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Deformations of asymptotically cylindrical coassociative submanifolds with fixed boundary

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Abstract

McLean proved that the moduli space of coassociative deformations of a compact coassociative 4–submanifold C in a G_2–manifold (M,φ,g) is a smooth manifold of dimension equal to $b^2_+(C)$. In this paper, we show that the moduli space of coassociative deformations of a noncompact, *asymptotically cylindrical* coassociative 4–fold C in an asymptotically cylindrical G_2–manifold (M,φ,g) is also a smooth manifold. Its dimension is the dimension of the positive subspace of the image of $H^2_{\rm cs}(C,\mathbb{R})$ in $H^2(C,\mathbb{R})$.

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1 Introduction

Let (M,g) be a Riemannian 7–manifold whose holonomy group $\mathrm{Hol}(g)$ is the exceptional holonomy group G_2 (or, more generally, a subgroup of G_2). Then M is naturally equipped with a constant 3–form φ and 4–form $*\varphi$. We call (M,φ,g) a *G_2–manifold*. *Complete* examples of Riemannian 7–manifolds with holonomy G_2 were constructed by Bryant and Salamon [3], and *compact* examples by Joyce [7] and Kovalev [11].

Now φ and $*\varphi$ are *calibrations* on M, in the sense of Harvey and Lawson [5]. The corresponding calibrated submanifolds in M are called *associative* 3*–folds* and *coassociative* 4*–folds*, respectively. They are distinguished classes of minimal 3– and 4–submanifolds in (M,g) with a rich structure, that can be thought of as analogous to complex curves and surfaces in a Calabi–Yau 3–fold.

Harvey and Lawson [5] introduced four types of calibrated geometries. Special Lagrangian submanifolds of Calabi–Yau manifolds, associative and coassociative submanifolds of G_2 manifolds and Cayley submanifolds of Spin(7) manifolds. Calibrated geometries have been of growing interest over the past few years and represent one of the most mysterious classes of minimal submanifolds [12], [13]. A great deal of progress has been made recently in the field of special Lagrangian submanifolds that arise in mirror symmetry for Calabi–Yau manifolds and plays a significant role in string theory, for references see [8]. As one might expect, another promising direction for future investigation is calibrated submanifolds in G_2 and Spin(7) manifolds. Recently, some progress has been made in constructing such submanifolds [6, 17, 18] and in understanding their deformations [1, 14].

The *deformation theory* of *compact* calibrated submanifolds was studied by McLean [22]. He showed that if C is a compact coassociative 4–fold in a G_2–manifold (M,φ,g), then the moduli space $\mathcal{M}_C$ of coassociative deformations of C is smooth, with dimension $b^2_+(C)$.

This paper proves an analogue of McLean's theorem for a special class of *noncompact* coassociative 4–folds. The situation we are interested in is when (M,φ,g) is an *asymptotically cylindrical* G_2–manifold, that is, it is a noncompact 7–manifold with one end asymptotic to the cylinder $X\times\mathbb{R}$ on a *Calabi–Yau* 3*–fold* X. The natural class of noncompact coassociative 4–folds in M are *asymptotically cylindrical* coassociative 4–folds C, asymptotic at infinity in M to a cylinder $L\times\mathbb{R}$, where L is a *special Lagrangian* 3*–fold* in X, with phase i. Understanding the deformations of such submanifolds when the ambient G_2–manifold decomposes into connected sum of two pieces will provide the necessary technical framework towards completing the Floer homology program for coassociative submanifolds, [13].

In particular, we prove the following theorem.

Theorem 1.1 *Let (M, φ, g) be a G_2–manifold asymptotic to $X \times (R, \infty)$ with decay rate $\alpha < 0$, where X is a Calabi–Yau 3–fold. Let C be a coassociative 4–fold in M asymptotic to $L \times (R', \infty)$ for $R' > R$ with decay rate β for $\alpha \leqslant \beta < 0$, where L is a special Lagrangian 3–fold in X with phase i.*

If $\gamma < 0$ is small enough then the moduli space $\mathcal{M}_C^\gamma$ of asymptotically cylindrical coassociative submanifolds in M close to C, and asymptotic to $L \times (R', \infty)$ with decay rate γ, is a smooth manifold of dimension $\dim V_+$, where V_+ is the positive subspace of the image of $H^2_{\mathrm{cs}}(C, \mathbb{R})$ in $H^2(C, \mathbb{R})$.

The principal analytic tool we shall use to prove this is the theory of *weighted Sobolev spaces* on manifolds with ends, developed by Lockhart and McOwen [15, 16]. The important fact is that elliptic partial differential operators on exterior forms such as $\mathrm{d} + \mathrm{d}^*$ or $\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*$ on the noncompact 4–manifold C are *Fredholm operators* between appropriate Banach spaces of forms, and we can describe their kernels and cokernels.

Results similar to Theorem 1.1 on the deformations of classes of noncompact *special Lagrangian m–folds* were proved by Marshall [19] and Pacini [24] for *asymptotically conical* special Lagrangian m–folds, and by Joyce [7, 9] for special Lagrangian m–folds with *isolated conical singularities.* Marshall and Joyce also use the Lockhart–McOwen framework, but Pacini uses a different analytical approach due to Melrose [20, 21]. Note also that Kovalev [11] constructs compact G_2–manifolds by gluing together two noncompact, asymptotically cylindrical G_2–manifolds.

We begin in Section 2 with an introduction to G_2–manifolds and coassociative submanifolds, including a sketch of the proof of McLean's theorem on deformations of compact coassociative 4–folds, and the definitions of *asymptotically cylindrical* G_2–manifolds and coassociative 4–folds. Section 3 introduces the weighted Sobolev spaces of Lockhart and McOwen, and determines the kernel and cokernel of the elliptic operator $\mathrm{d}_+ + \mathrm{d}^*$ on C used in the proof. Finally, Section 4 proves Theorem 1.1, using Banach space techniques and elliptic regularity.

Remark 1.2 In [11], Kovalev constructs asymptotically cylindrical manifolds X with holonomy $SU(3)$. Then $X \times \mathcal{S}^1$ is an asymptotically cylindrical G_2–manifold, though with holonomy $SU(3)$ rather than G_2. One can find examples of asymptotically cylindrical coassociative 4–folds C in $X \times \mathcal{S}^1$ of two types:

(a) $C = C' \times \mathrm{pt}$, for C' an asymptotically cylindrical complex surface in X; or

(b) $C = L \times \mathcal{S}^1$, for L an asymptotically cylindrical special Lagrangian 3–fold in X, with phase i.

Examples of type (a) can be constructed using algebraic geometry: if $X = \overline{X} \setminus D$ for $\overline{X}$ a Fano 3–fold and D a smooth divisor in $\overline{X}$, then we can take $C = \overline{C} \setminus D$ for $\overline{C}$ a smooth divisor in $\overline{X}$ intersecting D transversely. Examples of type (b) can be found by choosing the Calabi–Yau 3–fold (X, J, ω, Ω) to have an *antiholomorphic involution* $\sigma\colon X \to X$ with $\sigma^*(J) = -J$, $\sigma^*(\omega) = -\omega$ and $\sigma^*(\Omega) = -\overline{\Omega}$. Then the fixed points L of σ are a special Lagrangian 3–fold with phase i, and each infinite end of L is asymptotically cylindrical.

We can then apply Theorem 1.1 to these examples. One can show that if $\widetilde{C}$ is a small deformation of a coassociative 4–fold C of type (a) or (b) then $\widetilde{C}$ is also of type (a) or (b) and thus, Theorem 1.1 implies analogous results on the deformation theory of asymptotically cylindrical complex surfaces and special Lagrangian 3–folds in asymptotically cylindrical Calabi–Yau 3–folds.

2 Introduction to G_2 geometry

We now give background material on G_2–manifolds and their coassociative submanifolds that will be needed later. A good reference on G_2 geometry is Joyce [7, Sections 10–12], and a good reference on calibrated geometry is Harvey and Lawson [5].

2.1 G_2–manifolds and coassociative submanifolds

Let $(x_1, \ldots, x_7)$ be coordinates on $\mathbb{R}^7$. Write $\mathrm{d}\mathbf{x}_{ij\ldots l}$ for the exterior form $\mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \cdots \wedge \mathrm{d}x_l$ on $\mathbb{R}^7$. Define a metric g_0, a 3–form φ_0 and a 4–form $*\varphi_0$ on $\mathbb{R}^7$ by $g_0 = \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_7^2$,

$$\begin{aligned} \varphi_0 &= \mathrm{d}\mathbf{x}_{123} + \mathrm{d}\mathbf{x}_{145} + \mathrm{d}\mathbf{x}_{167} + \mathrm{d}\mathbf{x}_{246} - \mathrm{d}\mathbf{x}_{257} - \mathrm{d}\mathbf{x}_{347} - \mathrm{d}\mathbf{x}_{356} \ \text{ and} \\ *\varphi_0 &= \mathrm{d}\mathbf{x}_{4567} + \mathrm{d}\mathbf{x}_{2367} + \mathrm{d}\mathbf{x}_{2345} + \mathrm{d}\mathbf{x}_{1357} - \mathrm{d}\mathbf{x}_{1346} - \mathrm{d}\mathbf{x}_{1256} - \mathrm{d}\mathbf{x}_{1247}. \end{aligned} \tag{1}$$

The subgroup of $GL(7, \mathbb{R})$ preserving φ_0 is the *exceptional Lie group* G_2. It also preserves $g_0, *\varphi_0$ and the orientation on $\mathbb{R}^7$. It is a compact, semisimple, 14–dimensional Lie group, a subgroup of $\mathrm{SO}(7)$.

A G_2*–structure* on a 7–manifold M is a principal subbundle of the frame bundle of M, with structure group G_2. Each G_2–structure gives rise to a 3–form φ and a metric g on M, such that every tangent space of M admits an isomorphism with $\mathbb{R}^7$ identifying φ and g with φ_0 and g_0 respectively. By an abuse of notation, we will refer to (φ, g) as a G_2–structure.

Proposition 2.1 *Let M be a 7–manifold and (φ, g) a G_2–structure on M. Then the following are equivalent:*

(i) $\mathrm{Hol}(g) \subseteq G_2$, *and φ is the induced 3–form,*
(ii) $\nabla\varphi = 0$ *on M, where ∇ is the Levi–Civita connection of g, and*
(iii) $\mathrm{d}\varphi = \mathrm{d}^*\varphi = 0$ *on M.*

We call $\nabla\varphi$ the *torsion* of the G_2–structure (φ, g), and when $\nabla\varphi = 0$ the G_2–structure is *torsion-free.* A triple (M, φ, g) is called a *G_2–manifold* if M is a 7–manifold and (φ, g) a torsion-free G_2–structure on M. If g has holonomy $\mathrm{Hol}(g) \subseteq G_2$, then g is Ricci-flat. For explicit, *complete* examples of G_2–manifolds see Bryant and Salamon [3], and for *compact* examples see Joyce [7] and Kovalev [11]. Here are the basic definitions in *calibrated geometry*, due to Harvey and Lawson [5].

Definition 2.2 Let (M, g) be a Riemannian manifold. An *oriented tangent k–plane* V on M is a vector subspace V of some tangent space T_xM to M with $\dim V = k$, equipped with an orientation. If V is an oriented tangent k–plane on M then $g|_V$ is a Euclidean metric on V, so combining $g|_V$ with the orientation on V gives a natural *volume form* vol_V on V, which is a k–form on V.

Now let φ be a closed k–form on M. We say that φ is a *calibration* on M if for every oriented k–plane V on M we have $\varphi|_V \leqslant \mathrm{vol}_V$. Here $\varphi|_V = \alpha \cdot \mathrm{vol}_V$ for some $\alpha \in \mathbb{R}$, and $\varphi|_V \leqslant \mathrm{vol}_V$ if $\alpha \leqslant 1$. Let N be an oriented submanifold of M with dimension k. Then each tangent space T_xN for $x \in N$ is an oriented tangent k–plane. We call N a *calibrated submanifold* if $\varphi|_{T_xN} = \mathrm{vol}_{T_xN}$ for all $x \in N$.

Calibrated submanifolds are automatically *minimal submanifolds* (see [5, Theorem II.4.2]). There are two natural classes of calibrated submanifolds in G_2–manifolds.

Definition 2.3 Let (M, φ, g) be a *G_2–manifold*, as above. Then the 3–form φ is a *calibration* on (M, g). We define an *associative 3–fold* in M to be a 3–submanifold of M calibrated with respect to φ. Similarly, the Hodge star $*\varphi$ of φ is a calibration 4–form on (M, g). We define a *coassociative 4–fold* in M to be a 4–submanifold of M calibrated with respect to $*\varphi$.

McLean [22, Prop. 4.4] gives an alternative description of coassociative 4–folds:

Proposition 2.4 *Let (M,φ,g) be a G_2–manifold, and C a 4–dimensional submanifold of M. Then C admits an orientation making it into a coassociative 4–fold if and only if $\varphi|_C\equiv 0$.*

2.2 Deformations of compact coassociative 4–folds

Here is the main result in the *deformation theory* of coassociative 4–folds, proved by McLean [22, Theorem 4.5]. As our sign conventions for $\varphi_0, *\varphi_0$ in (1) are different to McLean's, we use self-dual 2–forms in place of McLean's anti-self-dual 2–forms.

Theorem 2.5 *Let (M,φ,g) be a G_2–manifold, and C a compact coassociative 4–fold in M. Then the moduli space $\mathcal{M}_C$ of coassociative 4–folds isotopic to C in M is a smooth manifold of dimension $b^2_+(C)$.*

Sketch proof Suppose for simplicity that C is an embedded submanifold. There is a natural orthogonal decomposition $TM|_C = TC\oplus\nu$, where $\nu\rightarrow C$ is the *normal bundle* of C in M. There is a natural isomorphism $\nu\cong\Lambda^2_+T^*C$, constructed as follows. Let $x\in C$ and $V\in\nu_x$. Then V lies in T_xM, so $V\cdot\varphi|_x\in\Lambda^2T^*_xM$, and $(V\cdot\varphi|_x)|_{T_xC}\in\Lambda^2T^*_xC$. Moreover $(V\cdot\varphi|_x)|_{T_xC}$ actually lies in $\Lambda^2_+T^*_xC$, the bundle of *self-dual 2–forms* on C, and the map $V\mapsto(V\cdot\varphi|_x)|_{T_xC}$ defines an *isomorphism* $\nu\xrightarrow{\cong}\Lambda^2_+T^*C$.

For small $\epsilon>0$, write $B_\epsilon(\nu)$ for the subbundle of ν with fibre at x the open ball about 0 in $\nu|_x$ with radius ϵ. Then the exponential map $\exp\colon\nu\rightarrow M$ induces a diffeomorphism between $B_\epsilon(\nu)$ and a small *tubular neighbourhood* T_C of C in M. The isomorphism $\nu\cong\Lambda^2_+T^*C$ gives a diffeomorphism $\exp\colon B_\epsilon(\Lambda^2_+T^*C)\rightarrow T_C$. Let $\pi\colon T_C\rightarrow C$ be the obvious projection.

