal{F}}(I, -)$ vanish for every diagonalizable functor (8.9 (e)), so $\mathrm{Hom}^\bullet_{\mathcal{F}}(P_*, -)$ takes diagonalizable functors to acyclic complexes. The sparseness condition (c) is proved in [18, Prop. 2.1].

The homomorphism complex $\mathrm{Hom}^\bullet_{\mathcal{F}}(P_*, -)$ is the only functor to which we apply Theorem 12.3; nevertheless we state and prove it in the general form because we think it makes the proof more understandable.

Theorem 12.3 *Let $T\colon \mathrm{Ch}^+(\mathcal{F}) \longrightarrow \mathrm{coCh}$ be a functor from the category of bounded below chain complexes in the abelian category $\mathcal{F}$ to the category of (not necessarily bounded) cochain complexes of abelian groups. Suppose furthermore that*

(a) *T is additive, exact and preserves inverse limits and quasi-isomorphisms,*

(b) *the complex $T(D)$ is acyclic for every diagonalizable functor $D \in \mathcal{F}$ (8.9 (e)), considered as a complex concentrated in dimension 0, and*

(c) *T is* sparse on symmetric powers *in the sense that for all $k \geq 1$ the cohomology of the complex $T(\mathcal{S}^k)$ is concentrated in dimensions congruent to 1 modulo $2k$.*

Then for all abelian groups A, and all $k \geq 1$ the natural map

$$T(A \otimes \mathcal{S}^k) \longrightarrow T(Q(A \otimes \mathcal{S}^k))$$

is a quasi-isomorphism.

Remark 12.4 The heart of Theorem 12.3 is a convergence issue, or a question to what extent the functor T commutes with infinite sums (up to quasi-isomorphism). Indeed, for any functor $G \in \mathcal{F}$, the cokernel of the stabilization map $G \longrightarrow QG$ is a bounded below complex of diagonalizable functors [27, 7.4], usually with non-trivial homology in arbitrarily high dimensions. So the cokernel QG/G can be written as the colimit of a sequence of bounded complex of diagonalizable functors. So if a functor T as in Theorem 12.3 commutes with filtered colimits or infinite sums, then properties (a) and (b) already imply that $T(QG/G)$ is acyclic, and so $T(G) \longrightarrow T(QG)$ is a quasi-isomorphism, for all $G \in \mathcal{F}$. In the case of interest for us, namely $T = \mathrm{Hom}^{\bullet}_{\mathcal{F}}(P_*, -)$, the functor T fails to commute with infinite sums, essentially because the inclusion functor I is not a small (or compact) object in the derived category of $\mathcal{F}$. And indeed, $\mathrm{Hom}^{\bullet}_{\mathcal{F}}(P_*, QG/G)$ fails to be acyclic in general as Example 11.3 shows. However the sparseness condition (c) makes it possible to obtain the desired conclusion for the symmetric power functors.

The following observation goes back, at least, to Dold and Puppe [11]. For every $k \geq 1$ let d_k denote the greatest common divisor of the binomial coefficients $\binom{k}{i}$ for $1 \leq i \leq k-1$. Then

$$d_k = \begin{cases} p & \text{if } k = p^h \text{ for a prime } p \text{ and } h > 0, \\ 1 & \text{else.} \end{cases}$$

Lemma 12.5 [11, 10.9] *For every $k \geq 1$ and every abelian group A, multiplication by the number d_k on the functor $A \otimes \mathcal{S}^k$ factors over a diagonalizable functor (8.9 (e)). In particular, if k is not a prime power, then $A \otimes \mathcal{S}^k$ is a retract of a diagonalizable functor.*

Proof For every $1 \leq i \leq k-1$ the comultiplication of the symmetric algebra gives a map of functors

$$\Delta_{i,k-i} : \; A \otimes \mathcal{S}^k \longrightarrow A \otimes \mathcal{S}^i \otimes \mathcal{S}^{k-i} \; ;$$

the explicit formula for this map is given by

$$a \otimes x_1 \cdot \dots \cdot x_k \longmapsto a \otimes \sum_{T \subset \{1,2,\dots,k\}, \; |T|=i} \Big(\prod_{j \in T} x_j\Big) \otimes \Big(\prod_{j \notin T} x_j\Big) \; .$$

The composite of $\Delta_{i,k-i}$ with the natural projection $A \otimes \mathcal{S}^i \otimes \mathcal{S}^{k-i} \longrightarrow A \otimes \mathcal{S}^k$ is multiplication by the binomial coefficient $\binom{k}{i}$. So if we choose a presentation

$$d_k = \sum_{i=1}^{k-1} \lambda_i \cdot \binom{k}{i}$$

for suitable integers λ_i then the composition

$$A \otimes \mathcal{S}^k \xrightarrow{\sum \lambda_i \cdot \Delta_{i,k-i}} \bigoplus_{i=1}^{k-1} A \otimes \mathcal{S}^i \otimes \mathcal{S}^{k-i} \longrightarrow A \otimes \mathcal{S}^k$$

is multiplication by the number d_k. □

Finally, we give the **proof of Theorem 12.3**. For a functor $G \in \mathcal{F}$ we use the notation $\bar{Q}G$ for the quotient complex QG/G. By the exactness of T we then have to show that for all abelian groups A, and all $k \geq 1$ the complex $T(\bar{Q}(A \otimes \mathcal{S}^k))$ is acyclic.

