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Hadamard spaces with isolated flats

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Abstract

We explore the geometry of nonpositively curved spaces with isolated flats, and its consequences for groups that act properly discontinuously, cocompactly, and isometrically on such spaces. We prove that the geometric boundary of the space is an invariant of the group up to equivariant homeomorphism. We also prove that any such group is relatively hyperbolic, biautomatic, and satisfies the Tits Alternative. The main step in establishing these results is a characterization of spaces with isolated flats as relatively hyperbolic with respect to flats. Finally we show that a CAT(0) space has isolated flats if and only if its Tits boundary is a disjoint union of isolated points and standard Euclidean spheres.

In an appendix written jointly with Hindawi, we extend many of the results of this article to a more general setting in which the isolated subspaces are not required to be flats.

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1 Introduction

In this article, we explore the large scale geometry of CAT(0) spaces with isolated flats and its implications for groups that act geometrically, ie, properly discontinuously, cocompactly, and isometrically, on such spaces. Spaces with isolated flats have many features in common with Gromov-hyperbolic spaces and can be viewed as the nonhyperbolic CAT(0) spaces that are closest to being hyperbolic.

Throughout this article, a *k–flat* is an isometrically embedded copy of Euclidean space $\mathbb{E}^k$ for $k \geq 2$. In particular a geodesic line is not considered to be a flat. Let $\mathrm{Flat}(X)$ denote the space of all flats in X with the topology of Hausdorff convergence on bounded sets (see Definition 2.1.1 for details). A CAT(0) space with a geometric group action has *isolated flats* if it contains an equivariant collection $\mathcal{F}$ of flats such that $\mathcal{F}$ is closed and isolated in $\mathrm{Flat}(X)$ and each flat $F \subseteq X$ is contained in a uniformly bounded tubular neighborhood of some $F' \in \mathcal{F}$. As with the notion of Gromov-hyperbolicity, the notion of isolated flats can be characterized in many equivalent ways.

1.1 Examples

The prototypical example of a CAT(0) space with isolated flats is the truncated hyperbolic space associated to a finite volume cusped hyperbolic manifold M. Such a space is obtained from hyperbolic space $\mathbb{H}^n$ by removing an equivariant collection of disjoint open horoballs that are the lifts of the cusps of M and endowing the resulting space with the induced length metric. The truncated space is nonpositively curved,[1] and its only flats are the boundaries of the deleted horoballs. It is easy to see that the truncated space has isolated flats. Furthermore, the fundamental group of M acts cocompactly on the truncated space.

Other examples of groups that act geometrically on CAT(0) spaces with isolated flats include the following:

- Geometrically finite Kleinian groups [8].
- Fundamental groups of compact manifolds obtained by gluing finite volume hyperbolic manifolds along cusps (Heintze, see [23, §3.1]).

[1] This fact is a special case of a result stated by Gromov in [23, §2.2]. A proof of Gromov's theorem was provided by Alexander–Berg–Bishop in [1]. For a proof tailored to this special case, see [11, II.11.27].

- Fundamental groups of compact manifolds obtained by the cusp closing construction of Thurston–Schroeder [40].
- Limit groups (also known as ω–residually free groups), which arise in the study of equations over free groups [2].

Examples in the 2–dimensional setting include the fundamental group of any compact nonpositively curved 2–complex whose 2–cells are isometric to regular Euclidean hexagons [4, 44]. Ballmann–Brin showed that such 2–complexes exist in abundance and can be constructed with arbitrary local data [4]. For instance, for each simplicial graph L there is a CAT(0) hexagonal 2–complex X such that the link of every vertex in X is isomorphic to the graph L ([35], see also [27] and [7]).

As indicated in [32], the notion of isolated flats has a natural generalization where the family $\mathcal{F}$ is a collection of closed convex subspaces rather than a collection of flats. In an appendix by Hindawi, Hruska, and Kleiner, we extend the results of this article to this more general setting.

In particular, this generalization includes the universal covers of compact 3–manifolds whose geometric decomposition contains at least one hyperbolic component (see [32]), and the universal covers of closed, real analytic, nonpositively curved 4–manifolds whose Tits boundary does not contain a nonstandard component (see [29] for details).

1.2 Main results

The following theorem proves that several other conditions are equivalent to having isolated flats. Further equivalent geometric notions are discussed in Theorem 1.2.3 below.

Theorem 1.2.1 *Let X be a* CAT(0) *space and Γ a group acting geometrically on X. The following are equivalent.*

(1) *X has isolated flats.*

(2) *Each component of the Tits boundary $\partial_T X$ is either an isolated point or a standard Euclidean sphere.*

(3) *X is a relatively hyperbolic space with respect to a family of flats $\mathcal{F}$.*

(4) *Γ is a relatively hyperbolic group with respect to a collection of virtually abelian subgroups of rank at least two.*

Several different nomenclatures exist in the literature for relative hyperbolicity. For groups, we use the terminology of Bowditch throughout. In Farb's terminology, the groups we call "relatively hyperbolic" are called "relatively hyperbolic with Bounded Coset Penetration." For metric spaces, the property we call "relatively hyperbolic" was introduced by Druțu–Sapir, although they used the term "asymptotically tree-graded" for such spaces. Druțu–Sapir proved the equivalence of the metric and group theoretic notions of relative hyperbolicity for a finitely generated group with the word metric [17].

The implications (3) $\Longleftrightarrow$ (4) and (3) $\Longrightarrow$ (1) follow in a straightforward fashion from work of Druțu–Osin–Sapir [17]. A large part of this article consists of establishing the remaining implications (1) $\Longrightarrow$ (3) and (1) $\Longleftrightarrow$ (2).

A result analogous to (1) $\Longrightarrow$ (3) is established by Kapovich–Leeb in [32] using a more restrictive notion of isolated flats, discussed in more detail in the next subsection. This notion, while sufficiently general for their purposes, is tailored to make their proof go smoothly. A major goal of this article is to generalize Kapovich–Leeb's results to a more natural class of spaces. A portion of our proof can be viewed as a streamlined version of Kapovich–Leeb's proof.

Theorem 1.2.1 has the following consequences for groups acting on spaces with isolated flats, using some existing results from the literature.

Theorem 1.2.2 *Let Γ act geometrically on a* CAT(0) *space X with isolated flats. Then X and Γ have the following properties.*

1. *Quasi-isometries of X map maximal flats to maximal flats.*
2. *A finitely generated subgroup $H \leq \Gamma$ is undistorted if and only if it is quasiconvex (with respect to the* CAT(0) *action).*
3. *The geometric boundary ∂X is a group invariant of Γ.*
4. *Γ satisfies the Strong Tits Alternative. In other words, every subgroup of Γ either is virtually abelian or contains a free subgroup of rank two.*
5. *Γ is biautomatic.*

Property (1) is a direct consequence of Theorem 1.2.1 and a result of Druțu–Sapir [17, Proposition 5.4]. Properties (2) and (3) are consequences of Theorem 1.2.1 and results proved by Hruska in [31]. Property (4) follows from Theorem 1.2.1 together with work of Gromov and Bowditch [24, 9]. Property (5) is proved using Theorem 1.2.1 and a result of Rebbechi [38].

In addition, Chatterji–Ruane have used Theorem 1.2.1 to show that a group acting geometrically on a space with isolated flats satisfies the property of Rapid

Decay as well as the Baum–Connes conjecture [13] (cf [16]). Groves has also used Theorem 1.2.1 in his proof that toral, torsion free groups acting geometrically on spaces with isolated flats are Hopfian [26].

In proving Theorem 1.2.1, we found it useful to clarify the notion of isolated flats by exploring various geometric formulations and proving their equivalence. Indeed, we consider this exploration to be the most novel contribution of this article. Each of these formulations has advantages in certain situations. We explain these various formulations below.

Let X be a CAT(0) space and Γ a group acting geometrically on X. We say that X has (IF1), or *isolated flats in the first sense*, if it satisfies the definition of isolated flats given above. A flat in X is *maximal* if it is not contained in a finite tubular neighborhood of any higher dimensional flat. The space X has *thin parallel sets* if for each geodesic line γ in a maximal flat F the parallel set $\mathbb{P}(\gamma)$ lies in a finite tubular neighborhood of F. The space has *uniformly thin parallel sets* if there is a uniform bound on the thickness of these tubular neighborhoods. The space X has *slim parallel sets* if for each geodesic γ in a maximal flat F, the Tits boundary of the parallel set $\mathbb{P}(\gamma)$ is equal to the Tits boundary of F.

We say that X has (IF2), or *isolated flats in the second sense*, if there is a Γ–invariant set $\mathcal{F}$ of flats in X such that the following properties hold.

(1) There is a constant $D < \infty$ so that each flat in X lies in a D–tubular neighborhood of some flat $F \in \mathcal{F}$.

(2) For each positive $\rho < \infty$ there is a constant $\kappa = \kappa(\rho) < \infty$ so that for any two distinct flats $F, F' \in \mathcal{F}$ we have

$$\operatorname{diam}\bigl(\mathcal{N}_\rho(F) \cap \mathcal{N}_\rho(F')\bigr) < \kappa.$$

Theorem 1.2.3 *Let X be a CAT(0) space and Γ a group acting geometrically on X. The following are equivalent.*

(1) *X has (IF1).*

(2) *X has uniformly thin parallel sets.*

(3) *X has thin parallel sets.*

(4) *X has slim parallel sets.*

(5) *X has (IF2).*

Of these formulations, (IF1) seems the most natural and also appears to be the most easily verified in practice. For instance, (IF1) is evidently satisfied for the cusp gluing manifolds of Heintze and the cusp closing manifolds of Thurston–Schroeder. The notion of slim parallel sets plays a crucial role in the proof of (2) $\Longrightarrow$ (1) from Theorem 1.2.1. The notion of thin parallel sets is important in establishing the equivalence of the remaining formulations. The formulation (IF2), while rather technical in appearance, seems to be the strongest and most widely applicable formulation, giving precise control over the coarse intersections of any pair of maximal flats. This formulation is used throughout this paper and has previously been used in the literature as a definition of isolated flats (see [30, 31]).

1.3 Historical background

CAT(0) spaces with isolated flats were first considered by Kapovich–Leeb and Wise, independently. Kapovich–Leeb [32] study a class of CAT(0) spaces in which the maximal flats are disjoint and separated by regions of strict negative curvature, ie, the separating regions are locally CAT(-1). It is clear that such a space satisfies (IF1). As mentioned above, they prove that such a space is relatively hyperbolic, although they use a different terminology, and they conclude that quasi-isometries of such spaces coarsely preserve the set of all flats. The latter result is a key step in their quasi-isometry classification of nongeometric Haken 3–manifold groups.

The Flat Closing Problem asks the following: If Γ acts geometrically on a CAT(0) space X, is it true that either Γ is word hyperbolic or Γ contains a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$? If X has isolated flats, an easy geometric argument solves the Flat Closing Problem in the affirmative. Such an argument was discovered by Ballmann–Brin in the context of CAT(0) hexagonal 2–complexes [4] and independently by Wise for arbitrary CAT(0) spaces satisfying the weaker formulation (IF1) of isolated flats [44, Proposition 4.0.4].

The Flat Plane Theorem states that a proper, cocompact CAT(0) space either is δ–hyperbolic or contains a flat plane [18, 24, 28, 10]. In the 2–dimensional setting, Wise proved an analogous Flat Triplane Theorem, which states that a proper, cocompact CAT(0) 2–complex either has isolated flats or contains an isometrically embedded triplane. A *triplane* is the space formed by gluing three Euclidean halfplanes isometrically along their boundary lines. A proof, due to Wise, of the Flat Triplane Theorem first appeared in [30], however the ideas are implicit in Wise's article [43], which has been circulated since 1998. Some of the ideas in the proof of Theorem 1.2.3 have combinatorial analogues in the proof of the Flat Triplane Theorem, although the 2–dimensional situation is substantially simpler than the general case.

