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An exotic smooth structure on $\mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$

András I Stipsicz
Zoltán Szabó

Rényi Institute of Mathematics, Hungarian Academy of Sciences
H-1053 Budapest, Reáltanoda utca 13–15, Hungary
and
Institute for Advanced Study, Princeton, NJ 08540, USA
Email: `stipsicz@renyi.hu`, `stipsicz@math.ias.edu`

Department of Mathematics, Princeton University
Princeton, NJ 08544, USA
Email: `szabo@math.princeton.edu`

Abstract

We construct smooth 4–manifolds homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$.

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1 Introduction

Based on work of Freedman [7] and Donaldson [2], in the mid 80's it became possible to show the existence of exotic smooth structures on closed simply connected 4–manifolds. On one hand, Freedman's classification theorem of simply connected, closed topological 4–manifolds could be used to show that various constructions provide homeomorphic 4–manifolds, while the computation of Donaldson's instanton invariants provided a smooth invariant distinguishing appropriate examples up to diffeomorphism, see [3] for the first such computation. For a long time the pair $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$ (the complex projective plane blown up at eight points) and a certain algebraic surface (the Barlow surface) provided such a simply connected pair with smallest Euler characteristic [12]. Recently, by a clever application of the rational blow-down operation originally introduced by Fintushel and Stern [4], Park found a smooth 4–manifold homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#7\overline{\mathbb{CP}^2}$ [17]. Applying a similar rational blow-down construction we show the following:

Theorem 1.1 *There exists a smooth 4–manifold X which is homeomorphic to $\mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$ but not diffeomorphic to it.*

Note that X has Euler characteristic $\chi(X)=9$, and thus provides the smallest known closed exotic simply connected smooth 4–manifold. The proof of Theorem 1.1 involves two steps. First we will construct a smooth 4–manifold X and determine its fundamental group and characteristic numbers. Applying Freedman's theorem, we conclude that X is homeomorphic to $\mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$. Then by computing the Seiberg–Witten invariants of X we show that it is not diffeomorphic to $\mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$. By determining all Seiberg–Witten basic classes of X we can also show that it is minimal. This result, in conjunction with the result of [15] gives:

Corollary 1.2 *Let $n\in\{6,7,8\}$. Then there are at least $n-4$ different smooth structures on the topological manifolds $\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$. The different smooth 4–manifolds $Z_1(n), Z_2(n), \ldots, Z_{n-4}(n)$ homeomorphic to $\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$ have $0, 2, \ldots, 2^{n-5}$ Seiberg–Witten basic classes, respectively.* □

In Section 2 we give several constructions of exotic smooth structures on the topological 4–manifold $\mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$ by rationally blowing down various configurations of chains of 2–spheres. Since the generalized rational blow-down operation is symplectic when applied along symplectically embedded spheres (see [19]), the 4–manifolds that are constructed here all admit symplectic structures.

The computation of their Seiberg–Witten basic classes show that they are all minimal symplectic 4–manifolds with isomorphic Seiberg–Witten invariants. It is not known whether these examples are diffeomorphic to each other.

It is interesting to note that any two minimal symplectic 4–manifolds on the topological manifold $\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$ $n \in \{1,\dots,8\}$ have (up to sign) identical Seiberg–Witten invariants. As a corollary, Seiberg–Witten invariants can tell apart only at most finitely many symplectic structures on the topological manifold $\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$ with $n \le 8$.

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2 The topological constructions

In constructing the 4–manifolds encountered in Theorem 1.1 we will apply the generalized rational blow-down operation [16] to certain configurations of spheres in rational surfaces. In order to locate the particular configurations, we start with a special elliptic fibration on $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$. The proof of the following proposition is postponed to Section 5. (For conventions and constructions see [9].)

Proposition 2.1 *There is an elliptic fibration* $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$ *with a singular fiber of type* III**, three fishtail fibers and two sections.* □

The type III* singular fiber (also known as the $\tilde{E}_7$ singular fiber) can be given by the plumbing diagram of Figure 1. (All spheres in the plumbing have self–intersection equal to -2.) If $h, e_1, \dots, e_9$ is the standard generating system of $H_2(\mathbb{CP}^2\#9\overline{\mathbb{CP}^2};\mathbb{Z})$ then the elliptic fibration can be arranged so that the homology classes of the spheres in the III* fiber are equal to the classes given in Figure 1. We also show in Section 5 that the two sections can be chosen to intersect the spheres in the left and the right ends of Figure 1, respectively.

[1] After the submission of this paper the results of Theorem 1.1 and Corollary 1.2 have been improved by finding infinitely many exotic smooth structures on $\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$ for $n \ge 5$, see [6, 18].

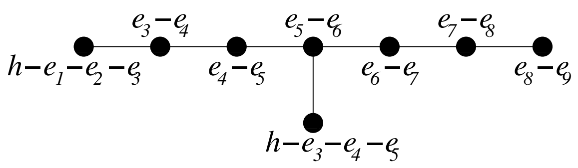

Figure 1: Plumbing diagram of the singular fiber of type III*

2.1 Generalized rational blow-down

Let $L_{p,q}$ denote the lens space $L(p^2, pq-1)$, where $p \geq q \geq 1$ and p, q are relatively prime. Let $C_{p,q}$ denote the plumbing 4–manifold obtained by plumbing 2–spheres along the linear graph with decorations $d_i \leq -2$ given by the continued fractions of $-\frac{p^2}{pq-1}$; we have the obvious relation $\partial C_{p,q} = L_{p,q}$, cf also [16]. Let $K \in H^2(C_{p,q}; \mathbb{Z})$ denote the cohomology class which evaluates on each 2–sphere of the plumbing diagram as $d_i + 2$.

Proposition 2.2 [1, 16, 19] *The 3–manifold $\partial C_{p,q} = L(p^2, pq-1)$ bounds a rational ball $B_{p,q}$ and the cohomology class $K|_{\partial C_{p,q}}$ extends to $B_{p,q}$.* □

The following proposition provides embeddings of some of the above plumbings into rational surfaces.

