ISSN 1364-0380 (on line) 1465-3060 (printed) **1881**

Geometry & Topology
Volume 9 (2005) 1881–1913
Published: 6 October 2005

Toward a general theory of linking invariants

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Abstract

Let N_1, N_2, M be smooth manifolds with $\dim N_1 + \dim N_2 + 1 = \dim M$ and let ϕ_i, for $i = 1, 2$, be smooth mappings of N_i to M where $\operatorname{Im}\phi_1 \cap \operatorname{Im}\phi_2 = \emptyset$. The classical linking number $\operatorname{lk}(\phi_1, \phi_2)$ is defined only when $\phi_{1*}[N_1] = \phi_{2*}[N_2] = 0 \in H_*(M)$.

The affine linking invariant alk is a generalization of lk to the case where $\phi_{1*}[N_1]$ or $\phi_{2*}[N_2]$ are not zero-homologous. In [7] we constructed the first examples of affine linking invariants of nonzero-homologous spheres in the spherical tangent bundle of a manifold, and showed that alk is intimately related to the causality relation of wave fronts on manifolds. In this paper we develop the general theory.

The invariant alk appears to be a universal Vassiliev–Goussarov invariant of order ≤ 1. In the case where $\phi_{1*}[N_1] = \phi_{2*}[N_2] = 0 \in H_*(M)$, it is a splitting of the classical linking number into a collection of independent invariants.

To construct alk we introduce a new pairing μ on the bordism groups of spaces of mappings of N_1 and N_2 into M, not necessarily under the restriction $\dim N_1 + \dim N_2 + 1 = \dim M$. For the zero-dimensional bordism groups, μ can be related to the Hatcher–Quinn invariant. In the case $N_1 = N_2 = S^1$, it is related to the Chas–Sullivan string homology super Lie bracket, and to the Goldman Lie bracket of free loops on surfaces.

AMS Classification numbers Primary: 57R19

Secondary: 14M07, 53Z05, 55N22, 55N45, 57M27, 57R40, 57R45, 57R52

Keywords: Linking invariants, winding numbers, Goldman bracket, wave fronts, causality, bordisms, intersections, isotopy, embeddings

Proposed: Steve Ferry — Received: 30 January 2004
Seconded: Ralph Cohen, Leonid Polterovich — Revised: 20 September 2005

1 Introduction

In this paper the word "smooth" means C^∞. Throughout this paper M is a smooth connected oriented manifold (not necessarily compact), and N_1, N_2 are smooth oriented closed manifolds. The dimensions of M, N_1, N_2 are denoted by m, n_1, n_2, respectively, and the one-point space is denoted by pt.

Let $\mathcal{N}_i$, for $i = 1, 2$, be a path-connected component of the space of all smooth mappings of N_i to M. (Thus the mappings in $\mathcal{N}_i, i = 1, 2$, are not assumed to be immersions.) Let $\mathcal{B} = \mathcal{B}_{\mathcal{N}_1,\mathcal{N}_2}$ be the space of quadruples $(\phi_1, \phi_2, \rho_1, \rho_2)$ where $\phi_i\colon N_i \to M, i = 1, 2$, belong to $\mathcal{N}_i$ and $\rho_i\colon \mathrm{pt} \to N_i$ are such that $\phi_1\rho_1 = \phi_2\rho_2$. Clearly, $\mathcal{B}$ can be regarded as a subset of $\mathcal{N}_1 \times \mathcal{N}_2 \times N_1 \times N_2$, and we equip $\mathcal{B}$ with the subspace topology.

The classical linking number lk is a $\mathbb{Z}$–valued invariant of a pair $(\phi_1, \phi_2) \in \mathcal{N}_1 \times \mathcal{N}_2$ with $n_1 + n_2 + 1 = m$ (and with $\phi_1(N_1) \cap \phi_2(N_2) = \emptyset$). The invariant $\mathrm{lk}(\phi_1, \phi_2)$ is defined only if $\phi_{1*}([N_1]), \phi_{2*}([N_2]) = 0 \in H_*(M)$ (or if $\phi_{1*}([N_1]), \phi_{2*}([N_2])$ are torsion classes, in which case lk takes values in $\mathbb{Q}$ or $\mathbb{Q}/\mathbb{Z}$). U Kaiser [16] generalized linking numbers to the case of arbitrary submanifolds of the linking dimension that are homologous into a boundary or into an end of the ambient manifold. (For M being the solid torus the similar approach to defining linking numbers was previously used by S Tabachnikov [30].)

The main goal of the paper is to construct a version of the linking invariant lk for pairs $(\phi_1, \phi_2) \in \mathcal{N}_1 \times \mathcal{N}_2$ with $n_1 + n_2 + 1 = m$ and $\phi_1(N_1) \cap \phi_2(N_2) = \emptyset$ without any restrictions on the homology classes $\phi_{1*}([N_1]), \phi_{2*}([N_2])$.

In greater detail, let $\Sigma \in \mathcal{N}_1 \times \mathcal{N}_2$ be the subset of pairs (ϕ_1, ϕ_2) with $\phi_1(N_1) \cap \phi_2(N_2) \neq \emptyset$. Fixing a pair $* \in \mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma$, we define an invariant

$$\mathrm{alk}\colon\ \mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma \to H_0(\mathcal{B})/\,\mathrm{Indet}$$

which is an invariant under link homotopy of pairs (ϕ_1, ϕ_2); here Indet is a certain indeterminacy subgroup. We call alk the *affine* linking invariant, since the change of the base point $*$ leads to changing of alk by an additive constant.

It turns out that the augmentation $\mathcal{B} \to \mathrm{pt}$ reduces our invariants to the classical ones (ie the linking numbers with values in $H_0(\mathrm{pt}) = \mathbb{Z}$) provided that the last ones are defined. In other words, here we have a splitting of the classical linking invariant.

Our constructions can be easily modified to yield affine linking type invariants under the *singular concordance* relation. In the case of 1–links in 3–manifolds this will give us the invariants constructed by Schneiderman in [26].

Our construction can be rather easily modified to give an invariant of a pair $(\phi_1, \phi_2) \in \mathcal{N}_1 \times \mathcal{N}_2$ with $\phi_1(N_1) \cap \phi_2(N_2) = \emptyset$ and without any restrictions on the dimensions n_1, n_2, m of N_1, N_2, M. In this case we also get an invariant of a link (ϕ_1, ϕ_2) considered up to the link homotopy. The invariant takes values in $\Omega_{n_1+n_2+1-m}(\mathcal{B})/\big(\operatorname{Im}\mu_{1,0} + \operatorname{Im}\mu_{0,1}\big)$, where $\mu_{i,j}\colon\ \Omega_i(\mathcal{N}_1) \otimes \Omega_j(\mathcal{N}_2) \to \Omega_{i+j+n_1+n_2-m}(\mathcal{B})$ is the pairing defined in Theorem 2.2. We plan to study this more general invariant in detail in another paper.

Most of our results are based on considering of a helpful pairing

$$\mu_{i,j}\colon\ \Omega_i(\mathcal{N}_1) \otimes \Omega_j(\mathcal{N}_2) \to \Omega_{i+j+n_1+n_2-m}(\mathcal{B})$$

where $\Omega_*(X)$ is the group of oriented bordisms of X. This pairing has many remarkable properties. For example:

(1) The pairings $\mu_{1,0}, \mu_{0,1}$ enable us to describe the indeterminacy for the invariant alk. (Note that $H_0(X) = \Omega_0(X)$ for all X.)

(2) The pairing $\mu_{0,0}$ tells us (in many cases) whether two C^∞ maps $f_1\colon\ N_1 \to M$ and $f_2\colon\ N_2 \to M$ can be deformed to maps with disjoint images, see Section 8. The case when two *immersions* f_1 and f_2 can be *regularly* homotoped to maps with disjoint images was considered by Hatcher and Quinn [15]. Concerning relations between $\mu_{0,0}$ and the Hatcher–Quinn invariant, see subsection 2.2. A conicidence problem for the case $N_1 = N_2$ was considered by Koschorke [18] via the approach of Hatcher–Quinn invariants.

(3) If $N_1 = N_2 = S^n$ and M is a $2n$–manifold then μ leads to a generalization of the Goldman bracket [12] of free loops on 2–surfaces, see subsection 2.1.

(4) In case of $N_1 = N_2 = S^1$ the pairing μ leads to a (graded) Lie algebra structure on $\Omega_*(\overline{\mathcal{N}})$ where $\overline{\mathcal{N}} = \overline{\mathcal{N}}_1 = \overline{\mathcal{N}}_2$ is the union of all the connected components of the space of mappings $S^1 \to M$. This Lie algebra structure is related to the string homology Lie bracket introduced by Chas and Sullivan [4], [5], cf also the work of A Voronov [36]. We are not able to discuss this algebra in detail here but intend to do it in the coming development of our work [9].

(5) In fact the mapping μ extends to a Lie bracket on the nonoriented bordism groups of mappings into M of garlands glued out of arbitrary manifolds. It also seems that for the appropriately chosen signs μ extends to a (super) Lie bracket even for oriented bordism groups, but we are still computing the appropriate signs in the graded Jacobi identity, see [9, Theorem 3.1].

