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$K-$ and L–theory of the semi-direct product of the discrete 3–dimensional Heisenberg group by $\mathbb{Z}/4$

Wolfgang Lück

Fachbereich Mathematik, Universität Münster
Einsteinstr. 62, 48149 Münster, Germany

Email: lueck@math.uni-muenster.de

URL: www.math.uni-muenster.de/u/lueck/

Abstract

We compute the group homology, the topological K–theory of the reduced C^*–algebra, the algebraic K–theory and the algebraic L–theory of the group ring of the semi-direct product of the three-dimensional discrete Heisenberg group by $\mathbb{Z}/4$. These computations will follow from the more general treatment of a certain class of groups G which occur as extensions $1 \to K \to G \to Q \to 1$ of a torsionfree group K by a group Q which satisfies certain assumptions. The key ingredients are the Baum–Connes and Farrell–Jones Conjectures and methods from equivariant algebraic topology.

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0 Introduction

The original motivation for this paper was the question of Chris Phillips how the topological K–theory of the reduced (complex) C^*–algebra of the semi-direct product $\mathrm{Hei} \rtimes \mathbb{Z}/4$ looks like. Here Hei is the *three-dimensional discrete Heisenberg group* which is the subgroup of $GL_3(\mathbb{Z})$ consisting of upper triangular matrices with 1 on the diagonals. The $\mathbb{Z}/4$–action is given by:

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -z & y - xz \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$$

The answer, which is proved in Theorem 2.6, consists of an explicit isomorphism

$$j_0 \bigoplus c[0]_0' \bigoplus c[2]_0' \colon K_0(\{*\}) \bigoplus \widetilde{R}_{\mathbb{C}}(\mathbb{Z}/4) \bigoplus \widetilde{R}_{\mathbb{C}}(\mathbb{Z}/2) \xrightarrow{\cong} K_0(C_r^*(\mathrm{Hei} \rtimes \mathbb{Z}/4))$$

and a short exact sequence

$$0 \to \widetilde{R}_{\mathbb{C}}(\mathbb{Z}/4) \bigoplus \widetilde{R}_{\mathbb{C}}(\mathbb{Z}/2) \xrightarrow{c[0]_1' \oplus c[2]_1'} K_1(C_r^*(\mathrm{Hei} \rtimes \mathbb{Z}/4)) \xrightarrow{c_1} \widetilde{K}_1(S^3) \to 0,$$

which splits since $\widetilde{K}_1(S^3) \cong \mathbb{Z}$. Here $\widetilde{R}_{\mathbb{C}}(\mathbb{Z}/m)$ is the kernel of the split surjective map $R_{\mathbb{C}}(\mathbb{Z}/m) \to R_{\mathbb{C}}(\{1\}) \cong \mathbb{Z}$ which sends the class of a complex $\mathbb{Z}/m$–representation to the class of $\mathbb{C} \otimes_{\mathbb{C}[\mathbb{Z}/m]} V$. As abelian group we get for $n \in \mathbb{Z}$

$$K_n(C_r^*(\mathrm{Hei} \rtimes \mathbb{Z}/4)) \cong \mathbb{Z}^5.$$

This computation will play a role in the paper by Echterhoff, Lück and Phillips [13], where certain C^*–algebras given by semi-direct products of rotation algebras with finite cyclic groups are classified.

Although the group $\mathrm{Hei} \rtimes \mathbb{Z}/4$ is very explicit, this computation is highly nontrivial and requires besides the Baum–Connes Conjecture a lot of machinery from equivariant algebraic topology. Even harder is the computation of the middle and lower K–theory. The result is (see Corollary 3.9)

$$\mathrm{Wh}_n(\mathrm{Hei} \rtimes \mathbb{Z}/4) \cong \begin{cases} NK_n(\mathbb{Z}[\mathbb{Z}/4]) \bigoplus NK_n(\mathbb{Z}[\mathbb{Z}/4]) & \text{for } n = 0, 1; \\ 0 & \text{for } n \leq -1, \end{cases}$$

where $NK_n(\mathbb{Z}[\mathbb{Z}/4])$ denotes the n-th Nil-group of $\mathbb{Z}[\mathbb{Z}/4]$ which appears in the Bass–Heller–Swan decomposition of $\mathbb{Z}[\mathbb{Z}/4 \times \mathbb{Z}]$. So the lower K–theory is trivial and the middle K–theory is completely made up of Nil-groups.

We also treat the L–groups. The answer and calculation is rather messy due to the appearance of UNil–terms and the structure of the family of infinite virtually cyclic subgroups (see Theorem 4.11). If one is willing to invert 2, these UNil–terms and questions about decorations disappear and the answer is given by the short split exact sequence:

$$\begin{aligned} 0 \to L_n(\mathbb{Z})\left[\frac{1}{2}\right] \bigoplus \widetilde{L}_n(\mathbb{Z}[\mathbb{Z}/2])\left[\frac{1}{2}\right] \bigoplus \widetilde{L}_{n-1}(\mathbb{Z}[\mathbb{Z}/2])\left[\frac{1}{2}\right] \\ \bigoplus \widetilde{L}_n(\mathbb{Z}[\mathbb{Z}/4])\left[\frac{1}{2}\right] \bigoplus \widetilde{L}_{n-1}(\mathbb{Z}[\mathbb{Z}/4])\left[\frac{1}{2}\right] \\ \xrightarrow{j} L_n(\mathbb{Z}[\mathrm{Hei}\rtimes\mathbb{Z}/4])\left[\frac{1}{2}\right] \to L_{n-3}(\mathbb{Z})\left[\frac{1}{2}\right] \to 0 \end{aligned}$$

Finally we will also compute the group homology (see Theorem 5.6)

$$\begin{aligned} H_n(G) &= \mathbb{Z}/2\times\mathbb{Z}/4 \text{ for } n\geq 1, n\neq 2,3; \\ H_2(G) &= \mathbb{Z}/2; \\ H_3(G) &= \mathbb{Z}\times\mathbb{Z}/2\times\mathbb{Z}/4. \end{aligned}$$

In turns out that we can handle a much more general setting provided that the Baum–Connes Conjecture or the Farell–Jones Conjecture is true for G. Namely, we will consider an extension of (discrete) groups

$$1 \to K \xrightarrow{i} G \xrightarrow{p} Q \to 1 \qquad (0.1)$$

which satisfies the following conditions:

(M) Each non-trivial finite subgroup of Q is contained in a unique maximal finite subgroup;

(NM) Let M be a maximal finite subgroup of Q. Then $N_QM = M$ unless G is torsionfree;

(T) K is torsionfree.

The special case, where K is trivial, is treated in [12, Theorem 5.1]. In [12, page 101] it is explained using [24, Lemma 4.5]), [24, Lemma 6.3] and [25, Propositions 5.17, 5.18 and 5.19 in II.5 on pages 107 and 108] why the following groups satisfy conditions (M) and (NM):

- Extensions $1\to\mathbb{Z}^n\to Q\to F\to 1$ for finite F such that the conjugation action of F on $\mathbb{Z}^n$ is free outside $0\in\mathbb{Z}^n$;
- Fuchsian groups;
- One-relator groups.

Of course Hei $\rtimes\mathbb{Z}/4$ is an example for G. For such groups G we will establish certain exact Mayer–Vietoris sequences relating the $K-$ or $L-$theory of G to the $K-$ and $L-$theory of $p^{-1}(M)$ for maximal finite subgroups $M\subseteq Q$ and terms involving the quotients $G\backslash \underline{E}G$ and $p^{-1}(M)\backslash \underline{E}p^{-1}(M)$. The classifying space $\underline{E}G$ for proper G–actions plays an important role and often there are nice small geometric models for them. One key ingredient in the computations for $\mathrm{Hei} \rtimes \mathbb{Z}/4$ will be to show that $G\backslash \underline{E}G$ in this case is S^3. For instance the computation of the group homology illustrates that it is often very convenient to work with the spaces $G\backslash \underline{E}G$ although one wants information about BG.

1 Topological K–theory

For a G–CW–complex X let $K_*^G(X)$ be its *equivariant K–homology theory*. If G is trivial, we abbreviate $K_*(X)$. For a C^*–algebra A let $K_*(A)$ be its topological K–theory. Recall that a model $\underline{E}G$ for the *classifying space for proper G–actions* is a G–CW–complex with finite isotropy groups such that $(\underline{E}G)^H$ is contractible for each finite subgroup $H \subseteq G$. It has the property that for any G–CW–complex X with finite isotropy groups there is precisely one G–map from X to $\underline{E}G$ up to G–homotopy. In particular two models for $\underline{E}G$ are G–homotopy equivalent. For more information about the spaces $\underline{E}G$ we refer for instance to [6], [20], [26]. [32]. Recall that the *Baum–Connes Conjecture* (see [6, Conjecture 3.15 on page 254]) says that the assembly map

$$\mathrm{asmb}\colon K_n^G(\underline{E}G) \xrightarrow{\cong} K_n(C_r^*(G))$$

is an isomorphism for each $n \in \mathbb{Z}$, where $C_r^*(G)$ is the *reduced group C^*–algebra* associated to G. (For an identification of the assembly map used in this paper with the original one we refer to Hambleton–Pedersen [17]). Let EG be a model for the *classifying space for free G–actions*, ie, a free G–CW–complex which is contractible (after forgetting the group action). Up to G–homotopy there is precisely one G–map $s\colon EG \to \underline{E}G$. The *classical assembly map a* is defined as the composition

$$a\colon K_n(BG) = K_n^G(EG) \xrightarrow{K_p^G(s)} K_n^G(\underline{E}G) \xrightarrow{\mathrm{asmb}} K_n(C_r^*(G)).$$

For more information about the Baum–Connes Conjecture we refer for instance to [6], [23], [26], [33].

From now on consider a group G as described in (0.1) We want to compute $K_n^G(\underline{E}G)$. If G satisfies the Baum–Connes Conjecture this is the same as $K_n(C_r^*(G))$.

First we construct a nice model for $\underline{E}Q$. Let $\{(M_i) \mid i \in I\}$ be the set of conjugacy classes of maximal finite subgroups of $M_i \subseteq Q$. By attaching free

Q–cells we get an inclusion of Q–CW–complexes $j_1\colon \coprod_{i\in I} Q\times_{M_i} EM_i \to EQ$. Define $\underline{E}Q$ as the Q–pushout

$$\begin{array}{ccc} \coprod_{i\in I} Q\times_{M_i} EM_i & \xrightarrow{j_1} & EQ \\ {\scriptstyle u_1}\big\downarrow & & \big\downarrow{\scriptstyle f_1} \\ \coprod_{i\in I} Q/M_i & \xrightarrow[k_1]{} & \underline{E}Q \end{array} \tag{1.1}$$

where u_1 is the obvious Q–map obtained by collapsing each EM_i to a point.

We have to explain why $\underline{E}Q$ is a model for the classifying space for proper actions of Q. Obviously it is a Q–CW–complex. Its isotropy groups are all finite. We have to show for $H\subseteq Q$ finite that $(\underline{E}Q)^H$ contractible. We begin with the case $H\neq\{1\}$. Because of conditions (M) and (NM) there is precisely one index $i_0\in I$ such that H is subconjugated to M_{i_0} and is not subconjugated to M_i for $i\neq i_0$ and we get

$$\left(\coprod_{i\in I} Q/M_i\right)^H \;=\; (Q/M_{i_0})^H \;=\; \{*\}.$$

It remains to treat $H=\{1\}$. Since u_1 is a non-equivariant homotopy equivalence and j_1 is a cofibration, f_1 is a non-equivariant homotopy equivalence and hence $\underline{E}Q$ is contractible (after forgetting the group action).

Let X be a Q–CW–complex and Y be a G–CW–complex. Then $X\times Y$ with the G–action given by $g\cdot(x,y)=(p(g)x,gy)$ is a G–CW–complex and the G–isotropy group $G_{(x,y)}$ of (x,y) is $p^{-1}(H_x)\cap G_y$. Hence $\underline{E}Q\times\underline{E}G$ is a G–CW–model for $\underline{E}G$ and $EQ\times\underline{E}G$ is a G–CW–model for EG, since $\ker(p\colon G\to Q)$ is torsionfree by assumption. Let Z be a M_i–CW–complex. Then there is a G–homeomorphism

$$G\times_{p^{-1}(M_i)}\left(Z\times\operatorname{res}_G^{p^{-1}(M_i)} Y\right)\xrightarrow{\cong}(Q\times_{M_i} Z)\times Y \qquad (g,(z,y))\mapsto((p(g),z),gy).$$

The inverse sends $((q,z),y)$ to $(g,(z,g^{-1}y)$ for any choice of $g\in G$ with $p(g)=q$. If we cross the Q–pushout (1.1) with $\underline{E}G$, then we obtain the following G–pushout:

$$\begin{array}{ccc} \coprod_{i\in I} G\times_{p^{-1}(M_i)} Ep^{-1}(M_i) & \xrightarrow{j_2} & EG \\ {\scriptstyle u_2}\big\downarrow & & \big\downarrow{\scriptstyle f_2} \\ \coprod_{i\in I} G\times_{p^{-1}(M_i)} \underline{E}p^{-1}(M_i) & \xrightarrow[k_2]{} & \underline{E}G \end{array} \tag{1.2}$$

If we divide out the G–action in the pushout (1.2) above we obtain the pushout:

$$\begin{tikzcd} \coprod_{i\in I} Bp^{-1}(M_i) \arrow[r, "j_3"] \arrow[d, "u_3"'] & BG \arrow[d, "f_3"] \\ \coprod_{i\in I} p^{-1}(M_i)\backslash \underline{E}p^{-1}(M_i) \arrow[r, "k_3"'] & G\backslash \underline{E}G \end{tikzcd} \tag{1.3}$$

If we divide out the Q–action in the pushout (1.1) we obtain the pushout:

$$\begin{tikzcd} \coprod_{i\in I} BM_i \arrow[r, "j_4"] \arrow[d, "u_4"'] & BQ \arrow[d, "f_4"] \\ \coprod_{i\in I} \{*\} \arrow[r, "k_4"'] & Q\backslash \underline{E}Q \end{tikzcd} \tag{1.4}$$

Theorem 1.5 *Let G be the group appearing in* (0.1) *and assume that conditions (M), (NM) and (T) hold. Assume that G and all groups $p^{-1}(M_i)$ satisfy the Baum–Connes Conjecture. Then the Mayer–Vietoris sequence associated to* (1.2) *yields the long exact sequence of abelian groups:*

$$\begin{aligned} \ldots &\xrightarrow{\partial_{n+1}} \bigoplus_{i\in I} K_n(Bp^{-1}(M_i)) \\ &\xrightarrow{\left(\bigoplus_{i\in I} K_n(Bl_i)\right)\oplus\left(\bigoplus_{i\in I} a[i]_n\right)} K_n(BG)\bigoplus\left(\bigoplus_{i\in I} K_n(C_r^*(p^{-1}(M_i))\right) \\ &\xrightarrow{a_n\oplus\left(\bigoplus_{i\in I} K_n(C_r^*(l_i))\right)} K_n(C_r^*(G)) \xrightarrow{\partial_n} \bigoplus_{i\in I} K_{n-1}(Bp^{-1}(M_i)) \\ &\xrightarrow{\left(\bigoplus_{i\in I} K_{n-1}(Bl_i)\right)\oplus\left(\bigoplus_{i\in I} a[i]_{n-1}\right)} K_{n-1}(BG)\bigoplus\left(\bigoplus_{i\in I} K_{n-1}(C_r^*(p^{-1}(M_i))\right) \\ &\xrightarrow{a_{n-1}\oplus\left(\bigoplus_{i\in I} K_{n-1}(C_r^*(l_i))\right)} \ldots \end{aligned}$$

Here the maps $a[i]_n$ and a are classical assembly maps and $l_i\colon p^{-1}(M_i)\to G$ is the inclusion.