Proposition 2.3
- *The 4–manifold $C_{28,9}$ embeds into $\mathbb{CP}^2\#17\overline{\mathbb{CP}^2}$;*
- *$C_{46,9}$ embeds into $\mathbb{CP}^2\#19\overline{\mathbb{CP}^2}$, and finally*
- *$C_{64,9}$ embeds into $\mathbb{CP}^2\#21\overline{\mathbb{CP}^2}$.*

Remark 2.4 The linear plumbings giving the configurations considered above are as follows:

- $C_{28,9} = (-2,-2,-12,-2,-2,-2,-2,-2,-2,-2,-4)$,
- $C_{46,9} = (-2,-2,-2,-2,-12,-2,-2,-2,-2,-2,-2,-2,-6)$ and
- $C_{64,9} = (-2,-2,-2,-2,-2,-2,-12,-2,-2,-2,-2,-2,-2,-2,-8)$.

Proof Let us consider an elliptic fibration on $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$ with a type III* singular fiber, three fishtail fibers F_1, F_2, F_3 and two sections s_1, s_2 as described by the schematic diagram of Figure 2. Let A_i denote the intersection of F_i with the section s_2, while B_i denotes the intersection of the fiber F_i with s_1 $(i = 1, 2, 3)$. First blow up the three double points (indicated by small circles)

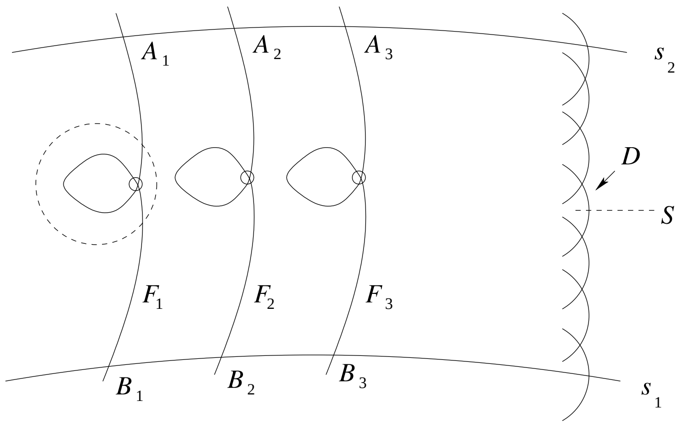

Figure 2: Singular fibers in the fibration

of the three fishtail fibers. To get the first configuration, further blow up at A_1, A_2, A_3 and smooth the transverse intersections B_1, B_2, B_3. Finally, apply two more blow-ups inside the dashed circle as shown by Figure 3. By counting the number of blow-ups, the desired embedding of $C_{28,9}$ follows.

In a similar way, now blow up A_1, A_2, B_3, and smooth B_1, B_2 and A_3. Four further blow-ups in the manner depicted by Figure 3 provides the embedding of $C_{46,9}$.

Finally, by blowing up A_1, B_2, B_3, and smoothing B_1, A_2 and A_3, and then performing six further blow-ups as before inside the dashed circle, we get the embedding of $C_{64,9}$ as claimed. □

Lemma 2.5 *For $i = 0, 1, 2$ the embedding $C_{28+18i,9} \subset \mathbb{CP}^2\#(17+2i)\overline{\mathbb{CP}^2}$ found above has simply connected complement.*

Proof Since rational surfaces are simply connected, the simple connectivity of the complement follows once we show that a circle in the boundary of the complement is homotopically trivial. Recall that, since the boundary of the complement is a lens space, it has cyclic fundamental group. In conclusion, homotopic triviality needs to be checked only for the generator of the fundamental group of the boundary. We claim that the normal circle to the (-2)–framed sphere D in the III* fiber intersected by the dashed (-2)–curve S of Figure 2 (which is in the III* fiber but not in our chosen configuration) is a generator of the fundamental group of the boundary 3–manifold. This observation easily

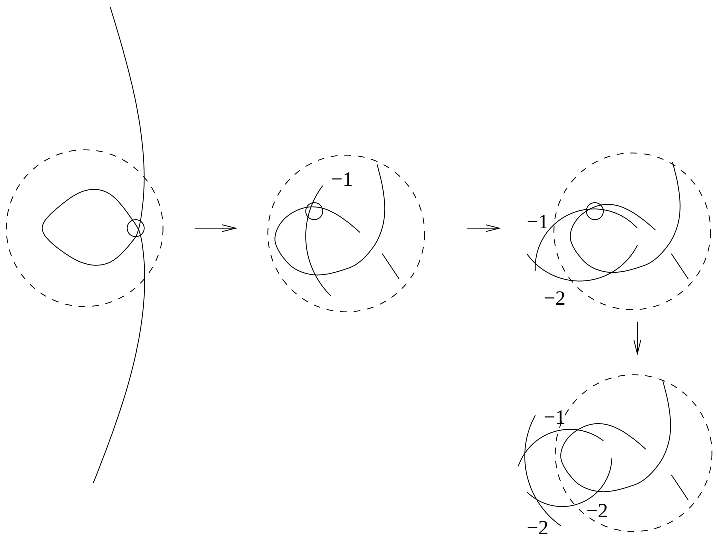

Figure 3: Further blow-ups of the fishtail fiber

follows from the facts that for the boundary lens space the first homology is naturally isomorphic to the fundamental group, and in the first homology the normal circle in question is $13+8i$ times the generator given by the linking normal circle of the last sphere of the configuration ($i=0,1,2$). Since for $i=0,1,2$ we have that $13+8i$ is relatively prime to $28+18i$, and the hemisphere of S in the complement shows that the normal circle of D is homotopically trivial in the complement, the proof of the lemma follows. □