The paper is organized as follows. In Section 2 we introduce the pairing μ. In Sections 3 and 4 we define affine linking invariants of pairs $(\phi_1, \phi_2) \in \mathcal{N}_1 \times \mathcal{N}_2$

(with $f_1(N_1) \cap f_2(N_2) = \emptyset$) as elements of the group $\Omega_0(\mathcal{B})$ modulo certain indeterminacy; the last one is described in terms of the pairing μ. In Section 5 we prove that the augmentation $\Omega_0(\mathcal{B}) = H_0(\mathcal{B}) \to H_0(\mathrm{pt}) = \mathbb{Z}$ induced by the mapping $\mathcal{B} \to \mathrm{pt}$ reduces our invariants to the classical linking invariant, when the last one is well-defined. In Section 6 we give conditions that guarantee the vanishing of the indeterminacy. In Section 7 we give an explicit description of $\pi_0(\mathcal{B})$ and $\Omega_0(\mathcal{B})$. In Section 8 we show that, in many cases, the pair $\bar{f}_1\colon N_1 \to M$, $\bar{f}_2\colon N_2 \to M$ of maps, with $n_1 + n_2 = m$ is homotopic to a pair (f_1, f_2) with disjoint images, $f_1(N_1) \cap f_2(N_2) = \emptyset$, if and only if the pairing μ takes the zero value on $(\bar{f}_1, \bar{f}_2)$. If $\bar{f}_1$ and $\bar{f}_2$ are homotopic to immersions, then the results of Section 8 follow immediately from Theorem 2.2 of Hatcher–Quinn [15].

Acknowledgements The first author was partially supported by the Walter and Constance Burke Research Initiation Award. The second author was partially supported by NSF, grant 0406311, and by MCyT, projects BFM 2002-00788 and MCyT BFM2003-02068/MATE, Spain; his visit to Dartmouth College was supported by the funds donated by Edward Shapiro to the Mathematics Department of Dartmouth College.

We are also grateful to the anonymous referee for the very valuable comments and suggestions.

2 The pairing $\mu_{i,j}\colon \Omega_i(\mathcal{N}_1) \otimes \Omega_j(\mathcal{N}_2) \to \Omega_{i+j+n_1+n_2-m}(\mathcal{B})$

In this section we do not assume that $\dim N_1 + \dim N_2 + 1 = \dim M$.

Given a space X, we denote by $\Omega_n(X)$ the n–dimensional oriented bordism group of X. Recall that $\Omega_n(X)$ is the set of the equivalence classes of (continuous) maps $f\colon V^n \to X$ where V is a closed oriented manifold. Here two maps $f_1\colon V_1 \to X$ and $f_2\colon V_2 \to X$ are equivalent if there exists a map $g\colon W^{n+1} \to X$, where W is a compact oriented manifold whose oriented boundary ∂W is diffeomorphic to $V_1 \sqcup (-V_2)$ and $g|_{\partial W} = f_1 \sqcup f_2$. Furthermore, the operation of disjoint union converts $\Omega_n(X)$ into an abelian group. See [25, 28, 29] for details.

Let $[V] \in H_n(V)$ be the fundamental class of a closed oriented n–dimensional manifold V. Every map $f\colon V \to X$ gives us an element $f_*[V] \in H_n(X)$, and the correspondence $(V, f) \mapsto f_*[V]$ yields the Steenrod–Thom homomorphism

$$\tau\colon \Omega_n(X) \to H_n(X).$$

It turns out [31] that this homomorphism is an isomorphism for $n \leq 3$ and an epimorphism for $n \leq 6$, see [25, 28] for modern proofs.

Let $\alpha_1\colon F_1 \to \mathcal{N}_1$ be a mapping representing $[\alpha_1] \in \Omega_i(\mathcal{N}_1)$ and let $\alpha_2\colon F_2 \to \mathcal{N}_2$ be a mapping representing $[\alpha_2] \in \Omega_j(\mathcal{N}_2)$. Let $\widetilde{\alpha}_i\colon F_i \times N_i \to M$, $i = 1, 2$, be such that $\widetilde{\alpha}_i(f, n) = (\alpha_i(f))(n)$. Let $v_1 \in F_1 \times N_1$ and $v_2 \in F_2 \times N_2$ be such that $\widetilde{\alpha}_1(v_1) = \widetilde{\alpha}_2(v_2)$. We say that $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ are *transverse* at (v_1, v_2) if $d\widetilde{\alpha}_2\big(T_{v_2}(F_2 \times N_2)\big)$ and $d\widetilde{\alpha}_1\big(T_{v_1}(F_1 \times N_1)\big)$ generate $T_{\widetilde{\alpha}_1(v_1)}M = T_{\widetilde{\alpha}_2(v_2)}M$. Following standard arguments we can assume that $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ are transverse, ie they are transverse at all (v_1, v_2) such that $\widetilde{\alpha}_1(v_1) = \widetilde{\alpha}_2(v_2)$.

Consider the pullback diagram

$$\begin{array}{ccc} V & \xrightarrow{j_1} & F_1 \times N_1 \\ \big\downarrow{\scriptstyle j_2} & & \big\downarrow{\scriptstyle \widetilde{\alpha}_1} \\ F_2 \times N_2 & \xrightarrow{\widetilde{\alpha}_2} & M \end{array} \tag{2.1}$$

of the maps $\widetilde{\alpha}_i$, for $i = 1, 2$.

Lemma 2.1 *If $\widetilde{\alpha}_1, \widetilde{\alpha}_2$ are transverse, then V is a closed oriented $(i{+}j{+}n_1{+}n_2{-}m)$–dimensional manifold.* □

Let $p_1\colon F_i \times N_i \to F_i$ and $p_2\colon F_i \times N_i \to N_i$, for $i = 1, 2$, be the obvious projections. Consider the mapping

$$\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2)\colon V \to \mathcal{B}, \quad v \mapsto \big(\phi_1^v, \phi_2^v, \rho_1^v, \rho_2^v\big)$$

where $\phi_i^v(n) = \widetilde{\alpha}_i(p_1(j_i(v)), n)$ for $n \in N_i$, and $\rho_i^v(\mathrm{pt}) = p_2(j_i(v))$.

Theorem 2.2 *The above construction yields a well-defined pairing*

$$\begin{aligned} \mu = \mu_{ij}\colon \Omega_i(\mathcal{N}_1) \otimes \Omega_j(\mathcal{N}_2) &\to \Omega_{i+j+n_1+n_2-m}(\mathcal{B}), \\ \mu\left([\alpha_1], [\alpha_2]\right) &= [V, \mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2)]. \end{aligned}$$

Proof The routine argument shows that the bordism class $[V, \mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2)]$ in $\Omega_{i+j+n_1+n_2-m}(\mathcal{B})$ depends only on $[\alpha_1] \in \Omega_i(\mathcal{N}_1)$, $[\alpha_2] \in \Omega_j(\mathcal{N}_2)$. The bilinearity of μ is obvious. □

2.1 Relation to the Goldman bracket

Let $\widetilde{\mathcal{N}}_1$ (respectively, $\widetilde{\mathcal{N}}_2$) be the topological space that is the union of all the connected components of the space of mappings of N_1 (respectively, N_2) into M. Let $\widetilde{\mathcal{B}}$ be the topological space that is the union of $\mathcal{B}_{\mathcal{N}_1,\mathcal{N}_2}$ over all the connected components $\mathcal{N}_1, \mathcal{N}_2$.

Let $\overline{\mathcal{N}}_i$ be the closure of $\widetilde{\mathcal{N}}_i$ in the space of all continuous maps. Similarly, define $\overline{\mathcal{B}}$ to be the closure of $\widetilde{\mathcal{B}}$.

Clearly, the pairing μ can be extended to

$$\overline{\mu}\colon\ \Omega_i(\overline{\mathcal{N}}_1)\otimes\Omega_j(\overline{\mathcal{N}}_2)\to\Omega_{i+j+n_1+n_2-m}(\overline{\mathcal{B}}).$$

Let M be a $2n$–dimensional manifold and let $N_1 = N_2 = S^n$. Given two points $x, y \in S^n$ (not necessarily distinct), choose an orientation preserving diffeomorphism $u_{x,y}\colon S^n \to S^n$ that maps x to y.

For M^{2n} and $N_1 = N_2 = S^n$ put $\overline{\mathcal{N}} = \overline{\mathcal{N}}_1 = \overline{\mathcal{N}}_2$ and consider the pairing

$$\overline{\mu}\colon\ \Omega_0(\overline{\mathcal{N}})\otimes\Omega_0(\overline{\mathcal{N}})\to\Omega_0(\overline{\mathcal{B}}). \tag{2.2}$$

Since $\Omega_0 = H_0$, the pairing (2.2) yields a pairing

$$H_0(\overline{\mathcal{N}})\otimes H_0(\overline{\mathcal{N}}) \xrightarrow{\ \overline{\mu}\ } H_0(\overline{\mathcal{B}}).$$

Every point $b = (\phi_1, \phi_2, \rho_1, \rho_2) \in \overline{\mathcal{B}}$ gives us a map $h_b\colon S^n \vee S^n \to M$ as follows. We regard the sphere S^n as a pointed space with the base point $*$. Clearly, both maps $\phi_i u_{*,\rho_i(\mathrm{pt})}\colon S^n \to M$, for $i = 1, 2$, map the point $*$ to the same point of M and therefore yield the map $S^n \vee S^n \to M$.

The pairing $\mu\colon H_0(\overline{\mathcal{N}})\otimes H_0(\overline{\mathcal{N}}) \to H_0(\overline{\mathcal{B}})$ has the following interpretation. Put $\widehat{\pi}_n$ to be the orbits of $\pi_n(M)$ under the natural action of $\pi_1(M)$. Then $H_0(\overline{\mathcal{N}})$ is naturally identified with $\mathbb{Z}\widehat{\pi}_n$, a free $\mathbb{Z}$–module over $\widehat{\pi}_n$. Observe that any two modules $\mathbb{Z}\widehat{\pi}_n(M, p)$ and $\mathbb{Z}\widehat{\pi}_n(M, q)$ are canonically isomorphic

Let $s_i\colon S^n \to M^{2n}$, for $i = 1, 2$, be two smooth generic mappings transverse to each other and realizing $[s_i] \in \widehat{\pi}_n$. Here genericity means that each intersection point of s_1 and s_2 is *not* a self-intersection point of s_1 or s_2. Put $P = \mathrm{Im}(s_1) \cap \mathrm{Im}(s_2)$. Then

$$\overline{\mu}(1[s_1]\otimes 1[s_2]) = \sum_{p\in P}\mathrm{sign}(p)[s_1, s_2, p], \tag{2.3}$$

where $\operatorname{sign} p$ is the natural orientation of p coming from the intersection pairing. Here $[s_1, s_2, p] \in \pi_0(\overline{\mathcal{B}})$ is the element that maps the first sphere as s_1, the second sphere as s_2, and maps pt to the preimages of p on the two spheres.