The question of whether (or when) the geometric boundary depends only on the group Γ was raised in [25, §6.B_4]; in the Gromov hyperbolic case the invariance follows from the stability of quasi-geodesics. Croke–Kleiner [14, 15] found examples which showed that this is not always the case, and then analyzed, for a class of examples including nonpositively curved graph manifolds, exactly what geometric structure determines the geometric boundary up to equivariant homeomorphism. We note that Buyalo, using a different geometric idea, later found examples showing that the equivariant homeomorphism type of the boundary is not a group invariant [12]. At the beginning of Croke–Kleiner's project, Kleiner studied, in unpublished work from 1997, the simpler case of spaces with isolated flats; he proved (a reformulation of) the implications (1) $\Longrightarrow$ (2), (3) of Theorem 1.2.1, using the strong hypothesis (IF2), and showed that equivariant quasi-isometries map geodesic segments close to geodesic segments and induce equivariant boundary homeomorphisms.

Hruska studied isolated flats in the 2–dimensional setting in [30]. He showed that in that setting, isolated flats is equivalent to the Relatively Thin Triangle Property, which states that each geodesic triangle is "thin relative to a flat," and also to the Relative Fellow Traveller Property, which states that quasigeodesics with common endpoints track close together "relative to flats." The Relative Fellow Traveller Property generalizes a phenomenon discovered by Epstein in the setting of geometrically finite Kleinian groups [19, Theorem 11.3.1]. In [31] Hruska then established parts (1), (2), and (3) of Theorem 1.2.2 for CAT(0) spaces satisfying both (IF2) and the Relative Fellow Traveller Property.

1.4 Summary of the sections

Section 2 contains general facts about the geometry of CAT(0) spaces and asymptotic cones. Subsection 2.1 is a brief review of facts from the literature, serving to establish the terminology and notation we use throughout. In the remaining subsections, we establish preliminary results that are not specific to the isolated flats setting. In Subsection 2.2, we prove several results about triangles in which one or more vertex angles have a small angular deficit. In Subsection 2.3, we prove a lemma characterizing periodic flats whose boundary sphere is isolated in the Tits boundary of the CAT(0) space. In Subsection 2.4, we prove a result about ultralimits of triangles in an ultralimit of CAT(0) spaces.

The goal of Section 3 is to prove the implication (1) $\Longrightarrow$ (3) from Theorem 1.2.1. The definition of isolated flats used in this section is the strong formulation

(IF2). Subsection 3.1 is a review of basic properties of spaces with isolated flats; in particular, we prove that maximal flats are periodic (this result is due to Ballmann–Brin and Wise, independently). In Subsection 3.2 we prove Proposition 3.2.5, which states that, in the presence of isolated flats, a large nondegenerate triangle with small angular deficit must lie close to a flat (cf Proposition 4.2 of [32]). This proposition can be viewed as a generalization of the well-known result that a triangle with zero angle deficit bounds a flat Euclidean region. In Subsection 3.3 we study the geometry of the asymptotic cones of a space with isolated flats. A key result is Proposition 3.3.2, which determines the space of directions at a point in an asymptotic cone. The space of directions consists of a disjoint union of spherical components corresponding to flats and a discrete set of isolated directions. Finally we prove Theorem 3.3.6, which establishes the implication (1) $\Longrightarrow$ (3) from Theorem 1.2.1. As mentioned above, the proof of (3) $\Longrightarrow$ (1) is immediate from work of Druţu–Sapir.

In Section 4, we focus on applications of Theorem 3.3.6. We briefly explain the equivalence (3) $\Longleftrightarrow$ (4) of Theorem 1.2.1 and explain the various parts of Theorem 1.2.2. Most of the results in this section are immediate consequences of Theorem 3.3.6 in conjunction with results from the literature. Consequently, must of this section is expository in nature. The applications are divided into two subsections, the first of which consists of geometric results relating to quasi-isometries and the second of which concerns consequences of relative hyperbolicity.

In Section 5 we study the various equivalent geometric formulations of isolated flats. In particular we give the proof of Theorem 1.2.3. We make use of these formulations in the proof of Theorem 5.2.5, which establishes the last remaining equivalence (1) $\Longleftrightarrow$ (2) of Theorem 1.2.1.

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2 Background and preliminaries

2.1 Notation and terminology

In this section, we establish notation and terminology that will be used throughout this article. We refer the reader to [11] and [3] for detailed introductions to CAT(0) spaces. For an introduction to asymptotic cones, we refer the reader to [25] and [33]. The notion of a tree-graded space is due to Druţu–Sapir [17].

Let X be a CAT(0) space. For points $p,q \in X$, we let $[p,q]$ denote the unique geodesic connecting p and q. If c and c' are geodesics emanating from a common point x, then $\angle_x(c,c')$ denotes the angle at x between c and c'. If $x \neq y$ and $x \neq z$, then $\angle_x(y,z)$ denotes the angle at x between $[x,y]$ and $[x,z]$. The comparison angle at x corresponding to the angle $\angle_x(y,z)$ is denoted $\tilde{\angle}_x(y,z)$.

For each $p \in X$, let $\Sigma_p X$ denote the space of directions at p in X and let $\overrightarrow{px}$ denote the direction at p corresponding to the geodesic $[p,x]$. The *logarithm map*

$$\log_p \colon X \setminus \{p\} \to \Sigma_p X$$

sends each point x to the corresponding direction $\overrightarrow{px}$.

The ideal boundary of X equipped with the cone topology is denoted ∂X and called the visual or geometric boundary. The Tits angle metric on ∂X is denoted by $\angle_T$ and the coresponding length metric by d_T. The Tits boundary $\partial_T X$ is the ideal boundary of X equipped with the metric d_T.

Definition 2.1.1 Let X be a proper metric space. The set of all closed subspaces of X has a natural topology of *Hausdorff convergence on bounded sets* defined as follows. For each closed set $C_0 \subseteq X$, each $x_0 \in X$, and each positive r and ϵ, define $U(C_0,x_0,r,\epsilon)$ to be the set of all closed subspaces $C \subseteq X$ such that the Hausdorff distance between $C \cap B(x_0,r)$ and $C_0 \cap B(x_0,r)$ is less than ϵ. The topology of Hausdorff convergence on bounded sets is the topology generated by the sets $U(\cdot,\cdot,\cdot,\cdot)$.

Definition 2.1.2 *Asymptotic cone* Let (X,d) be a metric space with a sequence of basepoints $(\star_n)$, and let (λ_n) be a sequence of numbers called *scaling constants* with $\omega\text{-}\lim \lambda_n = \infty$. The *asymptotic cone* $\mathrm{Cone}_\omega(X,\star_n,\lambda_n)$ of X with respect to $(\star_n)$ and (λ_n) is the ultralimit $\omega\text{-}\lim(X,\lambda_n^{-1}d,\star_n)$.

Any ultralimit of a sequence of CAT(0) spaces is a complete CAT(0) space. Any asymptotic cone of Euclidean space $\mathbb{E}^k$ is isometric to $\mathbb{E}^k$. Let X be any CAT(0) space with basepoints $(\star_n)$ and scaling constants (λ_n), and let (F_n) be a sequence of k-flats in X such that $\omega\text{-lim}\,\lambda_n^{-1}d(F_n,\star_n)$ is finite. Then the ultralimit of the sequence of embeddings

$$\big(F_n, \lambda_n^{-1}d_{F_n}, \pi_{F_n}(\star_n)\big) \hookrightarrow (X, \lambda_n^{-1}d, \star_n)$$

is a k-flat in $\mathrm{Cone}_\omega(X,\star_n,\lambda_n)$.

Definition 2.1.3 *Tree-graded* Let X be a complete geodesic metric space and let $\mathcal{P}$ be a collection of closed geodesic subspaces of X called *pieces*. We say that X is *tree-graded with respect to* $\mathcal{P}$ if the following two properties hold.

(1) Every two distinct pieces have at most one common point.

(2) Every simple geodesic triangle in X (a simple loop composed of three geodesics) lies inside one piece.

Definition 2.1.4 *Relatively hyperbolic spaces* Let X be a space and $\mathcal{A}$ a collection of subspaces of X. For each asymptotic cone $X_\omega = \mathrm{Cone}_\omega(X,\star_n,\lambda_n)$, let $\mathcal{A}_\omega$ be the collection of all subsets $A \subset X_\omega$ of the form $A = \omega\text{-lim}\,A_n$ where $A_n \in \mathcal{A}$ and $\omega\text{-lim}\,\lambda_n^{-1}\,d(A_n,\star_n) < \infty$. Then X is *relatively hyperbolic with respect to* $\mathcal{A}$ if, for every nonprincipal ultrafilter ω, each asymptotic cone X_ω is tree-graded with respect to A_ω.

2.2 Triangles with small angular deficit

In this subsection, we consider triangles whose angles are nearly the same as the corresponding comparison angles. We prove Proposition 2.2.1, which states that the family of such triangles is closed under certain elementary subdivisions, and Proposition 2.2.3, which provides a means of constructing numerous such triangles.

Proposition 2.2.1 *Let $\Delta(x,y,z)$ be a geodesic triangle in a* CAT(0) *space, and choose $p \in [y,z] \setminus \{x,y,z\}$. Suppose each angle of $\Delta(x,y,z)$ is within δ of the corresponding comparison angle. Then each angle of $\Delta(x,y,p)$ and each angle of $\Delta(x,p,z)$ is within 3δ of the corresponding comparison angle.*

Furthermore, if each comparison angle of $\Delta(x,y,z)$ is greater than θ for some $\theta > 0$, then $\angle_p(x,y)$, $\angle_p(x,z)$, and their comparison angles lie in $(\theta,\pi-\theta)$.

In order to prove the proposition, we first establish the following lemma concerning subdivisions of a triangle with one angle close to its comparison angle.

Lemma 2.2.2 *Choose points $x, y, z \in X$ and $p \in [y, z] \setminus \{x, y, z\}$.*

(1) *Suppose $\angle_y(x, z)$ is within δ of the corresponding comparison angle. Then the quantities $\angle_y(x, p)$, $\widetilde{\angle}_y(x, p)$, and $\widetilde{\angle}_y(x, z)$ are within δ.*

(2) *Suppose $\angle_x(y, z)$ is within δ of the corresponding comparison angle. Then the same is true of both $\angle_x(y, p)$ and $\angle_x(p, z)$. Furthermore, the quantities $\widetilde{\angle}_x(y, z)$ and $\widetilde{\angle}_x(y, p) + \widetilde{\angle}_x(p, z)$ are within δ.*

Proof The first assertion is immediate, since

$$\angle_y(x, z) = \angle_y(x, p) \le \widetilde{\angle}_y(x, p) \le \widetilde{\angle}_y(x, z) \le \angle_y(x, z) + \delta.$$

For the second assertion, observe that

$$\begin{aligned} \angle_x(y, z) &\le \angle_x(y, p) + \angle_x(p, z) \\ &\le \widetilde{\angle}_x(y, p) + \widetilde{\angle}_x(p, z) \\ &\le \widetilde{\angle}_x(y, z) \\ &\le \angle_x(y, z) + \delta, \end{aligned}$$

where the inequality on the third line is due to Alexandrov's Lemma. The assertion now follows from the fact that each angle is no greater than its comparison angle. □

Proof of Proposition 2.2.1 Lemma 2.2.2 shows that each of the angles

$$\angle_y(x, p), \quad \angle_z(x, p), \quad \angle_x(y, p), \quad \text{and} \quad \angle_x(p, z)$$

is within δ of its corresponding comparison angle. But since $p \in [y, z]$, we also have

$$\begin{aligned} \pi &\le \angle_p(x, y) + \angle_p(x, z) \\ &\le \widetilde{\angle}_p(x, y) + \widetilde{\angle}_p(x, z) \\ &= \big(\pi - \widetilde{\angle}_y(x, p) - \widetilde{\angle}_x(y, p)\big) + \big(\pi - \widetilde{\angle}_z(x, p) - \widetilde{\angle}_x(z, p)\big) \\ &\le 2\pi + 3\delta - \widetilde{\angle}_y(x, z) - \widetilde{\angle}_x(y, z) - \widetilde{\angle}_z(x, y) \\ &= \pi + 3\delta, \end{aligned}$$

where the inequality on the fourth line is a consequence of Lemma 2.2.2. It now follows easily that each of $\angle_p(x, y)$ and $\angle_p(x, z)$ is within 3δ of its comparison angle.