Remark 2.6 It is not hard to show that the 3–manifold $\partial C_{28,9}$ does bound a rational ball: we can embed $C_{28,9}$ into $11\overline{\mathbb{CP}^2}$ and the closure of the complement of the embedding (with reversed orientation) can be easily seen to be an appropriate rational ball. In turn, the embedding $C_{28,9} \subset 11\overline{\mathbb{CP}^2}$ results from the following observation. Attach a 4–dimensional 2–handle to $C_{28,9}$ along the (-1)–framed unknot K indicated by the plumbing diagram of Figure 4. By subsequently sliding down the (-1)–framed unknots we arrive at a 0–framed unknot, showing that the handle attachment along K embeds $C_{28,9}$ into a 4–manifold diffeomorphic to the connected sum $S^2\times D^2\#11\overline{\mathbb{CP}^2}$. By attaching a 3– and a 4–handle to this 4–manifold we get a closed 4–manifold diffeomorphic to $11\overline{\mathbb{CP}^2}$. The appropriate modification of the procedure gives the rational balls $B_{46,9}$ and $B_{64,9}$. In fact, by adding a cancelling 1–handle to K, doing surgery along it, following the resulting 0–framed unknot during the blow-

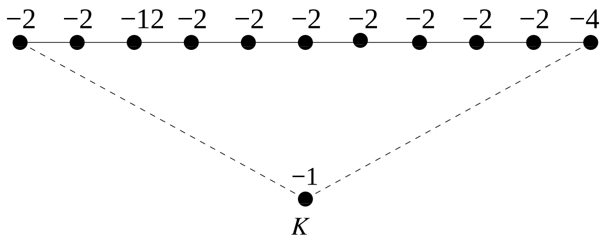

Figure 4: Plumbing diagram of the 4–manifold $C_{28,9}$

downs of the (-1)–spheres as described above, and then doing another surgery along the resulting 0–framed circle, we arrive at an explicit surgery description of the 4–manifold $B_{28,9}$ (and similarly of $B_{46,9}$ and $B_{64,9}$). Notice that this procedure also shows that the rational homology balls $B_{28+18i,9}$ we get in this way admit handlebody decompositions involving only handles in dimensions 0,1 and 2, hence the maps $\pi_1(\partial B_{28+18i,9}) \to \pi_1(B_{28+18i,9})$ induced by the natural embeddings are surjective. We just note here that the same argument works for all linear plumbings $(b_1,\dots,b_k)$ with $b_i \leq -2$ $(i=1,\dots,k)$ we get from the plumbing (-4) by the repeated applications of the following two transformation rules (cf also [11]):

- $(b_1,\dots,b_k) \longrightarrow (b_1-1,b_2,\dots,b_k,-2)$ and
- $(b_1,\dots,b_k) \longrightarrow (-2,b_1,\dots,b_{k-1},b_k-1)$.

We will give the details of the computation of Seiberg–Witten invariants in Section 3 only for the rational blow-down of $C_{28,9} \subset \mathbb{CP}^2\#17\overline{\mathbb{CP}^2}$. To make this computation explicit, we fix the convention that the second homology group $H_2(\mathbb{CP}^2\#17\overline{\mathbb{CP}^2};\mathbb{Z})$ is generated by the homology elements $h,e_1,\dots,e_{17}$ (with $h^2=1, e_i^2=-1$ $i=1,\dots,17$) and in this basis the homology classes of the spheres in $C_{28,9}$ can be given (from left to right on the linear plumbing of Figure 4) as

$$e_{16}-e_{17},\quad e_{10}-e_{16},$$
$$9h-2e_1-\sum_3^9 3e_i-\sum_{10}^{12} 2e_i-\sum_{13}^{17} e_i,\qquad h-e_1-e_2-e_3$$
$$e_3-e_4,\ e_4-e_5,\ e_5-e_6,\ e_6-e_7,\ e_7-e_8,\ e_8-e_9,\ e_9-e_{13}-e_{14}-e_{15},$$

where here the two sections s_1, s_2 represent e_1 and e_9, respectively.

Definition 2.7 Let us define X_1 as the rational blow-down of $\mathbb{CP}^2\#17\overline{\mathbb{CP}^2}$ along the copy of $C_{28,9}$ specified above, that is,

$$X_1 = (\mathbb{CP}^2\#17\overline{\mathbb{CP}^2} - \text{ int } (C_{28,9})) \cup_{L(784,251)} B_{28,9}.$$

Similarly, X_2, X_3 is defined as the rational blow-down of the configurations $C_{46,9}$ and $C_{64,9}$ in the appropriate rational surfaces.

As a consequence of Freedman's Classification of topological 4–manifolds we have:

Theorem 2.8 *The smooth 4–manifolds X_1, X_2, X_3 are homeomorphic to the rational surface $\mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$.*

Proof Since the complements of the configurations are simply connected and the fundamental group $\pi_1(\partial B_{p,q})$ surjects onto the fundamental group of $B_{p,q}$, simple connectivity of X_1, X_2, X_3 follows from Van Kampen's theorem. Computing the Euler characteristics and signatures of these 4–manifolds, Freedman's Theorem [7] implies the statement. □

2.2 A further example

A slightly different construction can be carried out as follows.

Lemma 2.9 *The plumbing 4–manifold $C_{32,15}$ embeds into $\mathbb{CP}^2\#16\overline{\mathbb{CP}^2}$.*

Recall that $C_{32,15}$ is equal to the 4–manifold defined by the linear plumbing with weights $(-2,-9,-5,-2,-2,-2,-2,-2,-2,-3)$.