Under this identification, submanifolds $\widetilde C$ in $T_C\subset M$ which are C^1 close to C are identified with the *graphs* $\Gamma_{\zeta^2_+}$ of small smooth sections ζ^2_+ of $\Lambda^2_+T^*C$ lying in $B_\epsilon(\Lambda^2_+T^*C)$. For each $\zeta^2_+\in C^\infty\bigl(B_\epsilon(\Lambda^2_+T^*C)\bigr)$ the graph $\Gamma_{\zeta^2_+}$ is a 4–submanifold of $B_\epsilon(\Lambda^2_+T^*C)$, and so $\widetilde C=\exp(\Gamma_{\zeta^2_+})$ is a 4–submanifold of T_C. We need to know: which 2–forms ζ^2_+ correspond to *coassociative* 4–folds $\widetilde C$ in T_C?

$\widetilde C$ is coassociative if $\varphi|_{\widetilde C}\equiv 0$. Now $\pi|_{\widetilde C}\colon\widetilde C\rightarrow C$ is a diffeomorphism, so we can push $\varphi|_{\widetilde C}$ down to C, and regard it as a function of ζ^2_+. That is, we define

$$Q\colon C^\infty\bigl(B_\epsilon(\Lambda^2_+T^*C)\bigr)\rightarrow C^\infty(\Lambda^3T^*C)\ \text{ by }\ Q(\zeta^2_+)=\pi_*(\varphi|_{\exp(\Gamma_{\zeta^2_+})}).$$

Then the moduli space $\mathcal{M}_C$ is locally isomorphic near C to the set of small self-dual 2–forms ζ^2_+ on C with $\varphi|_{\exp(\Gamma_{\zeta^2_+})} \equiv 0$, that is, to a neighborhood of 0 in $Q^{-1}(0)$.

To understand the equation $Q(\zeta^2_+) = 0$, note that at $x \in C$, $Q(\zeta^2_+)|_x$ depends on the tangent space to $\Gamma_{\zeta^2_+}$ at $\zeta^2_+|_x$, and so on $\zeta^2_+|_x$ and $\nabla\zeta^2_+|_x$. Thus the functional form of Q is

$$Q(\zeta^2_+)|_x = F\big(x, \zeta^2_+|_x, \nabla\zeta^2_+|_x\big) \quad \text{for } x \in C,$$

where F is a smooth function of its arguments. Hence $Q(\zeta^2_+) = 0$ is a *nonlinear first order PDE* in ζ^2_+. As φ is closed, $\varphi|_C \equiv 0$, and $\Gamma_{\zeta^2_+}$ is isotopic to C, we see that $\varphi|_{\Gamma_{\zeta^2_+}}$ is an exact 3–form on $\Gamma_{\zeta^2_+}$, so that $Q(\zeta^2_+)$ is *exact*. The *linearization* $\mathrm{d}Q(0)$ of Q at $\zeta^2_+ = 0$ is

$$\mathrm{d}Q(0)(\beta) = \lim_{\epsilon\to 0}\big(\epsilon^{-1}Q(\epsilon\beta)\big) = \mathrm{d}\beta.$$

Therefore $\mathrm{Ker}(\mathrm{d}Q(0))$ is the vector space $\mathcal{H}^2_+$ of *closed self-dual* 2*–forms* β on C, which by Hodge theory is a finite-dimensional vector space isomorphic to $H^2_+(C,\mathbb{R})$, with dimension $b^2_+(C)$. This is the *Zariski tangent space* of $\mathcal{M}_C$ at C, the *infinitesimal deformation space* of C as a coassociative 4–fold.

To complete the proof we must show that $\mathcal{M}_C$ is locally isomorphic to its Zariski tangent space $\mathcal{H}^2_+$, and so is a smooth manifold of dimension $b^2_+(C)$. To do this rigorously requires some technical analytic machinery, which is passed over in a few lines in [22, p. 731]. Here is one way to do it.

As Q maps from $\Lambda^2_+T^*C$ with fibre $\mathbb{R}^3$ to Λ^3T^*C with fibre $\mathbb{R}^4$, it is *overdetermined*, and not *elliptic*. To turn it into an elliptic operator, define

$$\begin{aligned} P\colon\ & C^\infty\big(B_\epsilon(\Lambda^2_+T^*C)\big) \times C^\infty(\Lambda^4T^*C) \to C^\infty(\Lambda^3T^*C) \\ \text{by} \qquad & P(\zeta^2_+, \zeta^4) = Q(\zeta^2_+) + \mathrm{d}^*\zeta^4. \end{aligned} \tag{2}$$

Then the linearization of P at $(0,0)$ is

$$\mathrm{d}P(0,0)\colon\ (\zeta^2_+, \zeta^4) \mapsto \mathrm{d}\zeta^2_+ + \mathrm{d}^*\zeta^4,$$

which is elliptic. Since ellipticity is an open condition, P is elliptic near $(0,0)$ in $C^\infty\big(B_\epsilon(\Lambda^2_+T^*C)\big) \times C^\infty(\Lambda^4T^*C)$.

Suppose $P(\zeta^2_+, \zeta^4) = 0$. Then $Q(\zeta^2_+) = -\mathrm{d}^*\zeta^4$, so

$$\|\mathrm{d}^*\zeta^4\|^2_{L^2} = -\big\langle \mathrm{d}^*\zeta^4, Q(\zeta^2_+)\big\rangle_{L^2} = -\big\langle \zeta^4, \mathrm{d}(Q(\zeta^2_+))\big\rangle_{L^2} = 0,$$

by integration by parts, since $Q(\zeta^2_+)$ is *exact*. Hence $P(\zeta^2_+, \zeta^4) = 0$ if and only if $Q(\zeta^2_+) = \mathrm{d}^*\zeta^4 = 0$. But 4–forms with $\mathrm{d}^*\zeta^4 = 0$ are constant, and the vector space of such ζ^4 is $H^4(C,\mathbb{R})$. Thus, $P^{-1}(0) = Q^{-1}(0) \times H^4(C,\mathbb{R})$.

Because $C^\infty(\Lambda^2_+T^*C), C^\infty(\Lambda^3T^*C)$ are not *Banach spaces*, we extend P in (2) to act on *Sobolev spaces* $L^p_{l+2}(\Lambda^2_+T^*C), L^p_{l+2}(\Lambda^3T^*C)$ for $p>4$ and $l\geqslant 0$, giving

$$\begin{aligned}\widetilde{P}&\colon L^p_{l+2}\bigl(B_\epsilon(\Lambda^2_+T^*C)\bigr)\times L^p_{l+2}(\Lambda^4T^*C)\to L^p_{l+1}(\Lambda^3T^*C),\\ \widetilde{P}&\colon (\zeta^2_+,\zeta^4)\mapsto \pi_*(\varphi|_{\Gamma_{\zeta^2_+}})+\mathrm{d}^*\zeta^4.\end{aligned}$$

Then $\widetilde{P}$ is a *smooth map of Banach manifolds.*

Let $\mathcal{H}^3$ be the vector space of closed and coclosed 3–forms on C, so that $\mathcal{H}^3\cong H^3(C,\mathbb{R})$ by Hodge theory, and V^p_{l+1} be the Banach subspace of $L^p_{l+1}(\Lambda^3T^*C)$ L^2–orthogonal to $\mathcal{H}^3$. Then one can show that $\widetilde{P}$ maps into V^p_{l+1}, and the linearization

$$\begin{aligned}\mathrm{d}\widetilde{P}(0,0)&\colon L^p_{l+2}(\Lambda^2_+T^*C)\times L^p_{l+2}(\Lambda^4T^*C)\to V^p_{l+1},\\ \mathrm{d}\widetilde{P}(0,0)&\colon (\zeta^2_+,\zeta^4)\mapsto \mathrm{d}\zeta^2_+ +\mathrm{d}^*\zeta^4\end{aligned}$$

is then *surjective* as a map of Banach spaces.

Thus, $\widetilde{P}\colon L^p_{l+2}\bigl(B_\epsilon(\Lambda^2_+T^*C)\bigr)\times L^p_{l+2}(\Lambda^4T^*C)\to V^p_{l+1}$ is a smooth map of Banach manifolds, with $\mathrm{d}\widetilde{P}(0,0)$ surjective. The *Implicit Mapping Theorem for Banach spaces* (Theorem 4.4) now implies that $\widetilde{P}^{-1}(0)$ is near 0 a smooth submanifold, locally isomorphic to $\mathrm{Ker}(\mathrm{d}\widetilde{P}(0))$. But $\widetilde{P}(\zeta^2_+,\zeta^4)=0$ is an *elliptic equation* for small ζ^2_+,ζ^4, and so elliptic regularity implies that solutions (ζ^2_+,ζ^4) are smooth.

Therefore $\widetilde{P}^{-1}(0)=P^{-1}(0)$ near 0, and also $\mathrm{Ker}(\mathrm{d}\widetilde{P}(0,0))=\mathrm{Ker}(\mathrm{d}P(0,0))$. Hence $P^{-1}(0)$ is, near $(0,0)$, a smooth manifold locally isomorphic to the kernel $\mathrm{Ker}(\mathrm{d}P(0,0))$. So from above $Q^{-1}(0)$ is near 0 a smooth manifold locally isomorphic to $\mathrm{Ker}(\mathrm{d}Q(0))$. Thus, $\mathcal{M}_C$ is near C a smooth manifold locally isomorphic to $H^2_+(C,\mathbb{R})$. This completes the proof. □

2.3 Asymptotically cylindrical G_2–manifolds and coassociative 4–folds

We first define *cylindrical* and *asymptotically cylindrical* G_2–manifolds.

Definition 2.6 A G_2–manifold (M_0,φ_0,g_0) is called *cylindrical* if $M_0=X\times\mathbb{R}$ and (φ_0,g_0) is compatible with this product structure, that is,

$$\varphi_0=\mathrm{Re}\,\Omega+\omega\wedge\mathrm{d}t\qquad\text{and}\qquad g_0=g_X+\mathrm{d}t^2,$$

where X is a (connected, compact) Calabi–Yau 3–fold with Kähler form ω, Riemannian metric g_X and holomorphic (3,0)-form Ω.

Definition 2.7 A connected, complete G_2–manifold (M,φ,g) is called *asymptotically cylindrical with decay rate* α if there exists a cylindrical G_2–manifold (M_0,φ_0,g_0) with $M_0 = X\times\mathbb{R}$ as above, a compact subset $K\subset M$, a real number R, and a diffeomorphism $\Psi\colon X\times(R,\infty)\to M\setminus K$ such that $\Psi^*(\varphi)=\varphi_0+\mathrm{d}\xi$ for some smooth 2–form ξ on $X\times(R,\infty)$ with $\left|\nabla^k\xi\right| = O(e^{\alpha t})$ on $X\times(R,\infty)$ for all $k\geqslant 0$, where ∇ is the Levi–Civita connection of the cylindrical metric g_0.

The point of this is that M has one end modelled on $X\times(R,\infty)$, and as $t\to\infty$ in (R,∞) the G_2–structure (φ,g) on M converges to order $O(e^{\alpha t})$ to the cylindrical G_2–structure on $X\times(R,\infty)$, with all of its derivatives. We suppose M and X are connected, that is, we allow M to have *only one end.*

This is because one can use Cheeger–Gromoll splitting theorem [4] to show that an orientable, connected, asymptotically cylindrical Riemannian manifold with Ricci-flat metric g can have at most 2 cylindrical ends. In the case when there are 2 cylindrical ends then there is reduction in the holonomy group $\mathrm{Hol}(g)$ and (M,g) is a cylinder. One can also show that reduction in holonomy can be obtained by just using the analytic set-up for Fredholm properties of an elliptic operator on noncompact manifolds, [25].

Here are the analogous definitions for coassociative submanifolds.

Definition 2.8 Let (M_0,φ_0,g_0) and X be as in Definition 2.6. A submanifold C_0 of M_0 is called *cylindrical* if $C_0 = L\times\mathbb{R}$ for some compact submanifold L in X, not necessarily connected. C_0 is *coassociative* if and only if L is a *special Lagrangian* 3*–fold with phase* i in the Calabi–Yau 3–fold X.

Definition 2.9 Let (M_0,φ_0,g_0), X, (M,φ,g), K,Ψ and α be as in Definitions 2.6 and 2.7, and let $C_0 = L\times\mathbb{R}$ be a cylindrical coassociative 4–fold in M_0, as in Definition 2.8.

A connected, complete coassociative 4–fold C in (M,φ,g) is called *asymptotically cylindrical with decay rate* β for $\alpha\leqslant\beta<0$ if there exists a compact subset $K'\subset C$, a normal vector field v on $L\times(R',\infty)$ for some $R'>R$, and a diffeomorphism $\Phi\colon L\times(R',\infty)\to C\setminus K'$ such that the diagram

$$\begin{tikzcd} X\times(R',\infty) \arrow[d, "\subset"] & L\times(R',\infty) \arrow[l, "\exp_v"'] \arrow[r, "\Phi"'] & (C\setminus K') \arrow[d, "\subset"] \\ X\times(R,\infty) \arrow[rr, "\Psi"] & & (M\setminus K) \end{tikzcd} \tag{3}$$

commutes, and $\left|\nabla^k v\right| = O(e^{\beta t})$ on $L\times(R',\infty)$ for all $k\geqslant 0$.

Here we require C but *not* L to be connected, that is, we allow C to have *multiple ends.* The point of Definition 2.9 is to find a good way to say that a submanifold C in M is asymptotic to the cylinder C_0 in $M_0 = X \times \mathbb{R}$ as $t \to \infty$ in $\mathbb{R}$, to order $O(e^{\beta t})$. We do this by writing C near infinity as the graph of a *normal vector field* v to $C_0 = L \times \mathbb{R}$ in $M_0 = X \times \mathbb{R}$, and requiring v and its derivatives to be $O(e^{\beta t})$.

3 Infinitesimal deformations of C

Let (M, φ, g) be an asymptotically cylindrical G_2–manifold asymptotic to $X \times (R, \infty)$, and C an asymptotically cylindrical coassociative 4–fold in M asymptotic to $L \times (R', \infty)$. We wish to study the moduli space $\mathcal{M}_C^\gamma$ of asymptotically cylindrical deformations $\widetilde{C}$ of C in M with rate γ. To do this we modify the proof of Theorem 2.5 in Section 2, for the case when C is compact. There we modelled $\mathcal{M}_C$ on $\widetilde{P}^{-1}(0)$ for a nonlinear map $\widetilde{P}$ between Banach spaces, whose linearization $\mathrm{d}\widetilde{P}(0,0)$ at 0 was the Fredholm map between Sobolev spaces

$$\mathrm{d}_+ + \mathrm{d}^* \colon L^p_{l+2}(\Lambda^2_+ T^*C) \times L^p_{l+2}(\Lambda^4 T^*C) \longrightarrow L^p_{l+1}(\Lambda^3 T^*C). \tag{4}$$

Now when C is not compact, as in the asymptotically cylindrical case, (4) is not in general Fredholm, and the proof of Theorem 2.5 fails. To repair it we use the analytical framework for asymptotically cylindrical manifolds developed by Lockhart and McOwen in [15, 16], involving *weighted Sobolev spaces* $L^p_{k,\gamma}(\Lambda^r T^*C)$. Roughly speaking, elements of $L^p_{k,\gamma}(\Lambda^r T^*C)$ are L^p_k r–forms on C which decay like $O(e^{\gamma t})$ on the end $L \times (R', \infty)$. This has the advantage of building the decay rate γ into the proof from the outset.