Let Λ be a ring with $\mathbb{Z}\subseteq\Lambda\subseteq\mathbb{Q}$ such that the order of each finite subgroup of G is invertible in Λ. Then the composition

$$\begin{aligned} \Lambda\otimes_{\mathbb{Z}} K_n(Bp^{-1}(M_i)) &\xrightarrow{\mathrm{id}_\Lambda\otimes_{\mathbb{Z}} a[i]_n} K_n(C_r^*(p^{-1}(M_i)) = K_n^{p^{-1}(M_i)}(\underline{E}p^{-1}(M_i)) \\ &\xrightarrow{\mathrm{id}_\Lambda\otimes_{\mathbb{Z}}\mathrm{ind}_{p^{-1}(M_i)\to\{1\}}} \Lambda\otimes_{\mathbb{Z}} K_n(p^{-1}(M_i)\backslash\underline{E}p^{-1}(M_i)) \end{aligned}$$

is an isomorphism, where ind *denotes the induction map. In particular the long exact sequence above reduces after applying* $\Lambda\otimes_{\mathbb{Z}}$ *to split exact short exact sequences of* Λ*–modules:*

$$0 \to \bigoplus_{i\in I} \Lambda \otimes_{\mathbb{Z}} K_n(Bp^{-1}(M_i)) \xrightarrow{\left(\bigoplus_{i\in I} \mathrm{id}_\Lambda \otimes_{\mathbb{Z}} K_n(Bl_i)\right) \bigoplus \left(\bigoplus_{i\in I} \mathrm{id}_\Lambda \otimes_{\mathbb{Z}} a[i]_n\right)}$$
$$\Lambda \otimes_{\mathbb{Z}} K_n(BG) \bigoplus \left(\bigoplus_{i\in I} \Lambda \otimes_{\mathbb{Z}} K_n(C_r^*(p^{-1}(M_i))) \right)$$
$$\xrightarrow{\bigoplus_{i\in I} \mathrm{id}_\Lambda \otimes_{\mathbb{Z}} K_n(C_r^*(l_i)) \bigoplus \mathrm{id}_\Lambda \otimes_{\mathbb{Z}} a_n} \Lambda \otimes_{\mathbb{Z}} K_n(C_r^*(G)) \to 0$$

Proof The Mayer Vietoris sequence is obvious using the fact that for a free G–CW–complex X there is a canonical isomorphism $K_n^G(X) \xrightarrow{\cong} K_n(G\backslash X)$. The composition

$$\Lambda \otimes_{\mathbb{Z}} K_n(Bp^{-1}(M_i)) \xrightarrow{\mathrm{id}_\Lambda \otimes_{\mathbb{Z}} a[i]_n} K_n(C_r^*(p^{-1}(M_i)) = K_n^{p^{-1}(M_i)}(\underline{E}p^{-1}(M_i))$$
$$\xrightarrow{\mathrm{id}_\Lambda \otimes_{\mathbb{Z}} \mathrm{ind}_{p^{-1}(M_i)\to\{1\}}} \Lambda \otimes_{\mathbb{Z}} K_n(p^{-1}(M_i)\backslash \underline{E}p^{-1}(M_i))$$

is bijective by [24, Lemma 2.8 (a)]. □

The advantage of the following version is that it involves the spaces $G\backslash\underline{E}G$ instead of the spaces BG, and these often have rather small geometric models. In the case $G = \mathrm{Hei} \rtimes \mathbb{Z}/4$ we will see that $G\backslash\underline{E}G$ is the three-dimensional sphere S^3 (see Lemma 2.4).

Theorem 1.6 *Let G be the group appearing in* (0.1) *and assume conditions (M), (NM) and (T) hold. Assume that G and all groups $p^{-1}(M_i)$ satisfy the Baum–Connes Conjecture. Then there is a long exact sequence of abelian groups:*

$$\ldots \xrightarrow{c_{n+1} \bigoplus_{i\in I} d[i]_{n+1}} K_{n+1}(G\backslash\underline{E}G) \xrightarrow{\partial_{n+1}} \bigoplus_{i\in I} K_n(C_r^*(p^{-1}(M_i)))$$
$$\xrightarrow{\left(\bigoplus_{i\in I} K_n(C_r^*(l_i))\right) \bigoplus \left(\bigoplus_{i\in I} c[i]_n\right)} K_n(C_r^*(G)) \bigoplus \left(\bigoplus_{i\in I} K_n(p^{-1}(M_i)\backslash\underline{E}p^{-1}(M_i)) \right)$$
$$\xrightarrow{c_n \bigoplus_{i\in I} d[i]_n} K_n(G\backslash\underline{E}G) \xrightarrow{\partial_n} \bigoplus_{i\in I} K_{n-1}(C_r^*(p^{-1}(M_i)))$$
$$\xrightarrow{\left(\bigoplus_{i\in I} K_{n-1}(C_r^*(l_i))\right) \bigoplus \left(\bigoplus_{i\in I} c[i]_{n-1}\right)} \ldots$$

Here the homomorphisms $d[i]_n$ come from the $\big(p^{-1}(M_i) \to G\big)$–equivariant maps $\underline{E}p^{-1}(M_i) \to \underline{E}G$ which are unique up to equivariant homotopy. The maps c_n and (analogously for $c[i]_n$) are the compositions

$$K_n(C_r^*(G)) \xrightarrow{\mathrm{asmb}^{-1}} K_n^G(\underline{E}G) \xrightarrow{\mathrm{ind}_{G\to\{1\}}} K_n(G\backslash\underline{E}G).$$

Let Λ be a ring with $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$ such that the order of each finite subgroup of G is invertible in Λ. Then the composition

$$\Lambda \otimes_{\mathbb{Z}} K_n(BG) \xrightarrow{\mathrm{id}_\Lambda \otimes_{\mathbb{Z}} a_n} \Lambda \otimes_{\mathbb{Z}} K_n(C_r^*(G)) \xrightarrow{\mathrm{id}_\Lambda \otimes_{\mathbb{Z}} c_n} \Lambda \otimes_{\mathbb{Z}} K_n(G\backslash\underline{E}G)$$

is an isomorphism of Λ–modules. In particular the long exact sequence above reduces after applying $\Lambda \otimes_{\mathbb{Z}} -$ to split exact short sequences of Λ–modules:

$$0 \to \bigoplus_{i\in I} \Lambda \otimes_{\mathbb{Z}} K_n(C_r^*(p^{-1}(M_i))) \xrightarrow{\left(\bigoplus_{i\in I} \mathrm{id}_\Lambda \otimes_{\mathbb{Z}} K_n(C_r^*(l_i))\right) \bigoplus \left(\bigoplus_{i\in I} \mathrm{id}_\Lambda \otimes_{\mathbb{Z}} c[i]_n\right)}$$

$$\Lambda \otimes_{\mathbb{Z}} K_n(C_r^*(G)) \bigoplus \left(\bigoplus_{i\in I} \Lambda \otimes_{\mathbb{Z}} K_n(p^{-1}(M_i)\backslash\underline{E}p^{-1}(M_i))\right)$$

$$\xrightarrow{\mathrm{id}_\Lambda \otimes_{\mathbb{Z}} c_n \bigoplus_{i\in I} \mathrm{id}_\Lambda \otimes_{\mathbb{Z}} d[i]_n} \Lambda \otimes_{\mathbb{Z}} K_n(G\backslash\underline{E}G) \to 0$$

Proof From the pushout (1.3) we get the long exact Mayer Vietoris sequence for (non-equivariant) topological K–theory

$$\ldots \xrightarrow{\partial_{n+1}} \bigoplus_{i\in I} K_n(Bp^{-1}(M_i)) \xrightarrow{\left(\bigoplus_{i\in I} K_n(Bl_i)\right) \bigoplus \left(\bigoplus_{i\in I} K_n((p^{-1}(M_i)\backslash s_i)\right)}$$

$$K_n(BG) \bigoplus \left(\bigoplus_{i\in I} K_n(p^{-1}(M_i)\backslash\underline{E}p^{-1}(M_i))\right) \xrightarrow{H_n(G\backslash s) \bigoplus \left(\bigoplus_{i\in I} d[i]_n\right)} K_n(G\backslash\underline{E}G)$$

$$\xrightarrow{\partial_n} \bigoplus_{i\in I} K_{n-1}(Bp^{-1}(M_i)) \xrightarrow{\left(\bigoplus_{i\in I} K_{n-1}(Bl_i)\right) \bigoplus \left(\bigoplus_{i\in I} K_{n-1}(p^{-1}(M_i)\backslash s_i)\right)}$$

$$K_{n-1}(BG) \bigoplus \left(\bigoplus_{i\in I} K_{n-1}(p^{-1}(M_i)\backslash\underline{E}p^{-1}(M_i))\right) \xrightarrow{K_{n-1}(G\backslash s) \bigoplus \left(\bigoplus_{i\in I} d[i]_{n-1}\right)} \ldots$$

where $s_i\colon Ep^{-1}(M_i) \to \underline{E}p^{-1}(M_i)$ and $s\colon EG \to \underline{E}G$ are (up to equivariant homotopy unique) equivariant maps. Now one splices the long exact Mayer–Vietoris sequences from above and from Theorem 1.5 together. □

2 The semi-direct product of the Heisenberg group and a cyclic group of order four

We want to study the following example. Let Hei be the *discrete Heisenberg group*. We will use the presentation

$$\mathrm{Hei} \quad = \quad \langle u, v, z \mid [u, z] = 1, [v, z] = 1, [u, v] = z \rangle. \tag{2.1}$$

Throughout this section let G be the semi-direct product

$$G = \mathrm{Hei} \rtimes \mathbb{Z}/4$$

with respect to the homomorphism $\mathbb{Z}/4 \to \mathrm{aut}(\mathrm{Hei})$ which sends the generator t of $\mathbb{Z}/4$ to the automorphism of Hei given on generators by $z \mapsto z$, $u \mapsto v$ and $v \mapsto u^{-1}$. Let Q be the semi-direct product $\mathbb{Z}^2 \rtimes \mathbb{Z}/4$ with respect to the automorphism $\mathbb{Z}^2 \to \mathbb{Z}^2$ which comes from multiplication with the complex number i and the inclusion $\mathbb{Z}^2 \subseteq \mathbb{C}$. Since the action of $\mathbb{Z}/4$ on $\mathbb{Z}^2$ is free outside 0, the group Q satisfies (M) and (NM) (see [24, Lemma 6.3]). The group G has the presentation

$$G = \langle u, v, z, t \mid [u, z] = [v, z] = [t, z] = t^4 = 1, [u, v] = z, tut^{-1} = v, tvt^{-1} = u^{-1} \rangle.$$

Let $i\colon \mathbb{Z} \to G$ be the inclusion sending the generator of $\mathbb{Z}$ to z. Let $p\colon G \to Q$ be the group homomorphism, which sends z to the unit element, u to $(1,0)$ in $\mathbb{Z}^2 \subseteq Q$, v to $(0,1)$ in $\mathbb{Z}^2 \subseteq Q$ and t to the generator of $\mathbb{Z}/4 \subseteq Q$. Then $1 \to \mathbb{Z} \to G \to Q \to 1$ is a central extension which satisfies the conditions (M), (NM) and (T) appearing in (0.1). Moreover, G is amenable and hence G and all its subgroups satisfy the Baum–Connes Conjecture [18].

In order to apply the general results above we have to figure out the conjugacy classes of finite subgroups of $Q = \mathbb{Z}^2 \rtimes \mathbb{Z}/4$ and among them the maximal ones. An element of order 2 in Q must have the form xt^2 for $x \in \mathbb{Z}^2$. In the sequel we write the group multiplication in Q and G multiplicatively and in $\mathbb{Z}^2$ additively. We compute $(xt^2)^2 = xt^2xt^2 = (x - x) = 0$. Hence the set of elements of order two in Q is $\{xt^2 \mid x \in \mathbb{Z}^2\}$. Consider $e_1 = (1,0)$ and $e_2 = (0,1)$ in $\mathbb{Z}^2$. We claim that up to conjugacy there are the following subgroups of order two: $\langle e_1t^2 \rangle, \langle e_1e_2t^2 \rangle, \langle t^2 \rangle$. This follows from the computations for $x, y \in \mathbb{Z}^2$

$$\begin{aligned} y(xt^2)y^{-1} &= yxyt^2 = (x + 2y)t^2; \\ t(xt^2)t^{-1} &= txt^{-1}t^2 = (ix)t^2. \end{aligned}$$

An element of order 4 must have the form xt for $x \in \mathbb{Z}^2$. We compute

$$(xt)^4 = xtxt^{-1}t^2xt^{-2}t^3xt^{-3} = (x + ix + i^2x + i^3x) = (1 + i + i^2 + i^3)x = 0x = 0.$$

Hence the set of elements of order four in Q is $\{xt \mid x \in \mathbb{Z}^2\}$. We claim that up to conjugacy there are the following subgroups of order four: $\langle e_1 t\rangle, \langle t\rangle$. This follows from the computations for $x, y \in \mathbb{Z}^2$

$$\begin{aligned} y(xt)y^{-1} &= (x + y - iy)t; \\ t(xt)t^{-1} &= (ix)t. \end{aligned}$$

We have $(e_1t)^2 = e_1te_1t = e_1ie_1t^2 = e_1e_2t^2$. The considerations above imply:

Lemma 2.2 *Up to conjugacy Q has the following non-trivial finite subgroups*

$$\langle e_1t^2\rangle, \langle e_1e_2t^2\rangle, \langle t^2\rangle, \langle e_1t\rangle, \langle t\rangle.$$

The maximal finite subgroups are up to conjugacy

$$M_0 = \langle t\rangle, M_1 = \langle e_1t\rangle, M_2 = \langle e_1t^2\rangle.$$

Since $t^4 = 1$, $(ut^2)^2 = ut^2ut^{-2} = uu^{-1} = 1$ and $(ut)^4 = utut^{-1}t^2ut^{-2}t^3ut^{-3} = uvu^{-1}v^{-1} = z$ hold in G, the preimages of these groups under $p\colon G \to Q$ are given by

$$\begin{aligned} p^{-1}(M_0) &= \langle t, z\rangle \cong \mathbb{Z}/4 \times \mathbb{Z}; \\ p^{-1}(M_1) &= \langle ut, z\rangle = \langle ut\rangle \cong \mathbb{Z}; \\ p^{-1}(M_2) &= \langle ut^2, z\rangle \cong \mathbb{Z}/2 \times \mathbb{Z}. \end{aligned}$$

One easily checks

Lemma 2.3 *Up to conjugacy the finite subgroups of G are $\langle t\rangle$, $\langle t^2\rangle$ and $\langle ut^2\rangle$.*

Next we construct nice geometric models for $\underline{E}G$ and its orbit space $G\backslash\underline{E}G$. Let $\mathrm{Hei}(\mathbb{R})$ be the *real Heisenberg group*, ie, the Lie group of real $(3,3)$–matrices of the special form:

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

In the sequel we identify such a matrix with the element $(x, y, z) \in \mathbb{R}^3$. Thus $\mathrm{Hei}(\mathbb{R})$ can be identified with the Lie group whose underlying manifold is $\mathbb{R}^3$ and whose group multiplication is given by

$$(a, b, c) \bullet (x, y, z) = (a + x, b + y + az, c + z).$$

The discrete Heisenberg group is given by the subgroup where all the entries x, y, z are integers. In the presentation of the discrete Heisenberg group (2.1) the elements u, v and z correspond to $(1,0,0)$, $(0,0,1)$ and $(0,1,0)$. Obviously

Hei is a torsionfree discrete subgroup of the contractible Lie group $\mathrm{Hei}(\mathbb{R})$. Hence $\mathrm{Hei}(\mathbb{R})$ is a model for $E\mathrm{Hei}$ and $\mathrm{Hei}\backslash\mathrm{Hei}(\mathbb{R})$ for $B\mathrm{Hei}$. We have the following $\mathbb{Z}/4$–action on $\mathrm{Hei}(\mathbb{R})$, with the generator t acting by $(x,y,z) \mapsto (-z, y-xz, x)$. This is an action by automorphisms of Lie groups and induces the homomorphism $\mathbb{Z}/4 \to \mathrm{aut}(\mathrm{Hei})$ on Hei which we have used above to define $G = \mathrm{Hei} \rtimes \mathbb{Z}/4$. The Hei–action and $\mathbb{Z}/4$–action on $\mathrm{Hei}(\mathbb{R})$ above fit together to a $G = \mathrm{Hei} \rtimes \mathbb{Z}/4$–action. The next result is the main geometric input for the desired computations.