Step 1 Diagonalizable functors

Suppose $D \in \mathcal{F}$ is diagonalizable (8.9 (e)). Then QD is an acyclic complex by 8.9 (e), and so $T(QD)$ is acyclic. By property (b) of the functor T the complex $T(D)$, and hence, by exactness, the quotient complex $T(\bar{Q}D)$ is also acyclic.

If k is not a prime power then $A \otimes \mathcal{S}^k$ is a retract of a diagonalizable functor by Lemma 12.5, so Theorem 12.3 holds for such exponents. From now on we assume that the exponent is of the form $k = p^h$ for a prime number p and some $h \geq 0$.

Step 2 Reduction to the case $A=\mathbb{Z}/p$

For the course of this proof we call an abelian group A *good* if the complex

$$T(\bar{Q}(A\otimes\mathcal{S}^{p^h}))$$

is acyclic. We show that if the group $\mathbb{Z}/p$ is good, then every abelian group is good.

Multiplication by the number p is an epimorphism on the functor $\mathbb{Q}/\mathbb{Z}\otimes\mathcal{S}^{p^h}$ with kernel isomorphic to $\mathbb{Z}/p\otimes\mathcal{S}^{p^h}$. Since the cubical construction and the functor T are exact, multiplication by p on the complex $T(\bar{Q}(\mathbb{Q}/\mathbb{Z}\otimes\mathcal{S}^{p^h}))$ is surjective and has kernel isomorphic to $T(\bar{Q}(\mathbb{Z}/p\otimes\mathcal{S}^{p^h}))$, which is acyclic since $\mathbb{Z}/p$ was assumed to be good. So multiplication by p is a quasi-isomorphism on the complex $T(\bar{Q}(\mathbb{Q}/\mathbb{Z}\otimes\mathcal{S}^{p^h}))$. On the other hand, multiplication by p on $\mathbb{Q}/\mathbb{Z}\otimes\mathcal{S}^{p^h}$ factors over a diagonalizable functor D, say, by Lemma 12.5. So multiplication by p on $T(\bar{Q}(\mathbb{Q}/\mathbb{Z}\otimes\mathcal{S}^{p^h}))$ factors through the complex $T(\bar{Q}D)$, which is acyclic by Step 1. Since multiplication by p on $T(\bar{Q}(\mathbb{Q}/\mathbb{Z}\otimes\mathcal{S}^{p^h}))$ is both a quasi-isomorphism and factors through an acyclic complex, $T(\bar{Q}(\mathbb{Q}/\mathbb{Z}\otimes\mathcal{S}^{p^h}))$ must itself be acyclic, and so the group $\mathbb{Q}/\mathbb{Z}$ is good.

The $\bar{Q}$–construction and the functor T commute with products. So the product of a family of good abelian groups is again good. In particular a product of any number of copies of the group $\mathbb{Q}/\mathbb{Z}$ is good. Every injective abelian group is a summand of a product of copies of $\mathbb{Q}/\mathbb{Z}$, hence injective abelian groups are good.

If A is a subgroup of an abelian group B, then the sequence of functors

$$0\longrightarrow A\otimes\mathcal{S}^{p^h}\longrightarrow B\otimes\mathcal{S}^{p^h}\longrightarrow (B/A)\otimes\mathcal{S}^{p^h}\longrightarrow 0$$

is exact. Since the cubical construction and the functor T are also exact, A is good as soon as B and B/A are. Since an arbitrary abelian group can be embedded into an injective abelian group with injective cokernel, every abelian group is good.

Step 3 Reformulation in terms of exterior power functors

Let $\Lambda^{p^h}\in\mathcal{F}$ denote the exterior power functor of degree p^h. The Koszul complex (see [25, 4.3.1.7] or [17, 3.2]) is an extension of length p^h-1 of the functor $\mathbb{Z}/p\otimes\Lambda^{p^h}$ by the functor $\mathbb{Z}/p\otimes\mathcal{S}^{p^h}$ with the special property that all functors occurring in the extension are diagonalizable. By Step 1, all these intermediate terms are sent to acyclic complexes by $T(\bar{Q}-)$. Since both T and

$\bar{Q}$ are exact, the complex $T(\bar{Q}(\mathbb{Z}/p \otimes \mathcal{S}^{p^h}))$ is acyclic if and only if the complex $T(\bar{Q}(\mathbb{Z}/p \otimes \Lambda^{p^h}))$ is acyclic.

The spareness assumption 12.3 (c) on the cohomology of $T(\mathcal{S}^{p^h})$ and the short exact sequence of functors

$$0 \longrightarrow \mathcal{S}^{p^h} \xrightarrow{\times p} \mathcal{S}^{p^h} \longrightarrow \mathbb{Z}/p \otimes \mathcal{S}^{p^h} \longrightarrow 0$$

imply that the cohomology of the complex $T(\mathbb{Z}/p \otimes \mathcal{S}^{p^h})$ is concentrated in dimensions congruent to 0 or 1 mod $2p^h$. The existence of the Koszul complex then shows that the cohomology of the complex $T(\mathbb{Z}/p \otimes \Lambda^{p^h})$ is concentrated in dimensions congruent to $p^h - 1$ or p^h mod $2p^h$.