This section will study the weighted analogue of (4),

$$\mathrm{d}_+ + \mathrm{d}^* \colon L^p_{l+2,\gamma}(\Lambda^2_+ T^*C) \times L^p_{l+2,\gamma}(\Lambda^4 T^*C) \longrightarrow L^p_{l+1,\gamma}(\Lambda^3 T^*C), \tag{5}$$

for small $\gamma < 0$. It will be shown in Section 4 to be the linearization at 0 of a nonlinear operator P for which $\mathcal{M}_C^\gamma$ is locally modelled on $P^{-1}(0)$.

Section 3.1 introduces weighted Sobolev spaces, and the Lockhart–McOwen theory of elliptic operators between them. Then Sections 3.2 and 3.3 compute the *kernel* and *cokernel* of (5) for small $\gamma < 0$, and Section 3.4 determines the set of rates γ for which (5) is *Fredholm.*

3.1 Elliptic operators on asymptotically cylindrical manifolds

We now sketch parts of the theory of analysis on manifolds with cylindrical ends due to Lockhart and McOwen [15, 16]. We begin with some elementary definitions.

Definition 3.1 Let (C, g) be an *asymptotically cylindrical Riemannian manifold.* That is, there is a Riemannian cylinder $(L \times \mathbb{R}, g_0)$ with L compact, a compact subset $K' \subset C$ and a diffeomorphism $\Phi\colon C \setminus K' \to L \times (R', \infty)$ such that

$$\nabla_0^k\big(\Phi_*(g) - g_0\big) = O(e^{\beta t}) \quad \text{for all } k \geqslant 0$$

for some rate $\beta < 0$, where ∇_0 is the Levi–Civita connection of g_0 on $L \times \mathbb{R}$.

Let E_0 be a *cylindrical vector bundle* on $L \times \mathbb{R}$, that is, a vector bundle on $L \times \mathbb{R}$ invariant under translations in $\mathbb{R}$. Let h_0 be a *smooth family of metrics* on the fibres of E_0 and ∇_{E_0} a *connection* on E_0 preserving h_0, with h_0, ∇_{E_0} invariant under translations in $\mathbb{R}$.

Let E be a vector bundle on C equipped with metrics h on the fibres, and a connection ∇_E on E preserving h. We say that E, h, ∇_E are *asymptotic to* E_0, h_0, ∇_{E_0} if there exists an identification $\Phi_*(E) \cong E_0$ on $L \times (R', \infty)$ such that $\Phi_*(h) = h_0 + O(e^{\beta t})$ and $\Phi_*(\nabla_E) = \nabla_{E_0} + O(e^{\beta t})$ as $t \to \infty$. Then we call E, h, ∇_E *asymptotically cylindrical.*

Choose a smooth function $\rho\colon C \to \mathbb{R}$ such that $\Phi^*(\rho) \equiv t$ on $L \times (R', \infty)$. This prescribes ρ on $C \setminus K'$, so we only have to extend ρ over the compact set K'. For $p \geq 1$, $k \geq 0$ and $\gamma \in \mathbb{R}$ we define the *weighted Sobolev space* $L^p_{k,\gamma}(E)$ to be the set of sections s of E that are locally integrable and k times weakly differentiable and for which the norm

$$\|s\|_{L^p_{k,\gamma}} = \Big(\sum_{j=0}^{k} \int_C e^{-\gamma\rho} \big|\nabla_E^j s\big|^p \mathrm{d}V\Big)^{1/p} \tag{6}$$

is finite. Then $L^p_{k,\gamma}(E)$ is a Banach space. Since ρ is uniquely determined except on the compact set K', different choices of ρ give the same space $L^p_{k,\gamma}(E)$, with equivalent norms.

For instance, the r–forms $E = \Lambda^r T^*C$ on C with metric g and the Levi–Civita connection are automatically asymptotically cylindrical, and if C is an oriented 4–manifold then the self-dual 2–forms $\Lambda^2_+ T^*C$ are also asymptotically cylindrical. We consider *partial differential operators* on asymptotically cylindrical manifolds.

Definition 3.2 In the situation of Definition 3.1, suppose E, F are two asymptotically cylindrical vector bundles on C, asymptotic to cylindrical vector bundles E_0, F_0 on $L \times \mathbb{R}$. Let $A_0 \colon C^\infty(E_0) \to C^\infty(F_0)$ be a linear partial differential operator of order k which is *cylindrical*, that is, invariant under translations in $\mathbb{R}$.

Suppose $A \colon C^\infty(E) \to C^\infty(F)$ is a linear partial differential operator of order k on C. We say that A is *asymptotic to* A_0 if under the identifications $\Phi_*(E) \cong E_0$, $\Phi_*(F) \cong F_0$ on $L \times (R', \infty)$ we have $\Phi_*(A) = A_0 + O(e^{\beta t})$ as $t \to \infty$ for $\beta < 0$. Then we call A an *asymptotically cylindrical* operator. It is easy to show that A extends to bounded linear operators

$$A^p_{k+l,\gamma} \colon L^p_{k+l,\gamma}(E) \longrightarrow L^p_{l,\gamma}(F) \tag{7}$$

for all $p > 1$, $l \geqslant 0$ and $\gamma \in \mathbb{R}$.

Now suppose A is an *elliptic* operator. (7) is *Fredholm* if and only if γ does not lie in a discrete set $\mathcal{D}_{A_0} \subset \mathbb{R}$, which we now define.

Definition 3.3 In Definition 3.2, suppose A and A_0 are *elliptic* operators on C and $L \times \mathbb{R}$, so that E, F have the same fibre dimensions. Extend A_0 to the complexifications $A_0 \colon C^\infty(E_0 \otimes_{\mathbb{R}} \mathbb{C}) \to C^\infty(F_0 \otimes_{\mathbb{R}} \mathbb{C})$. Define $\mathcal{D}_{A_0}$ to be the set of $\gamma \in \mathbb{R}$ such that for some $\delta \in \mathbb{R}$ there exists a nonzero section $s \in C^\infty(E_0 \otimes_{\mathbb{R}} \mathbb{C})$ invariant under translations in $\mathbb{R}$ such that $A_0(e^{(\gamma + i\delta)t} s) = 0$.

Then Lockhart and McOwen prove [16, Theorem 1.1]:

Theorem 3.4 *Let (C, g) be a Riemannian manifold asymptotic to $(L \times \mathbb{R}, g_0)$, and $A \colon C^\infty(E) \to C^\infty(F)$ an elliptic partial differential operator on C of order k between vector bundles E, F on C, asymptotic to the cylindrical elliptic operator $A_0 : C^\infty(E_0) \to C^\infty(F_0)$ on $L \times \mathbb{R}$. Define $\mathcal{D}_{A_0}$ as above.*

Then $\mathcal{D}_{A_0}$ is a discrete subset of $\mathbb{R}$, and for $p > 1$, $l \geqslant 0$ and $\gamma \in \mathbb{R}$, the extension $A^p_{k+l,\gamma} \colon L^p_{k+l,\gamma}(E) \to L^p_{l,\gamma}(F)$ is Fredholm if and only if $\gamma \notin \mathcal{D}_{A_0}$.

Suppose $\gamma \notin \mathcal{D}_{A_0}$. Then $A^p_{k+l,\gamma}$ is Fredholm, so its kernel $\operatorname{Ker}(A^p_{k+l,\gamma})$ is finite-dimensional. Let $e \in \operatorname{Ker}(A^p_{k+l,\gamma})$. Then by an elliptic regularity result [15, Theorem 3.7.2] we have $e \in L^p_{k+m,\gamma}(E)$ for all $m \geqslant 0$. The *weighted Sobolev Embedding Theorem* [15, Theorem 3.10] then implies that $e \in L^r_{k+m,\delta}(E)$ for all $r > 1$, $m \geqslant 0$ and $\delta > \gamma$, and e is smooth. But $\operatorname{Ker}(A^p_{k+1,\gamma})$ is invariant under small changes of γ in $\mathbb{R} \setminus \mathcal{D}_{A_0}$, so $e \in L^r_{k+m,\gamma}(E)$ for all $r > 1$ and $m \geqslant 0$. This proves:

Proposition 3.5 *For $\gamma \notin \mathcal{D}_{A_0}$ the kernel $\mathrm{Ker}(A^p_{k+l,\gamma})$ is independent of p,l, and is a finite-dimensional vector space of smooth sections of E.*

When $\gamma \notin \mathcal{D}_{A_0}$, as $A^p_{k+l,\gamma}$ is Fredholm the *cokernel*

$$\mathrm{Coker}(A^p_{k+l,\gamma}) = L^p_{l,\gamma}(F)/A^p_{k+l,\gamma}\big(L^p_{k+l,\gamma}(E)\big)$$

of $A^p_{k+l,\gamma}$ is also finite-dimensional. To understand it, consider the *formal adjoint* $A^*\colon C^\infty(F) \to C^\infty(E)$ of A. This is also an asymptotically cylindrical linear elliptic partial differential operator of order k on C, with the property that

$$\langle Ae, f\rangle_{L^2(F)} = \langle e, A^* f\rangle_{L^2(E)}$$

for compactly-supported $e \in C^\infty(E)$ and $f \in C^\infty(F)$.

Then for $p > 1$, $l \geqslant 0$ and $\gamma \notin \mathcal{D}_{A_0}$, the *dual operator* of (7) is

$$(A^*)^q_{-l,-\gamma}\colon L^q_{-l,-\gamma}(F) \longrightarrow L^q_{-k-l,-\gamma}(E), \tag{8}$$

where $q > 1$ is defined by $\frac{1}{p} + \frac{1}{q} = 1$. Here we mean that $L^q_{-k-l,-\gamma}(E)$, $L^q_{-l,-\gamma}(F)$ are isomorphic to the Banach space duals of $L^p_{k+l,\gamma}(E)$, $L^p_{l,\gamma}(F)$, and these isomorphisms identify $(A^*)^q_{-l,-\gamma}$ with the dual linear map to (7).

Now there is a *problem* with (8), as it involves Sobolev spaces with *negative numbers of derivatives* $-l, -k-l$. Such Sobolev spaces exist as spaces of *distributions*. But we can avoid defining or using these spaces, by the following trick. We are interested in $\mathrm{Ker}\big((A^*)^q_{-l,-\gamma}\big)$, as it is dual to $\mathrm{Coker}(A^p_{k+l,\gamma})$. The elliptic regularity argument above showing $\mathrm{Ker}(A^p_{k+l,\gamma})$ is independent of l also holds for negative differentiability, so we have $\mathrm{Ker}\big((A^*)^q_{-l,-\gamma}\big) = \mathrm{Ker}\big((A^*)^q_{k+m,-\gamma}\big)$ for $m \in \mathbb{Z}$, and in particular for $m \geqslant 0$. So we deduce:

Proposition 3.6 *In Theorem 3.4, let A^* be the formal adjoint of A. Then for all $\gamma \notin \mathcal{D}_{A_0}$, $p,q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $l,m \geqslant 0$ there is a natural isomorphism*

$$\mathrm{Coker}(A^p_{k+l,\gamma}) \cong \mathrm{Ker}\big((A^*)^q_{k+m,-\gamma}\big)^*. \tag{9}$$

When $\gamma \notin \mathcal{D}_{A_0}$ we see from (9) that the *index* of $A^p_{k+l,\gamma}$ is

$$\mathrm{ind}(A^p_{k+l,\gamma}) = \dim \mathrm{Ker}(A^p_{k+l,\gamma}) - \dim \mathrm{Ker}\big((A^*)^q_{k+m,-\gamma}\big). \tag{10}$$

Lockhart and McOwen show [16, Theorem 6.2] that for $\gamma, \delta \in \mathbb{R} \setminus \mathcal{D}_{A_0}$ with $\gamma \leqslant \delta$ we have

$$\mathrm{ind}(A^p_{k+l,\delta}) - \mathrm{ind}(A^p_{k+l,\gamma}) = \sum_{\epsilon \in \mathcal{D}_{A_0}:\gamma<\epsilon<\delta} d(\epsilon), \tag{11}$$

where $d(\epsilon) \geqslant 1$ is the dimension of the a vector space of solutions $s \in C^\infty(E_0 \otimes_{\mathbb{R}} \mathbb{C})$ of a prescribed form with $A_0(s) = 0$.

3.2 $\mathrm{d}+\mathrm{d}^*$ and $\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*$ on an asymptotically cylindrical manifold

Let (C, g) be an oriented asymptotically cylindrical Riemannian n–manifold asymptotic to a Riemannian cylinder $(L \times \mathbb{R}, g_0)$, where $g_0 = g_L + \mathrm{d}t^2$ and (L, g_L) is a compact oriented Riemannian $(n-1)$–manifold. Consider the *asymptotically cylindrical linear elliptic operators*

$$\mathrm{d}+\mathrm{d}^* \quad \text{and} \quad \mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*\colon \ \textstyle\bigoplus_{k=0}^n C^\infty(\Lambda^k T^*C) \longrightarrow \bigoplus_{k=0}^n C^\infty(\Lambda^k T^*C). \tag{12}$$

We shall apply the theory of Section 3.1 to study the extensions

$$(\mathrm{d}+\mathrm{d}^*)^p_{l+2,\gamma}\colon \ \textstyle\bigoplus_{k=0}^n L^p_{l+2,\gamma}(\Lambda^k T^*C) \longrightarrow \bigoplus_{k=0}^n L^p_{l+1,\gamma}(\Lambda^k T^*C), \tag{13}$$

$$(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)^p_{l+2,\gamma}\colon \ \textstyle\bigoplus_{k=0}^n L^p_{l+2,\gamma}(\Lambda^k T^*C) \longrightarrow \bigoplus_{k=0}^n L^p_{l,\gamma}(\Lambda^k T^*C), \tag{14}$$

for $p > 1$, $l \geqslant 0$ and $\gamma \in \mathbb{R}$, and their *kernels* and *cokernels*.

Lemma 3.7 *We have* $\mathrm{Ker}\big((\mathrm{d}+\mathrm{d}^*)^p_{l+2,\gamma}\big) \subseteq \mathrm{Ker}\big((\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)^p_{l+2,\gamma}\big)$ *for all* $p > 1$, $l \geqslant 0$ *and* $\gamma \in \mathbb{R}$, *and equality holds if* $\gamma < 0$.

Proof Since $\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^* = (\mathrm{d}+\mathrm{d}^*)^2$ we have $\mathrm{Ker}(\mathrm{d}+\mathrm{d}^*) \subseteq \mathrm{Ker}(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)$ on any space of twice differentiable forms, giving the inclusion. Suppose $\gamma < 0$ and $\chi \in \mathrm{Ker}\big((\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)^p_{l+2,\gamma}\big)$. Write $\chi = \sum_{k=0}^n \chi_k$ for $\chi_k \in L^p_{l+2,\gamma}(\Lambda^k T^*C)$. Then $\chi_k \in \mathrm{Ker}\big((\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)^p_{l+2,\gamma}\big)$, as $\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*$ takes k–forms to k–forms.