Proof We start again with a fibration $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$ with a singular fiber of type III*, three fishtails and two sections, as shown by Figure 2. After blowing up the double points of the three fishtail fibers, blow up at A_1, A_2, smooth the intersections at B_1, B_2, A_3 and keep the transverse intersection at B_3. One further blow-up as it is described by Figure 3 (performed inside the dashed circle of Figure 2) and finally the blow-up of the transverse intersection of the section s_1 with the singular fiber of type III* provides the desired configuration $C_{32,15}$ in $\mathbb{CP}^2\#16\overline{\mathbb{CP}^2}$. □

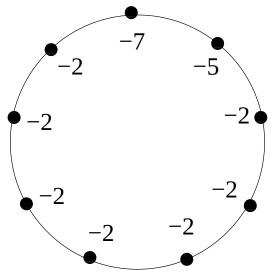

Figure 5: "Necklace" of spheres in $\mathbb{CP}^2\#14\overline{\mathbb{CP}^2}$

Remark 2.10 Consider the configuration of curves in $\mathbb{CP}^2\#14\overline{\mathbb{CP}^2}$ given above without the two last blow-ups. This configuration provides a "necklace" of spheres as shown in Figure 5. Now $C_{32,15}$ can be given from this picture by blowing up the intersection of the (-7)– and the (-2)–framed circles, and then blowing up the resulting (-8)–sphere appropriately one more time. Notice that by blowing up the intersection of the (-7)– and the (-5)–curves instead, we can get two disjoint configurations of $(-8,-2,-2,-2,-2)$ and $(-6,-2,-2)$, ie, two "classical rational blow-down" configurations. Blowing them down we would recover the existence of an exotic smooth structure on $\mathbb{CP}^2\#7\overline{\mathbb{CP}^2}$.

Define Y as the generalized rational blow-down of $\mathbb{CP}^2\#16\overline{\mathbb{CP}^2}$ along the configuration $C_{32,15}$ specified above.

Lemma 2.11 *The 4–manifold* Y *is homeomorphic to* $\mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$.

Proof Let α be the homology element represented by the circle in $\partial C_{32,15}$ we get by intersecting the boundary of the neighborhood of the plumbing with the sphere S in the III* fiber not used in the construction, cf Figure 2. Clearly α is 9 times the normal circle of the last sphere in the configuration. It follows that this circle generates $\pi_1(\partial C_{32,15})$. Since it also bounds a disk in $\mathbb{CP}^2\#16\overline{\mathbb{CP}^2}$ − int $C_{32,15}$, the complement of the configuration is simply connected, and so Van Kampen's theorem and the fact that the fundamental group $\pi_1(\partial B_{32,15})$ surjects onto the fundamental group of $B_{32,15}$ shows simple connectivity. As before, the computation of the Euler characteristics and signature, together with Freedman's Theorem provides the result. □

3 Seiberg–Witten invariants

In order to prove Theorem 1.1, we will compute the Seiberg–Witten invariants of the 4–manifolds constructed above. In order to make our presentation complete, we briefly recall basics of Seiberg–Witten theory for 4–manifolds with $b_2^+=1$. (For a more thorough introduction to Seiberg–Witten theory, with a special emphasis on the case of $b_2^+=1$, see [16, 17, 22].)

Suppose that X is a simply connected, closed, oriented 4–manifold with $b_2^+>0$ and odd, and fix a Riemannian metric g on X. Let $L\to X$ be a given complex line bundle with $c_1(L)\in H^2(X;\mathbb{Z})$ characteristic, ie, $c_1(L)\equiv w_2(TX)$ (mod 2). Through its first Chern class, the bundle L determines a spin^c structure $\mathbf{s}$ on X. The associated spinor $U(2)$–bundles $W_{\mathbf{s}}^{\pm}$ satisfy $L\cong \det(W_{\mathbf{s}}^{\pm})$. A connection $A\in\mathcal{A}_L$ on L, together with the Levi–Civita connection on TX and the Clifford multiplication on the spinor bundles induces a twisted Dirac operator

$$D_A\colon \Gamma(W_{\mathbf{s}}^+)\to\Gamma(W_{\mathbf{s}}^-).$$

For a connection $A\in\mathcal{A}_L$, section $\Psi\in\Gamma(W_{\mathbf{s}}^+)$ and g–self–dual 2–form $\eta\in\Omega_g^+(X;\mathbb{R})$ consider the *perturbed Seiberg–Witten equations*

$$D_A\Psi=0,\qquad F_A^+=i(\Psi\otimes\Psi^*)_0+i\eta,$$

where F_A^+ is the self–dual part of the curvature F_A of the connection A and $(\Psi\otimes\Psi^*)_0$ is the frace–free part of the endomorphism $\Psi\otimes\Psi^*$. For generic choice of the self–dual 2–form η the Seiberg–Witten moduli space — which is the quotient of the solution space to the above equations under the action of the gauge group $\mathcal{G}=\mathrm{Aut}(L)=\mathrm{Maps}(X;\mathbb{R})$ — is a smooth, compact manifold of dimension

$$d_L=\frac{1}{4}(c_1^2(L)-3\sigma(X)-2\chi(X))$$

(provided $d_L\geq 0$). By fixing a 'homology orientation' on X, that is, orienting $H_+^2(X;\mathbb{R})$, the moduli space can be equipped with a natural orientation. A natural 2–cohomology class β can be defined in the cohomology ring of the moduli space, and by integrating $\beta^{\frac{d_L}{2}}$ on the fundamental cycle of the moduli space we get the *Seiberg–Witten invariant* $SW_{X,g,\eta}(L)$. This value is independent of the choice of the metric g and perturbation 2–form η provided the manifold X satisfies $b_2^+(X)>1$. In case of $b_2^+(X)=1$, however, this independence fails to hold. Let ω_g denote the unique self–dual 2–form inducing the chosen homology orientation. It can be shown that $SW_{X,g,h}(L)$ depends only on the sign of the expression

$$(2\pi c_1(L)+[\eta])\cdot[\omega_g].$$

By fixing a sign for the above expression we say that we fixed a *chamber* for L, and going from one chamber to the other we cross a *wall.* It has been shown [13] that by crossing a wall the value of the Seiberg–Witten invariant changes by ± 1. To specify a chamber, we need to fix a cohomology class (or its Poincaré dual) with nonnegative square which can play the role of $[\omega_g]$ for some metric. To remove the ambiguity on sign, we require this element to pair positively with the element representing the given homology orientation.