Lemma 2.4 *The manifold* $\mathrm{Hei}(\mathbb{R})$ *with the* G*–action above is a model for* $\underline{E}G$*. The quotient space* $G\backslash\underline{E}G$ *is homeomorphic to* S^3*.*

Proof Let $\mathbb{R} \subseteq \mathrm{Hei}(\mathbb{R})$ be the subgroup of elements $\{(0,y,0) \mid y \in \mathbb{R}\}$. This is the center of $\mathrm{Hei}(\mathbb{R})$. The intersection $\mathbb{R} \cap \mathrm{Hei}$ is $\mathbb{Z} \subseteq \mathbb{R}$. Thus we get a $\mathbb{R}/\mathbb{Z} = S^1$–action on $\mathrm{Hei}\backslash\mathrm{Hei}(\mathbb{R})$. One easily checks that this S^1–action and the $\mathbb{Z}/4$–action above commute so that we see a $S^1 \times \mathbb{Z}/4$–action on $\mathrm{Hei}\backslash\mathrm{Hei}(\mathbb{R})$. The S^1–action is free, but the $S^1 \times \mathbb{Z}/4$–action is not. Next we figure out its fixed points.

Obviously t^2 sends (x,y,z) to $(-x,y,-z)$. We compute for $(a,b,c) \in \mathrm{Hei}$, $u \in \mathbb{R}$ and $(x,y,z) \in \mathrm{Hei}(\mathbb{R})$

$$\begin{aligned}
(a,b,c)\cdot(0,u,0)\cdot t\cdot(x,y,z) &= (a-z, u+b+y-xz-ax, c+x);\\
(a,b,c)\cdot(0,u,0)\cdot t^2\cdot(x,y,z) &= (a-x, u+b+y-az, c-z);\\
(a,b,c)\cdot(0,u,0)\cdot(x,y,z) &= (a+x, u+b+y+az, c+z).
\end{aligned}$$

Hence the isotropy group of $\mathrm{Hei}\cdot(x,y,z) \in \mathrm{Hei}\backslash\mathrm{Hei}(\mathbb{R})$ under the $S^1 \times \mathbb{Z}/4$–action contains $(\exp(2\pi iu), t)$ in its isotropy group under the $S^1 \times \mathbb{Z}/4$–action if and only if $(a-z, u+b+y-xz-ax, c+x) = (x,y,z)$ holds for some integers a,b,c. The last statement is equivalent to the condition that $2x$ and $x+z$ are integers, y is an arbitrary real number and $u - 3x^2 \in \mathbb{Z}$.

The isotropy group of $\mathrm{Hei}\cdot(x,y,z) \in \mathrm{Hei}\backslash\mathrm{Hei}(\mathbb{R})$ contains $(\exp(2\pi iu), t^2)$ in its isotropy group under the $S^1 \times \mathbb{Z}/4$–action if and only if $(a-x, u+b+y-az, c-z) = (x,y,z)$ holds for some integers a,b,c. Obviously the last statement is equivalent to the condition that $2x$, $2z$ and $u-2xz$ are integers and y is an arbitrary real number.

The isotropy group of $\mathrm{Hei}\cdot(x,y,z) \in \mathrm{Hei}\backslash\mathrm{Hei}(\mathbb{R})$ contains $(\exp(2\pi iu), 1)$ in its isotropy group under the $S^1 \times \mathbb{Z}/4$–action if and only if $(a+x, u+b+y+az, c+z) = (x,y,z)$ holds for some integers a,b,c. The last statement is equivalent to the condition that $x = 0$, $z = 0$, u is an integer and y is an arbitrary real number.

This implies that the orbits under the $S^1 \times \mathbb{Z}/4$–action on $\mathrm{Hei} \setminus \mathrm{Hei}(\mathbb{R}^3)$ are free except the orbits through $\mathrm{Hei} \cdot (1/2, 0, 1/2)$, whose isotropy group is the cyclic subgroup of order four generated by $(\exp(3\pi i/4), t)$, and the orbits though $\mathrm{Hei} \cdot (0,0,0)$, whose isotropy group is the cyclic subgroup of order four generated by $(\exp(0), t)$, and the orbits though $\mathrm{Hei} \cdot (1/2, 0, 0)$ and $\mathrm{Hei} \cdot (0, 0, 1/2)$, whose isotropy groups are the cyclic subgroup of order two generated by $(\exp(0), t^2)$. By the slice theorem any point $p \in \mathrm{Hei} \setminus \mathrm{Hei}(\mathbb{R})$ has a neighborhood of the form $S^1 \times \mathbb{Z}/4 \times_{H_p} U_p$, where H_p is its isotropy group and U_p a 2–dimensional real H_p–representation, namely the tangent space of $\mathrm{Hei} \setminus \mathrm{Hei}(\mathbb{R})$ at p. Since there are only finitely $S^1 \times \mathbb{Z}/4$–orbits which are non-free, the H_p–action on U_p is free outside the origin for each $p \in \mathrm{Hei} \setminus \mathrm{Hei}(\mathbb{R})$. In particular $H_p \backslash U_p$ is a manifold without boundary. If the isotropy group H_p is mapped under the projection $\mathrm{pr}\colon S^1 \times \mathbb{Z}/4 \to S^1$ to the trivial group, then $\mathbb{Z}/4\backslash \left(S^1 \times \mathbb{Z}/4 \times_{H_p} U_p\right)$ is S^1–homeomorphic to $S^1 \times H_p\backslash U_p$ and hence a free S^1–manifold without boundary. If the projection $\mathrm{pr}\colon S^1 \times \mathbb{Z}/4 \to S^1$ is injective on H_p, then $\mathbb{Z}/4\backslash \left(S^1 \times \mathbb{Z}/4 \times_{H_p} U_p\right)$ is the S^1–manifold $S^1 \times_{H_p} U_p$ with respect to the free H–action on S^1 induced by p which has no boundary and precisely one non-free S^1–orbit. This shows that the quotient of $\mathrm{Hei} \setminus \mathrm{Hei}(\mathbb{R}^3)$ under the $\mathbb{Z}/4$–action is a closed S^1–manifold with precisely one non-free orbit.

The fixed point set of any finite subgroup of G of the G–space $\mathrm{Hei}(\mathbb{R}) = \mathbb{R}^3$ is a non-empty affine real subspace of $\mathrm{Hei}(\mathbb{R}) = \mathbb{R}^3$ and hence contractible. This shows that $\mathrm{Hei}(\mathbb{R})$ with its G–action is a model for $\underline{E}G$. Hence $G\backslash\underline{E}G$ is a closed S^1–manifold with precisely one non-free orbit, whose quotient space under the S^1–action is the orbit space of T^2 under the $\mathbb{Z}/4$–action. One easily checks for the rational homology

$$H_n\left((\mathbb{Z}/4)\backslash T^2;\mathbb{Q}\right) \cong_{\mathbb{Q}} H_n(T^2) \otimes_{\mathbb{Z}[\mathbb{Z}/4]} \mathbb{Q} \cong H_n(S^2;\mathbb{Q}).$$

This implies that the S^1–space $G\backslash\underline{E}G$ is a Seifert bundle over $(\mathbb{Z}/4)\backslash T^2 \cong S^2$ with precisely one singular fiber. Since the orbifold fundamental group of this orbifold S^2 with precisely one cone point vanishes, the map $e\colon \pi_1(S^1) \to \pi_1(G\backslash\underline{E}G)$ given by evaluating the S^1–action at some base point is surjective by [30, Lemma 3.2]. The Hurewicz map $h\colon \pi_1(G\backslash\underline{E}G) \to H_1(G\backslash\underline{E}G)$ is bijective since $\pi_1(G\backslash\underline{E}G)$ is a quotient of $\pi_1(S^1)$ and hence is abelian. The composition

$$\pi_1(S^1) \xrightarrow{e} \pi_1(G\backslash\underline{E}G) \xrightarrow{h} H_1(G\backslash\underline{E}G)$$

agrees with the composition

$$\pi_1(S^1) \xrightarrow{h'} H_1(S^1) = H_1(\mathbb{Z}\backslash\mathbb{R}) \xrightarrow{e'} H_1(\mathrm{Hei} \setminus \mathrm{Hei}(\mathbb{R})) \xrightarrow{H_1(\mathrm{pr})} H_1(G\backslash\underline{E}G),$$

where h' is the Hurewicz map, e' given by evaluating the S^1–operation and pr is the obvious projection. The map $H_1(\mathbb{Z}\backslash\mathbb{R}) \to H_1(\mathrm{Hei}\setminus\mathrm{Hei}(\mathbb{R}))$ is trivial since the element $z \in \mathrm{Hei}$ is a commutator, namely $[u,v]$. Hence $G\backslash\underline{E}G$ is a simply connected closed Seifert fibered 3–manifold. We conclude from [30, Lemma 3.1] that $G\backslash\underline{E}G$ is homeomorphic to S^3. □

Next we investigate what information Theorem 1.6 gives in combination with Lemma 2.4.

We have to analyze the maps

$$c[i]_n\colon K_n(C_r^*(p^{-1}(M_i))) \to K_n(p^{-1}(M_i)\backslash\underline{E}p^{-1}(M_i)),$$

which are defined as the compositions

$$\begin{aligned} K_n(C_r^*(p^{-1}(M_i))) &\xrightarrow{\mathrm{asmb}^{-1}} K_n^{p^{-1}(M_i)}(\underline{E}p^{-1}(M_i)) \\ &\xrightarrow{\mathrm{ind}_{p^{-1}(M_i)\to\{1\}}} K_n(p^{-1}(M_i)\backslash\underline{E}p^{-1}(M_i)). \end{aligned}$$

For $i=1$ the group $p^{-1}(M_i)$ is isomorphic to $\mathbb{Z}$ and hence the maps $c[1]_n$ are all isomorphisms. In the case $i=0,2$ the group $p^{-1}(M_i)$ looks like $H_i\times\mathbb{Z}$ for $H_0=\langle t\rangle\cong\mathbb{Z}/4$ and $H_2=\langle ut^2\rangle\cong\mathbb{Z}/2$. The following diagram commutes:

$$\begin{tikzcd} K_n(C_r^*(H_i\times\mathbb{Z})) \arrow[d, "K_n(C_r^*(\mathrm{pr}_i))"'] & K_n^{H_i\times\mathbb{Z}}(\underline{E}H_i\times\mathbb{Z}) \arrow[l, "\mathrm{asmb}"', "\cong"] \arrow[d, "\mathrm{ind}_{H_i\times\mathbb{Z}\to\mathbb{Z}}"] & K_n^{H_i}(\{*\})\bigoplus K_{n-1}^{H_i}(\{*\}) \arrow[l, "\cong"'] \arrow[d, "\mathrm{ind}_{H_i\to\{1\}}\bigoplus\mathrm{ind}_{H_i\to\{1\}}"] \\ K_n(C_r^*(\mathbb{Z})) & K_n^{\mathbb{Z}}(E\mathbb{Z}) \arrow[l, "\mathrm{asmb}"', "\cong"] & K_n(\{*\})\bigoplus K_{n-1}(\{*\}) \arrow[l, "\cong"'] \end{tikzcd}$$

The map $\mathrm{ind}_{H_i\to\{1\}}\colon K_n^{H_i}(\{*\}) \to K_n(\{*\})$ is the map $\mathrm{id}\colon 0\to 0$ for n odd. For n even it can be identified with the homomorphism $\epsilon\colon R_{\mathbb{C}}(H_i)\to\mathbb{Z}$ which sends the class of a complex H_i–representation V to the complex dimension of $\mathbb{C}\otimes_{\mathbb{C}H_i}V$. This map is split surjective. The kernel of ϵ is denoted by $\widetilde{R}_{\mathbb{C}}(H_i)$. Define for $i=0,2$ maps

$$c[i]_n'\colon \widetilde{R}_{\mathbb{C}}(H_i) \;\to\; K_n(C_r^*(G)) \tag{2.5}$$

as follows. For n even it is the composition

$$\widetilde{R}_{\mathbb{C}}(H_i)\subseteq R_{\mathbb{C}}(H_i)=K_n(C_r^*(H_i))\xrightarrow{K_n(C_r^*(l_i'))}K_n(C_r^*(G)),$$

where $l_i'\colon H_i\to G$ is the inclusion. For n odd it is the composition

$$\widetilde{R}_{\mathbb{C}}(H_i)\subseteq R_{\mathbb{C}}(H_i)=K_{n-1}(C_r^*(H_i))\xrightarrow{x_i}K_n(C_r^*(H_i\times\mathbb{Z}))\xrightarrow{K_n(C_r^*(l_i))}K_n(C_r^*(G)),$$

where $l_i\colon H_i \times \mathbb{Z} = p^{-1}(M_i) \to G$ is the inclusion and

$$x_i \bigoplus K_n(y_i)\colon K_{n-1}(C_r^*(H_i)) \bigoplus K_n(C_r^*(H_i)) \xrightarrow{\cong} K_n(C_r^*(H_i \times \mathbb{Z}))$$

is the canonical isomorphism for $y_i\colon H_i \to H_i \times \mathbb{Z}$ the inclusion. The map

$$\partial_n\colon K_n(G\backslash\underline{E}G) \to \bigoplus_{i\in I} K_{n-1}(C_r^*(p^{-1}(M_i)))$$

appearing in Theorem 1.6 vanishes after applying $\mathbb{Q}\otimes_{\mathbb{Z}} -$. Since the target is a finitely generated torsionfree abelian group, the map itself is trivial. Hence we obtain from Theorem 1.6 short exact sequences for $n \in \mathbb{Z}$

$$0 \to \widetilde{R}_{\mathbb{C}}(\mathbb{Z}/4) \bigoplus \widetilde{R}_{\mathbb{C}}(\mathbb{Z}/2) \xrightarrow{c[0]'_n \oplus c[2]'_n} K_n(C_r^*(G)) \xrightarrow{c_n} K_n(S^3) \to 0,$$

where we identify $H_0 = \langle t\rangle = \mathbb{Z}/4$ and $H_2 = \langle ut^2\rangle = \mathbb{Z}/2$ and $G\backslash\underline{E}G = S^3$ using Lemma 2.4. If $j_n\colon K_n(\{*\}) = K_n(C_r^*(\{1\})) \to K_n(C_r^*(G))$ is induced by the inclusion of the trivial subgroup, we can rewrite the sequence above as the short exact sequence

$$0 \to K_n(\{*\}) \bigoplus \widetilde{R}_{\mathbb{C}}(\mathbb{Z}/4) \bigoplus \widetilde{R}_{\mathbb{C}}(\mathbb{Z}/2) \xrightarrow{j_n \oplus c[0]'_n \oplus c[2]'_n} K_n(C_r^*(G)) \xrightarrow{c_n} \widetilde{K}_n(S^3) \to 0,$$

where $\widetilde{K}_n(Y)$ is for a path connected space Y the cokernel of the obvious map $K_n(\{*\}) \to K_n(Y)$. We have $\widetilde{K}_0(S^3) = 0$ and $\widetilde{K}_1(S^3) \cong \mathbb{Z}$. Thus we get

Theorem 2.6 *We have the isomorphism*

$$j_0 \bigoplus c[0]'_0 \bigoplus c[2]'_0\colon K_0(\{*\}) \bigoplus \widetilde{R}_{\mathbb{C}}(\mathbb{Z}/4) \bigoplus \widetilde{R}_{\mathbb{C}}(\mathbb{Z}/2) \xrightarrow{\cong} K_0(C_r^*(\mathrm{Hei} \rtimes \mathbb{Z}/4))$$

and the short exact sequence

$$0 \to \widetilde{R}_{\mathbb{C}}(\mathbb{Z}/4) \bigoplus \widetilde{R}_{\mathbb{C}}(\mathbb{Z}/2) \xrightarrow{c[0]'_1 \oplus c[2]'_1} K_1(C_r^*(G)) \xrightarrow{c_1} \widetilde{K}_1(S^3) \to 0,$$

where the maps $c[i]'_n$ have been defined in (2.5). *In particular $K_n(C_r^*(G))$ is a free abelian group of rank five for all n.*

Remark 2.7 These computations are consistent with the computation of $K_n(C_r^*(G))\left[\frac{1}{2}\right]$ coming from the Chern character constructed in [22].

Remark 2.8 One can also use these methods to compute the topological K–theory of the real reduced group C^*–algebra $C_r^*(\mathrm{Hei} \rtimes \mathbb{Z}/4;\mathbb{R})$. One obtains the short exact sequence

$$\begin{aligned} 0 \to KO_n(\{*\}) \bigoplus \widetilde{K}_n(C_r^*(\mathbb{Z}/2;\mathbb{R})) \bigoplus \widetilde{K}_{n-1}(C_r^*(\mathbb{Z}/2;\mathbb{R})) \\ \bigoplus \widetilde{K}_n(C_r^*(\mathbb{Z}/4;\mathbb{R})) \bigoplus \widetilde{K}_{n-1}(C_r^*(\mathbb{Z}/4;\mathbb{R})) \\ \to K_n(C_r^*(\mathrm{Hei} \rtimes \mathbb{Z}/4)) \to \widetilde{KO}_n(S^3) \to 0, \end{aligned}$$

which splits after inverting 2.