Now we set up an induction on the exponent h. For the inductive step we use a certain complex of functors which relates the exterior power functor of degree p^h to that of degree p^{h-1}. Here and in what follows we extend the cubical construction to bounded below chain complexes of functors by applying the functor Q dimensionwise and taking the total complex of the resulting bicomplex. This extended Q–construction is still exact and preserves quasi-isomorphisms.

Step 4 There exists a complex C of functors from $\mathcal{F}$, concentrated in non-negative dimensions, with the following properties:

(a) In dimension zero, $C_0 = \mathbb{Z}/p \otimes \Lambda^{p^h}$ and the inclusion $\mathbb{Z}/p \otimes \Lambda^{p^h} \longrightarrow C$ induces quasi-isomorphisms

$$Q(\mathbb{Z}/p \otimes \Lambda^{p^h}) \longrightarrow QC \qquad \text{and} \qquad T(\mathbb{Z}/p \otimes \Lambda^{p^h}) \longrightarrow T(C) .$$

(b) All non-trivial homology functors of C are isomorphic to $\mathbb{Z}/p \otimes \Lambda^{p^{h-1}}$.

We let $\mathcal{F}(\mathbb{F}_p)$ denote the category of reduced functors from finitely generated $\mathbb{F}_p$–vector spaces to $\mathbb{F}_p$–vector spaces. We first construct a complex in the category $\mathcal{F}(\mathbb{F}_p)$; the desired complex is then obtained by composition with

$$- \otimes \mathbb{F}_p : \text{ (f.g. free abelian groups)} \longrightarrow \text{(f.g. } \mathbb{F}_p\text{–vector spaces)}$$

and the inclusion of $\mathbb{F}_p$–vector spaces into abelian groups.

For every $\mathbb{F}_p$–vector space V let $L^*(V)$ denote the quotient of the symmetric algebra on V by the ideal generated by all p-th powers of elements. Then $L^*(V)$ inherits the grading from the symmetric algebra and we let $L^n(V)$ denote the summand of homogenous degree n. If $p = 2$, then the functors L^n coincide with the exterior power functors Λ^n. By [16, 1.3.1] there exists a complex X (a quotient of the deRham complex) with

$$X_i \;=\; \begin{cases} L^i \otimes \Lambda_{\mathbb{F}_p}^{p^h-i} & \text{for } 0 \le i \le p^h \\ 0 & \text{else.} \end{cases}$$

whose only non-trivial homology is in dimension $p^{h-1}(p-1)$ where we have

$$H_{p^{h-1}(p-1)}X \;\cong\; \Lambda_{\mathbb{F}_p}^{p^{h-1}}$$

(we have reversed the grading of [16, 1.3.1] so that the differential decreases the dimension and X is a chain complex as opposed to a cochain complex). The complex X is part of the complex we are looking for, and we obtain the other part by dualization as follows.

The dual DF of a functor $F \in \mathcal{F}(\mathbb{F}_p)$ is defined by $DF(V) \;=\; F(V^\vee)^\vee$, where $V^\vee$ refers to the dual vector space of V. DF is again an object of the category $\mathcal{F}(\mathbb{F}_p)$. Dualization is contravariant and exact in F and it satisfies $D(F \otimes G) \cong DF \otimes DG$. The exterior power functors and the functors L^n are self-dual, ie, there are isomorphisms $D\Lambda^n_{\mathbb{F}_p} \cong \Lambda^n_{\mathbb{F}_p}$ and $DL^n \cong L^n$. So if we dualize the complex X we obtain a complex DX which is concentrated in dimensions $-p^h$ through 0, which satisfies $(DX)_0 \cong \Lambda^{p^h}_{\mathbb{F}_p}$, $(DX)_{-p^h} \cong L^{p^h}$ and whose only non-trivial homology is

$$H_{-p^{h-1}(p-1)}DX \;\cong\; D\Lambda_{\mathbb{F}_p}^{p^{h-1}} \;\cong\; \Lambda_{\mathbb{F}_p}^{p^{h-1}} \,.$$

The desired complex C is now obtained by splicing infinitely many copies of the complexes X and DX alternatingly at L^{p^h} and $\Lambda^{p^h}_{\mathbb{F}_p}$, and then passing from the complex in $\mathcal{F}(\mathbb{F}_p)$ to a complex in $\mathcal{F}$. More precisely,

$$\begin{aligned} C_0 \;&=\; \Lambda^{p^h}_{\mathbb{F}_p} \\ C_{n(p^h-1)+i} \;&=\; \begin{cases} X_i & \text{for } 1 \le i \le p^h-1 \text{ and } n \ge 0 \text{ even,} \\ DX_{p^h-i} & \text{for } 1 \le i \le p^h-1 \text{ and } n \ge 0 \text{ odd.} \end{cases} \end{aligned}$$

The map $Q(\mathbb{Z}/p \otimes \Lambda^{p^h}) \longrightarrow QC$: Let C/C_0 denote the quotient complex which is equal to C except in dimension zero, where it is trivial. The complex C/C_0 is bounded below and consists entirely of diagonalizable functors. Since the Q–construction of a diagonalizable functor is acyclic (8.9 (e)), the complex $Q(C/C_0)$ is also acyclic. Since Q is exact, the map $Q(\mathbb{Z}/p \otimes \Lambda^{p^h}) \longrightarrow QC$ is a quasi-isomorphism.