It is not hard to see that if $b_2^+(X)=1$ and $b_2^-(X)\leq 9$ then $d_L\geq 0$ implies that $c_1^2(L)\geq 0$, hence for choosing the perturbation term η small in norm, the sign of $(2\pi c_1(L)+[\eta])\cdot[\omega_g]$ will be independent of the choice of the metric g. Consequently, by restricting ourselves to Seiberg–Witten invariants with small perturbation, on a manifold X homeomorphic to $\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$ with $n\leq 9$ the function

$$SW_X\colon H^2(X;\mathbb{Z})\to\mathbb{Z}$$

is a diffeomorphism invariant. For such a manifold a cohomology class $K\in H^2(X;\mathbb{Z})$ is called a *Seiberg–Witten basic class* if $SW_X(K)\neq 0$.

It is a standard fact that, because of the presence of a metric with positive scalar curvature, the Seiberg–Witten map vanishes for the smooth 4–manifolds $\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$ with $n\leq 9$. Therefore in order to show that the manifolds X_i $(i=1,2,3)$ given in Definition 2.7 provide exotic structures on $\mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$ we only need to show that $SW_{X_i}\neq 0$. We will go through the computation of the invariants of X_1 only, the other cases follow similar patterns.

Theorem 3.1 *There is a characteristic cohomology class* $\tilde{K}\in H^2(X_1;\mathbb{Z})$ *with* $SW_X(\tilde{K})\neq 0$.

Corollary 3.2 *The 4–manifold* X_1 *is not diffeomorphic to* $\mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$.

Proof The corollary easily follows from Theorem 3.1, together with the facts that $SW_{\mathbb{CP}^2\#6\overline{\mathbb{CP}^2}}\equiv 0$ and that the Seiberg–Witten function is a diffeomorphism invariant for manifolds homeomorphic to $\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$ with $n\leq 9$. □

Proof of Theorem 3.1 Let $K\in H^2(\mathbb{CP}^2\#17\overline{\mathbb{CP}^2};\mathbb{Z})$ denote the characteristic cohomology class which satisfies

$$K(h)=3 \qquad\text{and}\qquad K(e_i)=1\ (i=1,\dots,17).$$

(The Poincaré dual of K is equal to $3h-\sum_{i=1}^{17}e_i$.) It can be shown that the restriction $K|_{\mathbb{CP}^2\#17\overline{\mathbb{CP}^2}-\operatorname{int}C_{28,9}}$ extends as a characteristic cohomology class to X_1: if μ denotes the generator of $H_1(\partial C_{28,9};\mathbb{Z})$ which is the boundary of a normal disk to the left–most circle in the plumbing diagram of Figure 4 then

$$PD(K|_{\partial C}) = 532\mu = 19\cdot(28\mu) \in H_1(\partial C_{28,9};\mathbb{Z});$$

since $H_1(B_{28,9};\mathbb{Z})$ is of order 28, the extendability trivially follows. (See also Proposition 2.2.) Let $\tilde{K}$ denote the extension of $K|_{\mathbb{CP}^2\#17\overline{\mathbb{CP}^2}-\ \text{int}\ C_{28,9}}$ to X_1. Using the gluing formula for Seiberg–Witten invariants along lens spaces, see eg [4, 16], and the fact that the dimensions of the moduli spaces defined by K and $\tilde{K}$ are equal, we have that the invariant $SW_{X_1}(\tilde{K})$ is equal to the Seiberg–Witten invariant of $\mathbb{CP}^2\#17\overline{\mathbb{CP}^2}$ evaluated on K, in the chamber corresponding to a metric which we get by pulling out $C_{28,9}$ along the 'neck' $L(784,251)\times[-T,T]$ far enough. For such a metric the period point provided by the harmonic 2–form ω_g will be orthogonal to the configuration $C_{28,9}$, hence the chamber can be represented by the Poincaré dual of any homology element $\alpha\in H_2(\mathbb{CP}^2\#17\overline{\mathbb{CP}^2};\mathbb{Z})$ of nonnegative square represented in $\mathbb{CP}^2\#17\overline{\mathbb{CP}^2}-$ int $C_{28,9}$. For example,

$$\alpha = 7h - 2e_1 - 3e_2 - \sum_3^9 2e_i - e_{10} - e_{12} - 2e_{13} - e_{16} - e_{17}$$

is such an element. (Simple computation shows that α is orthogonal to all second homology elements in $C_{28,9}$, $\alpha\cdot\alpha=0$ and $\alpha\cdot h=7$.)

It is known that in the chamber corresponding to $PD(h)$ the Seiberg–Witten invariant of $\mathbb{CP}^2\#17\overline{\mathbb{CP}^2}$ vanishes, since this is the chamber containing the period point of a positive scalar curvature metric, which prevents the existence of Seiberg–Witten solutions. Since the wall–crossing phenomenon is well–understood in Seiberg–Witten theory (the invariant changes by one once a wall is crossed), the proof of the theorem reduces to determine whether $PD(\alpha)$ and $PD(h)$ are in the same chamber with respect to K or not. Since $K(h)=3>0$ and $h\cdot\alpha>0$, the inequality $K(\alpha)<0$ would imply the existence of a wall between $PD(h)$ and $PD(\alpha)$, hence $SW_{X_1}(\tilde{K}) = SW_{\mathbb{CP}^2\#17\overline{\mathbb{CP}^2}}(K)\neq 0$, where the invariant of $\mathbb{CP}^2\#17\overline{\mathbb{CP}^2}$ is computed in the chamber containing $PD(\alpha)$. Simple computation shows that $K(\alpha)=-4$, concluding the proof. $\square$

Proof of Theorem 1.1 Now Theorem 2.8 and Corollary 3.2 provide a proof of the main theorem of the paper. $\square$