If $\gamma < 0$ then each χ_k lies in $L^2_2(\Lambda^k T^*C)$, and

$$\|\mathrm{d}\chi_k\|^2_{L^2} + \|\mathrm{d}^*\chi_k\|^2_{L^2} = \langle \mathrm{d}\chi_k, \mathrm{d}\chi_k\rangle_{L^2} + \langle \mathrm{d}^*\chi_k, \mathrm{d}^*\chi_k\rangle_{L^2} = \langle \chi_k, (\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)\chi_k\rangle_{L^2} = 0.$$

Thus $\mathrm{d}^*\chi_k = \mathrm{d}\chi_k = 0$, so that χ_k and hence χ lies in $\mathrm{Ker}\big((\mathrm{d}+\mathrm{d}^*)^p_{l+2,\gamma}\big)$. □

For $|\gamma|$ close to zero we can say more about the kernels of (13) and (14).

Proposition 3.8 *Suppose* $p, q > 1$, $l, m \geqslant 0$ *and* $\gamma < 0$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $[\gamma, -\gamma] \cap \mathcal{D}_{(\mathrm{d}+\mathrm{d}^*)_0} = [\gamma, -\gamma] \cap \mathcal{D}_{(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)_0} = \{0\}$. *Then*

$$\mathrm{Ker}\big((\mathrm{d}+\mathrm{d}^*)^p_{l+2,\gamma}\big) = \mathrm{Ker}\big((\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)^p_{l+2,\gamma}\big), \tag{15}$$

$$\mathrm{Ker}\big((\mathrm{d}+\mathrm{d}^*)^q_{m+2,-\gamma}\big) = \mathrm{Ker}\big((\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)^q_{m+2,-\gamma}\big), \qquad \textit{and} \tag{16}$$

$$\dim \mathrm{Ker}\big((\mathrm{d}+\mathrm{d}^*)^q_{m+2,-\gamma}\big) = \dim \mathrm{Ker}\big((\mathrm{d}+\mathrm{d}^*)^p_{l+2,\gamma}\big) + \textstyle\sum_{k=0}^{n-1} b^k(L). \tag{17}$$

Moreover all four kernels consist of smooth closed and coclosed forms.

Proof As $[\gamma,-\gamma]\cap\mathcal{D}_{(\mathrm{d}+\mathrm{d}^*)_0}=\{0\}$,

$$\mathrm{ind}\big((\mathrm{d}+\mathrm{d}^*)^q_{m+2,-\gamma}\big)-\mathrm{ind}\big((\mathrm{d}+\mathrm{d}^*)^p_{l+2,\gamma}\big)=2\textstyle\sum_{k=0}^{n-1}b^k(L). \tag{18}$$

This is because from (11), the l.h.s. of (18) is the dimension of the solution space of $(\mathrm{d}+\mathrm{d}^*)_0\chi=0$ on $L\times\mathbb{R}$ for χ independent of $t\in\mathbb{R}$. The space of such χ is the direct sum over $k=0,\ldots,n-1$ of the spaces of k–forms η and $(k{+}1)$–forms $\eta\wedge\mathrm{d}t$ for $\eta\in C^\infty(\Lambda^kT^*L)$ with $\mathrm{d}\eta=\mathrm{d}^*\eta=0$. By Hodge theory we deduce (18).

Now $\mathrm{d}{+}\mathrm{d}^*$ is *formally self adjoint*, that is, $A^*=A$ in the notation of Section 3.1. Thus

$$\mathrm{ind}\big((\mathrm{d}+\mathrm{d}^*)^q_{m+2,-\gamma}\big)=-\,\mathrm{ind}\big((\mathrm{d}+\mathrm{d}^*)^p_{l+2,\gamma}\big)=$$
$$\dim\mathrm{Ker}\big((\mathrm{d}+\mathrm{d}^*)^q_{m+2,-\gamma}\big)-\dim\mathrm{Ker}\big((\mathrm{d}+\mathrm{d}^*)^p_{l+2,\gamma}\big)$$

by (10), and equation (17) follows from (18). As $[\gamma,-\gamma]\cap\mathcal{D}_{(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)_0}=\{0\}$, the same proof shows that

$$\dim\mathrm{Ker}\big((\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)^q_{m+2,-\gamma}\big)=\dim\mathrm{Ker}\big((\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)^p_{l+2,\gamma}\big)+\textstyle\sum_{k=0}^{n-1}b^k(L), \tag{19}$$

since the solutions of $(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)_0\chi=0$ and $(\mathrm{d}+\mathrm{d}^*)_0\chi=0$ for χ on $L\times\mathbb{R}$ independent of t coincide. Lemma 3.7 proves (15), and combining this with (17) and (19) yields

$$\dim\mathrm{Ker}\big((\mathrm{d}+\mathrm{d}^*)^q_{m+2,-\gamma}\big)=\dim\mathrm{Ker}\big((\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)^q_{m+2,-\gamma}\big).$$

As the right hand side of (16) contains the left by Lemma 3.7, this implies (16).

It remains to show the four kernels consist of smooth closed and coclosed forms. Let χ lie in one of the kernels, and write $\chi=\sum_{k=0}^n\chi_k$ for χ_k a k–form. Since $(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)\chi=0$ we have $(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)\chi_k=0$, as $\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*$ takes k–forms to k–forms. Thus χ_k lies in the same kernel, so $(\mathrm{d}+\mathrm{d}^*)\chi_k=0$ by (15) or (16). But $\mathrm{d}\chi_k$ and $\mathrm{d}^*\chi_k$ lie in different vector spaces, so $\mathrm{d}\chi_k=\mathrm{d}^*\chi_k=0$ for all k. Hence $\mathrm{d}\chi=\mathrm{d}^*\chi=0$, and χ is closed and coclosed. Smoothness follows by elliptic regularity. □

As the forms χ in $\mathrm{Ker}\big((\mathrm{d}+\mathrm{d}^*)^p_{l+2,\gamma}\big)$ are closed we can map them to de Rham cohomology $H^*(C,\mathbb{R})$ by $\chi\mapsto[\chi]$. We identify the kernel and image of this map.

Proposition 3.9 *Suppose $p>1$, $l\geqslant 0$ and $\gamma<0$ with $[\gamma,-\gamma]\cap\mathcal{D}_{(\mathrm{d}+\mathrm{d}^*)_0}=[\gamma,-\gamma]\cap\mathcal{D}_{(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)_0}=\{0\}$. Then the map $\mathrm{Ker}\big((\mathrm{d}+\mathrm{d}^*)^p_{l+2,\gamma}\big)\to H^*(C,\mathbb{R})$ given by $\chi\mapsto[\chi]$ is injective, with image that of the natural map $H^*_{\mathrm{cs}}(C,\mathbb{R})\to H^*(C,\mathbb{R})$.*

Proof Lockhart [15, Ex. 0.14] shows that the vector space $\mathcal{H}^2(\Lambda^k T^*C, g)$ of closed, coclosed k–forms in $L^2(\Lambda^k T^*C)$ on an asymptotically cylindrical Riemannian manifold (C,g) is isomorphic under $\chi \mapsto [\chi]$ with the image of $H^k_{\mathrm{cs}}(C,\mathbb{R})$ in $H^k(C,\mathbb{R})$. Taking the direct sum over $k=0,\ldots,n$, this implies that for $l \geqslant 0$ the map

$$\mathrm{Ker}\big((\mathrm{d}+\mathrm{d}^*)^2_{l+2,0}\big) \to H^*(C,\mathbb{R}), \qquad \chi \mapsto [\chi]$$

is injective, with image that of the natural map $H^*_{\mathrm{cs}}(C,\mathbb{R}) \to H^*(C,\mathbb{R})$.

Using Proposition 3.5, $\gamma \notin \mathcal{D}_{(\mathrm{d}+\mathrm{d}^*)_0}$ and $\gamma < 0$ we have

$$\mathrm{Ker}\big((\mathrm{d}+\mathrm{d}^*)^p_{l+2,\gamma}\big) = \mathrm{Ker}\big((\mathrm{d}+\mathrm{d}^*)^2_{l+2,\gamma}\big) \subseteq \mathrm{Ker}\big((\mathrm{d}+\mathrm{d}^*)^2_{l+2,0}\big).$$

Therefore $\chi \mapsto [\chi]$ is *injective* on $\mathrm{Ker}\big((\mathrm{d}+\mathrm{d}^*)^p_{l+2,\gamma}\big)$, with image *contained in* that of $H^*_{\mathrm{cs}}(C,\mathbb{R}) \to H^*(C,\mathbb{R})$. It remains to show $\chi \mapsto [\chi]$ is *surjective* on this image.

Suppose $\eta \in H^j(C,\mathbb{R})$ lies in the image of $H^j_{\mathrm{cs}}(C,\mathbb{R})$. Then we may write $\eta = [\phi]$ for ϕ a smooth, closed, compactly-supported j–form on C. Hence $\mathrm{d}^*\phi \in \bigoplus_{k=0}^n L^p_{l+1,\gamma}(\Lambda^k T^*C)$. We shall show that $\mathrm{d}^*\phi$ lies in the image of (14) with $l+1$ in place of l. Since $\gamma \notin \mathcal{D}_{(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)_0}$, as in Section 3.1 this holds if and only if $\langle \mathrm{d}^*\phi, \xi\rangle_{L^2} = 0$ for all ξ in $\mathrm{Ker}\big((\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)^q_{m+2,-\gamma}\big)$.

But all such ξ are *closed* by Proposition 3.8, so $\langle \mathrm{d}^*\phi, \xi\rangle_{L^2} = \langle \phi, \mathrm{d}\xi\rangle_{L^2} = 0$. Therefore $\mathrm{d}^*\phi = (\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)\psi$ for some $\psi \in L^p_{l+3,\gamma}(\Lambda^{j-1}T^*C)$. Hence

$$(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)(\phi - \mathrm{d}\psi) = \mathrm{d}\big(\mathrm{d}^*\phi - (\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)\psi\big) = 0,$$

and $\phi - \mathrm{d}\psi$ lies in $\mathrm{Ker}\big((\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)^p_{l+2,\gamma}\big)$, which is $\mathrm{Ker}\big((\mathrm{d}+\mathrm{d}^*)^p_{l+2,\gamma}\big)$ by (15). As $[\phi - \mathrm{d}\psi] = [\phi] = \eta$ we have proved the surjectivity we need. □

3.3 $\mathrm{d}_+ + \mathrm{d}^*$ on a 4–manifold

Now we restrict to $\dim C = 4$, so that (C,g) is an oriented asymptotically cylindrical Riemannian 4–manifold asymptotic to a Riemannian cylinder $(L\times\mathbb{R}, g_0)$. In Section 4 we will take C to be an *asymptotically cylindrical coassociative* 4–*fold.* Consider the *asymptotically cylindrical linear elliptic operator*

$$\mathrm{d}_+ + \mathrm{d}^* \colon\ C^\infty(\Lambda^2_+ T^*C) \oplus C^\infty(\Lambda^4 T^*C) \longrightarrow C^\infty(\Lambda^3 T^*C).$$

Here d_+ is the restriction of d to the self-dual 2–forms. We use this notation to distinguish $\mathrm{d}_+ + \mathrm{d}^*$ from $\mathrm{d}+\mathrm{d}^*$ in (12). Roughly speaking, $\mathrm{d}_+ + \mathrm{d}^*$ is a *quarter* of $\mathrm{d}+\mathrm{d}^*$ in (12), as it acts on half of the even forms, rather than on all forms. Its *formal adjoint* is

$$\mathrm{d}_+^* + \mathrm{d}\colon\ C^\infty(\Lambda^3T^*C) \longrightarrow C^\infty(\Lambda^2_+T^*C)\oplus C^\infty(\Lambda^4T^*C),$$

where d_+^* is the projection of d^* to the self-dual 2–forms. We shall apply the results of Section 3.2 to study the extension

$$(\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2,\gamma}\colon\ L^p_{l+2,\gamma}(\Lambda^2_+T^*C)\oplus L^p_{l+2,\gamma}(\Lambda^4T^*C) \longrightarrow L^p_{l+1,\gamma}(\Lambda^3T^*C), \quad (20)$$

for $p>1$, $l\geqslant 0$ and $\gamma\in\mathbb{R}$. We begin with some algebraic topology.

Suppose for simplicity that C has no compact connected components, so that $H^4(C,\mathbb{R}) = H^0_{\mathrm{cs}}(C,\mathbb{R}) = 0$. Then L is a compact, oriented 3–manifold, and C is the interior $(\overline{C})^\circ$ of a compact, oriented 4–manifold $\overline{C}$ with boundary $\partial\overline{C} = L$. Thus we have a long exact sequence in cohomology:

$$\begin{array}{c} 0 \to H^0(C) \to H^0(L) \to H^1_{\mathrm{cs}}(C) \to H^1(C) \to H^1(L) \to H^2_{\mathrm{cs}}(C) \\ \downarrow \\ 0 \leftarrow H^4_{\mathrm{cs}}(C) \leftarrow H^3(L) \leftarrow H^3(C) \leftarrow H^3_{\mathrm{cs}}(C) \leftarrow H^2(L) \leftarrow H^2(C) \end{array} \quad (21)$$

where $H^k(C) = H^k(C,\mathbb{R})$ and $H^k(L) = H^k(L,\mathbb{R})$ are the de Rham cohomology groups, and $H^k_{\mathrm{cs}}(C,\mathbb{R})$ is *compactly-supported de Rham cohomology.* Let $b^k(C)$, $b^k(L)$ and $b^k_{\mathrm{cs}}(C)$ be the corresponding Betti numbers.

By *Poincaré duality* we have $H^k(C)\cong H^{4-k}_{\mathrm{cs}}(C)^*$ and $H^k(L)\cong H^{3-k}(L)^*$, so that $b^k(C) = b^{4-k}_{\mathrm{cs}}(C)$ and $b^k(L) = b^{3-k}(L)$. Note that (21) is written so that each vertically aligned pair of spaces are dual vector spaces, and each vertically aligned pair of maps are dual linear maps.

Let $V\subseteq H^2(C,\mathbb{R})$ be the image of the natural map $H^2_{\mathrm{cs}}(C,\mathbb{R})\to H^2(C,\mathbb{R})$. Taking alternating sums of dimensions in (21) shows that

$$\begin{aligned}\dim V &= b^2_{\mathrm{cs}}(C) - b^1(L) + b^1(C) - b^1_{\mathrm{cs}}(C) - b^0(L) + b^0(C)\\ &= b^0(C)+b^1(C)+b^2(C)-b^3(C)-b^0(L)-b^1(L).\end{aligned}$$

Now the cup product $\cup\colon\ H^2_{\mathrm{cs}}(C,\mathbb{R})\times H^2(C,\mathbb{R})\to\mathbb{R}$ restricted to $H^2_{\mathrm{cs}}(C,\mathbb{R})\times V$ is zero on the product of the kernel of $H^2_{\mathrm{cs}}(C,\mathbb{R})\to H^2(C,\mathbb{R})$ with V. Hence it pushes forward to a quadratic form $\cup\colon\ V\times V\to\mathbb{R}$, which is *symmetric* and *nondegenerate.*

Suppose $V = V_+\oplus V_-$ is a decomposition of V into subspaces with $\cup$ positive definite on V_+ and negative definite on V_-. Then $\dim V_+$ and $\dim V_-$ are *topological invariants* of C,L. That is, they depend only on C as an oriented 4–manifold, and not on the choice of subspaces $V_\pm$.

We now identify the kernel and cokernel of $(\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2,\gamma}$ in (20) for small $\gamma<0$.