Now, using the coproduct $S^n \to S^n \vee S^n$, we conclude that every $b \in \overline{\mathcal{B}}$ gives us a map

$$S^n \longrightarrow S^n \vee S^n \xrightarrow{h_b} M$$

So, we get a map $\varphi\colon \overline{\mathcal{B}} \to \overline{\mathcal{N}}$. Notice that this map is *not* continuous, but it induces a well-defined map $\pi_0(\overline{\mathcal{B}}) \to \pi_0(\overline{\mathcal{N}})$ since each $u_{x,y}$ is homotopic to the identity. Furthermore, since $\Omega_0 = H_0$, the pairing (2.2) yields a pairing

$$\alpha\colon H_0(\overline{\mathcal{N}}) \otimes H_0(\overline{\mathcal{N}}) \xrightarrow{\overline{\mu}} H_0(\overline{\mathcal{B}}) \xrightarrow{\varphi_*} H_0(\overline{\mathcal{N}}).$$

Now, because of the equality (2.3), we conclude that

$$\alpha(1[s_1] \otimes 1[s_2]) = \Big(\sum_{p \in P} \operatorname{sign}(p)[s_1 s_2 \in \pi_n(M,p)]\Big) \in \mathbb{Z}\widehat{\pi}_n. \tag{2.4}$$

Here the element $[s_1 s_2 \in \pi_n(M,p)]$ is the class in $\widehat{\pi}_n$ that is the product of the elements of $\pi_n(M,p)$ realized by s_1 and s_2. More accurately, we have to compose s_i with an automorphism u_{*,x_i} of S^n that maps the base point $*$ to the preimage $x_i = s_i^{-1}(p)$ of p.

For $n = 1$ the action of $\pi_1(M)$ on $\pi_n(M) = \pi_1(M)$ is given by the conjugation. So $\mathbb{Z}\widehat{\pi}_n = \mathbb{Z}\widehat{\pi}_1$ is a free $\mathbb{Z}$–module generated by the homotopy classes of free loops on M. Because of the explanations above $[s_1 s_2 \in \pi_n(M,p)]$ is a well-defined element of $\mathbb{Z}\widehat{\pi}_1$.

Now formula (2.4) is identical to the definition of Goldman's Lie bracket on $\mathbb{Z}\widehat{\pi}_1(M^2)$, see Goldman [12, page 267]. Since both α and Goldman's Lie bracket are bilinear and coincide on the generators of $\mathbb{Z}\widehat{\pi}_1$, they are equal. Thus for $N_1 = N_2 = S^1$ and M^2 our pairing μ generalizes Goldman's Lie bracket discovered in stages by Goldman [12] and Turaev [32].

2.2 Relation to the Hatcher–Quinn invariant

Consider two maps $f\colon N_1 \to M$ and $g\colon N_2 \to M$ in $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively. Hatcher and Quinn [15] considered the homotopy pullback diagram

$$\begin{array}{ccc} E(f,g) & \xrightarrow{f_E} & N_1 \\ {\scriptstyle g_E}\downarrow & & \downarrow{\scriptstyle f} \\ N_2 & \xrightarrow{g} & M \end{array}$$

For $i = 1, 2$, let ν_i be the stable normal vector bundle over N_i and let ξ be the stable vector bundle $f_E^*\nu_1 \oplus g_E^*\nu_2 \oplus f_E^* f^* \tau_M$. Let E denote $E(f,g)$ and let

$\Omega_k^{\mathrm{fr}}(E,\xi)$ denote the bordism group based on singular manifolds $h\colon V^k \to E$ together with a stable bundle isomorphism of the normal bundle of V with $h^*\xi$. The $\Omega_k^{\mathrm{fr}}(E,\xi)$ groups are the homotopy groups of the Thom spectrum $T\xi$.

Given two transverse maps $f_1\colon N_1 \to M$ and $f_2\colon N_2 \to M$ homotopic to f and g respectively, consider the pullback V (rather than the homotopy pullback) of f_1 and f_2. Consider the obvious maps $v_i\colon V \to N_i$ and construct a map $h\colon V \to E$ with $f_E \circ h = v_1$ and $g_E \circ h = v_2$. In fact, this h determines and is determined by homotopies from f_1 to f and from f_2 to g. Then (V,h) yields an element of $\Omega_{n_1+n_2-m}^{\mathrm{fr}}(E,\xi)$, and this element does not depend on the choice of the above described homotopies. So, we have the Hatcher–Quinn map

$$\mathrm{hq}\colon \pi_0(\mathcal{N}_1)\times\pi_0(\mathcal{N}_2) \longrightarrow \Omega_{n_1+n_2-m}^{\mathrm{fr}}(E,\xi).$$

Now, assume that $n_1+n_2=m$. Take a point of $\mathcal{B}$ and represent it by a commutative diagram

$$\begin{array}{ccc} \mathrm{pt} & \longrightarrow & N_1 \\ \big\downarrow & & \big\downarrow{\scriptstyle f_1} \\ N_2 & \xrightarrow{f_2} & M \end{array}$$

with f_1, f_2 transverse and $f_1 \simeq f$, $f_2 \simeq g$. Clearly, this diagram gives us an element of the group $\Omega_0^{\mathrm{fr}}(E,\xi)$, and in fact we have a map $\varphi\colon \Omega_0(\mathcal{B}) \to \Omega_0^{\mathrm{fr}}(E,\xi)$. It is easy to see that the diagram

$$\begin{array}{ccc} \pi_0(\mathcal{N}_1)\times\pi_0(\mathcal{N}_2) & \xrightarrow{\mathrm{hq}} & \Omega_0^{\mathrm{fr}}(E,\xi) \\ \big\downarrow & & \big\uparrow{\scriptstyle\varphi} \\ \Omega_0(\mathcal{N}_1)\otimes\Omega_0(\mathcal{N}_2) & \xrightarrow{\mu_{0,0}} & \Omega_0(\mathcal{B}) \end{array}$$

commutes. So in the case $n_1+n_2=m$ the pairing $\mu_{0,0}$ can be regarded as a version of the Hatcher–Quinn invariant.

3 Affine linking invariants

From here and till the end of the paper we assume that $n_1+n_2+1=m$ (unless the opposite is explicitly stated).

Put Σ to be the discriminant in $\mathcal{N}_1\times\mathcal{N}_2$, ie the subspace that consists of pairs (f_1,f_2) such that there exist $y_1\in N_1, y_2\in N_2$ with $f_1(y_1)=f_2(y_2)$. (We do not include into Σ the maps that are singular in the common sense but do not involve double points between $f_1(N_1)$ and $f_2(N_2)$.)

Definition 3.1 We define Σ_0 to be the subset (stratum) of Σ consisting of all the pairs (f_1, f_2) such that there exists precisely one pair (y_1, y_2) of points $y_1 \in N_1, y_2 \in N_2$ such that

(a) $f_1(y_1) = f_2(y_2)$;

(b) y_i is a regular point of f_i, for $i = 1, 2$;

(c) $(df_1)(T_{y_1}N_1) \cap (df_2)(T_{y_2}N_2) = 0$.

Construction 3.2 Let $\gamma\colon (a,b) \to \mathcal{N}_1 \times \mathcal{N}_2$ be a path which intersects Σ_0 in a point $\gamma(t_0)$. We also assume that

$$\gamma(t_0 - \delta, t_0 + \delta) \cap \Sigma = \gamma(t_0)$$

for δ small enough. We construct a vector $\mathbf{v} = \mathbf{v}(\gamma, t_0, \delta)$ as follows. We regard $\gamma(t_0)$ as a pair $(f_1, f_2) \in \mathcal{N}_1 \times \mathcal{N}_2$ and consider the points y_1, y_2 as in Definition 3.1. Set $z = f_1(y_1) = f_2(y_2)$. Choose a small $\delta > 0$ and regard $\gamma(t_0 + \delta)$ as a pair $(g_1, g_2) \in \mathcal{N}_1 \times \mathcal{N}_2$. Set $z_i = g_i(y_i), i = 1, 2$. Take a chart for M that contains z and z_i, $i = 1, 2$, and set

$$\mathbf{v}(\gamma, t_0, \delta) := \overrightarrow{zz_1} - \overrightarrow{zz_2} \in T_zM.$$

Definition 3.3 Let $\gamma\colon (a,b) \to \mathcal{N}_1 \times \mathcal{N}_2$ be a path as in Construction 3.2. We say that γ *intersects* Σ_0 *transversally for* $t = t_0$ if there exists $\delta_0 > 0$ such that

$$\mathbf{v}(\gamma, t_0, \delta) \notin (df_1)(T_{y_1}N_1) \oplus (df_2)(T_{y_2}N_2) \subset T_zM$$

for all $\delta \in (0, \delta_0)$.

It is easy to see that the concept of transverse intersection does not depend on the choice of the chart.

Definition 3.4 A path $\gamma\colon (a,b) \to \mathcal{N}_1 \times \mathcal{N}_2, -\infty \le a < b \le \infty$ is said to be *generic* if

(a) $\gamma(a,b) \cap \Sigma = \gamma(a,b) \cap \Sigma_0$;

(b) the set $J = \{t | \gamma(t) \cap \Sigma_0 \neq \emptyset\} \subset (a,b)$ is an isolated subset of $\mathbb{R}$;

(c) the path γ intersects Σ_0 transversally for all $t \in J$.

As one can expect, every path can be turned into a generic one by a small deformation. We leave the proof to the reader.