In fact, with a little more effort we can determine all the Seiberg–Witten basic classes of X_1:

Proposition 3.3 *If $L \in H^2(X_1;\mathbb{Z})$ is a Seiberg–Witten basic class of X_1 then L is equal to $\pm\tilde{K}$. Consequently X_1 is a minimal 4–manifold.*

Proof We start by studying $H_2(X_1 - B_{28,9};\mathbb{Z}) = H_2(\mathbb{CP}^2\#17\overline{\mathbb{CP}^2} - C_{28,9};\mathbb{Z})$. Clearly this is given by the subgroup of elements of $H_2(\mathbb{CP}^2\#17\overline{\mathbb{CP}^2};\mathbb{Z})$ that have trivial intersection in homology with all the spheres in $C_{28,9}$. A quick computation gives the following basis for this subgroup:

$$A_1 = e_{13} - e_{14},\ A_2 = e_{14} - e_{15},\ A_3 = e_{11} - e_{12},\ A_4 = 3h - e_{13} - \sum_1^9 e_i,$$

$$A_5 = -2e_1 + 2e_2 - e_{11},\ A_6 = 4h - e_1 - 2e_2 - \sum_3^9 e_i - 2e_{11} - e_{12} - e_{13},$$

$$A_7 = h - e_2 - e_{10} - e_{11} - e_{16} - e_{17}.$$

Let L be a Seiberg–Witten basic class of X_1; then L is uniquely determined by its restriction $L' \in H^2(X_1 - B_{28,9};\mathbb{Z})$. Following the argument in [15] we determine the basic classes of X_1 in two steps.

First we select some smoothly embedded spheres and tori in $\mathbb{CP}^2\#17\overline{\mathbb{CP}^2}$ that have trivial intersection number in homology with all the spheres in the embedded configuration $C_{28,9}$. To this end note that A_1, A_2, A_3, A_5, A_7 can be represented by spheres and A_4, A_6 by tori. In addition, we will also use the classes $A_8 = A_1 + A_4$ and $A_9 = A_1 + A_2 + A_4$ — these classes can be represented by tori. In this first round we only search for basic classes L that satisfy the additional adjunction inequalities

$$(A_i)^2 + |L(A_i)| \leq 0 \tag{3.1}$$

for $1 \leq i \leq 9$. L is determined by its evaluation on $A_1, \dots, A_7$, so the adjunction inequality on these elements leaves 8100 characteristic classes $L' \in H^2(X_1 - B_{28,9};\mathbb{Z})$ to consider. By the dimension formula, for a Seiberg–Witten basic class of X_1 we have $L^2 = (L')^2 \geq 3$, and $(L')^2 \equiv 3 \bmod 8$. This test weeds out most of the classes: among the 8100 classes there are only 22 with the right square. Among these 22 there are 20 that violate the adjunction inequality (3.1) along A_8 or A_9. The remaining 2 classes evaluate as $\pm(0,0,0,1,1,2,2)$ on $A_1, \dots, A_7$ and thus correspond to $\mp K$.

To finish the computation let us assume that there is a Seiberg–Witten basic class L of X_1 that violates the adjunction inequality (3.1) with one of A_i. Note that any spin^c structure on $\partial C_{28,9}$ that extends to $B_{28,9}$ has an extension to $C_{28,9}$ with square equal to $-b_2(C_{28,9}) = -11$. Using such an extension and the gluing formula along the lens space $\partial C_{28,9}$ we get a Seiberg–Witten basic class L_1 of $\mathbb{CP}^2\#17\overline{\mathbb{CP}^2}$ in a chamber perpendicular to the $C_{28,9}$ configuration, satisfying $d_L = d_{L_1}$ (where d_L, d_{L_1} denote the formal dimensions of the Seiberg–Witten moduli spaces). Now using the adjunction relation for spheres and tori of negative self–intersection [5, 14] we get another basic class L_2 of $\mathbb{CP}^2\#17\overline{\mathbb{CP}^2}$ in a similar chamber with $d(L_2) > d(L_1)$. By the gluing formula again, L_2 gives rise to a basic class L_3 of X_1 with $d(L_3) > d(L)$. Since we consider Seiberg–Witten invariants with small perturbation term only, X_1 has a unique chamber. Therefore it has only finitely many basic classes, consequently the above process has to stop, see also [15]. It can stop only at a basic class that satisfies all the adjunction inequalities for embedded spheres and tori and has positive formal dimension. Since $d_K = d_{-K} = 0$, our previous search rules this case out. □

Remark 3.4 A similar computation applied to X_2, X_3 and Y provides the same result, hence these manifolds are also minimal, homeomorphic to $\mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$ but not diffeomorphic to it. In particular, Seiberg–Witten invariants do not distinguish these 4–manifolds from each other.

Proof of Corollary 1.2 According to Proposition 3.3 and [15, Theorems 1.1, 1.2], there are 4–manifolds X, P, Q homeomorphic to $\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$ with $n = 6, 7, 8$, resp., admitting exactly two Seiberg–Witten basic classes. Blowing up X in at most two and P in at most one point, the application of the blow-up formula for Seiberg–Witten invariants implies the corollary. □

4 Symplectic structures

Since our operation is a special case of the generalized rational blow-down process, which is proved to be symplectic when performed along symplectically embedded spheres [19], we conclude:

Theorem 4.1 *The 4–manifolds X_1, X_2, X_3 and Y constructed above admit symplectic structures.*

Proof The 2–spheres in the configurations are either complex submanifolds or given by smoothings of transverse intersections of complex submanifolds, which are known to be symplectic. Furthermore, all geometric intersections are positive, hence the result of [19] applies. □

In the rest of the section we study the limit of Seiberg–Witten invariants in detecting exotic smooth 4–manifolds with symplectic structures which are homeomorphic to small rational surfaces.