3 Algebraic K–theory

In this section we want to describe what the methods above yield for the algebraic K–theory provided that instead of the Baum–Connes Conjecture the relevant version of the Farrell–Jones Conjecture for algebraic K–theory (see [14]) is true. The L–theory will be treated in the next section. We want to prove the following:

Theorem 3.1 *Let R be a regular ring, for instance $R = \mathbb{Z}$. Let G be the group appearing in* (0.1) *and assume that conditions (M), (NM), and (T) are satisfied. Suppose that G and all subgroups $p^{-1}(M_i)$ satisfy the Farrell–Jones Conjecture for algebraic K–theory with coefficients in R. Then we get for $n \in \mathbb{Z}$ the isomorphism*

$$\bigoplus_{i\in I} \mathrm{Wh}_n(Rl_i)\colon \ \mathrm{Wh}_n(R[p^{-1}(M_i)]) \xrightarrow{\cong} \mathrm{Wh}_n(RG),$$

where $l_i\colon p^{-1}(M_i) \to G$ is the inclusion.

Notice that in the context of the Farrell–Jones Conjecture one has to consider the *family of virtually cyclic subgroups* $\mathcal{VCY}$ and only under special assumptions it suffices to consider the *family* $\mathcal{FIN}$ *of finite subgroups*. Recall that a family $\mathcal{F}$ of subgroups is a set of subgroups closed under conjugation and taking subgroups and that a model for *the classifying space $E_{\mathcal{F}}(G)$ for the family $\mathcal{F}$* is a G–CW–complex whose isotropy groups belong to $\mathcal{F}$ and whose H–fixed point set is contractible for each $H \in \mathcal{F}$. It is characterized up to G–homotopy by the property that any G–CW–complex, whose isotropy groups belong to $\mathcal{F}$, possesses up to G–homotopy precisely one G–map to $E_{\mathcal{F}}(G)$. In particular two models for $E_{\mathcal{F}}(G)$ are G–homotopy equivalent and for an inclusion of families $\mathcal{F} \subset \mathcal{G}$ there is up to G–homotopy precisely one G–map $E_{\mathcal{F}}(G) \to E_{\mathcal{G}}(G)$. The space $\underline{E}G$ is the same as $E_{\mathcal{FIN}}(G)$.

Let $\mathcal{H}_*^G(X;\mathbf{K}(R?))$ and $\mathcal{H}_*^G(X;\mathbf{L}^{\langle -\infty\rangle}(R?))$ be the G–homology theories associated to the algebraic K and L–theory spectra over the orbit category $\mathbf{K}(R?)$

and $\mathbf{L}^{\langle -\infty\rangle}(R?)$ (see [11]). They satisfy for each subgroup $H \subseteq G$

$$\begin{aligned} \mathcal{H}_n^G(G/H;\mathbf{K}(R?)) &\cong K_n(RH);\\ \mathcal{H}_n^G(G/H;\mathbf{L}^{\langle -\infty\rangle}(R?)) &\cong L_n^{\langle -\infty\rangle}(RH). \end{aligned}$$

The *Farrell–Jones Conjecture* (see [14, 1.6 on page 257]) says that the projection $E_{\mathcal{VCYC}}(G) \to G/G$ induces isomorphisms

$$\begin{aligned} \mathcal{H}_n^G(E_{\mathcal{VCYC}}(G);\mathbf{K}(R?)) &\xrightarrow{\cong} \mathcal{H}_n^G(G/G;\mathbf{K}(R?)) = K_n(RG);\\ \mathcal{H}_n^G(E_{\mathcal{VCYC}}(G);\mathbf{L}^{\langle -\infty\rangle}(R?)) &\xrightarrow{\cong} \mathcal{H}_n^G(G/G;\mathbf{L}^{\langle -\infty\rangle}(R?)) = L_n^{\langle -\infty\rangle}(RG). \end{aligned}$$

In the L–theory case one must use $L^{\langle -\infty\rangle}$. There are counterexamples to the Farrell–Jones Conjecture for the other decorations p, h and s (see [16]).

In the sequel we denote for a G–map $f\colon X \to Y$ by $\mathcal{H}_n^G(f\colon X \to Y;\mathbf{K}(R?))$ the value of $\mathcal{H}_n^G$ on the pair given by the mapping cylinder of f and Y viewed as a G–subspace. We will often use the long exact sequence associated to this pair:

$$\begin{aligned} \ldots \to \mathcal{H}_n^G(X;\mathbf{K}(R?)) \to \mathcal{H}_n^G(Y;\mathbf{K}(R?)) \to \mathcal{H}_n^G(f\colon X \to Y;\mathbf{K}(R?))&\\ \to \mathcal{H}_{n-1}^G(X;\mathbf{K}(R?)) \to \mathcal{H}_{n-1}^G(Y;\mathbf{K}(R?)) \to \ldots& \end{aligned}$$

The following result is taken from [4].

Theorem 3.2 *There are isomorphisms*

$$\begin{aligned} \mathcal{H}_n^G(\underline{E}G;\mathbf{K}(\mathbb{R}?)) \bigoplus \mathcal{H}_n^G\left(\underline{E}G \to E_{\mathcal{VCYC}}(G);\mathbf{K}(R?)\right)&\\ \xrightarrow{\cong} \mathcal{H}_n^G(E_{\mathcal{VCYC}}(G);\mathbf{K}(R?));& \end{aligned}$$

$$\begin{aligned} \mathcal{H}_n^G(\underline{E}G;\mathbf{L}^{\langle -\infty\rangle}(\mathbb{R}?)) \bigoplus \mathcal{H}_n^G\left(\underline{E}G \to E_{\mathcal{VCYC}}(G);\mathbf{L}^{\langle -\infty\rangle}(R?)\right)&\\ \xrightarrow{\cong} \mathcal{H}_n^G(E_{\mathcal{VCYC}}(G);\mathbf{L}^{\langle -\infty\rangle}(R?)),& \end{aligned}$$

where in the K–theory context G and R are arbitrary and in the L–theory context G is arbitrary and we assume for any virtually cyclic subgroup $V \subseteq G$ that $K_{-i}(RV) = 0$ for sufficiently large i.

For a virtually cyclic group V we have $K_{-i}(\mathbb{Z}V) = 0$ for $n \geq 2$ (see [15]).

The terms $\mathcal{H}_n^G\left(\underline{E}G \to E_{\mathcal{VCYC}}(G);\mathbf{K}(R?)\right)$ vanish for instance if R is a regular ring containing $\mathbb{Q}$. The terms $\mathcal{H}_n^G\left(\underline{E}G \to E_{\mathcal{VCYC}}(G);\mathbf{L}^{\langle -\infty\rangle}(R?)\right)$ vanish after inverting 2 (see Lemma 4.2). Recall that the Whitehead group $\mathrm{Wh}_n(RG)$ by definition is $\mathcal{H}_n^G(EG \to G/G;\mathbf{K}(R?))$. This implies that $\mathrm{Wh}_n(RG) =$

$\mathcal{H}_n^G(EG \to E_{\mathcal{VCYC}}(G); \mathbf{K}(R?))$ if the Farrell–Jones Isomorphism Conjecture for algebraic K–theory holds for RG. The group $\mathrm{Wh}_1(\mathbb{Z}G)$ is the classical Whitehead group $\mathrm{Wh}(G)$. If R is a principal ideal domain, then $\mathrm{Wh}_0(RG)$ is $\widetilde{K}_0(RG)$ and $\mathrm{Wh}_n(RG) = K_n(RG)$ for $n \leq -1$.

If we cross the Q–pushout (1.1) with $E_{\mathcal{VCYC}}(G)$ we obtain the G–pushout:

$$\begin{array}{ccc} \coprod_{i\in I} G \times_{p^{-1}(M_i)} E_{\mathcal{VCYC}(K\cap p^{-1}(M_i))}(p^{-1}(M_i)) & \longrightarrow & E_{\mathcal{VCYC}(K)}(G) \\ \downarrow & & \downarrow \\ \coprod_{i\in I} G \times_{p^{-1}(M_i)} E_{\mathcal{VCYC}}(p^{-1}(M_i)) & \longrightarrow & E_{\mathcal{VCYC}_f}(G) \end{array} \tag{3.3}$$

where $\mathcal{VCYC}(K\cap p^{-1}(M_i))$ is the family of virtually cyclic subgroups of $p^{-1}(M_i)$, which are contained in $K \cap p^{-1}(M_i)$, and $\mathcal{VCYC}(K)$ is the family of virtually cyclic subgroups of G, which are contained in K, and $\mathcal{VCYC}_f$ is the family of virtually cyclic subgroups of G, whose image under $p\colon G \to Q$ is finite. Since K is torsionfree, elements in $\mathcal{VCYC}(K \cap p^{-1}(M_i))$ and $\mathcal{VCYC}(K)$ are trivial or infinite cyclic groups. The following result is taken from [24, Theorem 2.3].

Theorem 3.4 *Let $\mathcal{F} \subset \mathcal{G}$ be families of subgroups of the group Γ. Let Λ be a ring with $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$ and N be an integer. Suppose for every $H \in \mathcal{G}$ that the assembly map induces for $n \leq N$ an isomorphism*

$$\Lambda \otimes_{\mathbb{Z}} \mathcal{H}_n^H(E_{H\cap\mathcal{F}}(H); \mathbf{K}(R?)) \to \Lambda \otimes_{\mathbb{Z}} \mathcal{H}_n^H(H/H; \mathbf{K}(R?)),$$

where $H \cap \mathcal{F}$ is the family of subgroups $K \subseteq H$ with $K \in \mathcal{F}$. Then the map

$$\Lambda \otimes_{\mathbb{Z}} H_n^\Gamma(E_{\mathcal{F}}(\Gamma); \mathbf{K}(R?)) \to \Lambda \otimes_{\mathbb{Z}} H_n^\Gamma(E_{\mathcal{G}}(\Gamma); \mathbf{K}(R?))$$

is an isomorphism for $n \leq N$. The analogous result is true for $\mathbf{L}^{\langle -\infty \rangle}(R?)$ instead of $\mathbf{K}(R?)$.

In the sequel we will apply Theorem 3.4 using the fact that for an infinite cyclic group or an infinite dihedral group H the map

$$\mathrm{asmb}\colon \mathcal{H}_n^H(\underline{E}H; \mathbf{K}(R?)) \ \to \ \mathcal{H}_n^K(H/H; \mathbf{K}(R?)) = K_n(RH)$$

is bijective for $n \in \mathbb{Z}$. This follows for the infinite cyclic group from the Bass–Heller-decomposition and for the infinite dihedral group from Waldhausen [34, Corollary 11.5 and the following Remark] (see also [3] and [23, Section 2.2]).

The Farrell–Jones Conjecture for algebraic K–theory for the trivial family $\mathcal{TR}$ consisting of the trivial subgroup only is true for infinite cyclic groups and regular rings R as coefficients. We conclude from Theorem 3.4 that for a regular ring R the maps

$$\mathcal{H}_n^{p^{-1}(M_i)}(E(p^{-1}(M_i));\mathbf{K}(R?)) \xrightarrow{\cong} \mathcal{H}_n^{p^{-1}(M_i)}(E_{\mathcal{VCYC}(K\cap p^{-1}(M_i))}(p^{-1}(M_i));\mathbf{K}(R?))$$

and

$$\mathcal{H}_n^G(EG;\mathbf{K}(R?)) \xrightarrow{\cong} \mathcal{H}_n^G(E_{\mathcal{VCYC}(K)}(G);\mathbf{K}(R?))$$

are bijective for all $n \in \mathbb{Z}$. Hence we obtain for a regular ring R from the G–pushout (3.3) an isomorphism

$$\bigoplus_{i\in I} \mathcal{H}_n^{p^{-1}(M_i)}\left(E(p^{-1}(M_i)) \to E_{\mathcal{VCYC}}(p^{-1}(M_i));\mathbf{K}(R?)\right) \xrightarrow{\cong} \mathcal{H}_n^G\left(EG \to E_{\mathcal{VCYC}_f}(G);\mathbf{K}(R?)\right). \quad (3.5)$$

Let $\mathcal{VCYC}_1$ be the family of virtually cyclic subgroups of G whose intersection with K is trivial. Since $\mathcal{VCYC}$ is the union $\mathcal{VCYC}_f \cup \mathcal{VCYC}_1$ and the intersection $\mathcal{VCYC}_f \cap \mathcal{VCYC}_1$ is $\mathcal{FIN}$, we obtain a G–pushout

$$\begin{array}{ccc} \underline{E}G & \longrightarrow & E_{\mathcal{VCYC}_1}(G) \\ \downarrow & & \downarrow \\ E_{\mathcal{VCYC}_f}(G) & \longrightarrow & E_{\mathcal{VCYC}}(G) \end{array} \quad (3.6)$$

The following conditions are equivalent for a virtually cyclic group V: i.) V admits an epimorphism to $\mathbb{Z}$ with finite kernel, ii.) $H_1(V;\mathbb{Z})$ is infinite, iii.) The center of V is infinite. A virtually cyclic subgroup does not satisfy these three equivalent conditions if and only if it admits an epimorphism onto D_∞ with finite kernel.

Lemma 3.7 *Any virtually cyclic subgroup of Q is finite, infinite cyclic or isomorphic to D_∞.*

Proof Suppose that $V \subseteq Q$ is an infinite virtually cyclic subgroup. Choose a finite normal subgroup $F \subseteq V$ such that V/F is $\mathbb{Z}$ or D_∞. We have to show that F is trivial. Suppose F is not trivial. By assumption there is a unique maximal finite subgroup $M \subseteq Q$ with $F \subseteq M$. Consider $q \in N_G F$. Then $F \subseteq q^{-1}Mq \cap M$. This implies $q \in N_G M = M$. Hence $N_G F$ is contained in the finite group M what contradicts $V \subseteq N_G F$. Hence F must be trivial. □

Now we can prove Theorem 3.1.

Proof Lemma 3.7 implies that any infinite subgroup appearing in $\mathcal{VCY}_1$ is an infinite cyclic group or an infinite dihedral group. Hence Theorem 3.4 implies that $\mathcal{H}_n^G(\underline{E}G \to E_{\mathcal{VCY}_1}(G); \mathbf{K}(R?))$ vanishes for $n \in \mathbb{Z}$. We conclude from the G–pushout (3.6) that $\mathcal{H}_n^G\left(E_{\mathcal{VCY}_f}(G) \to E_{\mathcal{VCY}}(G); \mathbf{K}(R?)\right)$ vanishes for $n \in \mathbb{Z}$. Now Theorem 3.1 follows from (3.5). □

Now let us investigate what the results above imply for the middle and lower algebraic K–theory with integral coefficients of the group $G = \mathrm{Hei} \rtimes \mathbb{Z}/4$ introduced in Section 2 and $R = \mathbb{Z}$. The Farrell–Jones Conjecture for algebraic K-theory is true for G and $R = \mathbb{Z}$ in the range $n \leq 1$ since G is a discrete cocompact subgroup of the virtually connected Lie group $\mathrm{Hei}(\mathbb{R}) \rtimes \mathbb{Z}/4$ (see [14]). Each group $p^{-1}(M_i)$ is virtually cyclic and satisfies the Farrell–Jones Conjecture for algebraic K-theory for trivial reasons. From Theorem 3.1 we get for $n \leq 1$ an isomorphism

$$\mathrm{Wh}_n(p^{-1}(M_0)) \bigoplus \mathrm{Wh}_n(p^{-1}(M_1)) \bigoplus \mathrm{Wh}_n(p^{-1}(M_2)) \xrightarrow{\cong} \mathrm{Wh}_n(G),$$

which comes from the various inclusions of subgroups and the subgroups M_0, M_1 and M_2 of Q have been introduced in Lemma 2.2. The Bass–Heller–Swan decomposition yields an isomorphism for any group H

$$\mathrm{Wh}_n(H \times \mathbb{Z}) \;\cong\; \mathrm{Wh}_{n-1}(H) \bigoplus \mathrm{Wh}_n(H) \bigoplus NK_n(\mathbb{Z}H) \bigoplus NK_n(\mathbb{Z}H). \quad (3.8)$$

The groups $\mathrm{Wh}_n(\mathbb{Z}^k)$ and $\mathrm{Wh}_n(\mathbb{Z}/2 \times \mathbb{Z}^k)$ vanish for $n \leq 1$ and $k \geq 0$. The groups $\mathrm{Wh}_n(\mathbb{Z}/4)$ are trivial for $n \leq 1$. References for these claims are given in the proof of [24, Theorem 3.2]. The groups $\mathrm{Wh}_n(\mathbb{Z}/4 \times \mathbb{Z}^k)$ vanish for $n \leq -1$ and $k \geq 0$. This follows from [15]. Thus we get the following:

Corollary 3.9 *Let G be the group* Hei $\rtimes \mathbb{Z}/4$ *introduced in Section 2. Then*

$$\mathrm{Wh}_n(G) \;\cong\; \begin{cases} NK_n(\mathbb{Z}[\mathbb{Z}/4]) \bigoplus NK_n(\mathbb{Z}[\mathbb{Z}/4]) & \text{for } n = 0, 1; \\ 0 & \text{for } n \leq -1. \end{cases} \quad (3.10)$$

where the isomorphism for $n = 0, 1$ comes from the inclusions of the subgroup $p^{-1}(M_0) = \langle t, z\rangle = \mathbb{Z} \times \mathbb{Z}/4$ into G and the Bass–Heller–Swan decomposition (3.8).