The map $T(\mathbb{Z}/p \otimes \Lambda^{p^h}) \longrightarrow T(C)$: The fact that this map is a quasi-isomorphism is a consequence of the sparseness assumption 12.3 (c) on T. In more detail: the complex C/C_0 can be written as the inverse limit of the tower of truncated complexes K^n defined by

$$(K^n)_i \;=\; \begin{cases} C_i & \text{if } 1 \le i \le 2n(p^h-1), \\ \mathbb{Z}/p \otimes \Lambda^{p^h} & \text{if } i = 2n(p^h-1)+1, \\ 0 & \text{else.} \end{cases}$$

The boundary maps of K^n are those of C and the inclusion of the cycles

$$\mathbb{Z}/p \otimes \Lambda^{p^h} \;\cong\; D(X_0) \;\longrightarrow\; D(X_1) \;=\; C_{2n(p^h-1)}\;.$$

The chain map $K^{n+1} \longrightarrow K^n$ is the identity up to dimension $2n(p^h-1)$ and the epimorphism

$$C_{2n(p^h-1)+1} \;=\; X_1 \;\longrightarrow\; X_0 \;=\; \mathbb{Z}/p \otimes \Lambda^{p^h}\;.$$

in dimension $2n(p^h-1)+1$.

For each n, the kernel of the projection $K^n \longrightarrow (\mathbb{Z}/p \otimes \Lambda^{p^h})[2n(p^h-1)+1]$ onto the top functor is a bounded complex of diagonalizable functors. Since T is exact and takes diagonalizable functors to acyclic complexes, the induced map

$$T(K^n) \;\xrightarrow{\;\sim\;}\; T((\mathbb{Z}/p \otimes \Lambda^{p^h})[2n(p^h-1)+1])$$

is a quasi-isomorphism.

By the sparseness assumptions and Step 3, the cohomology of $T(\mathbb{Z}/p \otimes \Lambda^{p^h})$ is concentrated in dimensions congruent to p^h-1 and p^h mod $2p^h$. Hence the cohomology of $T(K^n)$ is concentrated in dimensions congruent to $-2n+p^h$ and $-2n+p^h+1$ mod $2p^h$. Hence the surjective map $T(K^{n+1}) \longrightarrow T(K^n)$ induces trivial maps on cohomology groups for dimensional reasons. Since T commutes with inverse limits, $T(C/C_0)$ is the inverse limit of the tower of complexes $T(K^n)$ for $n \ge 1$, so $T(C/C_0)$ is acyclic.

Example 12.6 It might be instructive to describe the complex C just constructed in the smallest non-trivial case, namely for $p=2$ and $h=1$. Then C is the mod 2 reduction of an 'integral' complex $\widetilde{C}$ defined by

$$\widetilde{C}_i \;=\; \begin{cases} 0 & \text{if } i < 0, \\ \Lambda^2 & \text{if } i = 0, \text{ and} \\ I \otimes I & \text{if } i > 0. \end{cases}$$

The differential $d_i\colon \widetilde{C}_i \longrightarrow \widetilde{C}_{i-1}$ is given by

$$d_i(x \otimes y) \;=\; \begin{cases} x \wedge y & \text{if } i = 1, \\ x \otimes y + y \otimes x & \text{if } i \ge 2 \text{ and } i \text{ is even}, \\ x \otimes y - y \otimes x & \text{if } i \ge 2 \text{ and } i \text{ is odd.} \end{cases}$$

The homology of the complex $\widetilde{C}$ is given by

$$H_i\widetilde{C} \;\cong\; \begin{cases} \mathbb{Z}/2\otimes I & \text{if } i\geq 1 \text{ and } i \text{ is odd,}\\ 0 & \text{else.}\end{cases}$$

Since the homology functors of $\widetilde{C}$ are additive and since $\widetilde{C}$ is diagonalizable in positive dimensions, $\widetilde{C}$ is quasi-isomorphic to the cubical construction $Q\Lambda^2$. Hence by 8.9 (b) we can read off the Dold–Puppe stable derived functors of Λ^2 as

$$L_i^{\mathrm{st}}\Lambda^2 \;\cong\; H_i\widetilde{C}(\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}/2 & \text{if } i\geq 1 \text{ and } i \text{ is odd,}\\ 0 & \text{else,}\end{cases}$$

compare e.g. [50]. The complex C constructed in Step 4 for $p=2$ and $h=1$ is isomorphic to the reduction $\mathbb{Z}/2\otimes\widetilde{C}$; by the universal coefficient theorem, the homology functors of C are thus isomorphic to the functor $\mathbb{Z}/2\otimes I$ in every positive dimension, and trivial otherwise.

Step 5 The complex $T(\bar{Q}(\mathbb{Z}/p\otimes\Lambda^{p^h}))$ is acyclic for all $h\geq 0$.