Proposition 4.2 *Suppose that the smooth 4–manifold X is homeomorphic to $S^2 \times S^2$ or $\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$ with $n \leq 8$ and X admits a symplectic structure ω. If X has more than one pair of Seiberg–Witten basic classes then X is not minimal.*

Proof By [13] we know that if $c_1(X)\cdot[\omega] > 0$ and X is simply connected then X is a rational surface, hence under the above topological constraint it admits no Seiberg–Witten basic classes. Therefore we can assume that $c_1(X)\cdot[\omega] < 0$. Suppose now that $\pm K$ and $\pm L$ are both pairs of basic classes and $K \neq \pm L$. Notice that by the dimension formula for the Seiberg–Witten moduli spaces it follows that $K^2 > 0$ and $L^2 > 0$. Suppose furthermore that X is minimal. By a theorem of Taubes [20] we can assume that $K = -c_1(X)$ and we can choose the sign of L to satisfy $L \cdot [\omega] > 0$. Let a denote the Poincaré dual of the cohomology class $\frac{1}{2}(K - L)$. By [21] and the fact that $SW_X(-L) \neq 0$, the nontrivial homology class a can be represented by a pseudo–holomorphic curves. It follows then that $(K - L)\cdot[\omega] > 0$.

Suppose first that $(K-L)^2 \geq 0$. Then the Light Cone Lemma [13, Lemma 2.6] implies that $K\cdot(K-L) > 0$ and $L\cdot(K-L) > 0$ unless $L = rK$ for some $r \in \mathbb{Q}$. The two inequalities imply $K^2 > L^2$, contradicting the fact that the moduli space corresponding to the spin^c structure determined by L is of nonnegative formal dimension. If $L = rK$ and $K\cdot(K-L) = 0$, then the fact $K^2 > 0$ implies that $r = 1$, hence $L = K$, contradicting our assumption $K \neq \pm L$.

Finally we have to examine the case when $(K-L)^2 < 0$. In this case, by [21, Proposition 7.1] for generic almost–complex structure the pseudo–holomorphic representative of the homology class $a = PD(\frac{1}{2}(K-L))$ contains a sphere component of square (-1), contradicting the minimality of X. □

Proposition 4.3 *Suppose that (X_1, ω_1) and (X_2, ω_2) are simply connected minimal symplectic 4–manifolds with $b_2^+ = 1$ and $b_2^- \leq 8$. If X_1 and X_2 are homeomorphic and have nonvanishing Seiberg–Witten invariants then we can choose the homeomorphism $f\colon X_1 \to X_2$ so that*

$$SW_{X_2}(L) = \pm SW_{X_1}(f^*(L))$$

for all characteristic classes $L \in H^2(X_2;\mathbb{Z})$.

Proof According to Proposition 4.2 both X_1 and X_2 has two basic classes $\pm c_1(X_i,\omega_i)$. According to Taubes' theorem [20] (using an appropriate homology orientation) we have

$$SW_{X_i}(c_1(X_i,\omega_i))=1,\quad SW_{X_i}(-c_1(X_i,\omega_i))=-1.$$

According to Freedman [7] the required homomorphism f can be induced by an isomorphism

$$g\colon\ H^2(X_2;\mathbb{Z})\longrightarrow H^2(X_1;\mathbb{Z})$$

that maps $c_1(X_2,\omega_2)$ to $c_1(X_1,\omega_1)$ and preserves the intersection form. The existence of such g is trivial when b_2^- is zero or one; the general case follows from the large automorphism group of the second cohomology group H^2 given by reflecting on cohomology classes with squares 1, -1 and -2. In particular, for the intersection form of $\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$ $(2\leq n\leq 8)$ it is easy to use reflections along the Poincaré duals of h, e_i, $h-e_i-e_j$ and $h-e_i-e_j-e_k$ to map a given characteristic class L with $L^2=9-n$ to $3h-e_1-\ldots-e_n$. Depending on whether g respects the chosen homology orientations on X_1 and X_2 or not, we have $SW_{X_2}(L)=\pm SW_{X_1}(f^*(L))$. □

The above result together with the blow-up formula for Seiberg–Witten invariants imply the following:

Corollary 4.4 *The Seiberg–Witten invariants can distinguish at most finitely many symplectic 4–manifolds homeomorphic to a rational surface X with Euler characteristic $e(X)<12$.* □

5 Appendix: singular fibers in elliptic fibrations

For the sake of completeness we give an explicit construction of the elliptic fibration $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}\to\mathbb{CP}^1$ used in the paper. The existence of such fibration is a standard result in complex geometry; in the following we will present it in a way useful for differential topological considerations.

Notice first that to verify the existence of a fibration with singular fibers of type III* (also known as the $\tilde{E}_7$–fiber) and three fishtail fibers is quite easy. As it is shown in [10] (see also [9, pp. 35–36]) the monodromy of an $\tilde{E}_7$–fiber can be chosen to be equal to $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$, while for a fishtail fiber the monodromy is conjugate to $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Since

$$\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix}^{-1}\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right)\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

the genus–1 Lefschetz fibration with the prescribed singular fibers over the disk extends to a fibration over the sphere S^2. Simple Euler characteristic computation and the classification of genus–1 Lefschetz fibrations show that the result is an elliptic fibration on $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$. The existence of the two sections positioned as required by the configuration of Figure 2 is, however, less apparent from this picture. One could repeat the above computation in the mapping class group of the typical fiber with appropriate marked points, arriving to the same conclusion. Here we rather use a more direct way of describing a pencil of curves in $\mathbb{CP}^2$ and following the blow-up procedure explicitly.