Definition 3.5 Let γ be a path in $\mathcal{N}_1 \times \mathcal{N}_2$ that intersects Σ transversally in one point $\gamma(t_0) \in \Sigma_0$. We associate a sign $\sigma(\gamma, t_0)$ to such a crossing as follows.

We regard $\gamma(t_0)$ as a pair $(f_1, f_2) \in \mathcal{N}_1 \times \mathcal{N}_2$ and consider the points $y_1 \in N_1, y_2 \in N_2$ such that $f_1(y_1) = f_2(y_2)$. Set $z = f_1(y_1) = f_2(y_2)$. Let $\mathfrak{r}_1$ and $\mathfrak{r}_2$ be frames which are tangent to N_1 and N_2 at y_1 and y_2, respectively, and both are assumed to be positive. Consider the frame

$$\{df_1(\mathfrak{r}_1), \mathbf{v}, df_2(\mathfrak{r}_2)\}$$

at $z \in M$, where $\mathbf{v}$ is a vector described in Construction 3.2. We put $\sigma(\gamma, t_0) = 1$ if this frame gives us the orientation of M, otherwise we put $\sigma(\gamma, t_0) = -1$. Because of the transversality and condition (c) from Definition 3.1, the family $\{df_1(\mathfrak{r}_1), \mathbf{v}, df_2(\mathfrak{r}_2)\}$ is indeed a frame. Note also that the vector $\mathbf{v}$ is not well-defined, but the above defined sign σ is.

Clearly if we traverse the path γ in the opposite direction then the sign of the crossing changes.

For every space X, the group $\Omega_0(X) = H_0(X)$ is the free abelian group with the base $\pi_0(X)$. So, every element of $\Omega_0(X)$ can be represented as a finite linear combination $\sum \lambda_k P_k$ with $\lambda_k \in \mathbb{Z}$ and $P_k \in X$, and every such linear combination gives us an element of $\Omega_0(X)$. Below in Section 7 we give examples of situations where $\pi_0(\mathcal{B})$ is an infinite set, in spite of the fact that $\mathcal{N}_1$ and $\mathcal{N}_2$ are path connected.

Definition 3.6 Let γ be a path in $\mathcal{N}_1 \times \mathcal{N}_2$ that intersects Σ transversally in one point $\gamma(t_0) \in \Sigma_0$. We define $[\gamma(t_0)] \in \Omega_0(\mathcal{B})$ as $\sigma(t_0)\gamma(t_0)$.

Clearly,

$$\varepsilon([\gamma(t_0])) = \sigma(\gamma, t_0), \tag{3.1}$$

where $\varepsilon\colon \Omega_0(\mathcal{B}) \to \mathbb{Z} = \Omega_0(\mathrm{pt})$ is the homomorphism induced by the map $\mathcal{B} \to \mathrm{pt}$.

Definition 3.7 Since $\Omega_i(X) = H_i(X)$ for $i = 0, 1$, by the Künneth formula we have the canonical isomorphism

$$\Omega_1(X \times Y) = \Omega_1(X) \otimes \Omega_0(Y) \oplus \Omega_0(X) \otimes \Omega_1(Y).$$

Now, the pairings

$$\mu_{10}\colon\ \Omega_1(\mathcal{N}_1) \otimes \Omega_0(\mathcal{N}_2) \to \Omega_0(\mathcal{B}) \text{ and } \mu_{01}\colon\ \Omega_0(\mathcal{N}_1) \otimes \Omega_1(\mathcal{N}_2) \to \Omega_0(\mathcal{B})$$

yield the homomorphism

$$\lambda\colon\ \Omega_1(\mathcal{N}_1 \times \mathcal{N}_2) \to \Omega_0(\mathcal{B}).$$

We define the *indeterminacy subgroup* $\mathrm{Indet} \subset \Omega_0(\mathcal{B})$ to be the image of the homomorphism λ. We define the abelian group $\mathbb{A} = \mathbb{A}(\mathcal{N}_1, \mathcal{N}_2)$ to be the quotient group of $\Omega_0(\mathcal{B})/\mathrm{Indet}$. Let $q = q_{\mathcal{N}_1,\mathcal{N}_2}\colon \Omega_0(\mathcal{B}) \to \mathbb{A}$ be the corresponding quotient homomorphism.

Examples 3.8 (1) Consider the case of M^3 being a lens space with a fundamental group $\mathbb{Z}_p$, $N_1 = S^2, N_2 = \mathrm{pt}$. In this case there is only one homotopy class of mappings $S^2 \to M$ and of $\mathrm{pt} \to M$. It is easy to see that $\Omega_0(\mathcal{B}) = \mathbb{Z}$.

Take $f\colon S^2 \to M$ and $g\colon \mathrm{pt} \times S^1 = S^1 \to M$. Then $\mu_{0,1}(f,g)$ equals to the intersection index between $f_*[S^2]$ and $g_*[S^1] \in H_*(M)$. Since $H_1(M) = \mathbb{Z}_p$ we get that $\mathrm{Im}\,\mu_{0,1} = 0 \in \mathbb{Z} = \Omega_0(\mathcal{B})$.

Take $r\colon S^2 \times S^1 \to M$ and $s\colon \mathrm{pt} \to M$. Then $\mu_{1,0}(r,s) \in \mathbb{Z}$ equals to the degree of the mapping $r\colon S^2 \times S^1 \to M$. Since $\pi_2(M) = 0$, the elementary obstruction theory shows that for a given image of $r_*(\cdot \times S^1) \in \pi_1(M)$ all the homotopy classes of mappings $S^2 \times S^1 \to M$ are classified by $\pi_3(M)$. Thus $\mathrm{Im}\,\mu_{1,0}$ coincides with the possible degrees of mappings $S^3 \to M$. Since all such mappings pass through the covering map $S^3 \to M$ that has degree p, we get $\mathrm{Im}\,\mu_{1,0} = p\mathbb{Z} \subset \mathbb{Z}$. Thus $\mathrm{Indet} = p\mathbb{Z} \subset \mathbb{Z} = \Omega_0(\mathcal{B})$.

(2) A harder example comes from $M = F_g \times S^1$, where F_g is an oriented surface of genus $g > 1$. Let $N_1 = N_2 = S^1$ and let $\alpha, \beta\colon S^1 \to M$ be linked embedded circles that project to two simple curves on F_g with one intersection point. Let $\mathcal{N}_1, \mathcal{N}_2$ be the connected components of the space of mappings $S^1 \to M$ that contain α and β, respectively. Let $r : (S^1, *) \to (\mathcal{N}_1, \alpha)$ be a mapping with the adjoint $\hat{r} : S^1 \times S^1 \to M$, $\hat{r}_{|_{S^1 \times 1}} = \alpha$.

If $\ker\big(\hat{r}_* : \pi_1(S^1 \times S^1) \to \pi_1(M)\big) \neq 1$, then using obstruction theory we get that $\hat{r}$ is homotopic to a mapping that passes through a mapping $S^1 \to M$. This mapping of S^1 can be made disjoint from β and $\mu_{1,0}(r,\beta) = 0$ for such r.

If $\hat{r}_*$ is injective then r is homotopic to the loop $\gamma_1^i \gamma_3^j \in (\mathcal{N}_1, \alpha)$, for some $i, j \in \mathbb{Z}$. Here γ_1 is a self homotopy of α induced by one full rotation of the parameterizing circle and γ_3 is a self homotopy of α under which every point of α slides one full turn along the S^1 fiber of $F_g \times S^1 \to F_g$ that contains the point, see the proof of [6, Lemma 6.11]. Clearly $\mu_{1,0}(\gamma_1, \beta) = 0$ and $\mu_{1,0}(\gamma_3, \beta)$ equals to a ± 1 times the class of the element of $\mathcal{B}$ that is obtained when α intersects β under the deformation γ_3. Thus $\mathrm{Im}\,\mu_{1,0} = \mathbb{Z} \subset \Omega_0(\mathcal{B})$.

Similarly one shows that $\mathrm{Im}\,\mu_{0,1} = \mathbb{Z} \subset \Omega_0(\mathcal{B})$ and that in fact $\mathrm{Im}\,\mu_{1,0} = \mathrm{Im}\,\mu_{0,1}$. Thus $\mathrm{Indet} = \mathbb{Z}$ for this example. One can also show that $\Omega_0(\mathcal{B}) = \bigoplus_1^\infty \mathbb{Z}$ in this case.

Similar construction for $F_g = S^1 \times S^1$ will give $\mathrm{Indet} = \mathbb{Z} = \Omega_0(\mathcal{B})$.

Theorem-Definition 3.9 *Let* $\mathbb{A}$ *be as above. Then there exists a function* $\mathrm{alk}\colon\ \mathcal{N}_1\times\mathcal{N}_2\setminus\Sigma\to\mathbb{A}$ *such that*

(a) alk *is constant on path connected components of* $\mathcal{N}_1\times\mathcal{N}_2\setminus\Sigma$;

(b) *if* $\gamma\colon\ [a,b]\to\mathcal{N}_1\times\mathcal{N}_2$ *is a generic path such that* $\gamma(a),\gamma(b)\notin\Sigma$ *and* $t_i, i\in I$, *are the moments when* $\gamma(t_i)\in\Sigma$ (*and hence* $\gamma(t_i)\in\Sigma_0$ *by the definition of the generic path*), *then*

$$\mathrm{alk}(\gamma(b))-\mathrm{alk}(\gamma(a))=q\Big(\sum_{i\in I}[\gamma(t_i)]\Big)\in\mathbb{A}.$$

Moreover, these properties determine the function alk *uniquely up to an additive constant. We call such a function an* affine linking invariant.

We prove Theorem 3.9 in Section 4.

Remark 3.10 After the first version [10] of this text appeared on the electronic archive, U Koschorke submitted the paper [19] where he used some invariants coming from the Hatcher–Quinn construction to study when two submanifolds of $M=S\times\mathbb{R}$ can be link homotoped to the disjoint S–levels.