To establish the last assertion, suppose each comparison angle of $\Delta(x,y,z)$ is greater than θ. Then

$$\angle_p(x,y) \le \tilde{\angle}_p(x,y) \le \pi - \tilde{\angle}_y(p,x) < \pi - \theta.$$

Similarly, we see that $\angle_p(x,z) \le \tilde{\angle}_p(x,z) < \pi - \theta$. But $\angle_p(x,y) + \angle_p(x,z)$ is at least π by the triangle inequality for $\angle_p$. Thus $\angle_p(x,y)$ and $\angle_p(x,z)$ are each greater than θ as desired. □

Proposition 2.2.3 *Let c and c' be geodesics with $c(0) = c'(0) = x$. Then for sufficiently small t and t', each angle of $\Delta\big(x, c(t), c'(t')\big)$ is within δ of the corresponding comparison angle.*

The proof of Proposition 2.2.3 uses the following lemma.

Lemma 2.2.4 *Choose $x,y,z \in X$ and let $c(t)$ be the geodesic $[x,y]$ parametrized so that $c(0) = x$. Then*

$$\lim_{t\to 0} \angle_{c(t)}(x,z) = \lim_{t\to 0} \tilde{\angle}_{c(t)}(x,z) = \pi - \angle_x(y,z).$$

Proof Consider the comparison triangle $\bar{\Delta}\big(x,z,c(t)\big)$. The comparison angle $\tilde{\angle}_x\big(c(t),z\big)$ tends to $\angle_x(y,z)$ as $t \to 0$ by the first variation formula [11, II.3.5]. Since $\tilde{\angle}_z\big(c(t),x\big)$ clearly tends to zero and the angles of $\bar{\Delta}$ sum to π, we see that

$$\limsup_{t\to 0} \angle_{c(t)}(x,z) \le \lim_{t\to 0} \tilde{\angle}_{c(t)}(x,z) = \pi - \angle_x(y,z).$$

But we also have

$$\begin{aligned} \liminf_{t\to 0} \angle_{c(t)}(x,z) &\ge \liminf_{t\to 0} \big(\pi - \angle_{c(t)}(y,z)\big) \\ &= \pi - \limsup_{t\to 0} \angle_{c(t)}(y,z) \\ &\ge \pi - \angle_x(y,z), \end{aligned}$$

where the first inequality follows from the triangle inequality for $\angle_{c(t)}$ and the last uses the upper semicontinuity of $\angle$. □

Proof of Proposition 2.2.3 By the definition of $\angle_x(c,c')$, all points y and z on c and c' sufficiently close to x have the property that $\tilde{\angle}_x(y,z)$ is within δ of $\angle_x(y,z)$. Furthermore, by Lemma 2.2.4 for each fixed z, sliding y toward x along c, we can guarantee that $\tilde{\angle}_y(x,z)$ is within δ of $\angle_y(x,z)$.

Thus we may assume that the angles of $\Delta(x,y,z)$ at x and y are within δ of the corresponding comparison angles. Applying Lemma 2.2.4 again, gives us a point $z' \in [x,z]$ so that the angles of $\Delta(x,y,z')$ at x and z' are within δ of their comparison angles. If we consider the subdivision of $\Delta(x,y,z)$ into subtriangles $\Delta(x,y,z')$ and $\Delta(z',y,z)$ and apply Lemma 2.2.2(2), it is clear that $\angle_y(x,z')$ is within δ of $\widetilde{\angle}_y(x,z')$ as well. □

2.3 Periodic Euclidean subspaces and the Tits boundary

Let Y be a proper CAT(0) space, and $F \subseteq Y$ a closed, convex subset. Let $G \subset \mathrm{Isom}(Y)$ be any group of isometries of Y. We say that F is *G–periodic* if $\mathrm{Stab}_G(F) := \{\, g \in G \mid g(F) = F \,\}$ acts cocompactly on F. We say that F is *periodic* if it is $\mathrm{Isom}(Y)$–periodic.

Lemma 2.3.1 *Let F be a periodic subspace of Y isometric to Euclidean space E^k for some $k \geq 1$. Then either $\partial_T F$ is isolated in $\partial_T Y$, or there is a flat half-plane $H \subset Y$ meeting F orthogonally.*

Proof Assume $\partial_T F$ is not isolated in $\partial_T Y$, ie, assume that there are sequences $\xi_k \in \partial_T Y \setminus \partial_T F$, $\eta_k \in \partial_T F$ such that

$$\angle_T(\xi_k, \eta_k) = \angle_T(\xi_k, \partial_T F) \to 0 \tag{2.3.2}$$

as $k \to \infty$. Pick $p \in F$. Let d_F denote the distance function from F. Then by (2.3.2) there exists $c_k \to 0$ such that for all $x \in [p, \xi_k]$ we have $d(x,F) \leq c_k\, d(x,p)$. Since d_F is convex, this also means that when we restrict d_F to the ray $[p,\xi_k]$ we get a function whose left and right derivatives are $\leq c_k$. Pick $R < \infty$, and let $x_k \in [p,\xi_k]$ be the point on $[p,\xi_k]$ where the distance to F is R. Now let $g_k \in \mathrm{Isom}(Y)$ be a sequence of isometries preserving F such that $g_k(x_k)$ remains in a bounded subset of Y, and pass to a subsequence so that $g_k(x_k)$ converges to some point x_∞ and the rays $g_k\big([p,\xi_k]\big)$ converge to a complete geodesic γ passing through x_∞. The derivative estimate on $d_F \big| [p,\xi_k]$ implies that d_F restricts to the constant function R on γ. By the Flat Strip Theorem, we obtain a flat strip of width R meeting F orthogonally. Since this works for any $R < \infty$, we can use the stabilizer of F again to obtain a flat half-plane leaving F orthogonally. □

2.4 Ultralimits of triangles

This subsection is devoted to the proof of the following proposition.

Proposition 2.4.1 *Let $X = \omega\text{-lim}(X_k, d_k, \star_k)$ be the ultralimit of a sequence of based* CAT(0) *spaces, and let x, y, and z be points of X with $x \neq y$ and $x \neq z$. Let (x_k), (y_k), and (z_k) be sequences representing x, y, and z respectively. Then there is a sequence (x'_k) representing x such that $x'_k \in [x_k, y_k]$ and such that, for all sequences (y'_k) and (z'_k) representing y and z, we have*

$$\omega\text{-lim} \angle_{x'_k}(y'_k, z'_k) = \angle_x(y, z).$$

Proposition 2.4.1 has the following immediate corollary regarding ultralimits of geodesic triangles.

Corollary 2.4.2 *Let $X = \omega\text{-lim}(X_k, d_k, \star_k)$ be the ultralimit of a sequence of* CAT(0) *spaces. Let (x_k), (y_k), and (z_k) be sequences representing three distinct points x, y, and z in X. Then there are sequences (x'_k), (y'_k), and (z'_k) also representing x, y, and z with the property that x'_k and y'_k lie on the segment $[x_k, y_k]$ and each angle of the triangle $\Delta(x, y, z)$ is equal to the ultralimit of the corresponding angles of $\Delta(x_k, y_k, z_k)$.* □

The proof of Proposition 2.4.1 uses the following two lemmas.

Lemma 2.4.3 *Let $X = \omega\text{-lim}(X_k, d_k, \star_k)$ as above, and choose points x, y, and z in X with $x \neq y$ and $x \neq z$. If (x_k), (y_k), and (z_k) represent x, y, and z respectively, then the quantity*

$$\omega\text{-lim} \angle_{x_k}(y_k, z_k)$$

depends only on the choice of (x_k) and not on the choice of (y_k) and (z_k).

Proof Fix a sequence (x_k) representing x, and choose sequences (y_k) and (y'_k) representing y. Observe that, since $x \neq y$ we have

$$\omega\text{-lim} \frac{d_k(y_k, y'_k)}{d_k(y_k, x_k)} = 0.$$

So the comparison angle $\tilde{\angle}_{x_k}(y_k, y'_k)$ has ultralimit zero, and therefore so does $\angle_{x_k}(y_k, y'_k)$. The lemma now follows easily from the triangle inequality for $\angle_{x_k}$. □

Lemma 2.4.4 *Suppose $X = \omega\text{-lim}(X_k, d_k, \star_k)$ as above, and we have (x_k), (y_k), and (z_k) representing points x, y, and z in X with $x \neq y$ and $x \neq z$. Then*

$$\omega\text{-lim} \angle_{x_k}(y_k, z_k) \leq \angle_x(y, z).$$

Proof Let c_k and c'_k be geodesic parametrizations of the segments $[x_k, y_k]$ and $[x_k, z_k]$ such that $c_k(0) = c'_k(0) = x_k$. Taking ultralimits of c_k and c'_k produces geodesic parametrizations c and c' of the segments $[x, y]$ and $[x, z]$.

For every positive t, we have

$$\angle_{x_k}(y_k, z_k) = \angle_{x_k}\bigl(c_k(t), c'_k(t)\bigr) \le \widetilde{\angle}_{x_k}\bigl(c_k(t), c'_k(t)\bigr).$$

Taking ultralimits of both sides of this inequality, we see that

$$\omega\text{-lim}\,\angle_{x_k}(y_k, z_k) \le \omega\text{-lim}\,\widetilde{\angle}_{x_k}\bigl(c_k(t), c'_k(t)\bigr) = \widetilde{\angle}_x\bigl(c(t), c'(t)\bigr).$$

Since the previous inequality holds for all $t > 0$, it remains true in the limit as $t \to 0$; ie,

$$\omega\text{-lim}\,\angle_{x_k}(y_k, z_k) \le \angle_x(y, z). \qquad \square$$

Proof of Proposition 2.4.1 Choose sequences (x_k), (y_k), and (z_k) representing x, y, and z respectively. The choice of (y_k) and (z_k) is irrelevant by Lemma 2.4.3. Thus by Lemma 2.4.4, it suffices to find another sequence (x'_k) representing x such that

$$\omega\text{-lim}\,\angle_{x'_k}(y_k, z_k) \ge \angle_x(y, z). \tag{2.4.5}$$

In X, choose points $w^n \in [x, y]$ with $w^n \to x$. Since $[x, y]$ is the ultralimit of the segments $[x_k, y_k]$ we can choose points $w^n_k \in [x_k, y_k]$ so that for each n the sequence (w^n_k) represents w^n.