Theorem 3.10 *Let (C,g) be an oriented, asymptotically cylindrical Riemannian 4–manifold asymptotic to $(L\times\mathbb{R},g_0)$, and use the notation of Sections 3.1–3.2. Suppose $\max\bigl(\mathcal{D}_{(\mathrm{d}_+ +\mathrm{d}^*)_0}\cap(-\infty,0)\bigr)<\gamma<0$, and let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $l,m\geqslant 0$. Then from Section 3.1 the operator $(\mathrm{d}_+ +\mathrm{d}^*)^p_{l+2,\gamma}$ of (20) is Fredholm with*

$$\operatorname{Coker}\bigl((\mathrm{d}_+ +\mathrm{d}^*)^p_{l+2,\gamma}\bigr)\cong\operatorname{Ker}\bigl((\mathrm{d}^*_+ +\mathrm{d})^q_{m+2,-\gamma}\bigr)^*. \tag{22}$$

The kernel $\operatorname{Ker}\bigl((\mathrm{d}_+ +\mathrm{d}^*)^p_{l+2,\gamma}\bigr)$ is a vector space of smooth, closed, self-dual 2–forms. The map $\operatorname{Ker}\bigl((\mathrm{d}_+ +\mathrm{d}^*)^p_{l+2,\gamma}\bigr)\to H^2(C,\mathbb{R})$, $\chi\mapsto[\chi]$ induces an isomorphism of $\operatorname{Ker}\bigl((\mathrm{d}_+ +\mathrm{d}^*)^p_{l+2,\gamma}\bigr)$ with a maximal subspace V_+ of the subspace $V\subseteq H^2(C,\mathbb{R})$ defined above on which the cup product $\cup\colon V\times V\to\mathbb{R}$ is positive definite. Hence

$$\dim\operatorname{Ker}\bigl((\mathrm{d}_+ +\mathrm{d}^*)^p_{l+2,\gamma}\bigr)=\dim V_+, \tag{23}$$

which is a topological invariant of C,L from above. Also, $\operatorname{Ker}\bigl((\mathrm{d}^*_+ +\mathrm{d})^q_{m+2,-\gamma}\bigr)$ is a vector space of smooth, closed and coclosed 3–forms.

Proof The first part follows immediately from Section 3.1. As $\operatorname{Ker}\bigl((\mathrm{d}_+ +\mathrm{d}^*)^p_{l+2,\gamma}\bigr)$ and $\operatorname{Ker}\bigl((\mathrm{d}^*_+ +\mathrm{d})^q_{m+2,-\gamma}\bigr)$ depend only on the connected component of $\mathbb{R}\setminus\mathcal{D}_{(\mathrm{d}_+ +\mathrm{d}^*)_0}$ containing γ, we can make $|\gamma|$ smaller if necessary to ensure that $[\gamma,-\gamma]\cap\mathcal{D}_{(\mathrm{d}+\mathrm{d}^*)_0}=[\gamma,-\gamma]\cap\mathcal{D}_{(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)_0}=\{0\}$, using the notation of Section 3.2.

Suppose $(\zeta^2_+,\zeta^4)\in\operatorname{Ker}\bigl((\mathrm{d}_+ +\mathrm{d}^*)^p_{l+2,\gamma}\bigr)$. Then $\mathrm{d}\zeta^2_+ +\mathrm{d}^*\zeta^4=0$, so applying $*$ and noting that $*\zeta^2_+=\zeta^2_+$ gives $\mathrm{d}^*\zeta^2_+ -\mathrm{d}(*\zeta^4)=0$. Hence $(\mathrm{d}+\mathrm{d}^*)(-*\zeta^4+\zeta^2_+ +\zeta^4)=0$, that is, the mixed form $-*\zeta^4+\zeta^2_+ +\zeta^4$ lies in the kernel of (13). Proposition 3.8 now implies that ζ^2_+ and ζ^4 are smooth, closed and coclosed, and therefore ζ^2_+ and ζ^4 also lie in the kernel, and $\operatorname{Ker}\bigl((\mathrm{d}_+ +\mathrm{d}^*)^p_{l+2,\gamma}\bigr)\subseteq\operatorname{Ker}\bigl((\mathrm{d}+\mathrm{d}^*)^p_{l+2,\gamma}\bigr)$, where $(\mathrm{d}+\mathrm{d}^*)^p_{l+2,\gamma}$ is as in Section 3.2.

Since $H^4(C,\mathbb{R})=0$, injectivity in Proposition 3.9 implies that $\zeta^4=0$. Thus $\operatorname{Ker}\bigl((\mathrm{d}_+ +\mathrm{d}^*)^p_{l+2,\gamma}\bigr)$ is a vector space of smooth, closed, self-dual 2–forms, as we have to prove. Write $\mathcal{H}^2$ for the space of 2–forms in $\operatorname{Ker}\bigl((\mathrm{d}+\mathrm{d}^*)^p_{l+2,\gamma}\bigr)$. Then by Proposition 3.9 the map $\mathcal{H}^2\to H^2(C,\mathbb{R})$, $\chi\mapsto[\chi]$ is injective with image that of $H^2_{\mathrm{cs}}(C,\mathbb{R})$ in $H^2(C,\mathbb{R})$, which is V in the notation above.

Under this isomorphism, the cup product on V is given by

$$\chi\cup\xi=\int_C\chi\wedge\xi=\int_C(\chi,*\xi)\mathrm{d}V_g=\langle\chi,*\xi\rangle_{L^2}, \tag{24}$$

for $\chi, \xi \in \mathcal{H}^2$. The Hodge star $*$ maps $\mathcal{H}^2 \cong V$ to itself with $*^2 = 1$. Let $V_\pm$ be the ± 1 eigenspaces of $*$ on V. Then $V = V_+ \oplus V_-$, and (24) implies that $\cup$ is positive definite on V_+ and negative definite on V_-. Hence $\dim V_+$ is a *topological invariant*, from above. But $\operatorname{Ker}\bigl((\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2,\gamma}\bigr)$ is the self-dual 2–forms in $\mathcal{H}^2$. Thus $\chi \mapsto [\chi]$ induces an isomorphism of $\operatorname{Ker}\bigl((\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2,\gamma}\bigr)$ with V_+, as we have to prove.

Finally, suppose $\zeta^3 \in \operatorname{Ker}\bigl((\mathrm{d}^*_+ + \mathrm{d})^q_{m+2,-\gamma}\bigr)$, so that $\mathrm{d}^*_+\zeta^3 = \mathrm{d}\zeta^3 = 0$. Then

$$0 = \mathrm{d}^*(\mathrm{d}\zeta^3) + 2\mathrm{d}(\mathrm{d}^*_+\zeta^3) = \mathrm{d}^*\mathrm{d}\zeta^3 + \mathrm{d}(\mathrm{d}^*\zeta^3 - \mathrm{d}(*\zeta^3)) = (\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)\zeta_3,$$

so ζ^3 lies in $\operatorname{Ker}\bigl((\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)^q_{m+2,-\gamma}\bigr)$, which consists of smooth closed and coclosed forms by Proposition 3.8. Thus $\operatorname{Ker}\bigl((\mathrm{d}^*_+ + \mathrm{d})^q_{m+2,-\gamma}\bigr)$ is a vector space of smooth, closed and coclosed 3–forms. □

We can say more about the cokernel (22). Its dimension is

$$\dim \operatorname{Ker}\bigl((\mathrm{d}^*_+ + \mathrm{d})^q_{m+2,-\gamma}\bigr) = b^0(L) - b^0(C) + b^1(C).$$

The map $\operatorname{Ker}\bigl((\mathrm{d}^*_+ + \mathrm{d})^q_{m+2,-\gamma}\bigr) \to H^3(C,\mathbb{R})$, $\chi \mapsto [\chi]$ is surjective, with kernel of dimension $b^0(L) - b^0(C) + b^1(C) - b^3(C) \geqslant 0$, which is the dimension of the kernel of $H^3_{\mathrm{cs}}(C,\mathbb{R}) \to H^3(C,\mathbb{R})$. So we can think of $\operatorname{Ker}\bigl((\mathrm{d}^*_+ + \mathrm{d})^q_{m+2,-\gamma}\bigr)$ as a space of closed and coclosed 3–forms filling out all of $H^3_{\mathrm{cs}}(C,\mathbb{R})$ and $H^3(C,\mathbb{R})$. But we will not need these facts, so we shall not prove them.

3.4 Conditions on the rate γ for $\mathrm{d}_+ + \mathrm{d}^*$ to be Fredholm

Finally we determine the set $\mathcal{D}_{(\mathrm{d}_+ + \mathrm{d}^*)_0}$ for the cylindrical operator $(\mathrm{d}_+ + \mathrm{d}^*)_0$ on $L \times \mathbb{R}$, which by Theorem 3.4 gives the set of γ for which (20) is not Fredholm.

Proposition 3.11 *Let (C,g) be an oriented, asymptotically cylindrical Riemannian 4–manifold asymptotic to $(L \times \mathbb{R}, g_0)$, let $p > 1$, $l \geqslant 0$ and $\gamma \in \mathbb{R}$, and define $(\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2,\gamma}$ as in (20). Then $(\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2,\gamma}$ is not Fredholm if and only if either $\gamma = 0$, or γ^2 is a positive eigenvalue of $\Delta = \mathrm{d}^*\mathrm{d}$ on functions on L, or γ is an eigenvalue of $-*\mathrm{d}$ on coexact 1–forms on L.*

Proof Throughout the proof $*$, d^* mean the Hodge star and d^* on L, not on $L \times \mathbb{R}$, and d^*_4 is d^* on $L \times \mathbb{R}$. An element of $C^\infty(\Lambda^2_+ T^*(L \times \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C})$ invariant under translations in $\mathbb{R}$ may be written uniquely in the form $\chi \wedge \mathrm{d}t + *\chi$ for $\chi \in C^\infty(T^*L \otimes_{\mathbb{R}} \mathbb{C})$. An element of $C^\infty(\Lambda^4 T^*(L \times \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C})$ invariant under translations in $\mathbb{R}$ may be written uniquely as $f\,\mathrm{d}V_L \wedge \mathrm{d}t$ for $f\colon L \to \mathbb{C}$ smooth, where $\mathrm{d}V_L$ is the volume form on L. By Definition 3.3 and Theorem 3.4, (20) is not Fredholm if and only if there exist $\delta \in \mathbb{R}$ and χ, f as above and not both zero, satisfying

$$\mathrm{d}\big(e^{(\gamma+i\delta)t}(\chi \wedge \mathrm{d}t + *\chi)\big) + \mathrm{d}_4^*\big(e^{(\gamma+i\delta)t} f\, \mathrm{d}V_L \wedge \mathrm{d}t\big) \equiv 0. \tag{25}$$

Expanding (25) yields

$$e^{(\gamma+i\delta)t}\big[\mathrm{d}\chi \wedge \mathrm{d}t + (\gamma+i\delta)(*\chi) \wedge \mathrm{d}t + \mathrm{d}(*\chi) - (*\mathrm{d}f) \wedge \mathrm{d}t + (\gamma+i\delta) f\, \mathrm{d}V_L\big] \equiv 0$$

on $L \times \mathbb{R}$, and separating components with and without $\mathrm{d}t$ gives

$$\mathrm{d}\chi + (\gamma+i\delta)(*\chi) - (*\mathrm{d}f) \equiv 0 \quad \text{and} \quad \mathrm{d}(*\chi) + (\gamma+i\delta) f\, \mathrm{d}V_L \equiv 0,$$

equations in 2– and 3–forms on L respectively. Applying the Hodge star $*$ on L shows that (25) is equivalent to the two equations

$$*\mathrm{d}\chi + (\gamma+i\delta)\chi - \mathrm{d}f \equiv 0 \quad \text{and} \quad \mathrm{d}^*\chi - (\gamma+i\delta) f \equiv 0 \tag{26}$$

in 1–forms and functions on L.

Since χ is a 1–form on L which is a compact manifold, one can use Hodge decomposition and write $\chi = \chi_0 \oplus \chi_1 \oplus \chi_2$ where χ_0 is a harmonic 1–form, $\chi_1 = \mathrm{d}f_1$ is an exact 1–form, $f_1 \in C^\infty(L)$ and $\chi_2 = \mathrm{d}^*\eta$ is a co-exact 1–form, $\eta \in C^\infty(\Lambda^2 T^*L)$. Dividing the first equation of (26) into harmonic, exact and coexact components, the system becomes

$$(\gamma+i\delta)\chi_0 = 0, \;\; (\gamma+i\delta)\mathrm{d}f_1 = \mathrm{d}f, \;\; -*\mathrm{d}\chi_2 = (\gamma+i\delta)\chi_2, \;\; \mathrm{d}^*\mathrm{d}f_1 = (\gamma+i\delta)f. \tag{27}$$

The second and fourth equations of (27) give $\mathrm{d}^*\mathrm{d}f = (\gamma+i\delta)^2 f$, and substituting the third equation into itself gives $\mathrm{d}^*\mathrm{d}\chi_2 = *\mathrm{d}*\mathrm{d}\chi_2 = (\gamma+i\delta)^2\chi_2$. But $\mathrm{d}^*\chi_2 = 0$ by definition. Thus $\chi_0, \chi_1, \chi_2, f$ satisfy the equations

$$\begin{aligned} (\gamma+i\delta)\chi_0 &= 0, & (\gamma+i\delta)\chi_1 &= \mathrm{d}f, \\ (\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)\chi_2 &= (\gamma+i\delta)^2\chi_2, & \mathrm{d}^*\mathrm{d}f &= (\gamma+i\delta)^2 f. \end{aligned} \tag{28}$$

When $\gamma = \delta = 0$, $\chi = 0$ and $f \equiv 1$ are a solution, so (20) is not Fredholm. So suppose $\gamma + i\delta \neq 0$. Then $\chi_0 = 0$, and $\chi_1 = 0$ if $f = 0$ by (28), so as χ, f are not both zero either $f \neq 0$ or $\chi_2 \neq 0$. If $f \neq 0$ then $(\gamma+i\delta)^2$ is a nonzero eigenvalue of $\Delta = \mathrm{d}^*\mathrm{d}$ on functions by the last equation of (28), so $\delta = 0$ as such eigenvalues are positive, and γ^2 is a positive eigenvalue of Δ as we want. Conversely, if $\Delta f = \gamma^2 f$ for nonzero γ, f then $\chi = \gamma^{-1}\mathrm{d}f$ satisfies the equations, so (20) is not Fredholm.

If $\chi_2 \neq 0$ then $(\gamma+i\delta)^2$ is a nonzero eigenvalue of $\Delta = \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*$ on coexact 1–forms by the third equation of (28), so $\delta = 0$ as above. The third equation of (27) then shows that γ is an eigenvalue of $-*d$ on coexact 1–forms on L. Conversely, taking χ_2 to be an eigenvector of $-*d$ on coexact 1–forms with eigenvalue γ and $\chi_0 = \chi_1 = f = 0$ solves the equations, so (20) is not Fredholm. □

4 Proof of Theorem 1.1

We now prove Theorem 1.1. Let (M, φ, g) be an asymptotically cylindrical G_2–manifold asymptotic to $X \times (R, \infty)$, $R > 0$, with decay rate $\alpha < 0$. Let C be an asymptotically cylindrical coassociative 4–fold in X asymptotic to $L \times (R', \infty)$ for $R' > R$ with decay rate β for $\alpha \leqslant \beta < 0$. Write $g_C = g|_C$ for the metric on C, g_X for the Calabi–Yau metric on X, and $g_L = g_X|_L$ for the metric on L. Then (C, g_C) is an *asymptotically cylindrical Riemannian* 4*–manifold*, with *rate* β.