Some information about $NK_n(\mathbb{Z}[\mathbb{Z}/4])$ is given in [5, Theorem 10.6 on page 695]. Their exponent divides 4^d for some natural number d.

4 L–theory

In this section we want to describe what the methods above yield for the algebraic L–theory provided that instead of the Baum–Connes Conjecture the relevant version of the Farrell–Jones Conjecture for algebraic L–theory (see [14]) is true.

Theorem 4.1 *Let G be the group appearing in (0.1) and assume that conditions (M), (NM), and (T) are satisfied. Suppose that G and all the groups $p^{-1}(M)$ for $M \subseteq Q$ maximal finite satisfy the Farrell–Jones Conjecture for L–theory with coefficients in R. Then:*

(i) *There is a long exact sequence of abelian groups*

$$\ldots \to \mathcal{H}_{n+1}(G\backslash \underline{E}G;\mathbf{L}^{\langle -\infty\rangle}(R)) \to \bigoplus_{i\in I} L_n^{\langle -\infty\rangle}(R[p^{-1}(M_i)])$$
$$\to \mathcal{H}_n^G(\underline{E}G;\mathbf{L}^{\langle -\infty\rangle}(R?)) \bigoplus \left(\bigoplus_{i\in I} \mathcal{H}_n(p^{-1}(M_i)\backslash \underline{E}p^{-1}(M_i);\mathbf{L}^{\langle -\infty\rangle}(R?))\right)$$
$$\to \mathcal{H}_n(G\backslash \underline{E}G;\mathbf{L}^{\langle -\infty\rangle}(R)) \to \bigoplus_{i\in I} L_{n-1}^{\langle -\infty\rangle}(R[p^{-1}(M_i)]) \to \ldots .$$

Let Λ be a ring with $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$ such that the order of each finite subgroup of G is invertible in Λ. Then the long exact sequence above reduces after applying $\Lambda\otimes_{\mathbb{Z}} -$ to short split exact sequences of Λ–modules

$$0 \to \bigoplus_{i\in I} \Lambda\otimes_{\mathbb{Z}} L_n^{\langle -\infty\rangle}(R[p^{-1}(M_i)]) \to \Lambda\otimes_{\mathbb{Z}} \mathcal{H}_n^G(\underline{E}G;\mathbf{L}^{\langle -\infty\rangle}(R?))$$
$$\bigoplus \left(\bigoplus_{i\in I} \Lambda\otimes_{\mathbb{Z}} \mathcal{H}_n(p^{-1}(M_i)\backslash \underline{E}p^{-1}(M_i);\mathbf{L}^{\langle -\infty\rangle}(R))\right)$$
$$\to \Lambda\otimes_{\mathbb{Z}} \mathcal{H}_n(G\backslash \underline{E}G;\mathbf{L}^{\langle -\infty\rangle}(R)) \to 0;$$

(ii) *Suppose for any virtually cyclic subgroup $V \subseteq G$ that $K_{-i}(RV) = 0$ for sufficiently large i. Then there is a canonical isomorphism*

$$\mathcal{H}_n^G(\underline{E}G;\mathbf{L}^{\langle -\infty\rangle}(\mathbb{R}?)) \bigoplus \mathcal{H}_n^G\left(\underline{E}G \to E_{\mathcal{VCYC}}(G);\mathbf{L}^{\langle -\infty\rangle}(R?)\right)$$
$$\xrightarrow{\cong} L_n^{\langle -\infty\rangle}(RG);$$

(iii) *We have*

$$\mathcal{H}_n^G\left(\underline{E}G \to E_{\mathcal{VCYC}}(G);\mathbf{L}^{\langle -\infty\rangle}(R?)\right)\left[\frac{1}{2}\right] = 0.$$

Proof (i) This is proved in a completely analogous way to Theorem 1.6.

(ii) This follows from Theorem 3.2.

(iii) This follows from the next Lemma 4.2. □

Lemma 4.2 *Let Γ be a group. Let $\mathcal{VCY}$ be the family of virtually cyclic subgroups of Γ and $\mathcal{VCY}_{\mathbb{Z}}$ be the subfamily of $\mathcal{VCY}$ consisting of subgroups of Γ which admit an epimorphism to $\mathbb{Z}$ with finite kernel. Let $\mathcal{F}$ and $\mathcal{G}$ be families of subgroups of Γ. If $\mathcal{FIN} \subseteq \mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{VCY}_{\mathbb{Z}}$ holds, then*

$$\mathcal{H}_n^{\Gamma}\left(E_{\mathcal{F}}(\Gamma) \to E_{\mathcal{G}}(\Gamma); \mathbf{L}^{\langle -\infty \rangle}(R?)\right) = 0.$$

If $\mathcal{FIN} \subseteq \mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{VCY}$ holds, then

$$\mathcal{H}_n^{\Gamma}\left(E_{\mathcal{F}}(\Gamma) \to E_{\mathcal{G}}(\Gamma); \mathbf{L}^{\langle -\infty \rangle}(R?)\right)\left[\frac{1}{2}\right] = 0.$$

Proof We conclude from Theorem 3.4 that it suffices to show for a virtually cyclic group V, which admits an epimorphism to $\mathbb{Z}$, that the map

$$\mathcal{H}_n^V(\underline{E}V; \mathbf{L}^{\langle -\infty \rangle}(R?)) \to L_n^{\langle -\infty \rangle}(RV)$$

is bijective and for a virtually cyclic group V, which admits an epimorphism to D_∞, that the map above is bijective after inverting two.

We begin with the case where $V = F \rtimes_\phi \mathbb{Z}$ for an automorphism $\phi\colon F \to F$ of a finite group F. There is a long exact sequence which can be derived from [27] and [28]:

$$\begin{aligned} \ldots \to L_n^{\langle -\infty \rangle}(RF) \xrightarrow{\mathrm{id} - L_n^{\langle -\infty \rangle}(R\phi)} L_n^{\langle -\infty \rangle}(RF) \to L_n^{\langle -\infty \rangle}(RV) \to \\ \to L_{n-1}^{\langle -\infty \rangle}(RF) \xrightarrow{\mathrm{id} - L_{n-1}^{\langle -\infty \rangle}(R\phi)} L_{n-1}^{\langle -\infty \rangle}(RF) \to \ldots \end{aligned}$$

Since $\mathbb{R}$ with the action of V coming from the epimorphism to $\mathbb{Z}$ and the action of $\mathbb{Z}$ by translation is a model for $\underline{E}V$, we also obtain a long exact Mayer–Vietoris sequence:

$$\begin{aligned} \ldots \to L_n^{\langle -\infty \rangle}(RF) \xrightarrow{\mathrm{id} - L_n^{\langle -\infty \rangle}(R\phi)} L_n^{\langle -\infty \rangle}(RF) \to \mathcal{H}_n^V(\underline{E}V; \mathbf{L}^{\langle -\infty \rangle}(R?)) \to \\ \to L_{n-1}^{\langle -\infty \rangle}(RF) \xrightarrow{\mathrm{id} - L_{n-1}^{\langle -\infty \rangle}(R\phi)} L_{n-1}^{\langle -\infty \rangle}(RF) \to \ldots \end{aligned}$$

These two sequences are compatible with the assembly map

$$\mathrm{asmb}\colon \mathcal{H}_n^V(\underline{E}V; \mathbf{L}^{\langle -\infty \rangle}(R?)) \to \mathcal{H}_n^V(V/V; \mathbf{L}^{\langle -\infty \rangle}(R?)) = L_n^{\langle -\infty \rangle}(RV),$$

which must be an isomorphism by the Five-Lemma.

Suppose that V admits an epimorphism onto $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2$. Then we can write V as an amalgamated product $F_1 *_{F_0} F_2$ for finite groups F_1 and F_2 and a common subgroup F_0. We can think of V as a graph of groups associated to a segment and obtain an action without inversions on a tree which yields a 1–dimensional model for $\underline{E}V$ with two equivariant 0–cells V/F_1 and V/F_2 and one equivariant one-cell $V/F_0 \times D^1$ (see [31, §5]). The associated long Mayer–Vietoris sequence looks like:

$$\begin{aligned}\ldots \to L_n^{\langle -\infty\rangle}(RF_0) \to L_n^{\langle -\infty\rangle}(RF_1) \bigoplus L_n^{\langle -\infty\rangle}(RF_2) \to \mathcal{H}_n^V(\underline{E}V;\mathbf{L}^{\langle -\infty\rangle}(R?))\\ \to L_{n-1}^{\langle -\infty\rangle}(RF_0) \to L_{n-1}^{\langle -\infty\rangle}(RF_1) \bigoplus L_{n-1}^{\langle -\infty\rangle}(RF_2) \to \ldots\end{aligned}$$

There is a corresponding exact sequence, where $\mathcal{H}_n^V(\underline{E}V;\mathbf{L}^{\langle -\infty\rangle}(R?))$ is replaced by $L_n^{\langle -\infty\rangle}(RV)$ and additional UNil–terms occur which vanish after inverting two (see for $\mathbb{Z} \subseteq R \subseteq \mathbb{Q}$ [8, Corollary 6] or see [29, Remark 8.7] and [27]). Now a Five-Lemma argument proves the claim. □

Theorem 4.3 *Let G be the group appearing in* (0.1) *and assume that conditions (M), (NM), and (T) are satisfied. Suppose that Q contains no element of order 2. Suppose that G and all the groups $p^{-1}(M)$ for $M \subseteq Q$ maximal finite satisfy the Farrell–Jones Conjecture for L–theory with coefficients in R. Then there is a long exact sequence of abelian groups:*

$$\begin{aligned}\ldots \to \mathcal{H}_{n+1}(G\backslash\underline{E}G;\mathbf{L}^{\langle -\infty\rangle}(R?)) \to \bigoplus_{i\in I} L_n^{\langle -\infty\rangle}(R[p^{-1}(M_i)])\\ \to L_n^{\langle -\infty\rangle}(RG) \bigoplus \left(\bigoplus_{i\in I} \mathcal{H}_n(p^{-1}(M_i)\backslash\underline{E}p^{-1}(M_i);\mathbf{L}^{\langle -\infty\rangle}(R?))\right)\\ \to \mathcal{H}_n(G\backslash\underline{E}G;\mathbf{L}^{\langle -\infty\rangle}(R?)) \to \ldots\end{aligned}$$

Let Λ be a ring with $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$ such that the order of each finite subgroup of G is invertible in Λ. Then the long exact sequence above reduces after applying $\Lambda \otimes_{\mathbb{Z}} -$ to short split exact sequences of Λ–modules

$$\begin{aligned}0 \to \bigoplus_{i\in I} \Lambda \otimes_{\mathbb{Z}} L_n^{\langle -\infty\rangle}(R[p^{-1}(M_i)]) \to\\ \Lambda \otimes_{\mathbb{Z}} L_n^{\langle -\infty\rangle}(RG) \bigoplus \left(\bigoplus_{i\in I} \Lambda \otimes_{\mathbb{Z}} \mathcal{H}_n(p^{-1}(M_i)\backslash\underline{E}p^{-1}(M_i);\mathbf{L}^{\langle -\infty\rangle}(R))\right)\\ \to \Lambda \otimes_{\mathbb{Z}} \mathcal{H}_n(G\backslash\underline{E}G;\mathbf{L}^{\langle -\infty\rangle}(R)) \to 0.\end{aligned}$$

Proof Because of Theorem 4.1 and Lemma 4.2 it suffices to prove that V admits an epimorphism to $\mathbb{Z}$ for an infinite virtually cyclic subgroup $V \subset G$. If $V \cap K$ is trivial, then V is an infinite virtually cyclic subgroup of Q and hence isomorphic to $\mathbb{Z}$ by Lemma 3.7. Suppose that $V \cap K$ is non-trivial. Then V can be written as an extension $1 \to K \cap V \to V \to p(V) \to 1$ for a finite subgroup $p(V) \subseteq Q$. The group $K \cap V$ is infinite cyclic and $p(V)$ must have odd order. Hence V contains a central infinite cyclic subgroup. This implies that V admits an epimorphism to $\mathbb{Z}$. □

From now on we assume that Q is an extension $1 \to \mathbb{Z}^n \to Q \to F \to 1$ for a finite group F such that the conjugation action of F on $\mathbb{Z}^n$ is free outside the origin.

Let $V \subseteq Q$ be infinite virtually cyclic. Either F is of odd order or F has a unique element f_2 of order two [24, Lemma 6.2]. Because of Lemma 3.7 either V is infinite cyclic or V is isomorphic to D_∞ and each element $v_2 \in V$ of order two is mapped under $Q \to F$ to the unique element $f_2 \in F$ of order two in F.

Suppose that $V \subseteq Q$ is isomorphic to D_∞. Then V contains at least one element $v_2 \in V$ of order two. Any other element of order two is of the shape $v_2 u$ for $u \in V \cap \mathbb{Z}^n$. Hence V is the subgroup $\langle v_2, V \cap \mathbb{Z}^n \rangle$ generated by v_2 and the infinite cyclic group $V \cap \mathbb{Z}^n$ regardless which element $v_2 \in V$ of order two we choose. For any infinite cyclic subgroup $C \subseteq \mathbb{Z}^n$ let $C_{\max}$ be the kernel of the projection $\mathbb{Z}^n \to (\mathbb{Z}^n/C)/\operatorname{tors}(\mathbb{Z}^n/C)$. This is the maximal infinite cyclic subgroup of $\mathbb{Z}^n$ which contains C. Define $V_{\max} \subseteq Q$ to be

$$V_{\max} := \langle v_2, (V \cap \mathbb{Z}^n)_{\max} \rangle.$$

The subgroup $V_{\max}$ is isomorphic to D_∞ and satisfies $V \subseteq V_{\max}$. Moreover, it is a maximal virtually cyclic subgroup, ie, $V_{\max} \subseteq W$ for a virtually cyclic subgroup $W \subseteq Q$ implies $V_{\max} = W$. Let $V \subseteq W$ be virtually cyclic subgroups of Q such that $V \cong D_\infty$. Then $W \cong D_\infty$ and $V_{\max} = W_{\max}$. Hence each virtually cyclic subgroup V with $V \cong D_\infty$ is contained in a unique maximal virtually cyclic subgroup of Q isomorphic to D_∞, namely $V_{\max}$.

Next we show $N_G V = V$ if V is a subgroup of Q with $V \cong D_\infty$ and $V = V_{\max}$. Let $v_2 \in V$ be an element of order two. Consider an element $q \in N_G V$. Then the conjugation action of q on $\mathbb{Z}^n$ sends $V \cap \mathbb{Z}^n$ to itself. Hence the conjugation action of q^2 on $\mathbb{Z}^n$ induces the identity on $V \cap \mathbb{Z}^n$. This implies that q is mapped under $Q \to F$ to the unit element or the unique element of order two f_2. Hence q is of the shape u or $v_2 u$ for some $u \in \mathbb{Z}^n$. Since $(v_2 u) v_2 (v_2 u)^{-1} = v_2 u v_2 u^{-1} v_2 = v_2 u^2$ and $u v_2 u^{-1} = v_2 u^2$, we conclude $v_2 u^2 \in V$. This implies that $u^2 \in V \cap \mathbb{Z}^n$. Since $V = V_{\max}$, we get $u \in V$ and hence $q \in V$.