The triangle inequality for $\angle_{w^n_k}$, together with Lemmas 2.4.4 and 2.2.4 give that

$$\begin{aligned} \omega\text{-}\lim_k \angle_{w^n_k}(y_k, z_k) &\ge \omega\text{-}\lim_k\bigl(\pi - \angle_{w^n_k}(x_k, z_k)\bigr) \\ &\ge \pi - \angle_{w^n}(x, z) \\ &= \angle_x(y, z) - \epsilon^{(n)}, \end{aligned}$$

where $\epsilon^{(n)}$ is a constant depending on n such that $\epsilon^{(n)} \to 0$ as $n \to \infty$. In other words, for each k and n there is a constant $\epsilon^{(n)}_k$ with $\omega\text{-lim}_k\, \epsilon^{(n)}_k = \epsilon^{(n)}$ such that

$$\angle_{w^n_k}(y_k, z_k) \ge \angle_x(y, z) - \epsilon^{(n)}_k.$$

The desired sequence (x'_k) is constructed using a diagonal argument as follows. Set $\mathcal{S}_0 = \mathbb{N}$ and recursively define sets $\mathcal{S}_n$ with $\omega(\mathcal{S}_n) = 1$ so that $\mathcal{S}_n$ is contained in $\mathcal{S}_{n-1}$ but does not contain the smallest element of $\mathcal{S}_{n-1}$, and so that for each $k \in \mathcal{S}_n$ we have $\epsilon^{(n)}_k < 2\epsilon^{(n)}$. Then $(\mathcal{S}_n)$ is a strictly decreasing sequence of sets with empty intersection.

Set $x'_k = w_k^{n(k)}$, where $n(k)$ is the unique natural number such that $k \in \mathcal{S}_{n(k)} \setminus \mathcal{S}_{n(k)+1}$. By construction, $\omega\text{-}\lim_k n(k) = \infty$. Thus (x'_k) represents x, and

$$\omega\text{-}\lim_k \epsilon_k^{(n(k))} \leq \omega\text{-}\lim_k 2\epsilon^{(n(k))} = 0,$$

so (x'_k) satisfies (2.4.5) as desired. □

3 Isolated flats and tree-graded spaces

The main goal of this section is Theorem 3.3.6, which establishes the equivalence (1) $\iff$ (3) of Theorem 1.2.1.

3.1 Isolated flats

Throughout this subsection, let X be a CAT(0) space and Γ be a group acting geometrically on X. For the purposes of this section, the term *isolated flats* refers to the formulation (IF2).

Lemma 3.1.1 *If X has isolated flats with respect to $\mathcal{F}$, then $\mathcal{F}$ is locally finite; in other words, only finitely many elements of $\mathcal{F}$ intersect any given compact set.*

Proof It suffices to show that only finitely many $F \in \mathcal{F}$ intersect each closed metric ball $\overline{B}(x,r)$. Let $\mathcal{F}_0$ be the collection of all flats $F \in \mathcal{F}$ intersecting this ball. Choose $\kappa = \kappa(1)$ so that any intersection of 1–neighborhoods of distinct elements of $\mathcal{F}$ has diameter less than κ. Any sequence of distinct elements in $\mathcal{F}_0$ contains a subsequence (F_i) that Hausdorff converges on bounded sets. In particular, whenever i and j are sufficiently large, there are closed discs $D_i \subset F_i$ and $D_j \subset F_j$ of radius κ whose Hausdorff distance is less than 1, contradicting our choice of κ. □

Lemma 3.1.2 *Any locally finite, Γ–invariant collection $\mathcal{F}$ of flats in X has the following properties.*

(1) *The elements of $\mathcal{F}$ lie in only finitely many Γ–orbits, and the stabilizers of the $F \in \mathcal{F}$ lie in only finitely many conjugacy classes.*

(2) *Each $F \in \mathcal{F}$ is Γ–periodic.*

In particular, the lemma holds if X has isolated flats with respect to $\mathcal{F}$.

Proof Let K be a compact set whose Γ–translates cover X. (1) follows easily from the fact that only finitely many flats in $\mathcal{F}$ intersect K. Thus we only need to establish (2). For each $F \in \mathcal{F}$, let $\{g_i\}$ be a minimal set of group elements such that the translates $g_i(K)$ cover F. If the flats $g_i^{-1}(F)$ and $g_j^{-1}(F)$ coincide, then $g_j g_i^{-1}$ lies in $H := \mathrm{Stab}_\Gamma(F)$. It follows that the g_i lie in only finitely many right cosets Hg_i. In other words, the sets $g_i(K)$ lie in only finitely many H–orbits. But any two H–orbits $Hg_i(K)$ and $Hg_j(K)$ lie at a finite Hausdorff distance from each other. Thus any $g_i(K)$ can be increased to a larger compact set K' so that the translates of K' under H cover F. So F is Γ–periodic, as desired. □

A flat in X is *maximal* if it is not contained in a finite tubular neighborhood of a higher dimensional flat. It follows easily from (IF2) that every flat in $\mathcal{F}$ is maximal.

Corollary 3.1.3 *The set of stabilizers of flats $F \in \mathcal{F}$ is precisely the set $\mathcal{A}$ of maximal virtually abelian subgroups of Γ with rank at least two.*

Proof By the Flat Torus Theorem, each subgroup $A \in \mathcal{A}$ stabilizes a flat. Lemma 3.1.2 and the Bieberbach Theorem show that the stabilizer of each flat $F \in \mathcal{F}$ is virtually abelian. The correspondence now follows easily from the maximality of elements of $\mathcal{F}$ and $\mathcal{A}$. □

3.2 Approximately Euclidean triangles

In this subsection we prove that, in a space with isolated flats, a sufficiently large triangle whose vertex angles are approximately equal to the corresponding comparison angles must lie close to a flat.

Lemma 3.2.1 *Let X be any proper, cocompact metric space. For each positive r and ϵ, there is a constant $a = a(r,\epsilon)$ so that for each isometrically embedded 2–dimensional Euclidean disc C in X of radius a, the central subdisc of C of radius r lies in an ϵ–tubular neighborhood of a flat.*

Proof If not, then there would be constants r and ϵ and a sequence of closed Euclidean 2–dimensional discs C_a for $a = r,\ r+1,\ r+2, \dots$, such that C_a has radius a and the central subdisc of C_a of radius r does not lie in the ϵ–tubular neighborhood of any flat. Applying elements of the cocompact isometry group and passing to a subsequence, we may arrange that the discs C_a Hausdorff converge on bounded sets to a 2–flat, contradicting our choice of r and ϵ. □

Lemma 3.2.2 *Suppose X has isolated flats. There is a decreasing function $D_1 = D_1(\theta) < \infty$ such that if $S \subset X$ is a flat sector of angle $\theta > 0$, then $S \subset \mathcal{N}_{D_1(\theta)}(F)$ for some $F \in \mathcal{F}$.*

Proof Pick positive constants θ and r, and set $\epsilon := 1$. Let $a = a(r,\epsilon)$ be the constant given by Lemma 3.2.1, and choose $\rho = \rho(\theta, a)$ so that, for any sector S with angle at least θ, the entire sector S lies inside a ρ–tubular neighborhood of the subsector

$$S' := \bigl\{\, s \in S \bigm| d(s, \partial S) \geq a \,\bigr\}.$$

Note that for fixed a, the quantity $\rho(\theta, a)$ is a decreasing function of θ. By Lemma 3.2.1, for each $s \in S'$ the intersection of S with $B(s,r)$ is a flat disc contained in the ϵ–neighborhood of a flat. By (IF2–1), this disc also lies in the $(D+\epsilon)$–neighborhood of a flat $F \in \mathcal{F}$. Then (IF2–2) shows that F will be independent of the choice of $s \in S'$ when r is sufficiently large. Setting $D_1(\theta) := D + \epsilon + \rho$ completes the proof. □

Lemma 3.2.3 *Let X have isolated flats. For all $\theta_0 > 0$, $R < \infty$, there exist $\delta_1 = \delta_1(\theta_0, R)$, $\rho_1 = \rho_1(\theta_0, R)$ such that if $p, x, y \in X$ satisfy $d(p,x), d(p,y) > \rho_1$ and*

$$\theta_0 < \angle_p(x,y) \leq \widetilde{\angle}_p(x,y) < \angle_p(x,y) + \delta_1 < \pi - \theta_0,$$

then there is a flat $F \in \mathcal{F}$ such that

$$\bigl([p,x] \cup [p,y]\bigr) \cap B(p,R) \subset \mathcal{N}_{D_1(\theta_0)}(F).$$

Proof If the lemma were false, we would have $\theta_0 > 0$, $R < \infty$ and sequences $p_k, x_k, y_k \in X$ such that $d(p_k, x_k), d(p_k, y_k) \to \infty$ and

$$\lim_{k\to\infty} \angle_{p_k}(x_k, y_k) = \lim_{k\to\infty} \widetilde{\angle}_{p_k}(x_k, y_k) =: \theta \in [\theta_0, \pi - \theta_0]$$

and such that there is no $F \in F$ with

$$\bigl([p_k, x_k] \cup [p_k, y_k]\bigr) \cap B(p_k, R) \subset \mathcal{N}_{D_1(\theta_0)}(F).$$

Applying the group Γ and passing to subsequences, we may assume that there exist $p \in X$, $\xi, \eta \in \partial_T X$ such that $[p_k, x_k] \to [p, \xi]$ and $[p_k, y_k] \to [p, \eta]$. By triangle comparison it follows that $\angle_p(\xi,\eta) = \angle_T(\xi,\eta) = \theta$, so $[p,\xi] \cup [p,\eta]$ bounds a flat sector; then Lemma 3.2.2 gives a contradiction. □

The following lemma is an easy consequence of the CAT(0) inequality and the Law of Cosines. The proof is left as an exercise for the reader.

Lemma 3.2.4 (Convexity) *Suppose $d(x,y)$ and $d(x,z)$ are at least D and $\angle_x(y,z)$ is at least θ. Then $d(y,z) \geq 2D\sin(\theta/2)$.* □

Proposition 3.2.5 *For all $\theta_0 > 0$ there are $\delta_2 = \delta_2(\theta_0) > 0$ and $\rho_2 = \rho_2(\theta_0)$ such that if $x,y,z \in X$, all vertex angles and comparison angles of $\Delta(x,y,z)$ lie in $(\theta_0, \pi - \theta_0)$, each vertex angle is within δ_2 of the corresponding comparison angle, and all distances are greater than ρ_2, then*

$$[x,y] \cup [x,z] \cup [y,z] \subset \mathcal{N}_{D_1(\theta_0)}(F)$$

for some flat $F \in \mathcal{F}$.

Proof Fix $\theta_0 > 0$ and let R be sufficiently large that each set of diameter at least $R/2$ lies in the $D_1(\theta)$–tubular neighborhood of at most one flat $F \in \mathcal{F}$. Let $\delta_1(\theta_0, R)$ and $\rho_1(\theta_0, R)$ be the constants guaranteed by Lemma 3.2.3, and define

$$\rho_2 := \max\left\{2\rho_1, \frac{\rho_1}{\sin(\theta_0/2)}\right\} \quad \text{and} \quad \delta_2 := \delta_1/3.$$

Pick any triangle $\Delta(x,y,z)$ such that each vertex angle is within δ_2 of the corresponding comparison angle, all vertex angles and comparison angles lie in $(\theta_0, \pi-\theta_0)$, and all side lengths are greater than ρ_2. Choose an arbitrary point on one of the sides of Δ, say $p \in [y,z]$. Then p divides $[y,z]$ into two segments, one of which, say $[p,y]$, has length greater than $\rho_2/2$.

We now verify that the points p, x, and y satisfy the hypothesis of Lemma 3.2.3. First note that $\angle_p(x,y)$ and $\widetilde{\angle}_p(x,y)$ are within $3\delta_2 = \delta_1$ of each other and both lie in $(\theta_0, \pi-\theta_0)$ by Proposition 2.2.1. By hypothesis $d(p,y) > \rho_2/2 \geq \rho_1$. Furthermore, observe that $d(p,y)$ and $d(x,y)$ are both greater than $\rho_1/2\sin(\theta_0/2)$, and that $\angle_p(x,y) \geq \theta_0$. Thus by Lemma 3.2.4, we have $d(p,y) > \rho_1$.