We proceed by induction. For $h=0$ we have $\mathbb{Z}/p\otimes\Lambda^1=\mathbb{Z}/p\otimes I$ which is an additive functor. Thus all cross-effect vanish and by property 8.9 (c) of the cubical construction, the complex $\bar{Q}(\mathbb{Z}/p\otimes\Lambda^1)$ is trivial. Hence $T(\bar{Q}(\mathbb{Z}/p\otimes\Lambda^1))$ is also trivial.

Now suppose that $h\geq 1$ and assume that $T(\bar{Q}(\mathbb{Z}/p\otimes\Lambda^{p^{h-1}}))$ is already known to be acyclic. Let C be any complex as in Step 4. We consider the commutative square of bounded below chain complexes of functors

$$\begin{tikzcd} \mathbb{Z}/p\otimes\Lambda^{p^h} \arrow[r] \arrow[d] & C \arrow[d] \\ Q(\mathbb{Z}/p\otimes\Lambda^{p^h}) \arrow[r] & QC \end{tikzcd}$$

where we view the functor $\mathbb{Z}/p\otimes\Lambda^{p^h}$ as a complex concentrated in dimension zero and the horizontal maps are induced by the inclusion $\mathbb{Z}/p\otimes\Lambda^{p^h}\longrightarrow C$.

By property (a) of the complex C and since T preserves quasi-isomorphisms, both rows of the square induce a quasi-isomorphism after applying T. Since T is exact, the map

$$T(\bar{Q}(\mathbb{Z}/p\otimes\Lambda^{p^h})) \;\longrightarrow\; T(\bar{Q}C)$$

is thus a quasi-isomorphism. So in order to finish the induction step, it remains to show that the complex $T(\bar{Q}C)$ is acyclic.

This last step is where the induction hypothesis is used. The complex C is the inverse limit of its Postnikov tower (homological truncations)

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_{-1} = 0 .$$

In the tower each map $P_n \longrightarrow P_{n-1}$ is a surjection whose kernel is quasi-isomorphic to the n-th homology functor of C concentrated in dimension n. By property (b) of the complex C all non-trivial homology functors are isomorphic to the exterior power functor $\mathbb{Z}/p \otimes \Lambda^{p^{h-1}}$, for which we already know that the map $T(\bar{Q}(\mathbb{Z}/p \otimes \Lambda^{p^{h-1}}))$ is acyclic. Since the Q–construction and T are exact we conclude by induction that for all $n \geq 0$ the complex $T(\bar{Q}P_n)$ is acyclic.

The Postnikov tower consists of bounded below chain complexes and it stabilizes in each dimension. So the complex $\bar{Q}C$ is the inverse limit of the complexes $\bar{Q}P_n$. Since T commutes with the inverse limits, $T(\bar{Q}C)$ is the inverse limits of the acyclic complexes $T(\bar{Q}P_n)$. Since T and $\bar{Q}$ also preserve epimorphisms, this inverse limit is acyclic.

13 Perspectives

We end the paper with an application of Theorem 5.2 which concerns an interesting homotopical property of the Gamma-ring DB. Then we discuss some variants of the Construction 3.5 of Gamma-ring maps from formal group laws, and some possible directions for further investigation.

For the application we use the conjugation action to obtain an obstruction to the existence of k–algebra structures on Gamma-rings. With this tool we then show that the Gamma-ring DB is not stably equivalent to the Eilenberg–MacLane Gamma-ring of any simplicial ring (unless B is a $\mathbb{Q}$–algebra). To motivate the criterion we look at the classical case of discrete rings first. If k is a commutative ring and A any associative ring, then the k–algebra structures on A correspond to the central ring maps $k \longrightarrow A$. In particular, the unit map of every such k–algebra structure gives an element of the set $\mathrm{Ring}(k, A)$ of ring maps which is a fixed point of the conjugation action of the units of A. Something similar happens for Gamma-rings. Suppose R is a Gamma-ring which is stably equivalent to an algebra over the commutative Gamma-ring k. Any chain of equivalences to a k–algebra determines a homotopy class of Gamma-ring maps $[\eta] \in [k, R]_{\mathrm{Ho}\,\mathcal{GR}}$ underlying the unit map of the algebra structure.

Theorem 13.1 *Suppose $\eta \in \mathrm{Ring}(k,R)$ is a Gamma-ring map whose homotopy class underlies a k–algebra structure of R. Then the conjugation action map*

$$UR^{\times} \longrightarrow \mathrm{Ring}(k,R) \quad , \quad u \mapsto u \cdot \eta \cdot u^{-1}$$

is null-homotopic. So if the conjugation action map is essential for every component of the space $\mathrm{Ring}(k,R)$ of Gamma-ring maps, then R is not stably equivalent to any k–algebra.