Let J be a complete system of representatives V for the set of conjugacy classes (V) of subgroups $V \subseteq Q$ with $V \cong D_\infty$ and $V = V_{\max}$. In the sequel let $\mathcal{ICOF}$ be the set of subgroups H with are infinite cyclic or finite. By attaching equivariant cells we construct a model for $E_{\mathcal{ICOF}}(Q)$ which contains $\coprod_{V\in J} Q \times_V E_{\mathcal{ICOF}}(V)$ as Q–CW–subcomplex. Define a Q–CW–complex $E_{\mathcal{VCY}}(Q)$ by the Q–pushout

$$\begin{array}{ccc} \coprod_{V\in J} Q \times_V E_{\mathcal{ICOF}}(V) & \longrightarrow & E_{\mathcal{ICOF}}(Q) \\ {\scriptstyle u_5}\downarrow & & \downarrow{\scriptstyle f_5} \\ \coprod_{V\in J} Q/V & \longrightarrow & E_{\mathcal{VCY}}(Q) \end{array} \tag{4.4}$$

where the map u_5 is the obvious projection and the upper horizontal arrow is the inclusion.

We have to show that $E_{\mathcal{VCY}}(Q)$ is a model for the classifying space for the family $\mathcal{VCY}$ of virtually cyclic subgroups of Q. Obviously all its isotropy groups belong to $\mathcal{VCY}$. Let $H \subseteq Q$ be a virtually cyclic group with $H \in \mathcal{ICOF}$. Choose a map of sets $s\colon Q/V \to Q$ such that its composition with the projection $Q \to Q/V$ is the identity. For any V–space X, there is a homeomorphism

$$(Q \times_V X)^H \xrightarrow{\cong} \coprod_{\substack{qV\in Q/V \\ s(qV)^{-1}Hs(qV)\subseteq V}} X^{s(qV)^{-1}Hs(qV)}, \quad (q,x) \mapsto s(qV)^{-1}qx,$$

whose inverse sends x of the summand $X^{s(qV)^{-1}Hs(qV)}$ belonging to $qV \in Q/V$ with $s(qV)^{-1}Hs(qV) \subseteq V$ to $(s(qV),x)$. Since $E_{\mathcal{ICOF}}(Q)^{s(qV)^{-1}Hs(qV)}$ and $(V/V)^{s(qV)^{-1}Hs(qV)}$ are contractible for each $qV \in Q/V$ with $s(qV)^{-1}Hs(qV) \subseteq V$, the map u_5^H is a homotopy equivalence. Hence f_5^H is a homotopy equivalence. The space $E_{\mathcal{ICOF}}(Q)^H$ is contractible. Therefore $E_{\mathcal{VCY}}(Q)^H$ is contractible. Let $H \subseteq Q$ be a virtually cyclic group with $H \notin \mathcal{ICOF}$. Then

$$\begin{aligned} \left(\coprod_{V\in J} Q/V\right)^H &= \{*\}; \\ (Q \times_V E_{\mathcal{ICOF}}(V))^H &= \emptyset; \\ E_{\mathcal{ICOF}}(Q)^H &= \emptyset. \end{aligned}$$

This implies that $E_{\mathcal{VCY}}(Q)^H = \{*\}$ is contractible.

Recall that $\mathcal{VCY}_f$ is the family of virtually cyclic subgroups of G whose image under $p\colon G \to Q$ is finite and $\mathcal{VCY}_1$ is the family of virtually cyclic subgroups of G whose intersection with $K = \ker(p)$ is trivial. Let $\mathcal{VCY}_{icof}$ be the family of subgroups of G whose image under $p\colon G\to Q$ is contained in $\mathcal{ICOF}$. If we cross the Q–pushout (4.4) with $E_{\mathcal{VCYC}}(G)$, we obtain the G–pushout:

$$\begin{tikzcd} \coprod_{V\in J} G\times_{p^{-1}(V)} E_{\mathcal{ICOF}}(p^{-1}(V)) \arrow[r] \arrow[d, "u_6"'] & E_{\mathcal{VCYC}_{icof}}(G) \arrow[d, "f_6"] \\ \coprod_{V\in J} G\times_{p^{-1}(V)} E_{\mathcal{VCYC}}(p^{-1}(V)) \arrow[r] & E_{\mathcal{VCYC}}(G) \end{tikzcd}$$

Because of Lemma 4.2 this G–pushout induces isomorphisms for $n\in\mathbb{Z}$

$$\bigoplus_{V\in J}\mathcal{H}_n^{p^{-1}(V)}\left(\underline{E}p^{-1}(V)\to E_{\mathcal{VCYC}}(p^{-1}(V));\mathbf{L}^{\langle-\infty\rangle}(R?)\right)$$
$$\xrightarrow{\cong}\mathcal{H}_n^G\left(E_{\mathcal{VCYC}_f}(G)\to E_{\mathcal{VCYC}}(G);\mathbf{L}^{\langle-\infty\rangle}(R?)\right).$$

We now conclude from Lemma 4.2:

Lemma 4.5 *Let $1\to\mathbb{Z}^n\to Q\to F\to 1$ be an extension such that the conjugation action of F on $\mathbb{Z}^n$ is free outside the origin. Let $1\to K\xrightarrow{i}G\xrightarrow{p}Q\to 1$ be an extension. Suppose that for any virtually cyclic group $V\subseteq G$ with $p(V)$ finite there exists an epimorphism $V\to\mathbb{Z}$. (This condition is satisfied if K is abelian and contained in the center of G.) Then there is an isomorphism*

$$\bigoplus_{V\in J}\mathcal{H}_n^{p^{-1}(V)}\left(\underline{E}p^{-1}(V)\to E_{\mathcal{VCYC}}(p^{-1}(V));\mathbf{L}^{\langle-\infty\rangle}(R?)\right)$$
$$\xrightarrow{\cong}\mathcal{H}_n^G\left(\underline{E}G\to E_{\mathcal{VCYC}}(G);\mathbf{L}^{\langle-\infty\rangle}(R?)\right).$$

Next we apply Theorem 4.1 and Lemma 4.5 to the special example $G=\mathrm{Hei}\rtimes\mathbb{Z}/4$ introduced in Section 2. We begin with constructing an explicit choice for J and determining the preimages $p^{-1}(V)$ for $V\in J$. Recall that J is a complete system of representatives of the conjugacy classes (V) of subgroups $V\subseteq Q$ with $V\cong D_\infty$ and $V=V_{\max}$. Let $IC(\mathbb{Z}^2)$ be the set of infinite cyclic subgroups L of $\mathbb{Z}^2$. Any subgroup $V\subseteq Q$ with $V\cong D_\infty$ can be written as $V=\langle v_2,V\cap\mathbb{Z}^2\rangle$ for $v_2\in V$ any element of order two. Hence we can write $V=\langle t^2a,L\rangle$ for $L\in IC(\mathbb{Z}^2)$ and $a\in\mathbb{Z}^2$. We have $\langle t^2a,L\rangle=\langle t^2a',L'\rangle$ if and only if $L=L'$ and $a-a'\in L=L'$. We have $V=V_{\max}$ for $V=\langle t^2a,L\rangle$ if and only if $L\subset\mathbb{Z}^2$ is maximal.

Let $IC^+(\mathbb{Z})$ be the subset for which $L\in IC(\mathbb{Z}^2)$ meets $\{(n_1,n_2)\in\mathbb{Z}^2\mid n_1\geq 0,n_2>0\}$. The $\mathbb{Z}/4$–action on $\mathbb{Z}^2$ induces a $\mathbb{Z}/2$–action on $L_1(\mathbb{Z}^2)$ by sending L to $i\cdot L$. Notice that $IC^+(\mathbb{Z}^2)$ is a fundamental domain for this action, ie, $IC(\mathbb{Z}^2)$ is the disjoint union of $IC^+(\mathbb{Z}^2)$ and its image under this involution.

We claim that a complete system of representatives of conjugacy classes (V) of subgroups V of $Q = \mathbb{Z}^2 \rtimes \mathbb{Z}/4$ with $V \cong D_\infty$ is

$$\begin{array}{ll} \langle t^2, (n_1,n_2)\rangle & n_1 \text{ even;} \\ \langle t^2(0,1), (n_1,n_2)\rangle & n_1 \text{ even;} \\ \langle t^2, (n_1,n_2)\rangle & n_2 \text{ even;} \\ \langle t^2(1,0), (n_1,n_2)\rangle & n_2 \text{ even;} \\ \langle t^2, (n_1,n_2)\rangle & n_1 \text{ and } n_2 \text{ odd;} \\ \langle t^2(1,0), (n_1,n_2)\rangle & n_1 \text{ and } n_2 \text{ odd,} \end{array} \tag{4.6}$$

where (n_1,n_2) runs through $IC^+ = \{(n_1,n_2) \in \mathbb{Z}^2 \mid n_1 > 0, n_2 \geq 0, (n_1,n_2) = 1\}$. This follows from the computations

$$\begin{aligned} (m_1,m_2)^{-1}(t^2(n_1,n_2))(m_1,m_2) &= t^2(n_1+2m_1, n_2+2m_2); \\ t(t^2(n_1,n_2))t^{-1} &= t^2(-n_2,n_1); \\ t^2(t^2(n_1,n_2))(t^2)^{-1} &= t^2(-n_1,-n_2). \end{aligned}$$

Now we list the preimages $p^{-1}(V)$ of these subgroups above and determine their isomorphism type. We claim that they can be described by the following generators

$$\begin{array}{ll} \langle t^2, t^2(n_1, n_1n_2/2, n_2), (0,1,0)\rangle & n_1 \text{ even;} \\ \langle t^2(0,0,1), t^2(n_1, n_1n_2/2, n_2), (0,1,0)\rangle & n_1 \text{ even;} \\ \langle t^2, t^2(n_1, n_1n_2/2, n_2), (0,1,0)\rangle & n_2 \text{ even;} \\ \langle t^2(1,0,0), t^2(n_1, n_1n_2/2, n_2), (0,1,0)\rangle & n_2 \text{ even;} \\ \langle t^2, t^2(2n_1, 2n_1n_2, 2n_2), (n_1, \frac{n_1n_2+1}{2}, n_2)\rangle & n_1 \text{ and } n_2 \text{ odd;} \\ \langle t^2(1,0,0), t^2(2n_1+1, 2n_1n_2+n_2, 2n_2), (n_1, \frac{n_1n_2+1}{2}, n_2)\rangle & n_1 \text{ and } n_2 \text{ odd,} \end{array} \tag{4.7}$$

where (n_1,n_2) runs through $IC^+ = \{(n_1,n_2) \in \mathbb{Z}^2 \mid n_1 > 0, n_2 \geq 0, (n_1,n_2) = 1\}$. This is obvious for the first four groups and follows for the last two groups from the computation

$$\begin{aligned} (n_1, \frac{n_1n_2+1}{2}, n_2)^2 &= (2n_1, 2n_1n_2, 2n_2) \cdot (0,1,0); \\ (2n_1+1, 2n_1n_2+n_2, 2n_2) &= (1,0,0) \cdot (2n_1, 2n_1n_2, 2n_2). \end{aligned}$$

The first four groups are isomorphic to $D_\infty \times \mathbb{Z}$ and the last two are isomorphic to the semi-direct product $D_\infty \rtimes_a \mathbb{Z}$ with respect to the automorphism a of $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2 = \langle s_1, s_2 \mid s_2^2 = s_2^2 = 1\rangle$ which send s_1 to s_2 and s_2 to s_1. For the first four groups there are explicit isomorphisms from $D_\infty \times \mathbb{Z} = \langle s_1, s_2, z \mid s_1^2 = s_2^2 = [s_1, z] = [s_2, z] = 1\rangle$ which send s_1, s_2, z to the three generators appearing in the presentation above. Similarly for the last two groups there are explicit isomorphisms from $D_\infty \rtimes_a \mathbb{Z} = \langle s_1, s_2, z \mid s_1^2 = s_2^2 = 1, z^{-1}s_1z = s_2\rangle$ which send s_1, s_2, z to the three generators appearing in the presentation above. We leave it to the reader to check that these generators appearing in the presentation above do satisfy the required relations.

Next we compute the groups $\mathcal{H}_n^{D_\infty \times \mathbb{Z}}(\underline{E}D_\infty \times \mathbb{Z} \to E_{\mathcal{VCYC}}(D_\infty \times \mathbb{Z}); \mathbf{L}^{-\infty}(\mathbb{Z}?))$ and $\mathcal{H}_n^{D_\infty \rtimes_a \mathbb{Z}}(\underline{E}D_\infty \rtimes_a \mathbb{Z} \to E_{\mathcal{VCYC}}(D_\infty \rtimes \mathbb{Z}); \mathbf{L}^{-\infty}(\mathbb{Z}?))$. There is an obvious model for $\underline{E}D_\infty$, namely $\mathbb{R}$ with the trivial $\mathbb{Z}$–action and the action of $D_\infty = \mathbb{Z} \rtimes_a \mathbb{Z}/2$, which comes from the $\mathbb{Z}$–action by translation and the $\mathbb{Z}/2$–action given by $-\operatorname{id}_{\mathbb{R}}$. From this we obtain an exact sequence

$$0 \to L_n^{\langle -\infty \rangle}(R) \xrightarrow{i} L_2^{\langle -\infty \rangle}(R[\mathbb{Z}/2]) \bigoplus L_2^{\langle -\infty \rangle}(R[\mathbb{Z}/2]) \xrightarrow{f} \mathcal{H}_n^{D_\infty}(\underline{E}D_\infty; \mathbf{L}^{-\infty}(R?)) \to 0$$

such that the composition of f with the obvious map

$$\mathcal{H}_n^{D_\infty}(\underline{E}D_\infty; \mathbf{L}(R?)) \to \mathcal{H}_n^{D_\infty}(\{*\}; \mathbf{L}(R?) = L_n^{\langle -\infty \rangle}(R[D_\infty])$$

is given by the two obvious inclusions $\mathbb{Z}/2 \to D_\infty = \langle s_1, s_2 \mid s_1 = s_2^2 = 1 \rangle$. Thus we obtain an isomorphism

$$\mathcal{H}_n^{D_\infty}\left(\underline{E}D_\infty \to E_{\mathcal{VCYC}}(D_\infty); \mathbf{L}^{\langle -\infty \rangle}(R?)\right) \quad = \quad \operatorname{UNil}_n(\mathbb{Z}/2 * \mathbb{Z}/2; R), \qquad (4.8)$$

where $\operatorname{UNil}_n(\mathbb{Z}/2 * \mathbb{Z}/2; R)$ is the UNil–term appearing in the short split exact sequence

$$0 \to L_n^{\langle -\infty \rangle}(R) \to L_n^{\langle -\infty \rangle}(R[\mathbb{Z}/2]) \oplus L_n^{\langle -\infty \rangle}(R[\mathbb{Z}/2]) \oplus \operatorname{UNil}_n(\mathbb{Z}/2 * \mathbb{Z}/2; R) \to L_n^{\langle -\infty \rangle}(R[\mathbb{Z}/2 * \mathbb{Z}/2]) \to 0$$

due to Cappell [8, Theorem 10]. For the computation of these terms $\operatorname{UNil}_n(\mathbb{Z}/2 * \mathbb{Z}/2; R)$ we refer to [2], [9] and [10]. They have exponent four and they are either trivial or are infinitely generated as abelian groups.