Therefore by Lemma 3.2.3 there is a flat $F_p \in \mathcal{F}$ such that

$$\big([p,x] \cup [p,y]\big) \cap B(p,R) \subset \mathcal{N}_{D_1(\theta_0)}\big(F_p\big).$$

But our choice of R guarantees that F_p is independent of the choice of $p \in [x,y] \cup [x,z] \cup [y,z]$. In other words, there is a single flat $F \in \mathcal{F}$ such that

$$[x,y] \cup [x,z] \cup [y,z] \subset \mathcal{N}_{D_1(\theta_0)}(F),$$

completing the proof. □

3.3 Asymptotic cones are tree-graded

Let X_ω be an asymptotic cone $\mathrm{Cone}_\omega(X, \star_n, \lambda_n)$ where X is CAT(0) with isolated flats. We let $\mathcal{F}_\omega$ denote the collection of flats F in X_ω of the form $F = \omega\text{-lim}\, F_n$ where $F_n \in \mathcal{F}$ and $\omega\text{-lim}\, \lambda_n^{-1}\, d(F_n, \star) < \infty$.

Lemma 3.3.1 *For all $\theta_0 > 0$ there is a $\delta_3 = \delta_3(\theta_0) > 0$ such that if $x, y, z \in X_\omega$ are distinct, all vertex angles and comparison angles of $\Delta(x,y,z)$ lie in $(\theta_0, \pi - \theta_0)$, and each vertex angle is within δ_3 of the corresponding comparison angle, then there is a flat $F \in \mathcal{F}_\omega$ containing $\{x,y,z\}$. Furthermore, if $F, F' \in \mathcal{F}_\omega$ and $F \cap F'$ contains more than one point, then $F = F'$.*

Proof Choose θ_0, and let δ_2 and ρ_2 be the constants provided by Proposition 3.2.5. Set $\delta_3 := \delta_2$. To prove the first assertion, choose $x, y, z \in X_\omega$ as above and apply Corollary 2.4.2 to get sequences (x_k), (y_k), and (z_k) representing x, y, and z such that each vertex angle of $\Delta(x,y,z)$ is the ultralimit of the corresponding angle of $\Delta(x_k, y_k, z_k)$. Then $\Delta(x_k, y_k, z_k)$ satisfies the hypothesis of Proposition 3.2.5 for ω–almost all k. Consequently, x, y, and z lie in a flat $F \in \mathcal{F}_\omega$.

To prove the second assertion, suppose flats $F, F' \in \mathcal{F}_\omega$ have distinct points x, y in common. Let (F_k) be a sequence in $\mathcal{F}$ with ultralimit F, and choose $z \in F'$ such that the triangle $\Delta(x,y,z)$ in F' is nondegenerate. Let (x_k), (y_k), and (z_k) be sequences in X representing x, y, and z, such that $x_k, y_k \in F_k$. Since $[x_k, y_k]$ is contained in F_k as well, we may assume by Corollary 2.4.2 that the angles of $\Delta(x,y,z)$ are ultralimits of the corresponding angles of $\Delta(x_k, y_k, z_k)$. Since $\Delta(x,y,z)$ is a flat triangle, its vertex angles are equal to their comparison angles. Therefore there exists $\theta_0 > 0$ such that for ω–almost all k the triangle $\Delta(x_k, y_k, z_k)$ satisfies the hypothesis of Proposition 3.2.5. Hence $\Delta(x_k, y_k, z_k)$ lies $D_1(\theta_0)$–close to a single flat F''_k. As $[x_k, y_k] \subset F_k$, we must have $F''_k = F_k$ for ω–almost all k by (IF2). Hence $z \in F$. Since a dense subset of F' lies in F, it follows that $F' \subseteq F$. □

Proposition 3.3.2 *If $x \in X_\omega$, then each connected component of $\Sigma_x X_\omega$ is either an isolated point or a sphere of the form $\Sigma_x F$ for some flat $F \in \mathcal{F}_\omega$ passing through x. The map $F \mapsto \Sigma_x F$ gives a one-to-one correspondence between flats of $\mathcal{F}_\omega$ passing through x and spherical components of $\Sigma_x F$. Furthermore, if a direction $\overrightarrow{xy}$ lies in a spherical component $\Sigma_x F$ then an initial segment of $[x,y]$ lies in F.*

Proof Suppose $y, z \in X_\omega$ and $0 < \angle_x(y,z) < \pi$. Then $\angle_x(y,z) \in (\theta, \pi - \theta)$ for some positive θ. Let $\delta_3 = \delta_3(\theta/8)$ be the constant given by Lemma 3.3.1, and let $\delta := \min\{\delta_3, \theta/4\}$. Sliding y and z toward x along the segments $[x,y]$ and $[x,z]$, we may assume by Proposition 2.2.3 that the angles of $\Delta(x,y,z)$ are within δ of their respective comparison angles, and also that $d(x,y) = d(x,z)$. Since $\delta \le \theta/4$, the angles of $\Delta(x,y,z)$ at y and z lie in the interval $(\theta/8, \pi/2)$. Lemma 3.3.1 now implies that x, y, and z lie in a common flat $F \in \mathcal{F}_\omega$. Thus the directions $\overrightarrow{xy}$ and $\overrightarrow{xz}$ lie in a sphere of the form $\Sigma_x F$ for some $F \in \mathcal{F}_\omega$, and each has an initial segment that lies in F.

If z' is any other point of X_ω with $0 < \angle_x(y,z') < \pi$, then $[x,y]$ and $[x,z']$ have initial segments in a flat $F' \in \mathcal{F}_\omega$. It follows from Lemma 3.3.1 that $F = F'$. Therefore $\Sigma_x F$ is a component of $\Sigma_x X_\omega$. Furthermore, every nontrivial component of $\Sigma_x X_\omega$ arises in this manner, and $\Sigma_x F = \Sigma_x F'$ implies $F = F'$. □

Corollary 3.3.3 *Let $\pi_F \colon X_\omega \to F$ be the nearest point projection of X_ω onto a flat $F \in \mathcal{F}_\omega$. Then π_F is locally constant on $X_\omega \setminus F$. In other words, π_F is constant on each component of $X_\omega \setminus F$.*

Proof Choose $s \in X_\omega \setminus F$ and let $x = \pi_F(s)$. Then $\angle_x(s,F)$ is at least $\pi/2$. So $\log_x(s)$ lies in a different component of $\Sigma_x X_\omega$ from the tangent sphere $\Sigma_x F$ by Proposition 3.3.2. By continuity of $\log_x$, if U is any connected neighborhood of s not containing x, the image $\log_x(U)$ is disjoint from the component $\Sigma_x F$. Thus each point of $\log_x(U)$ is at an angular distance π from $\Sigma_x F$. Hence for each $s' \in U$, we have $\pi_F(s') = x$. □

Lemma 3.3.4 *If p lies on the interior of the segment $[x,y] \subset X_\omega$, then x and y lie in distinct components of $X_\omega \setminus \{p\}$ unless p is contained in an open subarc of $[x,y]$ that lies in a flat $F \in \mathcal{F}_\omega$.*

Proof If the segments $[p,x]$ and $[p,y]$ do not have initial segments in a common flat, then the directions $\overrightarrow{px}$ and $\overrightarrow{py}$ lie in distinct components of $\Sigma_p X_\omega$. The result follows immediately from the continuity of $\log_p$. □

Lemma 3.3.5 *Every embedded loop in X_ω lies in some flat $F \in \mathcal{F}_\omega$.*

Proof Let γ be an embedded loop containing points $x \neq y$. By Lemma 3.3.4 and the fact that γ has no cut points, it follows that the geodesic $[x,y]$ lies in some flat $F \in \mathcal{F}_\omega$. Let β be a maximal open subpath of γ in the complement of F. It follows from Corollary 3.3.3 that β projects to a constant under π_F. Hence its endpoints coincide, which is absurd. □

Theorem 3.3.6 *Let X be a* CAT(0) *space admitting a proper, cocompact, isometric action of a group Γ. If X has isolated flats with respect to a family of flats $\mathcal{F}$, then X is relatively hyperbolic with respect to $\mathcal{F}$.*

Proof Each flat $F \in \mathcal{F}_\omega$ is a closed convex subspace of X_ω. By Lemma 3.3.1, any two flats $F \neq F' \in \mathcal{F}_\omega$ intersect in at most one point. Furthermore, Lemma 3.3.5 shows that every embedded geodesic triangle in X_ω lies in a flat $F \in \mathcal{F}_\omega$. □

4 Applications

Theorem 3.3.6 has many immediate consequences, which we examine in the present section. In particular, we show the equivalence (3) $\Longleftrightarrow$ (4) of Theorem 1.2.1 and prove the various parts of Theorem 1.2.2. These applications are divided into two subsections, the first of which deals with geometric properties of spaces with isolated flats. The second subsection is concerned with relative hyperbolicity and its ramifications.

4.1 Geometric invariants of spaces with isolated flats

The following theorem is an immediate consequence of Theorem 3.3.6 together with [17, Proposition 5.4].

Theorem 4.1.1 (Quasiflat) *Let X be a* CAT(0) *space with isolated flats. For each constants L and C, there is a constant $M = M(L, C)$ so that every quasi-isometrically embedded flat in X lies in an M–tubular neighborhood of some flat $F \in \mathcal{F}$.* □

We will now begin to explore the relationship between the geometry of X and the algebra of a group Γ acting geometrically on X.

Proposition 4.1.2 *Suppose Γ acts geometrically on a* CAT(0) *space X. The following are equivalent.*

1. *X is relatively hyperbolic with respect to a Γ–invariant collection $\mathcal{F}$ of flats.*
2. *Γ is a relatively hyperbolic metric space with respect to the left cosets of a collection $\mathcal{A}$ of virtually abelian subgroups of rank at least two.*

Proof The action of Γ on X induces a quasi-isometry $\Gamma \to X$. Since being relatively hyperbolic with respect to quasiflats is a geometric property [17, Theorem 5.1], it suffices to show that this quasi-isometry takes $\mathcal{F}$ to $\mathcal{A}$ if we assume either (1) or (2).

By Theorem 3.3.6, if X satisfies (1) then X has isolated flats with respect to $\mathcal{F}$. So by Lemma 3.1.2 and the Bieberbach Theorem, each $F \in \mathcal{F}$ is Γ–periodic with virtually abelian stabilizer. Let $\mathcal{A}$ be a collection of representatives of the conjugacy classes of stabilizers of flats $F \in \mathcal{F}$. Then the quasi-isometry $\Gamma \to X$ sends the elements of $\mathcal{F}$ to the left cosets of elements of $\mathcal{A}$.

Conversely, suppose X satisfies (2). Let $\mathcal{F}_0$ be a collection of flats stabilized by the elements of $\mathcal{A}$, as guaranteed by the Flat Torus Theorem. Then $\Gamma \to X$ again sends the Γ–translates of elements of $\mathcal{F}_0$ to the left cosets of elements of $\mathcal{A}$. □

Recall that being virtually abelian is a quasi-isometry invariant of finitely generated groups. Therefore, by [17, Corollary 5.19], satisfying Property (2) of Proposition 4.1.2 is also a quasi-isometry invariant of finitely generated groups. We therefore have the following corollary.