Remark 3.11 It is rather easy to prove (see Corollary 7.5) that the mapping $\mathrm{alk}\colon\ \pi_0(\mathcal{N}_1\times\mathcal{N}_2\setminus\Sigma)\to\mathbb{A}$ is always surjective.

3.1 Relations to front propagation and winding numbers

(a) Let STM denote the total space of the sphere tangent bundle over M, $\dim STM=2m-1$. In [7] we defined the affine linking invariant al for the mappings of $S^{m-1}\to STM$ that are homotopic to the inclusion of the fiber S^{m-1} to STM. Because of the orientability of the bundle $STM\to M$, the homotopy class of this inclusion is invariant under the $\pi_1(M)$–action on $[S^{m-1},STM]$. Using Theorem 7.4 we get that in this case $\Omega_0(\mathcal{B})=\mathbb{Z}$, and al is exactly alk for $N_1=N_2=S^{m-1}$ and the space $\mathcal{N}_1=\mathcal{N}_2$ consisting of mappings $S^{m-1}\to STM$ as above.

In [7] we have shown that in this case $\varepsilon(\mathrm{Indet})=\mathrm{Indet}=0$ when m is even or when m is odd and M is not a rational homology sphere. This shows that $\varepsilon(\mathrm{alk})$ can indeed be $\mathbb{Z}$–valued in many cases where the mappings are not zero-homologous.

This example is interesting since it is related to wave front propagation. Deep relations between front propagation and link theory were first discovered by

V Arnold [2], who observed that the isotopy type of the knot canonically associated to a front does not change under front propagation. A wave front on M is a singular spherical hypersurface equipped with a velocity vector field of the directions of the front propagation. The wave front represents the set of all points on M that a certain information has reached at the given moment. Wave fronts on M can be canonically lifted to STM by mapping a point of the front to the point of STM corresponding to the front velocity at this point. We assume that wave front propagation on M is given by a time-dependent flow on STM. (For example light propagation is indeed given by such a flow.)

Consider two wave fronts that originated at two different points on M. The value of the alk invariant on the canonical lifts to STM of the two wave fronts equals to the algebraic number of times the front of the earlier event has passed through the birth point of the later event before the later event occurred, see [7]. Thus if the alk invariant is nonzero we conclude that the later event occurred after the earlier born front passed through its birth point. Philosophically this means that the later event has obtained the information about the earlier event carried by its wave front. Such events are called causally related. Thus the alk invariant often allows one to detect that the events that created the wave fronts are causally related from the current picture of the wave front, without the knowledge of the time-dependent propagation law, times and places of events that created the wave fronts.

First examples relating causality to the linking number were constructed by R Low in the case of $M = \mathbb{R}^n$, see [20], [21], [22]. In this case the canonical lifts of the fronts are homologous into the end of $ST\mathbb{R}^n$. For such submanifolds the classical linking number can be defined as it was done by S Tabachnikov [30] for $M = \mathbb{R}^3$. (This construction of the classical linking number was later generalized by U Kaiser [16] to the case of arbitrary submanifolds of the linking dimension that are homologous into a boundary or into an end of the ambient manifold.) The modified Low conjecture says that two events are causally unrelated if and only if the lifts of their fronts are unlinked in the appropriate sense. For M being a 2–disk with holes strong results proving the conjecture for many cases were obtained by J Natario and P Tod [23].

(b) The classical winding number of a curve around a point p in $\mathbb{R}^2$ is the linking number between the curve and the 0–cycle $\{p,\infty\}$ in $S^2 = \mathbb{R}^2 \cup \{\infty\}$. In [8] we considered the generalizations $\mathrm{win}(F,p)$ and $\widetilde{\mathrm{win}}(F,p)$ of the winding numbers of the mapping $F\colon N_1^{m-1} \to M^m$ around a point $p\colon \mathrm{pt} = N_2 \to M$ to the case where $F_*([N_1]) \neq 0 \in H_*(M)$ and M does not have ends that could play the role of the infinity. (The invariants $\widetilde{\mathrm{win}}$ and win are the generalizations of the winding number to the case where the observable point p moves and is fixed in M, respectively.) We showed that affine winding numbers can be effectively used to estimate from below the number of times a wave front has passed through a point between two moments of time.

One can verify that the generalized winding number $\widetilde{\text{win}}$ also is included into the theory explored in this work, if we consider affine linking number for the case $N_2 = \text{pt}$. (It is clear, see Theorem 7.4, that in this case $\Omega_0(\mathcal{B}) = \mathbb{Z}$.)

It is easy to construct the version $\overline{\text{alk}}$ of the invariant alk constructed in this paper, where $\overline{\text{alk}}$ will be a function on $\pi_0(\mathcal{N}_1 \times \{*\} \setminus \Sigma)$ for some fixed mapping $* \in \mathcal{N}_2$. A straightforward verification shows that $\overline{\text{alk}}$ is well-defined provided it takes values in the quotient group of $\Omega_0(\mathcal{B})$ by $\text{Im}\big(\mu_{10}\colon\ \Omega_1(\mathcal{N}_1) \otimes \Omega_0(\mathcal{N}_2) \to \Omega_0(\mathcal{B})\big)$. The invariant win constructed in [8] is a particular case of such $\overline{\text{alk}}$ where $N_2 = \text{pt}$.

3.2 The invariant alk is the universal Vassiliev–Goussarov invariant of order ≤ 1

Fix a natural number n. Let $f = (f_1, f_2) \in \Sigma \subset \mathcal{N}_1 \times \mathcal{N}_2$ be such that $\text{Im}(f_1) \cap \text{Im}(f_2)$ consists of n distinct double points of transverse intersection. Each double point can be resolved in two essentially different ways: positive and negative, where the sign is as in Definition 3.5. Thus f with n such double points admits 2^n possible resolutions of the double points. A sign of the resolution is put to be $+$ if the number of negatively resolved double points is even, and it is put to be $-$ otherwise.

Let Γ be an abelian group. Let α be a Γ–valued *homotopy link invariant* of links from $\mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma$, ie a function $\alpha\colon \pi_0(\mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma) \to \Gamma$. (Thus α does not change under homotopies of a linked pair that do not involve double points between different components.) The *nth derivative $\alpha^{(n)}$ of α* is a function on singular links with exactly n distinct transverse intersection points between different component (and possibly many self-intersection points of the components). The value of $\alpha^{(n)}$ on such a singular link $f = (f_1, f_2)$ is defined as a sum (with appropriate signs) of the values of α on the nonsingular mappings obtained by the 2^n resolutions of double points. The invariant α is said to be of *order* $\leq n-1$ (or *Vassiliev–Goussarov invariant of order* $\leq n-1$, [35, 13, 14]) if $\alpha^{(n)}$ is identically zero.

The invariant alk is an $\mathbb{A}$–valued Vassiliev–Goussarov invariant of order ≤ 1. To see this consider a singular link $f = (f_1, f_2)$ with exactly two transverse double points between different components. We denote by $f(\pm, \pm)$ the four nonsingular links obtained by resolutions of the two double points. We denote by $f(\cdot,\pm)$ the singular link with one transverse double point between different components obtained from f by resolving the second singular point and keeping the first singular point intact. To prove that alk is an order ≤ 1 invariants it suffices to show that

$$\operatorname{alk}(f(+,+)) - \operatorname{alk}(f(+,-)) - \operatorname{alk}(f(-,+)) + \operatorname{alk}(f(-,-)) = 0.$$

Rewrite this expression as

$$\begin{aligned}\big(\operatorname{alk}(f(+,+)) - \operatorname{alk}(f(-,+))\big) - \big(\operatorname{alk}(f(+,-)) - \operatorname{alk}(f(-,-))\big)\\ = \operatorname{alk}^{(1)}(f(\cdot,+)) - \operatorname{alk}^{(1)}(f(\cdot,-)).\end{aligned}$$

By the definition of $\operatorname{alk}^{(1)}$ the values $\operatorname{alk}^{(1)}(f(\cdot,+)), \operatorname{alk}^{(1)}(f(\cdot,-))$ are equal to $+1[f(\cdot,+)]$, and $+1[f(\cdot,-)] \in \Omega_0(\mathcal{B})$. (Here $[f(\cdot,+)],[f(\cdot,-)] \in \pi_0(\mathcal{B})$ are the classes of the singular links with one transverse double point.) Clearly $[f(\cdot,+)] = [f(\cdot,-)] \in \pi_0(\mathcal{B})$. Thus

$$\begin{aligned}\operatorname{alk}(f(+,+)) - \operatorname{alk}(f(+,-)) - \operatorname{alk}(f(-,+)) + \operatorname{alk}(f(-,-))\\ = \operatorname{alk}^{(1)}(f(\cdot,+)) - \operatorname{alk}^{(1)}(f(\cdot,-)) = 0.\end{aligned}$$

Furthermore, if $\alpha\colon \pi_0(\mathcal{N}_1\times\mathcal{N}_2\setminus\Sigma) \to \Gamma$ is a Γ–valued Vassiliev–Goussarov invariant of order ≤ 1, then the increment $\alpha^{(1)}(\gamma(t_0)) = \Delta_\alpha(\gamma(t_0))$ of α under the positive crossing of Σ at $\gamma(t_0)\in\Sigma_0$ depends only on the element of $\pi_0(\mathcal{B})$ that corresponds to $\gamma(t_0)$. To see this consider a singular link f with two double points obtained from $\gamma(t_0)$ by preserving the existing double point and creating a second double point by a homotopy. Since α is an order ≤ 1 invariant we have

$$\begin{aligned}0 &= \alpha(f(+,+)) - \alpha(f(+,-)) - \alpha(f(-,+)) + \alpha(f(-,-))\\ &= \alpha^{(1)}(f(\cdot,+)) - \alpha^{(1)}(f(\cdot,-)).\end{aligned}$$

Thus the increments into α of the positive crossings of the discriminant at $\alpha^{(1)}(f(\cdot,+))$ and at $\alpha^{(1)}(f(\cdot,-))$ are equal. Clearly we can change $\gamma(t_0)\in\Sigma_0$ to any other element in Σ_0 that is in the same component of $\pi_0(\mathcal{B})$ by elementary homotopies that pass through an extra double point and by homotopies in $\pi_0(\mathcal{B})$ that do not create extra double points between the two components. These operations do not change the value of $\alpha^{(1)}$ on an element in $\mathcal{B}$ and we see that $\alpha^{(1)}(\gamma(t_0))$ indeed depends only on the element of $\pi_0(\mathcal{B})$ realized by $\gamma(t_0)$.