Suppose γ satisfies $\beta < \gamma < 0$, and $(0, \gamma^2]$ contains no eigenvalues of the Laplacian Δ_L on functions on L, and $[\gamma, 0)$ contains no eigenvalues of the operator $-*\mathrm{d}$ on coexact 1–forms on L. Let $p > 4$ and $l \geqslant 1$, and define

$$(\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2,\gamma}\colon L^p_{l+2,\gamma}(\Lambda^2_+ T^*C) \oplus L^p_{l+2,\gamma}(\Lambda^4 T^*C) \longrightarrow L^p_{l+1,\gamma}(\Lambda^3 T^*C) \qquad (29)$$

as in Section 3.3. Then Proposition 3.11 and the conditions on γ imply that $[\gamma, 0) \cap \mathcal{D}_{(\mathrm{d}_+ + \mathrm{d}^*)_0} = \emptyset$. Hence $\gamma \notin \mathcal{D}_{(\mathrm{d}_+ + \mathrm{d}^*)_0}$, so that $(\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2,\gamma}$ is *Fredholm* by Theorem 3.4. Also, Theorem 3.10 applies to $(\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2,\gamma}$.

Let ν_L be the normal bundle of L in X, regarded as the *orthogonal subbundle* to TL in $TX|_L$, and $\exp_L\colon \nu_L \to X$ the exponential map. For $r > 0$, write $B_r(\nu_L)$ for the subbundle of ν_L with fibre at x the open ball about 0 in $\nu_L|_x$ with radius r. Then for small $\epsilon > 0$, there is a *tubular neighbourhood* T_L of L in X such that $\exp_L\colon B_{2\epsilon}(\nu_L) \to T_L$ is a diffeomorphism. Also, $\nu_L \times \mathbb{R} \to L \times \mathbb{R}$ is the normal bundle to $L \times \mathbb{R}$ in $X \times \mathbb{R}$ with exponential map $\exp_L \times \mathrm{id}\colon \nu_L \times \mathbb{R} \to X \times \mathbb{R}$. Then $T_L \times \mathbb{R}$ is a tubular neighborhood of $L \times \mathbb{R}$ in $X \times \mathbb{R}$, and $\exp_L \times \mathrm{id}\colon B_{2\epsilon}(\nu_L) \times \mathbb{R} \to T_L \times \mathbb{R}$ is a diffeomorphism.

Let K, R, $\Psi\colon X \times (R, \infty) \to M \setminus K$, and $K', R' > R$, $\Phi\colon L \times (R', \infty) \to C \setminus K'$, and the normal vector field v on $L \times (R', \infty)$ be as in Section 2.3, so that (3) commutes. Then v is a section of $\nu_L \times (R', \infty) \to L \times (R', \infty)$, decaying at rate $O(e^{\beta t})$. Therefore making K' and R' larger if necessary, we can suppose the graph of v lies in $B_\epsilon(\nu_L) \times (R', \infty)$.

Write $\pi\colon B_\epsilon(\nu_L) \times (R', \infty) \to L \times (R', \infty)$ for the natural projection. Define

$$\Xi\colon B_\epsilon(\nu_L) \times (R', \infty) \to M \quad \text{by} \quad \Xi\colon w \mapsto \Psi\big[(\exp_L \times \mathrm{id})(v|_{\pi(w)} + w)\big]. \qquad (30)$$

Here w is a point in $B_\epsilon(\nu_L) \times (R', \infty)$, in the fibre over $\pi(w) \in L \times (R', \infty)$. Thus $v|_{\pi(w)}$ is a point in the same fibre, which is a ball of radius ϵ in a vector space.

Hence $v|_{\pi(w)} + w$ lies in the open ball of radius 2ϵ in the same vector space, that is, in the fibre of $B_{2\epsilon}(\nu_L) \times (R', \infty)$ over $\pi(w)$. Therefore $(\exp_L \times \mathrm{id})(v|_{\pi(w)} + w)$

is well-defined and lies in $T_L \times (R',\infty) \subset X \times (R',\infty)$, and $\Xi(w)$ is well-defined. Since $\exp_L \times \mathrm{id}\colon B_{2\epsilon}(\nu_L) \times \mathbb{R} \to T_L \times \mathbb{R}$ and $\Psi\colon X \times (R,\infty) \to M \setminus K$ are diffeomorphisms, we see that Ξ is a diffeomorphism with its image.

Identify $L \times (R',\infty)$ with the zero section in $B_\epsilon(\nu_L) \times (R',\infty)$. Then as (3) commutes we see that $\Xi|_{L\times(R',\infty)} \equiv \Phi$. Let ν_C be the normal bundle of C in M, which we regard not as the orthogonal subbundle to TC in $TM|_C$, but rather as the *quotient bundle* $TM|_C/TC$. Define an isomorphism ξ between the vector bundles $\nu_L \times (R',\infty)$ and $\Phi^*(\nu_C)$ over $L \times (R',\infty)$ as follows.

As Ξ is a diffeomorphism with its image, $\mathrm{d}\Xi\colon T\bigl(B_\epsilon(\nu_L)\times(R',\infty)\bigr) \to \Xi^*(TM)$ is an isomorphism. Restricting this to the zero section $L \times (R',\infty)$ and noting that $\Xi|_{L\times(R',\infty)} \equiv \Phi$, we see that

$$\mathrm{d}\Xi|_{L\times(R',\infty)}\colon T\bigl(B_\epsilon(\nu_L) \times (R',\infty)\bigr)|_{L\times(R',\infty)} \longrightarrow \Phi^*(TM) \tag{31}$$

is an isomorphism. As $\Phi\colon L \times (R',\infty) \to C \setminus K'$ is a diffeomorphism,

$$\mathrm{d}\Phi\colon T\bigl(L \times (R',\infty)\bigr) \longrightarrow \Phi^*(TN) \tag{32}$$

is an isomorphism. But (32) is the restriction of (31) to a vector subbundle. Quotienting (31) by (32) gives an isomorphism

$$\begin{aligned} \xi = \mathrm{d}\Xi|_{L\times(R',\infty)}\colon \nu_L \times (R',\infty) &\cong \frac{T\bigl(B_\epsilon(\nu_L) \times (R',\infty)\bigr)|_{L\times(R',\infty)}}{T\bigl(L \times (R',\infty)\bigr)} \\ &\longrightarrow \Phi^*(TM)/\Phi^*(TN) \cong \Phi^*(TM/TN) = \Phi^*(\nu_C). \end{aligned} \tag{33}$$

Now choose a small $\epsilon' > 0$, a tubular neighborhood T_C of C in M, and a diffeomorphism $\Theta\colon B_{\epsilon'}(\nu_C) \to T_C$ satisfying the conditions:

(i) $\Theta|_C \equiv \mathrm{id}_C\colon C \to C$, where $C \subset B_{\epsilon'}(\nu_C)$ is the zero section.

(ii) By (i), $\mathrm{d}\Theta|_C\colon T(B_{\epsilon'}(\nu_C))|_C \to TM|_C$ is an isomorphism, which restricts to the identity on the subbundles TC of each side. Hence it induces an isomorphism $\mathrm{d}\Theta|_C\colon \nu_C \cong T\bigl(B_\epsilon(\nu_C)\bigr)|_C/TC \to TM|_C/TC = \nu_C$. This isomorphism is the identity map.

(iii) ϵ' is small enough that $\xi^*\bigl(B_{\epsilon'}(\nu_C)\bigr) \subset B_\epsilon(\nu_L) \times (R',\infty) \subset \nu_L \times (R',\infty)$, and $\Theta \circ \xi \equiv \Xi$ on $\xi^*\bigl(B_{\epsilon'}(\nu_C)\bigr)$.

Notice that (iii) determines Θ and T_C uniquely on $B_{\epsilon'}(\nu_C)|_{C\setminus K'}$, and by construction here it satisfies (i) and (ii). Thus, it remains only to choose T_C and Θ satisfying (i), (ii) over the compact set $K' \subset C$, which is possible by standard differential topology.

The point of all this is that we have chosen a local identification Θ between ν_C and M near C that is compatible in a nice way with the asymptotic identifications Φ, Ψ of C, M with $L \times \mathbb{R}$ and $X \times \mathbb{R}$. Using Θ, submanifolds $\widetilde{C}$

of M close to C are identified with small sections s of ν_C, and importantly, the asymptotic convergence of $\widetilde{C}$ to C, and so to $L\times\mathbb{R}$, is reflected in the asymptotic convergence of s to 0.

As in the proof of Theorem 2.5, the map $V\mapsto (V\cdot\varphi|_x)|_{T_xC}$ defines an *isomorphism* $\nu_C\rightarrow\Lambda^2_+T^*C$. (Note that since $\varphi|_C\equiv 0$, this map is well-defined for $V\in T_xM/T_xC$, rather than just for $V\in T_xM$ orthogonal to T_xC.) We now *identify* ν_C with $\Lambda^2_+T^*C$, and regard Θ as a map $\Theta\colon B_{\epsilon'}(\Lambda^2_+T^*C)\rightarrow T_C\subset M$.

Write $L^p_{l+2,\gamma}\big(B_{\epsilon'}(\Lambda^2_+T^*C)\big)$ for the subset of $\zeta^2_+\in L^p_{l+2,\gamma}(\Lambda^2_+T^*C)$ which are sections of $B_{\epsilon'}(\Lambda^2_+T^*C)$, that is, $|\zeta^2_+|<\epsilon'$ on C. Since $L^p_{l+1,\gamma}\hookrightarrow C^0$ by Sobolev embedding this is an open condition on ζ^2_+, so $L^p_{l+2,\gamma}\big(B_{\epsilon'}(\Lambda^2_+T^*C)\big)$ is an open subset of the Banach space $L^p_{l+2,\gamma}(\Lambda^2_+T^*C)$.

Define $Q\colon L^p_{l+2,\gamma}\big(B_{\epsilon'}(\Lambda^2_+T^*C)\big)\rightarrow\{$3–forms on $C\}$ by $Q(\zeta^2_+)=(\Theta\circ\zeta^2_+)^*(\varphi)$. That is, we regard the section ζ^2_+ as a map $C\rightarrow B_{\epsilon'}(\Lambda^2_+T^*C)$, so $\Theta\circ\zeta^2_+$ is a map $C\rightarrow T_C\subset M$, and thus $(\Theta\circ\zeta^2_+)^*(\varphi)$ is a 3–form on C. The point of this definition is that if $\Gamma_{\zeta^2_+}$ is the *graph* of ζ^2_+ in $B_{\epsilon'}(\Lambda^2_+T^*C)$ and $\widetilde{C}=\Theta(\Gamma_{\zeta^2_+})$ its image in M, then $\widetilde{C}$ is *coassociative* if and only if $\varphi|_{\widetilde{C}}\equiv 0$, which holds if and only if $Q(\zeta^2_+)=0$. So $Q^{-1}(0)$ parametrizes coassociative 4–folds $\widetilde{C}$ close to C.

We now consider which class of 3–forms Q maps to.

Proposition 4.1 *$Q\colon L^p_{l+2,\gamma}\big(B_{\epsilon'}(\Lambda^2_+T^*C)\big)\longrightarrow L^p_{l+1,\gamma}(\Lambda^3T^*C)$ is a smooth map of Banach manifolds. The linearization of Q at 0 is $\mathrm{d}Q(0)\colon\zeta^2_+\mapsto\mathrm{d}\zeta^2_+$.*

Proof As in the proof of Theorem 2.5, the functional form of Q is

$$Q(\zeta^2_+)|_x=F\big(x,\zeta^2_+|_x,\nabla\zeta^2_+|_x\big)\quad\text{for } x\in C,$$

where F is a smooth function of its arguments. Since $p>4$ and $l\geqslant 1$ we have $L^p_{l+2,\gamma}(\Lambda^2_+T^*C)\hookrightarrow C^1_\gamma(\Lambda^2_+T^*C)$ by Sobolev embedding. General arguments then show that *locally* $Q(\zeta^2_+)$ is L^p_{l+1}.

To show $Q(\zeta^2_+)$ lies in $L^p_{l+1,\gamma}(\Lambda^3T^*C)$, we must know something of the asymptotic behavior of F at infinity. Essentially $Q(\zeta^2_+)$ is the restriction to $\Gamma(\zeta^2_+)$ of the 3–form $\Theta^*(\varphi)$ on $B_{\epsilon'}(\Lambda^2_+T^*C)$. Using the identifications $\Phi\colon L'\times(R'\infty)\rightarrow C\setminus K'$ and $\xi\colon\nu_L\times(R',\infty)\rightarrow\Phi^*(\Lambda^2_+T^*C)$ over $C\setminus K'$, by (iii) above this 3–form becomes $\Xi^*(\varphi)$ on $B_\epsilon(\nu_L)\times(R',\infty)$.

But the asymptotic conditions on Φ,Ψ and v imply that $\Xi^*(\varphi)$ is the sum of a translation-invariant 3–form on $B_\epsilon(\nu_L)\times\mathbb{R}$, the pullback of the cylindrical G_2

form φ_0 on $X\times\mathbb{R}$, and an error term which decays at rate $O(e^{\beta t})$, with all its derivatives. As $\beta<\gamma<0$, it is not difficult to see from this that Q maps to $L^p_{l+1,\gamma}(\Lambda^3T^*C)$. Smoothness of Q holds by general principles. Finally, the linearization of Q is d, by the calculation of McLean alluded to in Theorem 2.5. As the calculation is local, it does not matter that we are on a noncompact manifold C. □

Next we show that the image of Q consists of *exact* 3–forms. Notice that we lose one degree of differentiability: although the image of Q lies in $L^p_{l+1,\gamma}(\Lambda^3T^*C)$, we claim only that it lies in the exact 3–forms in $L^p_{l,\gamma}(\Lambda^3T^*C)$.

Proposition 4.2

$$Q\bigl(L^p_{l+2,\gamma}(B_{\epsilon'}(\Lambda^2_+T^*C))\bigr)\subseteq\mathrm{d}\bigl(L^p_{l+1,\gamma}(\Lambda^2T^*C)\bigr)\subset L^p_{l,\gamma}(\Lambda^3T^*C).$$

Proof Consider the restriction of the 3–form φ to the tubular neighborhood T_C of C. As φ is closed, and T_C retracts onto C, and $\varphi|_C\equiv 0$, we see that $\varphi|_{T_C}$ is *exact.* Thus we may write $\varphi|_{T_C}=\mathrm{d}\theta$ for $\theta\in C^\infty(\Lambda^2T^*T_C)$. Since $\varphi|_C\equiv 0$ we may choose $\theta|_C\equiv 0$. Also, as φ is asymptotic to $O(e^{\beta t})$ with all its derivatives to a translation-invariant 3–form φ_0 on $X\times\mathbb{R}$, we may take θ to be asymptotic to $O(e^{\beta t})$ with all its derivatives to a translation-invariant 2–form on $T_L\times\mathbb{R}$.