Corollary 4.1.3 *Let X_1 and X_2 be quasi-isometric* CAT(0) *spaces admitting geometric actions by groups Γ_1 and Γ_2 respectively. If X_1 has isolated flats, then X_2 also has isolated flats.* □

Definition 4.1.4 *Fellow travelling relative to flats.* A pair of paths

$$\alpha\colon [0,a] \to X \quad \text{and} \quad \alpha'\colon [0,a'] \to X$$

in a CAT(0) space *L–fellow travel relative to a sequence of flats* $(F_1, \dots, F_n)$ if there are partitions

$$0 = t_0 \le s_0 \le t_1 \le s_1 \le \cdots \le t_n \le s_n = a$$

and

$$0 = t'_0 \le s'_0 \le t'_1 \le s'_1 \le \cdots \le t'_n \le s'_n = a'$$

so that for $0 \le i \le n$ the Hausdorff distance between the sets $\alpha\big([t_i, s_i]\big)$ and $\alpha'\big([t'_i, s'_i]\big)$ is at most L, while for $1 \le i \le n$ the sets $\alpha\big([s_{i-1}, t_i]\big)$ and $\alpha'\big([s'_{i-1}, t'_i]\big)$ lie in an L–neighborhood of the flat F_i.

We will frequently say that paths *L–fellow travel relative to flats* if they L–fellow travel relative to some sequence of flats.

Definition 4.1.5 A CAT(0) space X satisfies the *Relative Fellow Traveller Property* if for each choice of constants λ and ϵ there is a constant $L = L(\lambda, \epsilon, X)$ such that (λ, ϵ)–quasigeodesics in X with common endpoints L–fellow travel relative to flats.

The following is an immediate consequence of Theorem 3.3.6 together with the Morse Property for relatively hyperbolic spaces proved by Druţu–Sapir in [17, Theorem 4.25].

Proposition 4.1.6 *If X has isolated flats, then it also has the Relative Fellow Traveller Property.* □

The Relative Fellow Traveller Property was previously established by Epstein for the truncated hyperbolic space associated to a finite volume cusped hyperbolic manifold [19, Theorem 11.3.1]. In the context of 2–dimensional CAT(0) spaces, Hruska showed that isolated flats is equivalent to the Relative Fellow Traveller Property and also to the Relatively Thin Triangle Property, which states that each geodesic triangle is thin relative to a flat in a suitable sense [30].

In [31], Hruska proved several results about CAT(0) spaces with isolated flats under the additional assumption that the Relative Fellow Traveller Property holds. With Proposition 4.1.6, we can now drop the Relative Fellow Traveller Property as a hypothesis in each of those theorems. In the remainder of this subsection, we list some immediate consequences of Proposition 4.1.6 together with [31].

A subspace Y of a geodesic space X is *quasiconvex* if there is a constant κ so that every geodesic in X connecting two points of Y lies in the κ–tubular neighborhood of Y. Let $\rho\colon G \to \mathrm{Isom}(X)$ be a geometric action of a group on a CAT(0) space. A subgroup $H \leq \Gamma$ is *quasiconvex with respect to ρ* if any (equivalently "every") H–orbit is a quasiconvex subspace of X.

In the word hyperbolic setting, Short has shown that any finitely generated $H \leq \Gamma$ is quasiconvex if and only if it is *undistorted* in Γ; in other words, the inclusion $H \hookrightarrow \Gamma$ is a quasi-isometric embedding [41]. It is easy to see that this equivalence does not extend to the general CAT(0) setting (see for instance [31]). However, in the presence of isolated flats, we have the following theorem.

Theorem 4.1.7 *If ρ is a geometric action of Γ on a CAT(0) space X with isolated flats, then a finitely generated subgroup $H \leq \Gamma$ is quasiconvex with respect to ρ if and only if H is undistorted in Γ.* □

We remark that Osin and Druţu–Sapir have examined related phenomena involving undistorted subgroups and relatively quasiconvex subgroups of relatively hyperbolic groups in [37, §4.2] and [17, §8.3]. In general, it is not true that all undistorted subgroups of a relatively hyperbolic group are quasiconvex. One can only conclude that they are "relatively quasiconvex" with respect to the parabolic subgroups. In two slightly different contexts, Osin and Druţu–Sapir study undistorted subgroups that have finite intersections with all parabolic subgroups, essentially proving that such subgroups are quasiconvex in the standard sense and are word hyperbolic.

The main difference between Theorem 4.1.7 and the work of Osin and Druţu–Sapir is that Theorem 4.1.7 applies to undistorted subgroups with arbitrary intersections with the parabolic subgroups. The $\mathrm{CAT}(0)$ geometry inside the flats plays a crucial role in proving quasiconvexity in the isolated flats setting in [31].

Now suppose Γ acts geometrically on two $\mathrm{CAT}(0)$ spaces X_1 and X_2. In the word hyperbolic setting, Gromov observed that the geometric boundary is a group invariant, in the sense that ∂X_1 and ∂X_2 are Γ–equivariantly homeomorphic. More generally, a quasiconvex subgroup of Γ is again word hyperbolic, and its boundary embeds equivariantly into the boundary of Γ [24].

Croke–Kleiner have shown that the homeomorphism type of the geometric boundary is not a group invariant in the general $\mathrm{CAT}(0)$ setting [14]. In fact Wilson has shown that the Croke–Kleiner construction produces a continuous family of homeomorphic 2–complexes whose universal covers have nonhomeomorphic geometric boundaries [42].

In the presence of isolated flats, the situation is quite similar to the hyperbolic setting. However, a subtle complication arises from the fact that it is currently unknown whether a quasiconvex subgroup of a $\mathrm{CAT}(0)$ group is itself $\mathrm{CAT}(0)$. (A group is $\mathrm{CAT}(0)$ if it admits a geometric action on a $\mathrm{CAT}(0)$ space.)

Theorem 4.1.8 (Boundary of a quasiconvex subgroup is well-defined) *Let ρ_1 and ρ_2 be geometric actions of groups Γ_1 and Γ_2 on $\mathrm{CAT}(0)$ spaces X_1 and X_2 both with isolated flats. For each i, let $H_i \leq \Gamma_i$ be a subgroup quasiconvex with respect to ρ_i. Then any isomorphism $H_1 \to H_2$ induces an equivariant homeomorphism $\Lambda H_1 \to \Lambda H_2$ between the corresponding limit sets.* □

Setting $H_i := \Gamma_i$ gives the following corollary.

Corollary 4.1.9 (Boundary is well-defined) *If Γ acts geometrically on a $\mathrm{CAT}(0)$ space X with isolated flats, then the geometric boundary of X is a group invariant of Γ.* □

4.2 Relative hyperbolicity and its consequences

The notion of a relatively hyperbolic group was first proposed by Gromov in [24]. A substantially different definition was proposed and studied by Farb in [20]. Gromov's approach was later formalized by Bowditch and shown to be equivalent to Farb's definition in [9]. For our purposes, the most useful characterization of relatively hyperbolic groups is given by the following theorem. (The converse implication is proved by Druțu–Sapir in [17]. The direct implication is proved by Osin–Sapir in an appendix to [17].)

Theorem 4.2.1 (Druțu–Osin–Sapir) *A finitely generated group Γ is relatively hyperbolic with respect to a collection $\mathcal{A}$ of finitely generated subgroups if and only if Γ is relatively hyperbolic as a metric space with respect to the set of left cosets of the subgroups $A \in \mathcal{A}$.*

The subgroups $A \in \mathcal{A}$ are called *peripheral subgroups* of the relatively hyperbolic structure.

The equivalence (3) $\Longleftrightarrow$ (4) of Theorem 1.2.1 follows immediately from Theorem 4.2.1 and Proposition 4.1.2.

A group Γ satisfies the Tits Alternative if every subgroup of Γ either is virtually solvable or contains a nonabelian free subgroup. For CAT(0) groups this property is equivalent to the Strong Tits Alternative, which states that every subgroup either is virtually abelian or contains a nonabelian free subgroup. The Strong Tits Alternative was proved for word hyperbolic groups by Gromov [24, 3.1.A], and has also been established for CAT(0) groups acting geometrically on certain real analytic 4–manifolds [46] or on cubical complexes (proved for special cases in [6] and [45], and in full generality in [39]). However, it is still unknown whether the Strong Tits Alternative holds for arbitrary CAT(0) groups.

Theorem 1.2.1, together with work of Gromov and Bowditch, allows us to extend the Strong Tits Alternative to the class of CAT(0) groups acting on spaces with isolated flats. More precisely, Gromov shows in [24, 8.2.F] that any properly discontinuous group of isometries of a proper δ–hyperbolic space either is virtually cyclic, is parabolic (ie, fixes a unique point at infinity), or contains a nonabelian free subgroup. According to Bowditch (elaborating on an idea due to Gromov), each relatively hyperbolic group acts properly discontinuously on a proper δ–hyperbolic space such that the maximal parabolic subgroups are exactly the conjugates of the peripheral subgroups [9]. The following result follows immediately.

Theorem 4.2.2 (Gromov–Bowditch) *Let* Γ *be relatively hyperbolic with respect to a collection of subgroups that each satisfy the* [*Strong*] *Tits Alternative. Then* Γ *also satisfies the* [*Strong*] *Tits Alternative.*

In the isolated flats setting, the peripheral subgroups are virtually abelian. We therefore conclude the following.

Theorem 4.2.3 (Isolated flats $\Longrightarrow$ Tits Alternative) *A group* Γ *that acts geometrically on a* $\mathrm{CAT}(0)$ *space with isolated flats satisfies the Strong Tits Alternative. In other words, every subgroup* $H \leq \Gamma$ *either is virtually abelian or contains a nonabelian free subgroup.* $\square$

All word hyperbolic groups are biautomatic [19]. As with the Tits Alternative, it is currently unknown whether this result also holds for arbitrary $\mathrm{CAT}(0)$ groups. In fact, the only nonhyperbolic spaces where biautomaticity was previously known are complexes built out of very restricted shapes of cells. For instance, Gersten–Short established biautomaticity for $\mathrm{CAT}(0)$ groups acting geometrically on $\mathrm{CAT}(0)$ 2–complexes of type $\tilde{A}_1 \times \tilde{A}_1$, $\tilde{A}_2$, $\tilde{B}_2$, and $\tilde{G}_2$ in [21, 22]. In particular, Gersten–Short's work includes $\mathrm{CAT}(0)$ square complexes and 2–dimensional Euclidean buildings. Niblo–Reeves proved biautomaticity for groups acting geometrically on arbitrary $\mathrm{CAT}(0)$ cube complexes in [36].

Rebbechi shows in [38] that a group which is hyperbolic relative to a collection of biautomatic subgroups is itself biautomatic. Since finitely generated, virtually abelian groups are biautomatic, we obtain the following immediate corollary to Theorem 1.2.1.

Theorem 4.2.4 (Isolated flats $\Longrightarrow$ biautomatic) *If* Γ *acts geometrically on a* $\mathrm{CAT}(0)$ *space with isolated flats, then* Γ *is biautomatic.* $\square$

Many $\mathrm{CAT}(0)$ 2–complexes with isolated flats have irregularly shaped cells. Such 2–complexes provide new examples of biautomatic groups. The following is an elementary construction of such a 2–complex.

Example 4.2.5 *Irregularly shaped cells* Let X be a compact hyperbolic surface and γ a geodesic loop in X representing a primitive conjugacy class. Let T be any flat 2–torus containing a simple geodesic loop γ'. Form a 2–complex Y by gluing X and T along the curve $\gamma = \gamma'$. Then the universal cover of Y has isolated flats. Typically $\gamma \subset X$ will intersect itself many times, subdividing X into a large number of irregularly shaped pieces.

5 Equivalent formulations of isolated flats

In this section we prove Theorem 1.2.3, which establishes the equivalence of the various geometric formulations of isolated flats discussed in the introduction. We then use Theorem 1.2.3 to prove the remaining equivalence (1) $\Longleftrightarrow$ (2) of Theorem 1.2.1, which characterizes spaces with isolated flats in terms of their Tits boundaries.