In particular, there exists a natural homomorphism $B\colon \Omega_0(\mathcal{B}) \to \Gamma$ that sends the bordism class of $(+1)\gamma(t_0)\in\Omega_0(\mathcal{B})$ to $\alpha^{(1)}(\gamma(t_0)) = \Delta_\alpha(\gamma(t_0))$. Moreover, this homomorphism B passes through the quotient homomorphism $q\colon \Omega_0(\mathcal{B}) \to \mathbb{A}$ as in 3.7, and therefore we get a homomorphism $A\colon \mathbb{A} \to \Gamma$ with $A \circ q = B$, cf Theorem 5.1 below. One verifies that

$$A(\mathrm{alk}(f) - \mathrm{alk}(f')) = \alpha(f) - \alpha(f'),$$

for all $f, f' \in (\mathcal{N}_1, \times \mathcal{N}_2 \setminus \Sigma)$.

Clearly if α and α' are Γ–valued Vassiliev–Goussarov invariants of order ≤ 1 such that $\alpha - \alpha'$ is a constant mapping, then the corresponding homomorphisms A and A' are equal. Let $f_0 \in \mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma$ be a chosen preferred point. Then for every $f \in \mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma$ we have $\alpha(f) = \alpha(f_0) + A(\mathrm{alk}(f)) - A(\mathrm{alk}(f_0))$. Thus alk completely determines the values of α on all f (modulo $\alpha(f_0)$), and hence alk is a universal Vassiliev–Goussarov invariant of order ≤ 1.

In particular, alk distinguishes all the elements $f, f' \in \mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma$ that can be distinguished via Vassiliev–Goussarov invariants of order ≤ 1 with values in an arbitrary group Γ.

4 Proof of Theorem 3.9

Definition 4.1 We define Σ_1 to be the subset (stratum) of Σ consisting of all the pairs (f_1, f_2) such that there exists precisely two pairs of points $y_1 \in N_1, y_2 \in N_2$ as in Definition 3.1. Here we assume that the two double points of the image are distinct. We also choose a base point $*$ of $\mathcal{N}_1 \times \mathcal{N}_2$ with $* \notin \Sigma$.

Notice that $\Sigma_i, i = 0, 1$, is the stratum of codimension i in Σ. In particular, a generic path in $\mathcal{N}_1 \times \mathcal{N}_2$ intersects Σ_0 in a finite number of points, and a generic disk in $\mathcal{N}_1 \times \mathcal{N}_2$ intersects Σ_1 in a finite number of points.

A generic path $\gamma\colon [0,1] \to \mathcal{N}_1 \times \mathcal{N}_2$ that connects two points in $\mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma$ intersects Σ_0 in finitely many points $\gamma(t_j), j \in J$, and all the intersection points are of the types described in 3.5. Put

$$\Delta_{\mathrm{alk}}(\gamma) = \sum_{j \in J} [\gamma(t_j)] \in \Omega_0(\mathcal{B}). \tag{4.1}$$

We let

$$\begin{aligned} A &= \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}, & B_1 &= \{(x,y) \in A \mid xy = 0\}, \\ B_2 &= \{(x,y) \in A \mid x = 0\}, & B_3 &= \{(0,0)\}, \\ B_4 &= \emptyset. \end{aligned}$$

We define a *regular disk* in $\mathcal{N}_1 \times \mathcal{N}_2$ to be a generically embedded disk D such that $D \cap \Sigma = D \cap (\Sigma_0 \cup \Sigma_1)$ and the triple $(D, D \cap \Sigma_0, D \cap \Sigma_1)$ is homeomorphic to a triple $(A, B, C), A \supset B \supset C$, where B is one of B_i's and C is equal to B_3 or B_4.

Lemma 4.2 *Let β be a generic loop that bounds a regular disk in $\mathcal{N}_1 \times \mathcal{N}_2$. Then $\Delta_{\mathrm{alk}}(\beta) = 0$.*

Proof It is easy to see that all the crossings of Σ_0 that happen along β can be subdivided into pairs, such that the elements of $\pi_0(\mathcal{B})$ corresponding to the two crossings in a pair are equal and the signs of the corresponding crossings of Σ_0 are opposite. Hence the inputs into $\Delta_{\mathrm{alk}}(\beta)$ of the elements of $\Omega_0(\mathcal{B})$ corresponding to the two crossings in a pair cancel out and $\Delta_{\mathrm{alk}}(\beta) = 0$. □

Lemma 4.3 *Let β be a generic loop that bounds a disk in $\mathcal{N}_1 \times \mathcal{N}_2$. Then $\Delta_{\mathrm{alk}}(\beta) = 0$.*

Proof Notice that the set $\Sigma \setminus (\Sigma_0 \cup \Sigma_1)$ is a subset of codimension ≥ 3 in $\mathcal{N}_1 \times \mathcal{N}_2$. So, without loss of generality we can (using a small deformation of the disk) assume that the disk is the union of regular ones, cf Arnold [1, 2]. Now the proof follows from Lemma 4.2. □

Corollary 4.4 *The invariant Δ_{alk} induces a well-defined homomorphism*

$$\Delta_{\mathrm{alk}}\colon\ \pi_1(\mathcal{N}_1 \times \mathcal{N}_2) = \pi_1(\mathcal{N}_1 \times \mathcal{N}_2, *) \to \Omega_0(\mathcal{B}).$$

Proof Since every element of $\pi_1(\mathcal{N}_1 \times \mathcal{N}_2, *)$ can be represented by a generic loop, the proof follows from Lemma 4.3. □

Now we give another description of Δ_{alk}. Let $\lambda\colon\ \Omega_1(\mathcal{N}_1 \times \mathcal{N}_2) \to \Omega_0(B)$ be the homomorphism from Definition 3.7.

Proposition 4.5 *The homomorphism $\Delta_{\mathrm{alk}}\colon\ \pi_1(\mathcal{N}_1 \times \mathcal{N}_2) \to \Omega_0(\mathcal{B})$ coincides with the homomorphism*

$$\pi_1(\mathcal{N}_1 \times \mathcal{N}_2) \xrightarrow{h} \Omega_1(\mathcal{N}_1 \times \mathcal{N}_2) \xrightarrow{\lambda} \Omega_0(\mathcal{B}),$$

where h is the Hurewicz homomorphism in the bordism theory $\Omega_*(-)$ and λ is the homomorphism from Definition 3.7.

Proof Given a generic loop α in $(\mathcal{N}_1, *)$ and the constant loop e in $(\mathcal{N}_2, *)$, let (α, e) be the corresponding loop in $(\mathcal{N}_1 \times \mathcal{N}_2, *)$. The homotopy class of (α, e) gives us an element $[(\alpha, e)] \in \pi_1(\mathcal{N}_1 \times \mathcal{N}_2, *)$. Similarly, a generic loop β in $(\mathcal{N}_2, *)$ gives us an element $[(e, \beta)] \in \pi_1(\mathcal{N}_1 \times \mathcal{N}_2, *)$.

Because of the isomorphism $\pi_1(\mathcal{N}_1 \times \mathcal{N}_2) = \pi_1(\mathcal{N}_1) \times \pi_1(\mathcal{N}_2)$, the classes $[(\alpha, e)]$ and $[(e, \beta)]$ generate the group $\pi_1(\mathcal{N}_1 \times \mathcal{N}_2, *)$. So, since Δ_{alk} is a homomorphism, it suffices to prove that $\Delta_{\mathrm{alk}}[(\alpha, e)] = (\lambda \circ h)[(\alpha, e)]$, and similarly for $[(e, \beta)]$. We do it here for the loops (α, e) only.

We calculate $\Delta_{\mathrm{alk}}[(\alpha, e)] \in \Omega_0(\mathcal{B})$. Fix a mapping $\overline{e}\colon N_2 \to M$ in $\mathcal{N}_2$ and consider the mapping

$$\overline{\alpha}\colon S^1 \times (N_1 \sqcup N_2) \to M$$

such that $\overline{\alpha}\big|_{S^1 \times N_1} = \alpha$ and $\overline{\alpha}\big|_{S^1 \times N_2}$ coincides with the composition

$$S^1 \times N_2 \xrightarrow{\text{projection}} N_2 \xrightarrow{\overline{e}} M.$$

Without loss of generality we may assume that $\overline{\alpha}\big|_{S^1 \times N_1}$ is transverse to $\overline{e}$. The inclusion $\mathcal{N}_1 \to \mathcal{N}_1 \times \mathcal{N}_2$, $x \mapsto (x, e)$ allows us to regard $h(\alpha) \in \Omega_1(\mathcal{N}_1)$ as an element of $\Omega_1(\mathcal{N}_1 \times \mathcal{N}_2)$.