Proof We can assume that R is itself a stably fibrant k–algebra. Then Construction 4.2 can be done in the category of k–algebras, as opposed to Gamma-rings, relative to the trivial group (Lemma 4.4 is also valid in the category of k–algebras). We obtain a diagram of k–algebras

$$\begin{tikzcd} k \arrow[r, equal] \arrow[d] & k \arrow[r] \arrow[d] & k[UR^{\times}] \arrow[d] \\ R & R_1 \arrow[l, two heads, "\sim"'] \arrow[r, "\sim"'] & R_3 . \end{tikzcd}$$

Ignoring the k–algebra structure we can use the objects in this diagram to model the space of Gamma-ring maps from k to R. More precisely, the space $\hom_{\mathcal{GR}}(k^{c}, R_3)$ admits a conjugation action by the simplicial group $UR^{\times}$ and this action is equivalent to the one in question (here k^{c} is a cofibrant replacement of k as a Gamma-ring). But in this model for the conjugation action, the composite of the approximation map $k^{c} \longrightarrow k$ with the unit map $k \longrightarrow R_3$ of the k–algebra structure on R_3 is a point-set level fixed point of the conjugation action of $UR^{\times}$. □

If we combine the previous result with Theorem 5.2 we can deduce that the Gamma-ring DB is not stably equivalent to the Eilenberg–MacLane Gamma-ring of any simplicial ring, unless B is an algebra over the rational numbers. This should be compared to Theorem 3.2 (b) which says that as a Γ–space, DB is stably equivalent to the smash product $H\mathbb{Z} \wedge^{L} HB$. In other words, DB 'additively' decomposes into a product of Eilenberg–MacLane Γ–spaces. In contrast the following corollary shows that the multiplicative structure of DB is genuinely homotopy-theoretic. The idea of the proof is that any stable equivalence between DB and a simplicial ring would give DB the structure of an $H\mathbb{Z}$–algebra. Such an algebra structure in turn gives rise to a "central" Gamma-ring map $H\mathbb{Z} \longrightarrow DB$ (in the sense of Theorem 13.1). But Theorem 5.2 identifies all Gamma-ring map from $H\mathbb{Z}$ to DB and shows that none of them is "central".

Corollary 13.2 *For a commutative ring B the following conditions are equivalent.*

(1) *B is an algebra over the rational numbers.*

(2) *The projection $DB \longrightarrow D_1B \cong HB$ is a stable equivalence of Gamma-rings.*

(3) *DB is stably equivalent (as a Gamma-ring) to the Eilenberg–MacLane Gamma-ring of a simplicial ring.*

Proof Condition (1) is equivalent to condition (2) by Theorem 3.2 (b), and condition (2) implies condition (3). The proof that condition (3) implies condition (2) is a combination of Theorem 13.1, in the case $k = H\mathbb{Z}$, with Theorem 5.2. Assume that condition (3) holds. Since the Eilenberg–MacLane Gamma-ring of a simplicial ring is an $H\mathbb{Z}$–algebra, there exist a component of the space $\mathrm{Ring}(H\mathbb{Z}, DB)$ of Gamma-ring maps for which the conjugation action map $DB^{\times} \longrightarrow \mathrm{Ring}(H\mathbb{Z}, DB)$ is homotopically trivial, by Theorem 13.1. But this map is part of a homotopy fiber sequence

$$(DB)_1^{\times} \longrightarrow \mathrm{Ring}(H\mathbb{Z}, DB) \longrightarrow \mathrm{Ring}(H\mathbb{Z}, DB)/\mathrm{conj}.$$

Since the base of this fibration is weakly equivalent to the classifying space of a groupoid (Theorem 5.2), its homotopy groups are trivial above dimension 1. So the conjugation action map is injective on homotopy groups in positive dimensions. Since the map is also null-homotopic, the space $(DB)_1^{\times}$ must be weakly contractible, which implies condition (2). □

13.3 Coordinate free definition The definition of the Gamma-ring DB and the Gamma-ring map F_* depended on a formal group *law* F, ie, on a 1–dimensional commutative formal group with a choice of coordinate. We will now describe coordinate-free versions of these constructions which at the same time are defined in a more general context.

As input we consider a category $\mathcal{C}$ which has a zero object and finite coproducts. The natural enrichment of $\mathcal{C}$ over the category Γ^{op} of finite pointed sets is given by

$$X \wedge k^+ \;=\; \underbrace{X \amalg \cdots \amalg X}_{k} \qquad (\text{coproduct in } \mathcal{C}).$$

Every object X of $\mathcal{C}$ has an *endomorphism Gamma-ring* [46, 4.6], denoted $\mathrm{End}_{\mathcal{C}}(X)$ and defined by

$$\mathrm{End}_{\mathcal{C}}(X)(k^+) \;=\; \mathrm{Hom}_{\mathcal{C}}(X, X \wedge k^+) \;.$$

The unit map $\mathbb{S} \longrightarrow \mathrm{End}_{\mathcal{C}}(X)$ comes from the identity map of X, viewed as a point in $\mathrm{End}_{\mathcal{C}}(X)(1^+)$, and the multiplication $\mathrm{End}_{\mathcal{C}}(X) \wedge \mathrm{End}_{\mathcal{C}}(X) \longrightarrow \mathrm{End}_{\mathcal{C}}(X)$ is induced by the composition product

$$\begin{array}{ccc} \mathrm{End}_{\mathcal{C}}(X)(k^+) \wedge \mathrm{End}_{\mathcal{C}}(X)(l^+) & \longrightarrow & \mathrm{End}_{\mathcal{C}}(X)(k^+ \wedge l^+) \ , \\ f \wedge g & \longmapsto & (f \wedge l^+) \circ g \ . \end{array}$$

As an example we can take $\mathcal{C}$ to be the category of commutative, complete augmented B–algebras. If we choose the object X to be the power series ring on one generator, then the endomorphism Gamma-ring of X is precisely DB.