We can take as model for $\underline{E}(D_\infty \times \mathbb{Z})$ the product $\underline{E}D_\infty \times \mathbb{R}$, where $\mathbb{Z}$ acts on $\mathbb{R}$ by translation. We get from (4.8) and Lemma 4.2 using a Mayer–Vietoris argument an isomorphism

$$\begin{aligned}
&\mathcal{H}_n^{\mathbb{Z} \times D_\infty}\left(\underline{E}(D_\infty \times \mathbb{Z}) \to E_{\mathcal{VCYC}}(D_\infty \times \mathbb{Z}); \mathbf{L}^{\langle -\infty \rangle}(R?)\right) \\
&\quad \cong \mathcal{H}_n^{\mathbb{Z} \times D_\infty}\left(\underline{E}D_\infty \times \mathbb{R} \to E_{\mathcal{VCYC}}(D_\infty) \times \mathbb{R}; \mathbf{L}^{\langle -\infty \rangle}(R?)\right) \\
&\quad \cong \mathcal{H}_n^{D_\infty}\left(\underline{E}D_\infty \times S^1 \to E_{\mathcal{VCYC}}(D_\infty) \times S^1; \mathbf{L}^{\langle -\infty \rangle}(R?)\right) \\
&\quad \cong \operatorname{UNil}_n(\mathbb{Z}/2 * \mathbb{Z}/2; R) \bigoplus \operatorname{UNil}_{n-1}(\mathbb{Z}/2 * \mathbb{Z}/2; R). \qquad (4.9)
\end{aligned}$$

Next we investigate $D_\infty \rtimes_a \mathbb{Z}$. Let $\mathcal{VCYC}(D_\infty)$ be the family of virtually cyclic subgroups of $D_\infty \rtimes_a \mathbb{Z}$ which lie in D_∞ and let $\mathcal{VCYC}_f$ be the family of virtually cyclic subgroups of $\mathcal{VCY}(D_\infty)$ whose intersection with D_∞ is finite. Then the family $\mathcal{VCY}$ of virtually cyclic subgroups of $D_\infty \rtimes_a \mathbb{Z}$ is the union of $\mathcal{VCY}(D_\infty)$ and $\mathcal{VCY}_f$ and the family $\mathcal{FIN}$ of finite subgroups of $D_\infty \rtimes_a \mathbb{Z}$ is the intersection of $\mathcal{VCY}(D_\infty)$ and $\mathcal{VCY}_f$. Hence we get a pushout of $D_\infty \rtimes_a \mathbb{Z}$–spaces

$$\begin{array}{ccc} \underline{E}(D_\infty \rtimes_a \mathbb{Z}) & \longrightarrow & E_{\mathcal{VCY}(D_\infty)}(D_\infty \rtimes_a \mathbb{Z}) \\ \downarrow & & \downarrow \\ E_{\mathcal{VCY}_f}(D_\infty \rtimes_a \mathbb{Z}) & \longrightarrow & E_{\mathcal{VCY}}(D_\infty \rtimes_a \mathbb{Z}) \end{array}$$

Any finite subgroup of D_∞ is trivial or isomorphic to $\mathbb{Z}/2$. Any group V which can be written as extension $1 \to \mathbb{Z}/2 \to V \to \mathbb{Z} \to 1$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}$. Hence any infinite group V occurring in $\mathcal{VCY}_f$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/2\times\mathbb{Z}/2$. We conclude from Lemma 4.2 and the $D_\infty \rtimes_a \mathbb{Z}$–pushout above

$$\begin{aligned} &\mathcal{H}_n^{D_\infty \rtimes_a \mathbb{Z})}\left(E_{\mathcal{VCY}(D_\infty)}(D_\infty \rtimes_a \mathbb{Z}) \to E_{\mathcal{VCY}}(D_\infty \rtimes_a \mathbb{Z}); \mathbf{L}^{\langle -\infty\rangle}(R?)\right) \\ &\qquad \cong \mathcal{H}_n^{D_\infty \rtimes_a \mathbb{Z})}\left(\underline{E}(D_\infty \rtimes_a \mathbb{Z}) \to E_{\mathcal{VCY}_f}(D_\infty \rtimes_a \mathbb{Z}); \mathbf{L}^{\langle -\infty\rangle}(R?)\right) \cong 0. \end{aligned}$$

Hence we get an isomorphism

$$\begin{aligned} &\mathcal{H}_n^{D_\infty \rtimes_a \mathbb{Z})}\left(\underline{E}(D_\infty \rtimes_a \mathbb{Z}) \to E_{\mathcal{VCY}}(D_\infty \rtimes_a \mathbb{Z}); \mathbf{L}^{\langle -\infty\rangle}(R?)\right) \\ &\qquad \cong \mathcal{H}_n^{D_\infty \rtimes_a \mathbb{Z})}\left(\underline{E}(D_\infty \rtimes_a \mathbb{Z}) \to E_{\mathcal{VCY}(D_\infty)}(D_\infty \rtimes_a \mathbb{Z}); \mathbf{L}^{\langle -\infty\rangle}(R?)\right). \end{aligned}$$

We can take as model for $\underline{E}(D_\infty \rtimes_a \mathbb{Z})$ the to both sides infinite mapping telescope of the $(a\colon D_\infty \to D_\infty)$–equivariant map $\underline{E}a\colon \underline{E}D_\infty \to \underline{E}D_\infty$ with the $D_\infty \rtimes_a \mathbb{Z}$–action for which $\mathbb{Z}$ acts by shifting the telescope to the right. A model for $E_{\mathcal{VCY}(D_\infty)}(D_\infty \rtimes_a \mathbb{Z})$ is the to both sides infinite mapping telescope of the $(a\colon D_\infty \to D_\infty)$–equivariant map $\{*\} \to \{*\}$. Of course this is the same as $\mathbb{R}$ with the $D_\infty \rtimes_a \mathbb{Z}$–action, for which D_∞ acts trivially and $\mathbb{Z}$ by translation. The long Mayer–Vietoris sequence together with (4.8) yields a long exact sequence:

$$\begin{aligned} \ldots \to \mathrm{UNil}_n(\mathbb{Z}/2 * \mathbb{Z}/2; R) &\xrightarrow{\mathrm{id} - \mathrm{UNil}_n(a)} \mathrm{UNil}_n(\mathbb{Z}/2 * \mathbb{Z}/2; R) \\ &\to \mathcal{H}_n^{D_\infty \rtimes_a \mathbb{Z}}\left(\underline{E}(D_\infty \rtimes_a \mathbb{Z}) \to E_{\mathcal{VCY}}(D_\infty \rtimes_a \mathbb{Z}); \mathbf{L}^{\langle -\infty\rangle}(R?)\right) \\ \to \mathrm{UNil}_{n-1}(\mathbb{Z}/2 * \mathbb{Z}/2; R) &\xrightarrow{\mathrm{id} - \mathrm{UNil}_{n-1}(a)} \mathrm{UNil}_{n-1}(\mathbb{Z}/2 * \mathbb{Z}/2; R) \to \ldots \end{aligned} \tag{4.10}$$

The homomorphism $\mathrm{UNil}_{n-1}(a)$ has been analyzed in [7].

Theorem 4.11 *Let G be the group* $\mathrm{Hei} \rtimes \mathbb{Z}/4$ *introduced in Section 2. Then*

(i) *There is a short exact sequence which splits after inverting* 2

$$0 \to L_n^{\langle -\infty\rangle}(\mathbb{Z}) \bigoplus \widetilde{L}_n^{\langle -\infty\rangle}(\mathbb{Z}[\mathbb{Z}/2]) \bigoplus \widetilde{L}_{n-1}^{\langle -\infty\rangle}(\mathbb{Z}[\mathbb{Z}/2]) \bigoplus \widetilde{L}_n^{\langle -\infty\rangle}(\mathbb{Z}[\mathbb{Z}/4]) \bigoplus \widetilde{L}_{n-1}^{\langle -\infty\rangle}(\mathbb{Z}[\mathbb{Z}/4]) \xrightarrow{j} \mathcal{H}^G(\underline{E}G;\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z})) \to L_{n-3}^{\langle -\infty\rangle}(\mathbb{Z}) \to 0;$$

(ii) *There is for* $n\in\mathbb{Z}$ *an isomorphism*

$$\mathcal{H}_n^G(\underline{E}G;\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z})) \bigoplus \mathcal{H}_n^G\left(\underline{E}G \to E_{\mathcal{VCYC}}(G);\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z}?)\right) \xrightarrow{\cong} L_n^{\langle -\infty\rangle}(\mathbb{Z}G);$$

(iii) *Let* IC^+ *be the set* $\{(n_1,n_2)\in\mathbb{Z}^2 \mid n_1>0, n_2\geq 0, (n_1,n_2)=1\}$*. Then there is an isomorphism*

$$\left(\bigoplus_{\substack{(n_1,n_2)\in IC^+,\\ n_1 \text{ or } n_2 \text{ even}}} \bigoplus_{i=1}^{4} \left(\mathrm{UNil}_n(\mathbb{Z}/2*\mathbb{Z}/2;\mathbb{Z}) \bigoplus \mathrm{UNil}_{n-1}(\mathbb{Z}/2*\mathbb{Z}/2;\mathbb{Z})\right)\right) \bigoplus$$
$$\left(\bigoplus_{\substack{(n_1,n_2)\in IC^+,\\ n_1 \text{ and } n_2 \text{ odd}}} \bigoplus_{i=1}^{2} \mathcal{H}_n^{D_\infty\rtimes_a\mathbb{Z}}\left(\underline{E}(D_\infty\rtimes_a\mathbb{Z}) \to E_{\mathcal{VCYC}}(D_\infty\rtimes_a\mathbb{Z});\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z}?)\right)\right)$$
$$\xrightarrow{\cong} \mathcal{H}^G\left(\underline{E}G \to E_{\mathcal{VCYC}}(G);\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z}?)\right),$$

where the term $\mathcal{H}_n^{D_\infty\rtimes_a\mathbb{Z}}\left(\underline{E}(D_\infty\rtimes_a\mathbb{Z}) \to E_{\mathcal{VCYC}}(D_\infty\rtimes_a\mathbb{Z});\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z}?)\right)$ *is analyzed in* (4.10)*;*

(iv) *The canonical map* $L^{\langle -\infty\rangle}(\mathbb{Z}G) \xrightarrow{\cong} L^{\epsilon}(\mathbb{Z}G)$ *is bijective for all decorations* $\epsilon = p,h,s$*.*

Proof (i) This follows from Theorem 4.1 (i) since the groups $\widetilde{L}_n^p(\mathbb{Z}/2) = \widetilde{L}_n^{\langle -\infty\rangle}(\mathbb{Z}/2)$ and $\widetilde{L}_n^p(\mathbb{Z}/4) = \widetilde{L}_n^{\langle -\infty\rangle}(\mathbb{Z}/4)$ are torsionfree [1, Theorem 1].

(ii) The Farrell–Jones Conjecture for algebraic L–theory with coefficients in $R=\mathbb{Z}$ is true for $G = \mathrm{Hei}\rtimes\mathbb{Z}/4$ and $p^{-1}(V)$ for $V\subseteq Q$ virtually cyclic since G is a discrete cocompact subgroup of the virtually connected Lie group $\mathrm{Hei}(\mathbb{R})\rtimes_a\mathbb{Z}/4$ (see [14]). Since for a virtually cyclic group V we have $K_n(\mathbb{Z}V)=0$ for $n\leq -2$ [15], we can apply Theorem 4.1 (ii).

(iii) This follows from Lemma 4.5, the lists (4.6) and (4.7) and the isomorphism (4.9).

(iv) Because of the Rothenberg sequences it suffices to show that the Tate cohomology groups $\widehat{H}^n(\mathbb{Z}/2,\mathrm{Wh}_q(G))$ vanish for $q\leq 1$ and $n\in\mathbb{Z}$. If $q\leq -1$, then

$\mathrm{Wh}_q(G) = 0$, and, if $q = 0,1$, then $\mathrm{Wh}_q(G) = NK_q(\mathbb{Z}[\mathbb{Z}/4]) \bigoplus NK_q(\mathbb{Z}[\mathbb{Z}/4])$ by Corollary 3.9. One easily checks that the involution on $\mathrm{Wh}_q(G)$ corresponds under this identification to the involution on $NK_q(\mathbb{Z}[\mathbb{Z}/4]) \bigoplus NK_q(\mathbb{Z}[\mathbb{Z}/4])$ which sends (x_1, x_2) to $(x_2, \tau(x_1))$ for $\tau\colon NK_q(\mathbb{Z}[\mathbb{Z}/4]) \to NK_q(\mathbb{Z}[\mathbb{Z}/4])$ the involution on the Nil-Term. Hence the $\mathbb{Z}[\mathbb{Z}/2]$–module $\mathrm{Wh}_q(G)$ is isomorphic to the $\mathbb{Z}[\mathbb{Z}/2]$–module $\mathbb{Z}[\mathbb{Z}/2] \otimes_{\mathbb{Z}} NK_q(\mathbb{Z}[\mathbb{Z}/4])$, which is obtained from the $\mathbb{Z}$–module $NK_q(\mathbb{Z}[\mathbb{Z}/4])$ by induction with the inclusion of the trivial group into $\mathbb{Z}/2$. This implies $\widehat{H}^n(\mathbb{Z}/2, \mathrm{Wh}_q(G)) = 0$ for $q \le 1$ and $n \in \mathbb{Z}$. $\square$

Remark 4.12 If one inverts 2, then the computation for $L_n(\mathbb{Z}[\mathrm{Hei} \rtimes \mathbb{Z}])$ simplifies drastically as explained in the introduction because of Lemma 4.2. In general this example shows how complicated it is to deal with the infinite virtually cyclic subgroups which admit an epimorphism to D_∞ and the resulting UNil–terms.

5 Group homology

Finally we explain what the methods above give for the group homology

Theorem 5.1 *Let G be the group appearing in* (0.1) *and assume that conditions (M), (NM), and (T) are satisfied. We then obtain a long exact Mayer–Vietoris sequence*

$$\begin{aligned}
\ldots \to H_{n+1}(G\backslash \underline{E}G) &\xrightarrow{\partial_{n+1}} \\
\bigoplus_{i\in I} H_n(p^{-1}(M_i)) &\xrightarrow{\left(\bigoplus_{i\in I} H_n(l_i)\right) \bigoplus \left(\bigoplus_{i\in I} H_n(p^{-1}(M_i)\backslash s_i)\right)} \\
H_n(G) \bigoplus &\left(\bigoplus_{i\in I} H_n(p^{-1}(M_i)\backslash \underline{E}p^{-1}(M_i))\right) \\
&\xrightarrow{H_n(G\backslash s) \bigoplus \left(\bigoplus_{i\in I} H_n(d_i)\right)} H_n(G\backslash \underline{E}G) \xrightarrow{\partial_n} \\
\bigoplus_{i\in I} H_{n-1}(p^{-1}(M_i)) &\xrightarrow{\left(\bigoplus_{i\in I} H_{n-1}(l_i)\right) \bigoplus \left(\bigoplus_{i\in I} H_{n-1}(p^{-1}(M_i)\backslash s_i)\right)} \ldots
\end{aligned}$$

from the pushout (1.3) *where $l_i\colon p^{-1}(M_i) \to G$ is the inclusion, $s_i\colon Ep^{-1}(M_i) \to \underline{E}p^{-1}(M_i)$, $s\colon EG \to \underline{E}G$ are the obvious equivariant maps and*

$$d_i\colon p^{-1}(M_i)\backslash \underline{E}p^{-1}(M_i) \to G\backslash \underline{E}G$$

is the map induced by the l_i–equivariant map $\underline{E}p^{-1}(M_i) \to \underline{E}G$.

Remark 5.2 There are often finite-dimensional models for $\underline{E}G$ as discussed in [19], [21]. If or instance, there is a k–dimensional model for BK and a m–dimensional model for $\underline{E}Q$ and d is a positive integer such that the order of any finite subgroup of Q divides d, then there is a $(dk+n)$–dimensional model for $\underline{E}G$ [21, Theorem 3.1]. If Q is an extension $0 \to \mathbb{Z}^n \to Q \to F \to 1$ for a finite group F and there is a k–dimensional model for BK, then there is a $(|F| \cdot k + n)$–dimensional model for $\underline{E}G$.