Now it is easy to see that

$$\Delta_{\mathrm{alk}}[(\alpha, e)] = \mu_{10}\big(\big[\overline{\alpha}\big|_{S^1 \times N_1}\big] \otimes [\overline{e}]\big) = \lambda(h(\alpha)) \in \Omega_1(\mathcal{B}), \tag{4.2}$$

where $\big[\overline{\alpha}\big|_{S^1 \times N_1}\big]$ and $[\overline{e}]$ are the bordism classes of corresponding maps. □

Corollary 4.6

$$\mathrm{Im}\{\Delta_{\mathrm{alk}}\colon \pi_1(\mathcal{N}_1 \times \mathcal{N}_2, *) \to \Omega_0(\mathcal{B})\} = \mathrm{Indet} \subset \Omega_0(\mathcal{B}).$$ □

Take an arbitrary point $f = (f_1^1, f_2^1) \in \mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma$ and choose a generic path γ going from $*$ to f. We set

$$\mathrm{alk}(f) = q\big(\Delta_{\mathrm{alk}}(\gamma)\big) \in \mathbb{A}$$

where q is the epimorphism from Definition 3.7.

Theorem 4.7 *The function*

$$\mathrm{alk}\colon\ \mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma \to \mathbb{A}$$

is an affine linking invariant. Furthermore, any other affine linking invariant $\widetilde{\mathrm{alk}}$ *coincides with* alk *if* $\widetilde{\mathrm{alk}}(*) = 0$.

Proof To show that alk is constant on path components we must verify that the definition of alk is independent on the choice of the path γ that goes from $*$ to f. This is the same as to show that $q(\Delta_{\mathrm{alk}}(\varphi)) = 0$ for every closed generic loop φ at $*$. But this follows from Corollary 4.6 directly.

Furthermore, it is clear that alk increases by $q([\gamma(t)]) \in \mathbb{A}$ under a transverse passage by a path γ through the stratum Σ_0 at the point $\gamma(t)$. This yields property (b) of alk from Theorem 3.9. The last claim is obvious. □

Remark 4.8 Clearly alk depends on the choice of $*$. On the other hand, if we change $*$, then new alk and the old alk will differ by an additive constant. This is the reason why we use the adjective "affine".

Clearly, Theorem 3.9 is a direct consequence of Theorem 4.7.

5 Relations between alk and the classical linking invariant lk

Given a closed oriented manifold N^n with the fundamental class $[N] \in H_n(M)$, we say that a map $f\colon N \to M$ is zero-homologous if $f_*([N]) = 0 \in H_n(M)$.

Let $\varepsilon\colon \Omega_0(\mathcal{B}) \to \mathbb{Z}$ be the homomorphism from (3.1).

Theorem 5.1 *Suppose that $\mathcal{N}_1$ and $\mathcal{N}_2$ consist of zero-homologous mappings. Then $\varepsilon(\mathrm{Indet}) = 0$. Furthermore, for all $f = (f_1, f_2), f' = (f_1', f_2') \in \mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma$, we have*

$$\varepsilon(\mathrm{alk}(f)) - \varepsilon(\mathrm{alk}(f')) = \mathrm{lk}(f_1, f_2) - \mathrm{lk}(f_1', f_2') \in \mathbb{Z}.$$

Proof Since $\mathcal{N}_1$ and $\mathcal{N}_2$ consist of zero-homologous maps, the classical linking invariant $\mathrm{lk}\colon \mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma \to \mathbb{Z}$ is well-defined. Now, similarly to Δ_{alk}, we define

$$\Delta_{\mathrm{lk}}\colon \pi_1(\mathcal{N}_1 \times \mathcal{N}_2, *) \to \mathbb{Z}, \quad \Delta_{\mathrm{lk}}(\gamma) = \sum_{i=1}^{k} \sigma(\gamma, t_i),$$

where the generic loop γ in $(\mathcal{N}_1 \times \mathcal{N}_2, *)$ intersects $\Sigma_0 \subset \Sigma \subset \mathcal{N}_1 \times \mathcal{N}_2$ in certain points $\gamma(t_1), \dots, \gamma(t_k)$. (Here we use the notation γ for the loop as well as for its homotopy class.) Since lk is well-defined, we conclude that $\Delta_{\mathrm{lk}}(\gamma) = 0$ for all γ, ie $\sum \sigma(\gamma, t_i) = 0$.

Now, we have

$$\Delta_{\mathrm{alk}}([\gamma]) = \sum [\gamma(t_i)] \in \Omega_0(\mathcal{B}).$$

So, in view of (3.1)

$$\varepsilon(\Delta_{\mathrm{alk}}([\gamma])) = \sum \varepsilon([\gamma(t_i)]) = \sum \sigma(\gamma, t_i) = 0.$$

Thus, by Corollary 4.6,

$$\varepsilon\,(\mathrm{Indet}) = \varepsilon(\mathrm{Im}(\Delta_{\mathrm{alk}})) = 0.$$

Now take a generic path γ which connects f and f'. Then

$$\begin{aligned}\varepsilon(\mathrm{alk}(f)) - \varepsilon(\mathrm{alk}(f')) &= \varepsilon\left(\sum [\gamma(t_i)]\right) = \sum \sigma(\gamma, t_i)\\ &= \mathrm{lk}(f_1, f_2) - \mathrm{lk}(f_1', f_2'),\end{aligned}$$

which proves the second claim of the theorem. □

Remarks 5.2 Theorem 5.1 demonstrates that, up to an additive constant, $\varepsilon \circ \mathrm{alk}$ is equal to the classical linking number lk whenever $\mathcal{N}_1$ and $\mathcal{N}_2$ consist of zero-homologous mappings. So, alk is an extension of the classical lk–invariant of zero-homologous submanifolds.

Since the homomorphism ε is the summation over the components of $\mathcal{B}$, we conclude that, for zero-homologous mappings, alk *can be regarded as a splitting of the classical linking invariant* into a collection of independent invariants. In many cases it can be shown that $\Omega_0(\mathcal{B})$ is an infinitely generated abelian group (see Theorem 7.4–Corollary 7.10) and that the indeterminacy subgroup Indet is zero (see Section 6). Since it is easy to show (see Corollary 7.5) that the mapping alk: $\pi_0(\mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma) \to \mathbb{A}$ is always surjective, we see that for these cases the classical lk invariant of zero-homologous submanifolds splits into infinitely many independent invariants.

On the other hand, as it was explained in this paper, the invariant alk exists regardless of whether the mappings are zero-homologous or not.

Also, there are many cases where $\mathcal{N}_1, \mathcal{N}_2$ do not consist of zero-homologous mappings while nevertheless $\varepsilon(\mathrm{Indet}) = 0 = \mathrm{Indet}$, and thus $\varepsilon \circ \mathrm{alk}$ is a $\mathbb{Z}$–valued invariant.

6 Examples where the indeterminacy subgroup vanishes

Given the manifolds M, N_1, N_2 as in Section 3, we assume in addition that N_1 and N_2 are connected, and that $n_1 n_2 > 0$.

Theorem 6.1 (Preissman) *Let M be a closed manifold that admits a Riemannian metric of negative sectional curvature. Then the following holds:*

(i) *Every nontrivial abelian subgroup of $\pi_1(M)$ is an infinite cyclic group.*

(ii) *For every nontrivial abelian subgroup A of π there exists a unique abelian subgroup B_A of π which contains A and is maximal with respect to this property. In fact, B_A is the centralizer $Z(A)$ of A in $\pi_1(M)$.*

Proof See do Carmo [11] or the original paper by Preissman [24]. □

Definition 6.2 A finitely generated group π is called a *Preissman group* if it satisfies the properties (i) and (ii) from Theorem 6.1.

Proposition 6.3 *Let π be a Preissman group. Let $\alpha, \beta \in \pi$ be such that $\alpha\beta \neq \beta\alpha$, and let $\gamma \in \pi$ be such that $\alpha\gamma = \gamma\alpha$ and $\beta\gamma = \gamma\beta$. Then $\gamma = e$.*

Proof Suppose that $\gamma \neq e$. Let $G = \{x\}$ be the (unique) maximal cyclic subgroup of π which contains γ. Since $\alpha\gamma = \gamma\alpha$, the subgroup $\{\alpha, \gamma\}$ of π is contained in G, and so $\alpha = x^m$ for some m. Similarly, $\beta = x^k$, and thus $\alpha\beta = \beta\alpha$. This is a contradiction. □

Theorem 6.4 *For M, N_1, N_2 as above, suppose that $\pi_1(M)$ is a Preissman group, and that $\pi_i(M) = 0$ for $2 \leq i \leq 1 + \max\{n_1, n_2\}$. Then the indeterminacy subgroup* $\mathrm{Indet} \subset \Omega_0(\mathcal{B})$ *is the zero subgroup,* $\mathrm{Indet} = \{0\} \subset \Omega_0(\mathcal{B})$.

Proof Throughout the proof we denote $\pi_1(M)$ by π. We must prove that, for every $\alpha \in \pi_1(\mathcal{N}_1)$ and $\beta \in \pi_1(\mathcal{N}_2)$,

$$\Delta_{\mathrm{alk}}[(\alpha, e)] = 0 = \Delta_{\mathrm{alk}}[(e, \beta)] \tag{6.1}$$

(see Lemma 4.6).

We prove the first equality from (6.1) only, the second equality can be proved in the similar way. Fix $\phi_i \in \mathcal{N}_i$ for $i = 1, 2$, and consider a loop α in $(\mathcal{N}_1, \phi_1)$. Let $\widetilde{\alpha}\colon S^1 \times N_1 \to M$ be the adjoint map, $\widetilde{\alpha}(t, n) = \alpha(t(n))$. Since $\pi_i(M) = 0$ for $2 \leq i \leq 1 + n_1$, it follows from the elementary obstruction theory that the homomorphism

$$\widetilde{\alpha}_*\colon\ \pi_1(S^1 \times N) \to \pi$$

completely determines the homotopy class of $\widetilde{\alpha}$. We use the isomorphism $\pi_1(S^1 \times N) \simeq \pi_1(S^1) \times \pi_1(N)$ and set

$$\gamma = \widetilde{\alpha}_*(\iota, e), \tag{6.2}$$

where $\iota \in \pi_1(S^1)$ is the generator.