5.1 Proof of Theorem 1.2.3

The implications (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are immediate from the definitions. We prove (1) $\Longrightarrow$ (3) in Proposition 5.1.4. We then show in Proposition 5.1.5 that (4) implies both (1) and (2). Finally we prove (4) $\Longrightarrow$ (5) and (5) $\Longrightarrow$ (1) in Propositions 5.1.7 and 5.1.8 respectively.

Lemma 5.1.1 *Any closed, isolated Γ–invariant subset $\mathcal{F} \subseteq \mathrm{Flat}(X)$ is locally finite.*

Proof If $K \subset X$ is compact, then

$$\{\, F \in \mathcal{F} \mid F \cap \bar{K} \neq \emptyset \,\}$$

is a closed subset of the compact set

$$\{\, F \in \mathrm{Flat}(X) \mid F \cap \bar{K} \neq \emptyset \,\},$$

and is therefore compact; but all its elements are isolated, so it is also finite. Thus $\mathcal{F}$ is a locally finite collection. □

The following is an immediate corollary of Lemmas 5.1.1 and 3.1.2.

Corollary 5.1.2 *Let $\mathcal{F}$ be a closed, isolated Γ–invariant subset of $\mathrm{Flat}(X)$. Then the elements of $\mathcal{F}$ lie in only finitely many Γ–orbits and each $F \in \mathcal{F}$ is Γ–periodic.* □

Lemma 5.1.3 *Suppose X has* (IF1) *with respect to the family $\mathcal{F}$. We may assume without loss of generality that each element of $\mathcal{F}$ is maximal.*

Proof By Corollary 5.1.2, the elements of $\mathcal{F}$ lie in only finitely many Γ–orbits. Thus we can delete Γ–orbits of flats from $\mathcal{F}$ until it becomes a minimal Γ–invariant subset satisfying (IF1). But each element of a minimal $\mathcal{F}$ is clearly maximal. □

Proposition 5.1.4 *If X has* (IF1) *then X has thin parallel sets.*

Proof It is sufficient to consider parallel sets of geodesics lying in flats $F \in \mathcal{F}$. Let Ξ be the collection of pairs (F, S) where $F \in \mathcal{F}$, $S \subset F$ is isometric to $\mathbb{E}^k$ for some $k \geq 1$, and the parallel set of S is not contained in a finite neighborhood of F. Suppose Ξ is nonempty, and choose an element $(F, S) \in \Xi$ which is "maximal" in the sense that if $(F', S') \in \Xi$ and $\dim F' \geq \dim F$ and $\dim S' \geq \dim S$, then $\dim F' = \dim F$ and $\dim S' = \dim S$.

Step 1. We first show that $S = F$. Assume by way of contradiction that $\dim S < \dim F$, so that $\dim \bar{F} \geq 1$.

We have a Euclidean product decomposition $\mathbb{P}(S) = S \times Y$ where $Y \subset X$ is convex. Let $\pi\colon \mathbb{P}(S) \to Y$ be the projection onto the second factor, and define $\bar{F} \subset Y = \pi(F)$. Then we have $F = S \times \bar{F}$. Note that Y is not contained in a finite neighborhood of $\bar{F}$. So by applying the cocompact stabilizer of F and a convergence argument, we may assume without loss of generality that $\partial_T Y \setminus \partial_T \bar{F} \neq \emptyset$.

Suppose $\partial_T \bar{F}$ is not a component of $\partial_T Y$. Then we may apply Lemma 2.3.1 to see that there is a flat half-plane $H \subset Y$ which meets $\bar{F}$ orthogonally. But then $S' := S \times \partial H \subset F$ is a subflat of F with $\dim S' > \dim S$ and $\mathbb{P}(S') \supset S \times H$ is not contained in a finite neighborhood of F. This contradicts the choice of the pair (F, S), and so $\partial_T \bar{F}$ must be a component of $\partial_T Y$.

Pick $\xi \in \partial_T Y \setminus \partial_T \bar{F}$. Then there is a geodesic $\gamma \subset Y$ with $\partial_T \gamma = \{\xi, \eta\}$ where $\eta \in \partial_T \bar{F}$. By (IF1) there is a flat $F' \in \mathcal{F}$ such that $S \times \gamma \subset \mathcal{N}_D(F')$; let $\bar{F}'$ be the projection of F' to Y. Let $\alpha \subset \bar{F}$ be a geodesic with $\eta \in \partial_T \alpha$, and pick $x \in F$ such that $\bar{x} := \pi(x) \in \alpha$. By the Bieberbach Theorem, the stabilizer of F contains a subgroup A which acts cocompactly by translations on F; therefore we can find an infinite sequence $g_k \in A$ such that $d\big(g_k(\alpha), \alpha\big)$ is uniformly bounded, and $g_k \bar{x}$ converges to $\eta_- \in \partial_\infty X$, where $\partial_T \alpha = \{\eta, \eta_-\}$. All the flats $g_k(F')$ pass through some ball around x, so the collection $\big\{g_k(F')\big\}$ is finite. It follows that for each $\epsilon > 0$, there is a translation $g \in A$ which preserves F', and which translates $\bar{F}$ in a direction ϵ–close to η_-; in particular g does not preserve S. If $\tau \subset \bar{F}$ is a geodesic translated by g, then $S' := S \times \tau$ is a Euclidean subset with $\dim S' > \dim S$, and $\mathbb{P}(S') \supset F'$ is not contained in a finite neighborhood of F. This contradicts the choice of (F, S). Therefore we have $S = F$.

Step 2. We now know that $S = F$. So $\mathbb{P}(F) = F \times Y$ where Y is unbounded. Pick $p \in F$, and let $[p, x_k] \subset \mathbb{P}(F)$ be a sequence of geodesics in the Y–factor through p such that $d(x_k, F) \to \infty$. Let y_k be the midpoint of the segment $[p, x_k]$, and let $F_k \subset \mathbb{P}(F)$ be the flat parallel to F passing through y_k. By Lemma 5.1.3, F is maximal. Thus the parallel flat F_k is maximal as well. Applying isometries $g_k \in \Gamma$ and passing to a subsequence, we can arrange that the maximal flats $g_k(F_k)$ converge to a flat $\bar{F}$ that is not maximal.

Each maximal flat $g_k(F_k)$ is at a Hausdorff distance at most D from some $\hat{F}_k \in \mathcal{F}$ with $\dim \hat{F}_k = \dim F$. Since $g_k(F_k)$ converges and $\mathcal{F}$ is locally finite, we can pass to a further subsequence so that the flats $\hat{F}_k$ are equal to a single flat $\hat{F} \in \mathcal{F}$. Thus the Hausdorff distance between $\hat{F}$ and $\bar{F}$ is at most D. But $\hat{F}$ is maximal and $\bar{F}$ is not, which is a contradiction. □

Proposition 5.1.5 *If X has slim parallel sets, then X has uniformly thin parallel sets and satisfies* (IF1)*.*

The proposition is an immediate consequence of the following lemma in the case $n = 2$.

Lemma 5.1.6 *Suppose X has slim parallel sets. Then for each $n \geq 2$ there is a closed, isolated Γ–invariant set $\mathcal{F}_n \subset \mathrm{Flat}(X)$ and a constant D_n with the following properties.*

1. *Each element of $\mathcal{F}_n$ is a maximal flat of dimension at least n.*
2. *Each k–flat in X with $k \geq n$ is contained in the D_n–tubular neighborhood of some $F \in \mathcal{F}_n$.*
3. *If $\gamma \subseteq F$ is a geodesic contained in a flat $F \in \mathcal{F}_n$ then the parallel set of γ lies in the D_n–neighborhood of F.*

Proof Since there is an upper bound on the dimensions of all flats in X, the lemma is satisfied for sufficiently large n by the set $\mathcal{F}_n := \emptyset$. We will prove the lemma by decreasing induction on n. Suppose the lemma is true for $n+1$.

If F is a maximal n–flat, then by hypothesis its parallel set has the form $F \times K_F$ for some compact set K_F. Define $\mathcal{F}_n$ to be the union of $\mathcal{F}_{n+1}$ and the set of all n–flats of the form $F \times \{c_F\}$, where F is maximal and c_F is the circumcenter of K_F. Note that $\mathcal{F}_n$ is Γ–invariant.

Step 1. We will first show that the set of all maximal flats of dimension n is closed in $\mathrm{Flat}(X)$. If not, then there is a sequence (F_i) of maximal n–flats converging to a nonmaximal n–flat F, which must lie in a finite tubular neighborhood of a higher dimensional flat $\hat{F} \in \mathcal{F}_{n+1}$. It follows from (3) that

$F \subset \mathcal{N}_{D_{n+1}}(\hat{F})$. Since F is not maximal, it is not parallel to any F_i. So F_i is not contained in any finite tubular neighborhood of $\hat{F}$. Consequently, for any $r > D_{n+1}$, if we set

$$C_i := F_i \cap \mathcal{N}_r(\hat{F}),$$

then C_i is a convex, open, proper subset of F_i whose inscribed radius ρ_i tends to infinity as $i \to \infty$. Let $p_i \in F_i$ be a point on the boundary of C_i that is also on the boundary of an inscribed n–dimensional Euclidean disc in C_i of radius ρ_i. By Corollary 5.1.2, we know that $\hat{F}$ has a cocompact stabilizer. So we may apply elements $g_i \in \mathrm{Stab}(\hat{F})$ and pass to a subsequence so that the $g_i(p_i)$ converge to a point p, the $g_i(F_i)$ converge to a flat $\bar{F}$ containing p, the $g_i(C_i)$ converge to an open halfspace $H \subset \bar{F}$ that lies in the closed r–tubular neighborhood of $\hat{F}$, and each point of the bounding hyperplane ∂H lies at a distance exactly r from $\hat{F}$. Therefore ∂H lies in the parallel set of a geodesic in $\hat{F}$ and $r > D_{n+1}$, which contradicts the inductive hypothesis applied to $\hat{F}$.

Step 2. Our next goal is to show that a sequence of pairwise nonparallel maximal flats cannot converge in $\mathrm{Flat}(X)$. Suppose by way of contradiction that (F_i) is such a sequence that converges in $\mathrm{Flat}(X)$ to F. By Step 1, we know that F is maximal. Fix a positive r, and define C_i and p_i in F_i as in Step 1. As F is n–dimensional, it is not yet known to be periodic. Nevertheless, we may apply isometries $g_i \in \Gamma$ and pass to a subsequence to arrange that $g_i(p_i)$ converges to a point p, that $g_i(F_i)$ converges to a flat $\bar{F}$, that $g_i(F)$ converges to a flat $\hat{F}$, and that $g_i(\partial C_i)$ converges to an $(n-1)$–dimensional subflat S of $\bar{F}$ each point of which is at a distance exactly r from $\hat{F}$. By Step 1, the limiting flats $\bar{F}$ and $\hat{F}$ are again maximal. But $\bar{F} \cup \hat{F}$ is contained in the parallel set of any geodesic line $\gamma \subset S$. It follows from slim parallel sets that $\partial_T \bar{F} = \partial_T \hat{F}$, and hence that $\bar{F}$ and $\hat{F}$ are parallel and separated by a distance exactly r.

Since r is arbitrary, the preceding paragraph produces for each positive r a pair of parallel maximal n–flats $\bar{F}_r$ and $\hat{F}_r$ at a Hausdorff distance exactly r. The convex hull of $\bar{F}_r \cup \hat{F}_r$ is isometric to a product $\bar{F}_r \times [-r/2, r/2]$. Applying isometries in Γ again, we can arrange that as $r \to \infty$ the maximal flats $\bar{F}_r \times \{0\}$ converge to an n–flat which is not maximal, contradicting Step 1.