Let

$$C_1 = \{[x:y:z] \in \mathbb{CP}^2 \mid p_1(x,y,z) = (x-z)z^2 = 0\} \qquad \text{and}$$

$$C_2 = \{[x:y:z] \in \mathbb{CP}^2 \mid p_2(x,y,z) = x^3 + zx^2 - zy^2 = 0\}$$

be two given complex curves in the complex projective plane $\mathbb{CP}^2$. The curve C_1 is the union of the lines $L_1 = \{(x-z)=0\}$ and $L_2 = \{z=0\}$, with the latter of multiplicity two. C_2 is an immersed sphere with one positive transverse double point — blowing this curve up nine times in its smooth points results a fishtail fiber, see also [8, Section 2.3]. L_2 intersects C_2 in a single point $P=[0:1:0]$ (hence this point is a triple tangency between the two curves), and L_1 (also passing through P) intersects C_2 in two further (smooth) points $R=[1:\sqrt{2}:1]$ and $Q=[1:-\sqrt{2}:1]$, cf Figure 6. Therefore the pencil

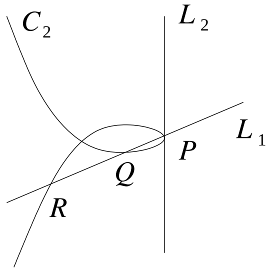

Figure 6: Curves generating the pencil

$$C_t = C_{[t_1:t_2]} = \{(t_1p_1 + t_2p_2)^{-1}(0)\} \qquad (t = [t_1:t_2] \in \mathbb{CP}^1)$$

of elliptic curves defined by C_1 and C_2 provides a map f from $\mathbb{CP}^2$ to $\mathbb{CP}^1$ well–defined away from the three base points P, Q, R. In order to get the desired fibration we will perform seven infinitely close blow-ups at the base point P and two further blow-ups at R and Q, resp. We will explain only the first blow-up at P, the rest follows a similar pattern. After the blow-up of P we would like to have a pencil on the blown-up manifold. We take $\tilde{C}_2$ to be the proper transform of C_2, while $\tilde{C}_1$ will be the proper transform of C_1 together with a certain multiple of the exceptional divisor, chosen so that the two curves represent the same homology class. Under this homological condition the two curves can be given as zero sets of holomorphic sections of the same holomorphic line bundle, hence $\tilde{C}_1$ and $\tilde{C}_2$ define a pencil on the blown-up manifold. Since $[C_1]=[C_2]=3h\in H_2(\mathbb{CP}^2;\mathbb{Z})$, it is easy to see that $[\tilde{C}_2]=3h-e_1\in H_2(\mathbb{CP}^2\#\overline{\mathbb{CP}^2};\mathbb{Z})$, where (as usual) e_1 denotes the homology class of the exceptional divisor of the blow-up. Now it is a simple matter to see that the proper transform of C_1 is the union of the proper transforms of L_1 and L_2, which transforms represent $h-e_1$ and $2h-2e_1$ in $H_2(\mathbb{CP}^2\#\overline{\mathbb{CP}^2};\mathbb{Z})$. Therefore in the pencil we need to take the curve given by the proper transform of C_1 together with the exceptional curve, the latter with multiplicity two. Since e_1 is part of $\tilde{C}_1$, we further have to blow up its intersection with $\tilde{C}_2$. The same principle shows that in the further blow-ups the exceptional divisors e_2,e_3,e_4,e_5,e_6 come with multiplicities $3,4,3,2$ and 1. Finally, after blowing up for the seventh time, the two curves defining the pencil get locally separated, and hence e_7 will not lie in any of the curves of the new pencil anymore — it will be a section, that is, it intersects all the curves in the pencil transversally in one point. (Notice that we used seven blow-ups to separate the curves C_1 and C_2 at P, where they intersected each other of order seven: L_2 being a linear curve of multiplicity two, intersected the cubic curve C_2 of order six, while L_1 simply passed through the intersection point P.) After blowing up the two further base points Q,R, we get a fibration on the nine–fold blow-up $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$ with a fishtail fiber, a singular fiber of type III* and three sections provided by the exceptional divisors e_7,e_8 and e_9. (To recover the homology classes of the spheres in the type III* fiber indicated by Figure 1 one needs to rename the exceptional divisors of the blow-ups; we leave this simple exercise to the reader.) A final simple calculation shows that the resulting fibration has three fishtail fibers:

Proposition 5.1 *The pencil*

$$\{C_{[t_1:t_2]}=(t_1p_1+t_2p_2)^{-1}(0)\mid [t_1:t_2]\in\mathbb{CP}^1\}$$

contains four singular curves: C_1,C_2 and C_3,C_4. Furthermore the latter two curves are homeomorphic to C_2 and give rise to fishtail fibers after blowing up the base points of the pencil.

Proof Since $L_2=\{z=0\}\subset C_1$, all other curves of the pencil are contained in $\{z=1\}\cup\{P\}$. The curve $C_t=C_{[t_1:t_2]}$ has a singular point if and only if for the polynomial $p_t(x,y) = t_1(x-1) + t_2(x^3+x^2-y^2)$ we can find $(x_0,y_0)\in\mathbb{C}^2$ with

$$p_t(x_0,y_0) = 0, \qquad \frac{\partial p_t}{\partial x}(x_0,y_0) = 0 \qquad \text{and} \qquad \frac{\partial p_t}{\partial y}(x_0,y_0) = 0.$$

Since $\frac{\partial p_t}{\partial y}(x,y) = -2t_2 y$, it vanishes if $t_2 = 0$ (providing C_1) or $y = 0$. In the latter case the above system reduces to

$$t_1(x-1) + t_2(x^3+x^2) = 0 \qquad \text{and} \qquad t_1 + t_2(3x^2+2x) = 0,$$

which admits a nontrivial solution (t_1,t_2) if and only if the determinant

$$(x-1)(3x^2+2x) - (x^3+x^2) = 2x(x^2-x-1)$$

vanishes. The solution $x = 0$ implies $t_1 = 0$, giving C_2, while $x = \frac{1}{2}(1\pm\sqrt{5})$ give the two singular points on the curves C_3 and C_4. Simple Euler characteristic computation shows that the two curves will give rise to fishtail fibers in the elliptic fibration. □

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