Now we suppose that the object X of $\mathcal{C}$ is equipped with the structure of abelian cogroup object. So there is a given co-addition map $X \longrightarrow X \amalg X$ and a co-inverse map $X \longrightarrow X$ which make the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ into an abelian group, natural for all objects Y of $\mathcal{C}$. Every abelian cogroup structure on X gives rise to a homomorphism of Gamma-rings $H\mathbb{Z} \longrightarrow \mathrm{End}_{\mathcal{C}}(X)$ as follows. At a finite pointed set k^+ the map

$$H\mathbb{Z}(k^+) \ = \ \widetilde{\mathbb{Z}}[k^+] \ \longrightarrow \ \mathrm{Hom}_{\mathcal{C}}(X, X \wedge k^+) \ = \ \mathrm{End}_{\mathcal{C}}(X)(k^+)$$

is the additive extension of the map that sends $i \in k^+$ to the i-th coproduct inclusion $X \longrightarrow X \wedge k^+$. When $\mathcal{C}$ is the category of commutative, complete augmented B–algebras and X is the power series ring on one generator, then making X into an abelian cogroup object is the same thing as giving a (1–dimensional, commutative) formal group law F over B. Furthermore, in this case the map $H\mathbb{Z} \longrightarrow \mathrm{End}_{\mathcal{C}}(X)$ arising from the abelian cogroup structure corresponds to the map F_* of Construction 3.5 under the identification $DB \cong \mathrm{End}_{\mathcal{C}}(X)$. So from this point of view construction 3.5 is just a special case of the fact that every abelian cogroup structures gives rise to a homomorphism from the Eilenberg–MacLane Gamma-ring $H\mathbb{Z}$ to an endomorphism Gamma-ring.

13.4 Non-commutative formal group laws Construction 3.5 can be modified to work for not necessarily commutative formal group laws, but this variant does not lead to any interesting phenomena. There is a Gamma-ring, denoted by Gp, which is constructed the same way $H\mathbb{Z}$ is, but with free groups instead of free abelian groups. So as a Γ–space, Gp takes a pointed set to the reduced free group it generates. The multiplication again comes from substitution, this times words in the generators of the free groups are substituted into each other. Abelianization gives a Gamma-ring map $Gp \longrightarrow H\mathbb{Z}$. If F is a 1–dimensional but not necessarily commutative formal group law over the commutative ring B, then it gives rise to a Gamma-ring map

$$F_* : \ Gp \ \longrightarrow \ DB$$

in much the same way as in Construction 3.5. The Gamma-ring map F_* factors over $H\mathbb{Z}$ if and only if the formal group law F is commutative.

While the construction makes sense, Gp is uninteresting as a source of Gamma-ring homomorphisms: we claim that the unit map $\mathbb{S} \longrightarrow Gp$ from the sphere Gamma-ring is a stable equivalence. This claim follows from the fact that the map from a high dimensional sphere into the free group it generates is an equivalence in the stable range. Since the Gamma-ring Gp is stably equivalent to the initial Gamma-ring, the derived space of homomorphisms into any other Gamma-ring is contractible.

13.5 Higher dimensional formal group laws Another variant of Construction 3.5 proceeds from an n–dimensional commutative formal group law F. This time the construction gives a weak Gamma-ring map

$$F_* : \; H\mathbb{Z} \;\longrightarrow\; M_n(DB)$$

into the Gamma-ring of $n \times n$–matrices over DB. For an arbitrary Gamma-ring R the Gamma-ring $M_n(R)$ of $n \times n$–matrices over R is defined as the endomorphism Gamma-ring of the free R–module on n generators, ie,

$$M_n(R) \;=\; \mathrm{Hom}_{R\text{-mod}}(R \wedge n^+, R \wedge n^+)$$

(here Hom_R refers to the internal homomorphism Γ–space in the category of R–modules). We define another Gamma-ring $\bar{M}_n(DB)$ as the endomorphism Gamma-ring, in the sense of 13.3, of the power series ring in n generators in the category of augmented, complete B–algebras. Then there is a stable equivalence $M_n(DB) \longrightarrow \bar{M}_n(DB)$. Since an n–dimensional commutative formal group law F is the same thing as an abelian cogroup structure on the power series ring in n variables, it leads to a map of Gamma-rings $F_*\colon H\mathbb{Z} \longrightarrow \bar{M}_n(DB)$.

13.6 Formal module structures Yet another variation of our main theme consists in considering formal module structures over an associative ring R. A *(1–dimensional) formal R–module (law)* over a commutative ring B consists of a (1–dimensional and commutative) formal group law F and a ring homomorphism from R into the endomorphism ring of the formal group law F. In the spirit of Construction 3.5, any formal R–module structure F over B gives rise to a Gamma-ring map $F_*\colon HR \longrightarrow DB$ with source the Eilenberg–MacLane Gamma-ring of R.