The proof of Proposition 4.1 now shows that the map $\zeta^2_+\mapsto(\Theta\circ\zeta^2_+)^*(\theta)$ maps $L^p_{l+2,\gamma}\bigl(B_{\epsilon'}(\Lambda^2_+T^*C)\bigr)\to L^p_{l+1,\gamma}(\Lambda^2T^*C)$. But

$$Q(\zeta^2_+)=(\Theta\circ\zeta^2_+)^*(\varphi)=(\Theta\circ\zeta^2_+)^*(\mathrm{d}\theta)=\mathrm{d}\bigl[(\Theta\circ\zeta^2_+)^*(\theta)\bigr],$$

so $Q(\zeta^2_+)\in\mathrm{d}\bigl(L^p_{l+1,\gamma}(\Lambda^2T^*C)\bigr)$ for $\zeta^2_+\in L^p_{l+2,\gamma}(B_{\epsilon'}(\Lambda^2_+T^*C))$. □

As in the proof of Theorem 2.5, we augment Q by a space of 4–forms on C to make it *elliptic*. Define

$$\begin{gathered}P\colon\ L^p_{l+2,\gamma}\bigl(B_{\epsilon'}(\Lambda^2_+T^*C)\bigr)\times L^p_{l+2,\gamma}(\Lambda^4T^*C)\longrightarrow L^p_{l+1,\gamma}(\Lambda^3T^*C)\\ \text{by}\qquad P(\zeta^2_+,\zeta^4)=Q(\zeta^2_+)+\mathrm{d}^*\zeta^4.\end{gathered}$$

Proposition 4.1 implies that the linearization $\mathrm{d}P(0,0)$ of P at 0 is the Fredholm operator $(\mathrm{d}_++\mathrm{d}^*)^p_{l+2,\gamma}$ of (29). Define $\mathcal{C}$ to be the image of $(\mathrm{d}_++\mathrm{d}^*)^p_{l+2,\gamma}$. Then $\mathcal{C}$ is a *Banach subspace* of $L^p_{l+1,\gamma}(\Lambda^3T^*C)$, since $(\mathrm{d}_++\mathrm{d}^*)^p_{l+2,\gamma}$ is Fredholm. We show P maps into $\mathcal{C}$.

Proposition 4.3 *P maps $L^p_{l+2,\gamma}\bigl(B_{\epsilon'}(\Lambda^2_+T^*C)\bigr)\times L^p_{l+2,\gamma}(\Lambda^4T^*C)\longrightarrow\mathcal{C}$.*

Proof Let $(\zeta^2_+,\zeta^4)\in L^p_{l+2,\gamma}\big(B_{\epsilon'}(\Lambda^2_+T^*C)\big)\times L^p_{l+2,\gamma}(\Lambda^4T^*C)$ so that $P(\zeta^2_+,\zeta^4)$ $=Q(\zeta^2_+)+\mathrm{d}^*\zeta^4$ lies in $L^p_{l+1,\gamma}(\Lambda^3T^*C)$. We must show it lies in $\mathcal{C}$. Since $\gamma\notin\mathcal{D}_{(\mathrm{d}_++\mathrm{d}^*)_0}$, from Section 3 this holds if and only if

$$\big\langle Q(\zeta^2_+)+\mathrm{d}^*\zeta^4,\chi\big\rangle_{L^2}=0\qquad\text{for all }\chi\in\operatorname{Ker}\big((\mathrm{d}^*_++\mathrm{d})^q_{m+2,-\gamma}\big),\tag{34}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $m\geqslant 0$.

By Theorem 3.10, $\operatorname{Ker}\big((\mathrm{d}^*_++\mathrm{d})^q_{m+2,-\gamma}\big)$ consists of *closed and coclosed* 3–forms χ. By Proposition 4.2 we have $Q(\zeta^2_+)=\mathrm{d}\lambda$ for $\lambda\in L^p_{l+1,\gamma}(\Lambda^2T^*C)$. So

$$\big\langle Q(\zeta^2_+)+\mathrm{d}^*\zeta^4,\chi\big\rangle_{L^2}=\big\langle\mathrm{d}\lambda,\chi\big\rangle_{L^2}+\big\langle\mathrm{d}^*\zeta^4,\chi\big\rangle_{L^2}=\big\langle\lambda,\mathrm{d}^*\chi\big\rangle_{L^2}+\big\langle\zeta^4,\mathrm{d}\chi\big\rangle_{L^2}=0$$

for $\chi\in\operatorname{Ker}\big((\mathrm{d}^*_++\mathrm{d})^q_{m+2,-\gamma}\big)$, as χ is closed and coclosed, and the inner products and integration by parts are valid because of the matching of rates $\gamma,-\gamma$ and L^p,L^q with $\frac{1}{p}+\frac{1}{q}=1$. So (34) holds, and P maps into $\mathcal{C}$. □

We now apply the *Implicit Mapping Theorem for Banach spaces*, [7, Theorem 1.2.5].

Theorem 4.4 *Let $\mathcal{A},\mathcal{B}$ and $\mathcal{C}$ be Banach spaces, and $\mathcal{U},\mathcal{V}$ open neighborhoods of 0 in $\mathcal{A}$ and $\mathcal{B}$. Suppose that the function $P\colon\mathcal{U}\times\mathcal{V}\to\mathcal{C}$ is smooth with $P(0,0)=0$, and that $\mathrm{d}P_{(0,0)}|_{\mathcal{B}}\colon\mathcal{B}\to\mathcal{C}$ is an isomorphism of $\mathcal{B},\mathcal{C}$ as vector and topological spaces. Then there exists a connected open neighbourhood $\mathcal{U}'\subset\mathcal{U}$ of 0 in $\mathcal{A}$ and a unique smooth map $G\colon\mathcal{U}'\to\mathcal{V}$ such that $G(0)=0$ and $P(x,G(x))=0$ for all $x\in\mathcal{U}'$.*

Define $\mathcal{A}$ to be $\operatorname{Ker}\big((\mathrm{d}_++\mathrm{d}^*)^p_{l+2,\gamma}\big)$, and $\mathcal{B}$ to be the subspace of $L^p_{l+2,\gamma}(\Lambda^2_+T^*C)$ $\oplus L^p_{l+2,\gamma}(\Lambda^4T^*C)$ which is L^2–orthogonal to $\mathcal{A}$. As $\mathcal{A}$ is finite-dimensional and the L^2 inner product is continuous on $L^p_{l+2,\gamma}(\Lambda^2_+T^*C)\oplus L^p_{l+2,\gamma}(\Lambda^4T^*C)$, both $\mathcal{A},\mathcal{B}$ are *Banach spaces*, and $\mathcal{A}\oplus\mathcal{B}=L^p_{l+2,\gamma}(\Lambda^2_+T^*C)\oplus L^p_{l+2,\gamma}(\Lambda^4T^*C)$. Choose open neighborhoods $\mathcal{U},\mathcal{V}$ of 0 in $\mathcal{A},\mathcal{B}$ such that $\mathcal{U}\times\mathcal{V}\subseteq L^p_{l+2,\gamma}\big(B_{\epsilon'}(\Lambda^2_+T^*C)\big)\times L^p_{l+2,\gamma}(\Lambda^4T^*C)$.

Let $P,\mathcal{C}$ be as above. Then by Propositions 4.1 and 4.3, $P\colon\mathcal{U}\times\mathcal{V}\to\mathcal{C}$ is a smooth map of Banach manifolds with $P(0,0)=0$, and linearization $\mathrm{d}P(0,0)=(\mathrm{d}_++\mathrm{d}^*)^p_{l+2,\gamma}\colon\mathcal{A}\oplus\mathcal{B}\to\mathcal{C}$. By definition $\mathcal{A}$ is the kernel and $\mathcal{C}$ is the image of $(\mathrm{d}_++\mathrm{d}^*)^p_{l+2,\gamma}$. Hence $\mathrm{d}P_{(0,0)}|_{\mathcal{B}}\colon\mathcal{B}\to\mathcal{C}$ is an isomorphism of vector spaces, and as it is a continuous linear map of Banach spaces, it is an isomorphism of topological spaces by the Open Mapping Theorem.

Thus Theorem 4.4 applies, and gives a connected open neighborhood $\mathcal{U}'$ of 0 in $\mathcal{U}$, and a smooth map $G\colon\mathcal{U}'\to\mathcal{V}$ such that $G(0)=0$ and $P(x,G(x))\equiv 0$.

Moreover $P^{-1}(0)$ coincides with $\{(x,G(x)) : x \in \mathcal{U}'\}$ near $(0,0)$, and so is smooth, finite-dimensional and locally isomorphic to $\mathcal{A} = \mathrm{Ker}\big((\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2,\gamma}\big)$.

Lemma 4.5 $P^{-1}(0) = Q^{-1}(0) \times \{0\}$.

Proof Clearly $Q^{-1}(0) \times \{0\} \subseteq P^{-1}(0)$. Suppose $(\zeta^2_+, \zeta^4) \in P^{-1}(0)$, so that $Q(\zeta^2_+) + \mathrm{d}^*\zeta^4 = 0$. We shall show that $\zeta^4 = 0$, so that $Q(\zeta^2_+) = 0$, and thus $P^{-1}(0) \subseteq Q^{-1}(0) \times \{0\}$. By Proposition 4.2 we have $Q(\zeta^2_+) = \mathrm{d}\lambda$ for $\lambda \in L^p_{l+1,\gamma}(\Lambda^2 T^*C)$, so $\mathrm{d}\lambda = -\mathrm{d}^*\zeta^4$. Hence

$$\|\mathrm{d}^*\zeta^4\|^2_{L^2} = \big\langle \mathrm{d}^*\zeta^4, \mathrm{d}^*\zeta^4\big\rangle_{L^2} = -\big\langle \mathrm{d}^*\zeta^4, \mathrm{d}\lambda\big\rangle_{L^2} = -\big\langle \zeta^4, \mathrm{d}^2\lambda\big\rangle_{L^2} = 0,$$

where the inner products and integration by parts are valid as $L^p_{l+2,\gamma} \hookrightarrow L^2_2$. Thus $Q(\zeta^2_+) = \mathrm{d}^*\zeta^4 = 0$. But $\mathrm{d}^*\zeta^4 \cong \nabla\zeta^4$ as ζ^4 is a 4–form, so ζ^4 is *constant.* Since also $\zeta^4 \to 0$ near infinity in C, we have $\zeta^4 \equiv 0$. □

Combining this with the previous description of $P^{-1}(0)$ shows that $Q^{-1}(0)$ is smooth, finite-dimensional and locally isomorphic to $\mathrm{Ker}\big((\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2,\gamma}\big)$. Next we show that $Q^{-1}(0)$ is independent of l, and so consists of smooth solutions.

Proposition 4.6 *If $\zeta^2_+ \in Q^{-1}(0)$ then $\zeta^2_+ \in L^p_{m+2,\gamma}(\Lambda^2_+ T^*C)$ for all $m \geqslant 1$.*

Proof Let $m \geqslant 1$ and $(\zeta^2_+, \zeta^4) \in L^p_{m+2,\gamma}\big(B_{\epsilon'}(\Lambda^2_+ T^*C)\big) \times L^p_{m+2,\gamma}(\Lambda^4 T^*C)$, so that $P(\zeta^2_+, \zeta^4) = Q(\zeta^2_+) + \mathrm{d}^*\zeta^4$ lies in $L^p_{m+1,\gamma}(\Lambda^3 T^*C)$. Write ∇ for the Levi–Civita connection of g_C on C. We shall apply the ∇–Laplacian $\Delta_C = g^{ab}_C \nabla_a \nabla_b$ to $P(\zeta^2_+, \zeta^4)$, using the index notation. Writing Q in terms of F as in Proposition 4.1, we have

$$\Delta_C\big[P(\zeta^2_+, \zeta^4)\big] = g^{ab}_C \nabla_a \nabla_b \big[F\big(x, \zeta^2_+|_x, \nabla\zeta^2_+|_x\big)\big] + \Delta_C \mathrm{d}^*\zeta^4. \tag{35}$$

We shall expand (35) in terms of the derivatives of F by the chain rule, using the following notation. Write $F = F(x,y,z)$. Write $\nabla^x F$ for the derivative of F 'in the x direction' using ∇. That is, $(\nabla^x F)(x', y', z') = \nabla\big(F(x, y(x), z(x))\big)|_{x'}$, where $y(x), z(x)$ satisfy $y(x') = y'$, $z(x') = z'$ and $(\nabla y)(x') = (\nabla z)(x') = 0$. Write $\partial^y F$ and $\partial^z F$ for the partial derivatives of F in the y, z directions. That is, with x *held constant* the domains $\Lambda^2_+ T^*_x C$ and $T^*_x C \otimes \Lambda^2_+ T^*_x C$ of y, z and the range $\Lambda^3 T^*_x C$ of F are *vector spaces*, so the map $(y,z) \mapsto F(x,y,z)$ is a smooth map between vector spaces, and has well-defined partial derivatives.

Then expanding (35) gives

$$\Delta_C\big[P(\zeta^2_+, \zeta^4)\big] = L_{\zeta^2_+}(\zeta^2_+, \zeta^4) + E(x, \zeta^2_+, \nabla\zeta^2_+, \nabla^2\zeta^2_+), \tag{36}$$

where $L_{\zeta^2_+}$ is the linear third-order operator for $0 \leqslant n \leqslant m+1$ given by

$$\begin{aligned} L_{\zeta^2_+} &: L^p_{n+3,\gamma}(\Lambda^2_+T^*C \oplus \Lambda^4T^*C) \longrightarrow L^p_{n,\gamma}(\Lambda^3T^*C), \\ L_{\zeta^2_+}(\xi^2_+,\xi^4)|_x &= (\partial^z F)(x,\zeta^2_+|_x,\nabla\zeta^2_+|_x)\cdot\Delta_C\nabla\xi^2_+|_x + \Delta_C\mathrm{d}^*\xi^4|_x, \end{aligned} \tag{37}$$

and $E(x,\zeta^2_+,\nabla\zeta^2_+,\nabla^2\zeta^2_+)$ is the 'error term' given by

$$\begin{aligned} E(x,\zeta^2_+,\nabla\zeta^2_+,\nabla^2\zeta^2_+) &= (\partial^y F)(x,y,z)\cdot\Delta_C\zeta^2_+ + g^{ab}_C(\nabla^x_a\nabla^x_b F)(x,y,z) \\ &+ g^{ab}_C(\nabla^x_a\partial^y F)(x,y,z)\cdot\nabla_b\zeta^2_+ + g^{ab}_C(\nabla^x_a\partial^z F)(x,y,z)\cdot\nabla_b\nabla\zeta^2_+ \\ &+ g^{ab}_C(\partial^y\nabla^x_b F)(x,y,z)\cdot\nabla_a\zeta^2_+ + g^{ab}_C(\partial^y\partial^y F)(x,y,z)\cdot(\nabla_a\zeta^2_+\otimes\nabla_b\zeta^2_+) \\ &+ g^{ab}_C(\partial^y\partial^z F)(x,y,z)\cdot(\nabla_a\zeta^2_+\otimes\nabla_b\nabla\zeta^2_+) + g^{ab}_C(\partial^z\nabla^x_b F)(x,y,z)\cdot\nabla_a\nabla\zeta^2_+ \\ &+ g^{ab}_C(\partial^z\partial^y F)(x,y,z)\cdot(\nabla_a\nabla\zeta^2_+\otimes\nabla_b\zeta^2_+) \\ &+ g^{ab}_C(\partial^z\partial^z F)(x,y,z)\cdot(\nabla_a\nabla\zeta^2_+\otimes\nabla_b\nabla\zeta^2_+). \end{aligned} \tag{38}$$

Here in (37) and (38) we have used ' $\cdot$ ' to denote various natural bilinear products, and in (38) we write $y = \zeta^2_+|_x$ and $z = \nabla\zeta^2_+|_x$ in the arguments of F.