Thus $\mathcal{F}_n$ is a closed, isolated, Γ–invariant set in $\mathrm{Flat}(X)$ containing exactly one flat from each parallel class of maximal flats of dimension $\geq n$. By Corollary 5.1.2, the elements of $\mathcal{F}_n$ lie in only finitely many Γ–orbits and are each Γ–periodic. Another convergence argument using the cocompact stabilizer of each $F \in \mathcal{F}_n$ can now be used to uniformly bound the thickness of the parallel sets $\mathbb{P}(\gamma)$ for each geodesic $\gamma \subset F$. The lemma follows immediately, since each k–flat with $k \geq n$ lies inside one of these parallel sets. $\square$

Proposition 5.1.7 *Let X have slim parallel sets. Then X has* (IF2).

Proof Consider the set $\mathcal{F} := \mathcal{F}_2$ given by Lemma 5.1.6. Then $\mathcal{F}$ is locally finite, consists of only maximal flats, and also has the property that no two flats in $\mathcal{F}$ are parallel. Pick $\rho < \infty$ and a sequence of pairs $F_k \neq F'_k$ in $\mathcal{F}$ such that $\operatorname{diam}\bigl(\mathcal{N}_\rho(F_k) \cap \mathcal{N}_\rho(F'_k)\bigr) \to \infty$. After passing to subsequences and applying a sequence of isometries, we may assume that $F_k \to F_\infty \in \mathcal{F}$, $F'_k \to F'_\infty \in \mathcal{F}$, and $\mathcal{N}_\rho(F_\infty) \cap \mathcal{N}_\rho(F'_\infty)$ contains a complete geodesic γ. Since $\mathcal{F}$ is locally finite, we in fact have $F_k = F_\infty$, $F'_k = F'_\infty$ for all sufficiently large k.

But $F_\infty \cup F'_\infty$ lies in the parallel set of γ. As in the proof of Lemma 5.1.6, it follows that F_∞ and F'_∞ are parallel, contradicting the fact that $F_k \neq F'_k$. $\square$

Proposition 5.1.8 *If X has* (IF2) *with respect to $\mathcal{F}$, then X also has* (IF1) *with respect to $\mathcal{F}$.*

Proof The only nontrivial fact to check is that a sequence of distinct flats in $\mathcal{F}$ cannot converge in $\operatorname{Flat}(X)$. But this follows immediately from Lemma 3.1.1. $\square$

5.2 The structure of the Tits boundary

In this section we show that X has isolated flats if and only if the components of its Tits boundary are isolated points and standard spheres.

Proposition 5.2.1 *If X has isolated flats, then for each $\theta_0 > 0$ there is a positive constant $\delta_4 = \delta_4(\theta_0)$ such that whenever $p \in X$ and $\xi, \eta \in \partial_T X$ satisfy*

$$\theta_0 \leq \angle_p(\xi,\eta) \leq \angle_T(\xi,\eta) \leq \angle_p(\xi,\eta) + \delta_4 \leq \pi - \theta_0, \tag{5.2.2}$$

then there is a flat $F \in \mathcal{F}$ so that

$$[p,\xi] \cup [p,\eta] \subset \mathcal{N}_{D_1(\theta_0)}(F).$$

Proof Choose θ_0 positive, and let R be sufficiently large that any set of diameter at least $R/4$ is contained in the $D_1(\theta_0)$–tubular neighborhood of at most one flat in $\mathcal{F}$. Let $\delta_1 = \delta_1(\theta_0, R)$ and $\rho_1 = \rho_1(\theta_0, R)$ be the constants given by Lemma 3.2.3, and set $\delta_4 := \delta_1$. Now suppose $p \in X$ and $\xi, \eta \in \partial_T X$ satisfy

$$\theta_0 \leq \angle_p(\xi,\eta) \leq \angle_T(\xi,\eta) \leq \angle_p(\xi,\eta) + \delta_1 \leq \pi - \theta_0. \tag{5.2.3}$$

Then we have

$$\angle_p(\xi,\eta) \le \widetilde{\angle}_p\big(c(\rho_1), c'(\rho_1)\big) \le \angle_T(\xi,\eta),$$

where c and c' are geodesic parametrizations of $[p,\xi]$ and $[p,\eta]$ respectively. So by Lemma 3.2.3 there is a flat $F_p \in \mathcal{F}$ so that

$$B(p,R) \cap \big([p,\xi] \cup [p,\eta]\big) \subset \mathcal{N}_{D_1(\theta_0)}\big(F\big).$$

Notice that (5.2.3) remains valid if we replace p with any point $x \in [p,\xi] \cup [p,\eta]$. Thus we can apply the preceding argument to the rays $[x,\xi]$ and $[x,\eta]$ to produce for each x a flat $F_x \in \mathcal{F}$ such that

$$B(x,R) \cap \big([x,\xi] \cup [x,\eta]\big) \subset \mathcal{N}_{D_1(\theta_0)}\big(F_x\big).$$

But our choice of R guarantees that all the flats F_x are the same. So $[p,\xi] \cup [p,\eta]$ lies in the $D_1(\theta)$–tubular neighborhood of a single flat $F \in \mathcal{F}$. □

The following proposition was established by Schroeder [5, Appendix 4.E] in the Riemannian setting and extended to arbitrary CAT(0) spaces by Leeb [34]. In the proposition, a *standard sphere* is a sphere isometric to a unit sphere in Euclidean space. A *standard hemisphere* is isometric to a hemisphere of a standard sphere.

Proposition 5.2.4 (Schroeder, Leeb) *Let X be a proper* CAT(0) *space, and $S = S^{k-1}$ an isometrically embedded standard sphere in $\partial_T X$. Then one of the following holds.*

(1) *There is a k–flat F in X such that $\partial_T F = S$.*

(2) *There is an isometric embedding of a standard hemisphere $H \subset S^k$ in $\partial_T X$ such that the topological boundary of H maps to S.*

Theorem 5.2.5 *Let X be any* CAT(0) *space admitting a geometric group action. Then X has isolated flats if and only if each connected component of $\partial_T X$ either is an isolated point or is isometric to a standard sphere. Furthermore, the spherical components of $\partial_T X$ are precisely the Tits boundaries of the flats $F \in \mathcal{F}$.*

Proof Suppose X has isolated flats. By (IF2), it is clear that if $F, F' \in \mathcal{F}$ are distinct then $\partial_T F \cap \partial_T F' = \emptyset$. If $\xi, \eta \in \partial_T X$ and $0 < \angle_T(\xi,\eta) < \pi$, then we can find $\theta_0 > 0$ and $p \in X$ such that (5.2.2) holds. Hence $[p,\xi] \cup [p,\eta] \subset \mathcal{N}_{D_1(\theta_0)}(F)$ for some $F \in \mathcal{F}$, which means that $\{\xi,\eta\} \subset \partial_T F$. Thus $\partial_T X$ has the desired structure.

Now suppose every component of $\partial_T X$ is either an isolated point or a standard sphere. We will show that X has slim parallel sets. By Proposition 5.2.4, any spherical component of $\partial_T X$ is the boundary of a maximal flat. It follows that the Tits boundary of any maximal flat F is a component of $\partial_T X$. But for each geodesic $\gamma \subset F$, the Tits boundary of $\mathbb{P}(\gamma)$ is a connected set containing $\partial_T F$. Hence $\partial_T \mathbb{P}(\gamma) = \partial_T F$. □

A Appendix: Isolated subspaces

By Mohamad Hindawi, G Christopher Hruska and Bruce Kleiner

Hindawi proved in [29] (independent of the main body of this article) that the universal cover of a closed, real analytic, nonpositively curved 4–manifolds whose Tits boundary has no nonstandard components exhibits similar behavior to CAT(0) spaces with isolated flats.

In this appendix, we show that many of the results of this article extend to a more general class of CAT(0) spaces, including the manifolds studied by Hindawi. The main idea is to weaken the hypothesis (IF2) so that the elements of $\mathcal{F}$, instead of being flats, are closed convex subspaces. This general idea was introduced as a generalization of isolated flats by Kapovich–Leeb in [32].

Theorem A.0.1 *Let X be a CAT(0) space, and $\Gamma \curvearrowright X$ be a geometric action. Suppose $\mathcal{F}$ is a Γ–invariant collection of closed convex subsets. Assume that*

(A) *There is a constant $D < \infty$ such that each flat $F \subseteq X$ lies in a D–tubular neighborhood of some $C \in \mathcal{F}$.*

(B) *For each positive $r < \infty$ there is a constant $\rho = \rho(r) < \infty$ so that for any two distinct elements $C, C' \in \mathcal{F}$ we have*

$$\operatorname{diam}\bigl(\mathcal{N}_r(C) \cap \mathcal{N}_r(C')\bigr) < \rho.$$

Then:

(1) *The collection $\mathcal{F}$ is locally finite, there are only finitely many Γ–orbits in $\mathcal{F}$, and each $C \in \mathcal{F}$ is Γ–periodic.*

(2) *Every connected component of $\partial_T X$ containing more than one point is contained in $\partial_T C$, for some $C \in \mathcal{F}$.*

(3) *For every asymptotic cone X_ω of X, and every $p \in X_\omega$, each connected component of $\Sigma_p X_\omega$ containing more than one point is contained in $\Sigma_p C_\omega$, for some $C_\omega \in F_\omega$.*

(4) *Every asymptotic cone X_ω is tree-graded with respect to the collection $\mathcal{F}_\omega$.*

(5) *Γ is hyperbolic relative to the collection of stabilizers of elements of $\mathcal{F}$.*

(6) *Suppose the stabilizers of elements of $C \in \mathcal{F}$ are* CAT(0) *groups with very well-defined boundary (see below). Then Γ has a very well-defined boundary.*

Definition A.0.2 A group Γ has a *very well-defined boundary* if, whenever Γ acts geometrically on a pair of CAT(0) spaces X and Y, any Γ–equivariant quasi-isometry $X \to Y$ sends each geodesic to within a uniformly bounded Hausdorff distance of a geodesic.

Examples of groups with very well-defined boundary include word hyperbolic groups, virtually abelian groups, and any group acting geometrically on a product $X_1 \times \cdots \times X_k$ where each X_i is either an irreducible higher rank symmetric space or an irreducible higher rank Euclidean building.

Proof of Theorem A.0.1 The point is that many of the arguments in this article do not use the assumption that the elements of $\mathcal{F}$ are flats in an essential way.

The proof of (1) is identical to that of Lemmas 3.1.1 and 3.1.2. The proof of (2) follows from the arguments in Sections 3.2 and 5.2. Assertions (3), (4), and (5) follow using arguments similar to those in Sections 3 and 4.

Assertion (6) follows using arguments similar to those in [31]. Note that the arguments in [31] use the Relative Fellow Traveller Property as a hypothesis. However, this property is a special case of Druţu–Sapir's Morse Property for relatively hyperbolic spaces [17]. The arguments of [31] go through essentially unchanged if one uses the Morse Property in place of the Relative Fellow Traveller Property throughout. □

A partial converse to the preceding theorem is given by the following result due to Druţu–Sapir [17].

Theorem A.0.3 *Suppose Γ is relatively hyperbolic with respect to a finite collection of subgroups $\{P_i\}$. Fix a finite generating set for Γ, and consider the corresponding word metric on Γ. Then each P_i is quasiconvex in Γ (with respect to the word metric). Furthermore,*

(1) *every quasiflat in Γ lies in a uniformly bounded tubular neighborhood of some coset gP_i, and*

(2) *for each positive $r < \infty$, there is a constant $\rho = \rho(r) < \infty$ so that for any two distinct cosets g_1P_1 and g_2P_2, we have*

$$\operatorname{diam}\bigl(\mathcal{N}_r(g_1P_1) \cap \mathcal{N}_r(g_2P_2)\bigr) < \rho.$$

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