By the same method as in Section 5 we obtain a map

$$\kappa_R : \; \widetilde{\mathcal{F}R\text{-mod}}^{\,\mathrm{str}}(B) \;\longrightarrow\; \mathrm{Ring}(HR, DB)/\mathrm{conj}.$$

from the groupoid of formal R–module structures over B and strict isomorphisms to the homotopy orbits of the derived space of Gamma-ring maps by the connected component of the homotopy units $DB^{\times}$. It seems reasonable to expect that the map κ_R is again a weak equivalence; whether this is the case depends on whether the appropriate analog of Theorem 12.1 holds over the ring R.

We denote by $R^{\mathrm{com}} = R/(rs - sr)$ the quotient of R by the commutator ideal. Then the arguments of Sections 6 through 11 can be adapted to show:

Theorem 13.7 *Suppose that the ring R has the following property: for all $m \in \mathbb{Z}$, all $k \geq 0$ and R^{com}–modules A the map*

$$\mathrm{Ext}^m_{\mathcal{F}(R^{\mathrm{com}})}(I, A \otimes_{R^{\mathrm{com}}} \mathcal{S}^k) \longrightarrow \mathbb{E}\mathrm{xt}^m_{\mathcal{F}(R^{\mathrm{com}})}(I, Q(A \otimes_{R^{\mathrm{com}}} \mathcal{S}^k))$$

is an isomorphism. Then the map

$$\kappa_R : \widetilde{\mathcal{F}R\text{-mod}}^{\mathrm{str}}(B) \longrightarrow \mathrm{Ring}(HR, DB)/\mathrm{conj}.$$

is a weak equivalence of simplicial sets.

The hypothesis is true for $R = \mathbb{Z}$ — this is the content of Theorem 12.1. The proof of Theorem 12.1 can be adapted to establish the hypothesis for $R = \mathbb{Z}/n$. We conjecture that indeed the hypothesis holds in general; we expect that a 'good' proof of Theorem 12.1, ie, a proof that does not use the calculations of the MacLane cohomology groups $\mathrm{Ext}^*_{\mathcal{F}}(I, \mathcal{S}^k)$ as input, would also work in the more general context. If this is the case, the map κ_R is a weak equivalence for any ring R.

Example 13.8 Suppose B is an $\mathbb{F}_p$–algebra and F a formal group law over F. In this case we can reinterpret the height of F in terms of the homotopy class of the Gamma-ring map F_*.

The p–series of F is either trivial or of the form

$$[p]_F(x) \;=\; u \cdot x^{p^h} \;+\; \text{terms of higher degree}$$

for some $h \geq 1$ and some non-zero $u \in B$. The number h is called the *height* of F. If $[p]_F = 0$, then F is isomorphic to the additive formal group law [19, III.1 Cor. 2], and the height of F is infinite.

Claim The height of F is equal to the largest number h such that

$$H\mathbb{Z} \xrightarrow{F_*} DB \longrightarrow D_{p^h-1}B$$

can be factored, in the homotopy category of Gamma-rings, over the Eilenberg–MacLane Gamma-ring for $\mathbb{F}_p$.

Indeed, if F has height h, then for every pointed set K the map

$$F_*(K): \ \widetilde{\mathbb{Z}}[K] = H\mathbb{Z}(K) \longrightarrow DB(K) \subseteq \widetilde{B}[\![K]\!]$$

satisfies

$$F_*(p \cdot x) = [p]_F(F_*(x)) \equiv 0 \qquad \text{modulo degree } p^h.$$

So the composite map

$$H\mathbb{Z} \xrightarrow{F_*} DB \longrightarrow D_{p^h-1}B$$

factors uniquely over $H\mathbb{F}_p$ on the point-set level. Conversely, suppose that there exists a commutative square in the homotopy category of Gamma-rings

$$\begin{tikzcd} H\mathbb{Z} \arrow[r, "F_*"] \arrow[d] & DB \arrow[d] \\ H\mathbb{F}_p \arrow[r, dashed] & D_{p^h-1}B \end{tikzcd}$$

By the analog of Theorem 6.4 for buds of formal $\mathbb{F}_p$–modules (which holds since the hypothesis of Theorem 13.7 are satisfied for $R = \mathbb{F}_p$), the maps from $H\mathbb{F}_p \longrightarrow D_{p^h-1}B$ in the homotopy category of Gamma-rings are in bijective correspondence with strict isomorphism classes of p^h–buds of formal $\mathbb{F}_p$–module structures on B. But over an $\mathbb{F}_p$–algebra every formal $\mathbb{F}_p$–module bud is strictly isomorphic to the additive formal group law [19, III.1 Cor. 2]. Hence the $(p^h - 1)$–buds of F and of the additive formal group law over B induce the same maps $H\mathbb{Z} \longrightarrow D_{p^h-1}B$ in the homotopy category of Gamma-rings. By Theorem 6.4, the $(p^h - 1)$–bud of F is thus strictly isomorphic to the $(p^h - 1)$–bud of the additive formal group law, and so $[p]_F \equiv 0$ modulo degree $p^h - 1$. So the height of F is at least h.

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