The point is that (36) splits $\Delta_C[P(\zeta^2_+,\zeta^4)]$ up into the piece $L_{\zeta^2_+}(\zeta^2_+,\zeta^4)$ containing all the *third derivative* terms of ζ^2_+,ζ^4, plus a term $E(x,\zeta^2_+,\nabla\zeta^2_+,\nabla^2\zeta^2_+)$ depending only on ζ^2_+ *up to second derivatives.* Furthermore, we may write $L_{\zeta^2_+}(\zeta^2_+,\zeta^4)$ as a *linear operator* $L_{\zeta^2_+}$ applied to (ζ^2_+,ζ^4), where the coefficients of $L_{\zeta^2_+}$ depend on ζ^2_+ only *up to first derivatives.*

The operator $L_{\zeta^2_+}$ is essentially $\Delta_C(\mathrm{d}_+ + \mathrm{d}^*)$, and so is a *linear third-order elliptic operator*. Note that we allow $0 \leqslant n \leqslant m+1$ in (37). The reason for this is that the *coefficients* of $L_{\zeta^2_+}$ depend on $\zeta^2_+,\nabla\zeta^2_+$, and so lie in L^p_{m+1} locally. So the maximum regularity we can expect for $L_{\zeta^2_+}(\xi^2_+,\xi^4)$ is L^p_{m+1}, forcing $n \leqslant m+1$.

The most obvious value for n in (37) is $n = m-1$, as then the domain of $L_{\zeta^2_+}$ is the space $L^p_{m+2,\gamma}$ containing (ζ^2_+,ζ^4). However, our next lemma is an *elliptic regularity result* for $L_{\zeta^2_+}$ when $n = m$. This is because we will use it to increase the regularity of (ζ^2_+,ζ^4) from $L^p_{m+2,\gamma}$ to $L^p_{m+3,\gamma}$ by 'bootstrapping'.

Lemma 4.7 *There exists $A > 0$ such that if $(\xi^2_+,\xi^4) \in L^p_{3,\gamma}(\Lambda^2_+T^*C \oplus \Lambda^4T^*C)$ with $L_{\zeta^2_+}(\xi^2_+,\xi^4) \in L^p_{m,\gamma}(\Lambda^3T^*C)$ then $(\xi^2_+,\xi^4) \in L^p_{m+3,\gamma}(\Lambda^2_+T^*C \oplus \Lambda^4T^*C)$, and*

$$\left\|(\xi^2_+,\xi^4)\right\|_{L^p_{m+3,\gamma}} \leqslant A\left(\left\|L_{\zeta^2_+}(\xi^2_+,\xi^4)\right\|_{L^p_{m,\gamma}} + \left\|(\xi^2_+,\xi^4)\right\|_{L^p_{0,\gamma}}\right). \tag{39}$$

Proof If $L_{\zeta^2_+}$ were a *smooth, asymptotically cylindrical* elliptic operator the lemma would follow from Lockhart and McOwen [15, Theorem 3.7.2], [16, Equation (2.4)]. We have to deal with the fact that the coefficients of $L_{\zeta^2_+}$ depend on $\zeta^2_+, \nabla\zeta^2_+$, and so are only L^p_{m+1} locally rather than smooth. Unfortunately, Lockhart and McOwen's proof [16, p420] is not very informative, saying only that (39) is established 'by standard parametric techniques'.

Local results of this form are proved by Morrey [23, Section 6.2]. Supposing only that the coefficients of the third-order elliptic operator $L_{\zeta^2_+}$ are C^m, Morrey [23, Theorem 6.2.5] implies that if (ξ^2_+, ξ^4) is locally L^p_3 and $L_{\zeta^2_+}(\xi^2_+, \xi^4)$ is locally L^p_m then (ξ^2_+, ξ^4) is locally L^p_{m+3}. Also, [23, Theorem 6.2.6] proves a local interior estimate of the form (39), where $A > 0$ depends on m, p, the domains involved, C^m bounds on the coefficients of $L_{\zeta^2_+}$, and a *modulus of continuity* for their m^{th} derivatives.

Such a modulus of continuity is provided by a Hölder $C^{0,\alpha}$ bound for the m^{th} derivatives, for $\alpha \in (0,1)$. Thus, we can prove local estimates of the form (39) provided we have local $C^{m,\alpha}$ bounds for the coefficients of $L_{\zeta^2_+}$, which follow from local $C^{m+1,\alpha}$ bounds for ζ^2_+. Now $\zeta^2_+ \in L^p_{m+2,\gamma}(\Lambda^2_+T^*C)$ and $p > 4$, so the Sobolev Embedding Theorem shows that L^p_{m+2} embeds in $C^{m+1,\alpha}$ for $\alpha = 1 - 4/p$. Therefore we do have $C^{m,\alpha}$ control on the coefficients of $L_{\zeta^2_+}$, and can show that $L_{\zeta^2_+}$ is asymptotic in a weighted Hölder $C^{m,\alpha}_\gamma$ sense to a cylindrical operator. So (39) holds as in [16], and the lemma is proved. □

Suppose now that $\zeta^2_+ \in Q^{-1}(0)$. Then for $m = l \geqslant 1$ we have $(\zeta^2_+, 0) \in L^p_{m+2,\gamma}\big(B_{\epsilon'}(\Lambda^2_+T^*C)\big) \times L^p_{m+2,\gamma}(\Lambda^4T^*C)$, with $P(\zeta^2_+, 0) \equiv 0$. So (36) gives

$$L_{\zeta^2_+}(\zeta^2_+, 0) = -E(x, \zeta^2_+, \nabla\zeta^2_+, \nabla^2\zeta^2_+). \tag{40}$$

As $\zeta^2_+ \in L^p_{m+2,\gamma}(\Lambda^2_+T^*C)$ we see from (38) and the asymptotic behavior of F that the right hand side of (40) lies in $L^p_{m,\gamma}(\Lambda^3T^*C)$. Lemma 4.7 with $(\xi^2_+, \xi^4) = (\zeta^2_+, 0)$ then shows that $\zeta^2_+ \in L^p_{m+3,\gamma}(\Lambda^2_+T^*C)$. So we have increased the regularity of ζ^2_+ by one derivative. By induction we have $\zeta^2_+ \in L^p_{m+2,\gamma}(\Lambda^2_+T^*C)$ for $m = l, l+1, l+2, \ldots$, and Proposition 4.6 is proved. □

Now let $\mathcal{M}^\gamma_C$ be the moduli space of asymptotically cylindrical coassociative submanifolds in M close to C, and asymptotic to $L \times (R', \infty)$ with decay rate γ, as in Theorem 1.1. Define a map $S\colon\ Q^{-1}(0) \to \{4\text{–submanifolds of } M\}$ by $S\colon\ \zeta^2_+ \to \Theta(\Gamma_{\zeta^2_+})$, where $\Gamma_{\zeta^2_+}$ is the *graph* of ζ^2_+ in $B_{\epsilon'}(\Lambda^2_+T^*C)$ as above.

Proposition 4.8 *S is a homeomorphism from $Q^{-1}(0)$ to a neighborhood of C in $\mathcal{M}_C^\gamma$.*

Proof First we must show that S maps $Q^{-1}(0)\to\mathcal{M}_C^\gamma$. Let $\zeta_+^2\in Q^{-1}(0)$, and set $\widetilde{C}=S(\zeta_+^2)$. Then $(\Theta\circ\zeta_+^2)^*(\varphi)=Q(\zeta_+^2)=0$, so $\varphi|_{\widetilde{C}}=0$ as $\widetilde{C}$ is the image of the map $\Theta\circ\zeta_+^2\colon C\to M$. Hence $\widetilde{C}$ is *coassociative.* Proposition 4.6 and Sobolev Embedding shows that ζ_+^2 is smooth, so $\widetilde{C}$ is a *smooth submanifold* of M.

To show $\widetilde{C}$ is *asymptotically cylindrical* with rate γ, note that $\Theta\circ\zeta_+^2\colon C\to\widetilde{C}$ is a diffeomorphism. Define $\widetilde{K}'=\Theta\circ\zeta_+^2(K')$ and $\widetilde{\Phi}\colon L\times(R',\infty)\to\widetilde{C}\setminus\widetilde{K}'$ by $\widetilde{\Phi}=\Theta\circ\zeta_+^2\circ\Phi$. Then $\widetilde{K}'$ is compact, and $\widetilde{\Phi}$ is a diffeomorphism. Now ζ_+^2 is a section of $\Lambda_+^2T^*C\cong\nu_C$, so $\Phi^*(\zeta_+^2)$ is a section of $\Phi^*(\nu_C)$ over $L\times(R',\infty)$. Pulling back by the isomorphism ξ gives a section $\xi^*\circ\Phi^*(\zeta_+^2)$ of the vector bundle $\nu_L\times(R',\infty)$ over $L\times(R',\infty)$.

Now the normal vector field v on $L\times(R',\infty)$ making (3) commute is also a section of $\nu_L\times(R',\infty)$. Define $\widetilde{v}=v+\xi^*\circ\Phi^*(\zeta_+^2)$. Then $\widetilde{v}$ is a section of $L\times(R',\infty)$, and the definition of Ξ in (30) and part (iii) of the definition of Θ after (33) show that (3) commutes with C,K',Φ,v replaced by $\widetilde{C},\widetilde{K}',\widetilde{\Phi},\widetilde{v}$. Therefore by Definition 2.9, $\widetilde{C}$ is asymptotically cylindrical with rate γ if $\left|\nabla^k\widetilde{v}\right|=O(e^{\gamma t})$ on $L\times(R',\infty)$ for all $k\geqslant 0$.

As C is asymptotically cylindrical with rate $\beta<\gamma$ we have $\left|\nabla^k v\right|=O(e^{\beta t})$ for all $k\geqslant 0$. Proposition 4.6 and Sobolev Embedding for weighted spaces, which holds as in Lockhart [15, Theorem 3.10] and Bartnik [2, Theorem 1.2], then imply that $\left|\nabla^k\zeta_+^2\right|=O(e^{\gamma\rho})$ on C for all $k\geqslant 0$, where ρ is as in Definition 3.1. Therefore $\nabla^k(\xi^*\circ\Phi^*(\zeta_+^2))=O(e^{\gamma t})$ on $L\times(R',\infty)$, as $\Phi^*(\rho)\equiv t$, and ξ,Φ are asymptotically cylindrical. As $\beta<\gamma$ this gives $\left|\nabla^k\widetilde{v}\right|=O(e^{\gamma t})$ for all $k\geqslant 0$, and $\widetilde{C}$ is asymptotically cylindrical with weight γ. Hence S maps $Q^{-1}(0)\to\mathcal{M}_C^\gamma$.

Next we reverse the argument. Suppose $\widetilde{C}$ is close to C in $\mathcal{M}_C^\gamma$. As $\widetilde{C},C$ are C^1 close there exists a unique smooth section ζ_+^2 of $B_{\epsilon'}(\Lambda_+^2T^*C)$ such that $\Theta\circ\zeta_+^2\colon C\to\widetilde{C}$ is a diffeomorphism. Since $\varphi|_{\widetilde{C}}\equiv 0$ we have $Q(\zeta_+^2)=(\Theta\circ\zeta_+^2)^*(\varphi)=0$. Let $\widetilde{C}$ have data $\widetilde{K}'$, $\widetilde{\Phi}$, $\widetilde{v}$ as in Definition 2.9. The argument above shows that $\widetilde{v}=v+\xi^*\circ\Phi^*(\zeta_+^2)$. But $\left|\nabla^k v\right|=O(e^{\gamma t})$ and $\left|\nabla^k\widetilde{v}\right|=O(e^{\gamma t})$ for all $k\geqslant 0$. Subtracting implies that $\left|\nabla^k\zeta_+^2\right|=O(e^{\gamma\rho})$ on C for all $k\geqslant 0$.

We need to show that $\zeta_+^2\in L_{l+2,\gamma}^p\bigl(B_{\epsilon'}(\Lambda_+^2T^*C)\bigr)$. The estimate $\left|\nabla^k\zeta_+^2\right|=O(e^{\gamma\rho})$ for $k\geqslant 0$ does not prove this, however it does imply that ζ_+^2 lies in

$L^p_{l+2,\gamma'}\big(B_{\epsilon'}(\Lambda^2_+T^*C)\big)$ for any $\gamma' > \gamma$. Hence ζ^2_+ lies in $Q'^{-1}(0)$, where Q' is Q defined with rate γ'.

Choose $\gamma' > \gamma$ such that $[\gamma,\gamma'] \cap \mathcal{D}_{(\mathrm{d}_+ + \mathrm{d}^*)_0} = \emptyset$. Then $Q^{-1}(0) \subseteq Q'^{-1}(0)$, and from above $Q^{-1}(0)$, $Q'^{-1}(0)$ are smooth, finite-dimensional and locally isomorphic to $\mathrm{Ker}\big((\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2,\gamma}\big)$, $\mathrm{Ker}\big((\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2,\gamma'}\big)$ respectively. But as $[\gamma,\gamma'] \cap \mathcal{D}_{(\mathrm{d}_+ + \mathrm{d}^*)_0} = \emptyset$ these kernels are equal, so $Q^{-1}(0)$ and $Q'^{-1}(0)$ coincide near 0. Thus if $\widetilde{C}$ is close enough to C in $\mathcal{M}^\gamma_C$ then ζ^2_+ lies in $Q^{-1}(0)$, as we want.

To show S is a *homeomorphism* requires us to specify the topology on $\mathcal{M}^\gamma_C$, which we have not done. The natural way to define a topology on a space of submanifolds is to identify submanifolds $\widetilde{C}$ near C with sections of the normal bundle ν_C to C, and induce the topology from some Banach norm on a space of sections of ν_C. In our case, this just means that the topology on $\mathcal{M}^\gamma_C$ is induced from some choice of Banach norm on sections ζ^2_+ of $\Lambda^2_+T^*C$, say the C^1_γ topology.

But since $Q^{-1}(0)$ with its $L^p_{l+2,\gamma}$ topology is locally homeomorphic to the *finite-dimensional* vector space $\mathrm{Ker}\big((\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2,\gamma}\big)$, *any* choice of topology on sections ζ^2_+ gives the same topology on $\mathcal{M}^\gamma_C$, as all Banach norms give the same topology on a finite-dimensional space. So S is a local homeomorphism. $\square$

As from above $Q^{-1}(0)$ is smooth, finite-dimensional and locally isomorphic to $\mathrm{Ker}\big((\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2,\gamma}\big)$, Theorem 3.10 and Proposition 4.8 now prove Theorem 1.1.

Remark 4.9 In a forthcoming paper, we study the deformation space of asymptotically cylindrical coassociative submanifolds with moving boundary. Similarly to the setting here, we work with an asymptotically cylindrical G_2–manifold M with a Calabi–Yau boundary X at infinity. We also assume that the boundary of the coassociative submanifold at infinity is a special Lagrangian submanifold of X and is allowed to move. The details of this construction will appear in [26].

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