Suppose that there is a N–dimensional model for $\underline{E}G$. Then there is also a N–dimensional model for $\underline{E}p^{-1}(M_i)$ for each $i \in I$ and under the assumptions of Theorem 5.6 we obtain for $n \geq N+1$ an isomorphism

$$\bigoplus_{i\in I} H_n(l_i)\colon \bigoplus_{i\in I} H_n(p^{-1}(M_i)) \xrightarrow{\cong} H_n(G).$$

Next we compute the group homology $H_*(\mathrm{Hei}\rtimes\mathbb{Z}/4)$. We start with the computation of $H_n(\mathrm{Hei})$. The Atiyah-Hirzebruch spectral sequence associated to the central extension $1 \to \mathbb{Z} \xrightarrow{i'} \mathrm{Hei} \xrightarrow{p'} \mathbb{Z}^2 \to 0$ yields the isomorphism

$$H_2(\mathbb{Z}^2) \xrightarrow{\cong} H_3(\mathrm{Hei})$$

and the long exact sequence

$$\begin{aligned} 0 \to H_1(\mathbb{Z}^2) \to H_2(\mathrm{Hei}) \xrightarrow{H_2(p')} H_2(\mathbb{Z}^2) \to H_0(\mathbb{Z}^2) = H_1(S^1) = H_1(\mathbb{Z}) \\ \xrightarrow{H_1(i')} H_1(\mathrm{Hei}) \xrightarrow{H_1(p')} H_1(\mathbb{Z}^2) \to 0. \end{aligned}$$

Since $z \in \mathrm{Hei}$ is a commutator, namely $[u,v]$, the map $H_1(i')\colon H_1(\mathbb{Z}) \to H_1(\mathrm{Hei})$ is trivial. This implies:

Lemma 5.3 *There are natural isomorphisms*

$$\begin{array}{rcll} H_1(p')\colon H_1(\mathrm{Hei}) & \xrightarrow{\cong} & H_1(\mathbb{Z}^2); & \\ H_1(\mathbb{Z}^2) & \xrightarrow{\cong} & H_2(\mathrm{Hei}); & \\ H_2(\mathbb{Z}^2) & \xrightarrow{\cong} & H_3(\mathrm{Hei}); & \\ H_n(\mathrm{Hei}) & = & 0 & \textit{for } n \geq 4. \end{array}$$

Next we analyze the Atiyah-Hirzebruch spectral sequence associated to the split extension $1 \to \mathrm{Hei} \xrightarrow{k} G := \mathrm{Hei}\rtimes\mathbb{Z}/4 \xrightarrow{\mathrm{pr}} \mathbb{Z}/4 \to 1$. The isomorphisms above appearing in the computation of the homology of Hei are compatible with the $\mathbb{Z}/4$–actions. Thus we get

$$\begin{array}{rcll} H_p(\mathbb{Z}/4; H_q(\mathrm{Hei})) & = & H_p(\mathbb{Z}/4; H_q(\mathbb{Z}^2)) = \mathbb{Z}/2 & \text{for } q = 1,2, p \geq 0, p \text{ even}; \\ H_p(\mathbb{Z}/4; H_q(\mathrm{Hei})) & = & H_p(\mathbb{Z}/4; H_q(\mathbb{Z}^2)) = 0 & \text{for } q = 1,2, p \geq 0, p \text{ odd}; \\ H_p(\mathbb{Z}/4; H_q(\mathrm{Hei})) & = & H_p(\mathbb{Z}/4) & \text{for } q = 0,3; \\ H_p(\mathbb{Z}/4; H_q(\mathrm{Hei})) & = & = 0 & \text{for } q \geq 4. \end{array}$$

Hence the E^2–term looks like:

$$\begin{array}{ccccccc}
\mathbb{Z} & \mathbb{Z}/4 & 0 & \mathbb{Z}/4 & 0 & \mathbb{Z}/4 & 0 \\
\mathbb{Z}/2 & 0 & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 \\
\mathbb{Z}/2 & 0 & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 \\
\mathbb{Z} & \mathbb{Z}/4 & 0 & \mathbb{Z}/4 & 0 & \mathbb{Z}/4 & 0
\end{array}$$

Using the model for $\underline{E}G$ of Lemma 2.4 we see that the map $B\mathrm{Hei} \to G\backslash\underline{E}G$ can be identified with the quotient map $B\mathrm{Hei} \to \mathbb{Z}/4\backslash B\mathrm{Hei}$ of an orientation preserving smooth $\mathbb{Z}/4$–action on the closed orientable 3–manifold $B\mathrm{Hei}$, where the quotient is again a closed orientable 3–manifold and the action has at least one free orbit. Since we can compute the degree of a map by counting preimages of a regular value, the degree must be ± 4. Recall that $G \to \mathbb{Z}/4$ is split surjective. These remarks imply together with the spectral sequence above

Lemma 5.4 *The composition $H_3(B\mathrm{Hei}) \xrightarrow{H_3(Bk)} H_3(BG) \xrightarrow{H_3(G\backslash s)} H_3(G\backslash\underline{E}G)$ is an injective map of infinite cyclic subgroups whose cokernel has order four. The map $H_3(\mathrm{Hei}) \to H_3(G)$ is injective and the order of the cokernel of the induced map $H_3(G)/\operatorname{tors}(H_3(G)) \to H_3(G\backslash\underline{E}G)$ divides four;*

Moreover, there are the following possibilities

(i) *The differential $d^2_{2,1}\colon E^2_{2,1} \cong \mathbb{Z}/2 \to E^2_{0,2} \cong \mathbb{Z}/2$ is trivial. Then $H_2(G)$ is $\mathbb{Z}/2$. Moreover, either the group $H_3(G)$ is $\mathbb{Z} \times \mathbb{Z}/4$ and the induced map $H_3(\mathrm{Hei}) \to H_3(G)/\operatorname{tors}(H_3(G))$ is an injective homomorphism of infinite cyclic groups whose cokernel has order two, or the group $H_3(G)$ is $\mathbb{Z} \times \mathbb{Z}/2 \times \mathbb{Z}/4$ and the induced map $H_3(\mathrm{Hei}) \to H_3(G)/\operatorname{tors}(H_3(G))$ is an isomorphism of infinite cyclic groups.*

(ii) *The differential $d^2_{2,1}\colon E^2_{0,2} \cong \mathbb{Z}/2 \to E^2_{2,1} \cong \mathbb{Z}/2$ is non-trivial. Then $H_2(G)$ is 0.*

It is not obvious how to compute the homology groups $H_n(G)$ for $G = \mathrm{Hei} \rtimes \mathbb{Z}/4)$ from the Atiyah-Hirzebruch spectral sequence. Let us try Theorem 5.1. It yields the long exact Mayer Vietoris sequence

$$\begin{aligned}
\ldots \to H_{n+1}(G\backslash\underline{E}G) &\xrightarrow{\partial_{n+1}} \bigoplus_{i=0}^{2} H_n(p^{-1}(M_i)) \\
&\xrightarrow{\left(\bigoplus_{i=0}^2 H_n(l_i)\right)\bigoplus\left(\bigoplus_{i=0}^2 H_n(p^{-1}(M_i)\backslash s_i)\right)} H_n(G) \bigoplus \left(\bigoplus_{i=0}^{2} H_n(p^{-1}(M_i)\backslash\underline{E}p^{-1}(M_i))\right) \\
&\xrightarrow{H_n(G\backslash s)\bigoplus\left(\bigoplus_{i=0}^2 H_n(d_i)\right)} H_n(G\backslash\underline{E}G) \xrightarrow{\partial_n}
\end{aligned}$$

$$\bigoplus_{i=0}^{2} H_{n-1}(p^{-1}(M_i)) \xrightarrow{(\bigoplus_{i=0}^2 H_{n-1}(l_i)) \bigoplus (\bigoplus_{i=0}^2 H_{n-1}(p^{-1}(M_i)\backslash s_i))} \ldots$$

where the maximal finite subgroups M_0, M_1 and M_2 of Q have been introduced in Lemma 2.2. The map $s_i\colon p^{-1}(M_i)\backslash Ep^{-1}(M_i) \to p^{-1}(M_i)\backslash \underline{E}p^{-1}(M_i)$ can be identified with

$$\begin{array}{rcl} s_0\colon B\langle t,z\rangle = B\langle t\rangle \times B\langle z\rangle & \xrightarrow{\mathrm{pr}} & B\langle z\rangle; \\ s_1\colon B\langle ut\rangle & \xrightarrow{\mathrm{id}} & B\langle ut\rangle; \\ s_2\colon B\langle ut^2,z\rangle = B\langle ut^2\rangle \times B\langle z\rangle & \xrightarrow{\mathrm{pr}} & B\langle z\rangle. \end{array}$$

Hence we obtain the exact sequence

$$\begin{gathered} \ldots \to H_{n+1}(G) \bigoplus H_{n+1}(\langle z\rangle) \bigoplus H_{n+1}(\langle ut\rangle) \bigoplus H_{n+1}(\langle z\rangle) \\ \xrightarrow{H_{n+1}(G\backslash s) \bigoplus_{i=0}^2 H_{n+1}(d_i)} H_{n+1}(G\backslash \underline{E}G) \\ \xrightarrow{\partial'_{n+1}} H_n(\langle t\rangle \times \langle z\rangle) \bigoplus H_n(\langle ut\rangle) \bigoplus H_n(\langle ut^2\rangle \times \langle z\rangle) \\ \xrightarrow{(H_n(\mathrm{incl}'_0) \bigoplus H_n(\mathrm{incl}'_1) \bigoplus H_n(\mathrm{incl}'_2)) \bigoplus H_n(\mathrm{pr}) \bigoplus \mathrm{id} \bigoplus H_n(\mathrm{pr})} \\ H_n(G) \bigoplus H_n(\langle z\rangle) \bigoplus H_n(\langle ut\rangle) \bigoplus H_n(\langle z\rangle) \\ \xrightarrow{H_n(G\backslash s) \bigoplus_{i=0}^2 H_n(d_i)} H_n(G\backslash \underline{E}G) \xrightarrow{\partial'_n} \ldots \end{gathered}$$

where

$$\begin{array}{rcl} \mathrm{incl}'_0\colon \langle t,z\rangle = \langle t\rangle \times \langle z\rangle & \to & G \\ \mathrm{incl}'_1\colon \langle ut\rangle & \to & G \\ \mathrm{incl}'_2\colon \langle ut,z\rangle = \langle ut\rangle \times \langle z\rangle & \to & G \end{array}$$

are the inclusions. This yields the exact sequence:

$$\begin{gathered} \ldots \to H_{n+1}(G) \xrightarrow{H_{n+1}(G\backslash s)} H_{n+1}(G\backslash \underline{E}G) \\ \xrightarrow{\partial''_n} \widetilde{H}_n(\langle t\rangle) \bigoplus \widetilde{H}_{n-1}(\langle t\rangle) \bigoplus \widetilde{H}_n(\langle ut^2\rangle) \bigoplus \widetilde{H}_{n-1}(\langle ut^2\rangle) \\ \to H_n(G) \xrightarrow{H_n(G\backslash s)} H_n(G\backslash \underline{E}G) \xrightarrow{\partial''_n} \ldots \end{gathered} \tag{5.5}$$

Recall that $G\backslash \underline{E}G$ is S^3. We conclude from Lemma 5.4 that the order of the cokernel of the map $H_3(G\backslash s)\colon H_3(BG) \to H_3(G\backslash \underline{E}G)$ divides four. Since the order of $\widetilde{H}_2(\langle t\rangle) \bigoplus \widetilde{H}_1(\langle t\rangle) \bigoplus \widetilde{H}_2(\langle e_1t^2\rangle) \bigoplus \widetilde{H}_2(\langle e_1t^2\rangle)$ is eight, the long exact sequence above implies that the group $H_2(G)$ is different from zero and that the group $H_3(G)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}/2 \times \mathbb{Z}/4$. Now Lemma 5.4 and the long exact sequence (5.5) above imply

Theorem 5.6 *For $G = \mathrm{Hei} \rtimes \mathbb{Z}/4$ we have isomorphisms*

$$\begin{aligned} H_n(G) &= \mathbb{Z}/2 \times \mathbb{Z}/4 \text{ for } n \geq 1, n \neq 2,3; \\ H_2(G) &= \mathbb{Z}/2; \\ H_3(G) &= \mathbb{Z} \times \mathbb{Z}/2 \times \mathbb{Z}/4. \end{aligned}$$

The map $H_3(\mathrm{Hei}) \to H_3(G)/\operatorname{tors}(H_3(G))$ is an isomorphism.

One can compute the group cohomology analogously or derive it from the homology by the universal coefficient theorem.

6 Survey over other extensions

There are other prominent extensions of Hei which can be treated analogously to the case $\mathrm{Hei} \rtimes \mathbb{Z}/4$. We give a brief summary of the topological K–theory and the algebraic K–theory below. In all cases $G\backslash \underline{E}G$ is S^3.

6.1 Order six symmetry

Consider the following automorphism $\omega\colon \mathrm{Hei} \to \mathrm{Hei}$ of order 6 which sends u to v, v to $u^{-1}v$, and z to z.

Theorem 6.1 *For the group*

$$\begin{aligned} G = \mathrm{Hei} \rtimes \mathbb{Z}/6 = \langle u,v,z,t \mid [u,v] = z, t^6 = 1, [u,z] = [v,z] = [t,z] = 1, \\ tut^{-1} = v, tvt^{-1} = u^{-1}v \rangle \end{aligned}$$

there is a short exact sequence

$$0 \to \widetilde{R}_{\mathbb{C}}(\langle t \rangle) \to K_1(C_r^*(G)) \to \widetilde{K}_1(S^3) \to 0$$

and an isomorphism

$$R_{\mathbb{C}}(\langle t \rangle) \xrightarrow{\cong} K_0(C_r^*(G)).$$

There are isomorphisms

$$\mathrm{Wh}_n(G) \cong \begin{cases} NK_1(\mathbb{Z}[\mathbb{Z}/6]) \bigoplus NK_1(\mathbb{Z}[\mathbb{Z}/6]) & \text{for } n = 1; \\ \mathrm{Wh}_{-1}(\mathbb{Z}/6) \cong \mathbb{Z} & \text{for } n = -1, 0; \\ 0 & \text{for } n \leq -2. \end{cases}$$

6.2 Order three symmetry

Next we deal with the $\mathbb{Z}/3$–action on Hei given by ω^2, where ω is the automorphism of order six investigated in Subsection 6.1.

Theorem 6.2 *For the group*

$$G = \text{Hei} \rtimes \mathbb{Z}/3 = \langle u, v, z, t \mid [u,v] = z, t^3 = 1, [u,z] = [v,z] = [t,z] = 1, \\ tut^{-1} = u^{-1}v, tvt^{-1} = u^{-1}z^{-1} \rangle$$

there is a short exact sequence

$$0 \to \widetilde{R}_{\mathbb{C}}(\langle t \rangle) \to K_1(C_r^*(G)) \to \widetilde{K}_1(S^3) \to 0$$

and an isomorphism

$$R_{\mathbb{C}}(\langle t \rangle) \xrightarrow{\cong} K_0(C_r^*(G)).$$

We have $\mathrm{Wh}_n(G) = 0$ *for* $n \leq 2$.

The L*–groups* $L_n\epsilon(\mathbb{Z}G)$ *are independent of the choice of decoration* $\epsilon = -\infty$, p, h, s *and the reduced ones fit into a short split exact sequence*

$$0 \;\to\; \widetilde{L}_n^{\langle -\infty \rangle}(\mathbb{Z}\langle t \rangle) \bigoplus \widetilde{L}_{n-1}^{\langle -\infty \rangle}(\mathbb{Z}\langle t \rangle) \;\to\; \widetilde{L}_n^{\langle -\infty \rangle}(\mathbb{Z}G) \;\to\; L_{n-3}^{\langle -\infty \rangle}(\mathbb{Z}) \;\to\; 0.$$

6.3 Order two symmetry

Next we deal with the $\mathbb{Z}/2$–action on Hei given by $u \mapsto u^{-1}$, $v \mapsto v^{-1}$ and $z \mapsto z$. This is the square of the automorphism of order four used in the $\mathbb{Z}/4$–case.

Theorem 6.3 *For the group*

$$G = \text{Hei} \rtimes \mathbb{Z}/2 = \langle u, v, z, t \mid [u,v] = z, t^2 = 1, [u,z] = [v,z] = [t,z] = 1, \\ tut^{-1} = u^{-1}, tvt^{-1} = v^{-1} \rangle$$

there is a short exact sequence

$$0 \to \oplus_{i=0}^2 \widetilde{R}_{\mathbb{C}}(M_i) \to K_1(C_r^*(G)) \to \widetilde{K}_1(S^3) \to 0$$

and an isomorphism

$$0 \to K_0(\{*\}) \bigoplus \bigoplus_{i=0}^{2} \widetilde{R}_{\mathbb{C}}(M_i) \xrightarrow{\cong} K_0(C_r^*(G)),$$

where

$$\begin{aligned} M_0 &= \langle t \rangle; \\ M_1 &= \langle ut \rangle; \\ M_2 &= \langle vt \rangle. \end{aligned}$$

We have $\mathrm{Wh}_n(G) = 0$ *for* $n \leq 2$.

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