Lemma 6.5 *If $\gamma = e$, then $\Delta_{\mathrm{alk}}[(\alpha, e)] = 0$.*

Proof Indeed, in this case $\widetilde{\alpha}\colon S^1 \times N_1 \to M$ is homotopic to the map

$$S^1 \times N_1 \xrightarrow{\text{proj}} N_1 \xrightarrow{\phi_1} M,$$

because both maps induce the same homomorphism of fundamental groups. So, since $n_1 + n_2 = m - 1 < m$, there is a generic map $\widehat{\alpha}$ homotopic to $\widetilde{\alpha}$ such that $\widehat{\alpha}(S^1 \times N_1)$ does not meet $\phi_2(N_2)$. □

Lemma 6.6 *If $\operatorname{Im}\widetilde{\alpha}_* = \mathbb{Z} \subset \pi$, then $\Delta_{\mathrm{alk}}[(\alpha, e)] = 0$.*

Proof If $\operatorname{Im}\widetilde{\alpha}_* = \mathbb{Z}$, then $\widetilde{\alpha}_*$ can be decomposed as

$$\pi_1(S^1 \times N_1) \to \mathbb{Z} \subset \pi. \tag{6.3}$$

Since $S^1 = K(\mathbb{Z}, 1)$, the map $\pi_1(S^1 \times N_1) \to \mathbb{Z}$ in (6.3) can be induced by a map $\varphi\colon S^1 \times N_1 \to S^1$. Furthermore, the inclusion $\mathbb{Z} \subset \pi$ in (6.3) can be induced by a map (inclusion) $\psi\colon S^1 \to M$, and we can assume that $\psi(S^1)$ does not meet $\phi_2(N_2)$ since $m - n_2 > 1$. Now, the map $\widetilde{\alpha}\colon S^1 \times N_1 \to M$ is homotopic to a map

$$\widehat{\alpha}\colon S^1 \times N_1 \xrightarrow{\varphi} S^1 \xrightarrow{\psi} M.$$

Clearly, $\widehat{\alpha}$ does not meet $\phi_2(N_2)$ and, moreover, any small perturbation of $\widehat{\alpha}$ does not. Thus, $\Delta_{\mathrm{alk}}[(\alpha, e)] = 0$. □

Now we finish the proof of Theorem 6.4. Consider the homomorphism

$$(\phi_1)_*\colon \pi_1(N_1) \to \pi.$$

If $\operatorname{Im}(\phi_1)_*$ is non-abelian, then $\gamma = e \in \pi$ by Proposition 6.3, because γ commutes with $\operatorname{Im}(\phi_1)_*$. So, $\Delta_{\mathrm{alk}}[(\alpha, e)] = 0$ by Lemma 6.5.

Furthermore, assume that $\operatorname{Im}(\phi_1)_*$ is abelian. Since γ commutes with $\operatorname{Im}(\phi_1)_*$ we conclude that $\operatorname{Im}(\widetilde{\alpha}_*)$ is an abelian subgroup of π. So, $\operatorname{Im}(\widetilde{\alpha}_*) = \mathbb{Z}$ or 0 because π is a Preissman group. If $\operatorname{Im}(\widetilde{\alpha}_*) = \mathbb{Z}$ then $\Delta_{\mathrm{alk}}[(\alpha, e)] = 0$ (by Lemma 6.6), and if $\operatorname{Im}(\widetilde{\alpha}_*) = 0$, then $\gamma = e \in \pi$ and $\Delta_{\mathrm{alk}}[(\alpha, e)] = 0$ (by Lemma 6.5). □

Remark 6.7 Assume that $n_1 + n_2 + 1 \neq m$ but otherwise the manifolds N_1, N_2, M satisfy all the conditions of Theorem 6.4. Assume moreover that $\max(n_1, n_2) < m - 1$ and that there exists $(f_1, f_2) \in \mathcal{N}_1 \times \mathcal{N}_2$ with $\operatorname{Im} f_1 \cap \operatorname{Im} f_2 = \emptyset$. Then the straightforward modifications of the proof of the Theorems 6.4 shows that $\operatorname{Im}\mu_{1,0} = \operatorname{Im}\mu_{0,1} = 0 \subset \Omega_{n_1+n_2+1+0-m}(\mathcal{B})$.

Theorem 6.8 *Let M, N_1 and N_2 be as above with an extra condition that $\pi_i(M)$ is finite for $i = 1, \dots, \max(n_1, n_2) + 1$, then the indeterminacy subgroup* $\mathrm{Indet} \subset \Omega_0(\mathcal{B})$ *is the zero subgroup.*

Proof Elementary obstruction theory implies that a mapping $N_i \times * \to M$ (for $i = 1, 2$) has are finitely many non-homotopic extensions to a mapping $N_i \times S^1 \to M$ (for $i = 1, 2$). Thus $\pi_1(\mathcal{N}_1 \times \mathcal{N}_2)$ is finite and hence

$$\mathrm{Im}\big(\Delta_{\mathrm{alk}} \colon \pi_1(\mathcal{N}_1 \times \mathcal{N}_2, *) \to \Omega_0(\mathcal{B})\big) = \mathrm{Indet} \subset \Omega_0(\mathcal{B})$$

is a finite subgroup. Since $\Omega_0(\mathcal{B})$ is torsion free, we get that $\mathrm{Indet} = 0$. □

Remark 6.9 Assume that $n_1 + n_2 + 1 \neq m$ but otherwise the manifolds N_1, N_2, M satisfy all the conditions of Theorem 6.8. Then the straightforward modification of the proof of Theorem 6.8 shows that $\mathrm{Im}\,\mu_{1,0} = \mathrm{Im}\,\mu_{0,1} = 0 \subset \Omega_{n_1+n_2+1+0-m}(\mathcal{B})$.

Examples 6.10 (1) It is well-known that every closed manifold M^m that admits a complete Riemannian metric of negative sectional curvature has the universal covering space homeomorphic to $\mathbb{R}^m$, and thus $\pi_i(M) = 0$, for $i > 1$, while $\pi_1(M)$ is a Preissman group.

Combining this with Theorems 6.1 and 6.4 we get that for such M the group $\mathrm{Indet} = 0$ for all $N_1, N_2, \mathcal{N}_1, \mathcal{N}_2$, and hence $\mathbb{A}$ is a free abelian group.

To construct more manifolds with Preissman fundamental group, notice that the total space of a locally trivial bundle $F \to E \to B$ has the Preismann fundamental group if F is simply-connected and $\pi_1(B)$ is Preissman.

(2) Let M^m be an oriented base space of a finite covering map $\Sigma^m \to M$, where Σ^m is a simply connected m–dimensional rational homology sphere. The Serre–Hurewicz theorem implies that $\pi_k(\Sigma^m)$ is finite, for $k = 2, \dots, m-1$, and therefore $\pi_k(M)$ is finite, for $k = 1, \dots, m-1$. Now Theorem 6.8 implies that $\mathrm{Indet} = 0$ for all $N_1, N_2, \mathcal{N}_1, \mathcal{N}_2$, provided that $n_1, n_2 > 0$.

Theorem 5.1 implies that in all these cases, if $\mathcal{N}_1, \mathcal{N}_2$ consist of zero-homologous mappings, then alk is a splitting of the classical linking invariant lk into a direct sum of independent $\mathbb{Z}$–valued invariants. Corollary 7.10 says that this direct sum is infinite when M is closed and admits a complete metric of negative sectional curvature; the images of $\pi_1(N_1), \pi_1(N_2)$ in $\pi_1(M)$ under the homomorphisms induced by the mappings $f_1 \in \mathcal{N}_1, f_2 \in \mathcal{N}_2$ are nontrivial; and $\big|H_1(M)/\big(\mathrm{Im}\, f_{1*}(H_1(N_1)) + \mathrm{Im}\, f_{2*}(H_1(N_2))\big)\big| = \infty$.

7 Description of $\pi_0(\mathcal{B})$ and $\Omega_0(\mathcal{B})$

In Section 7 we assume N_1 and N_2 are connected; *however we do not assume that* $n_1+n_2+1=m$. The goal of this section is to give a method of evaluation of $\pi_0(\mathcal{B})$ (and therefore of $\Omega_0(\mathcal{B})$ since the last group is the free abelian group with the base $\pi_0(\mathcal{B})$) and, in particular, to demonstrate that $\pi_0(\mathcal{B})$ can be infinite in spite of $\pi_0(\mathcal{N}_1)$ and $\pi_0(\mathcal{N}_2)$ being one-point sets.

We must recall some facts from elementary homotopy theory. Given two pointed spaces X and Y, let $[X,Y]^\bullet$ and $[X,Y]$ be the set of pointed homotopy classes of pointed maps $X\to Y$ and the set of unpointed homotopy classes, respectively. Then $\pi_1(Y)$ acts on $[X,Y]^\bullet$ in a usual way, and

$$[X,Y]^\bullet/\pi_1(Y)=[X,Y] \tag{7.1}$$

(see [27], for example).

Furthermore, $\pi_1(X)$ acts on $[X,X]$, and therefore we get a right $\pi_1(X)$–action on $[X,Y]$ via the composition map $[X,X]\times[X,Y]\to[X,Y]$. In greater detail, we apply $\alpha\in\pi_1(X)$ to the homotopy class $1_X\in[X,X]$ and get a map (homotopy class) $\alpha(1_X)$. Now, given $f\in[X,Y]$, we define $f\alpha=(f)\alpha\in[X,Y]$ to be the composition $f\circ\alpha(1_X)$. It is worthy to mention that, for the standard $\pi_1(Y)$–action on $[X,Y]$ we have

$$\beta(f)=\beta(1_Y)\circ f,\quad \beta\in\pi_1(Y), f\in[X,Y]. \tag{7.2}$$

Furthermore, given $\alpha\in\pi_1(X)$ and $f\in[X,Y]^\bullet$, we have

$$f\alpha=f_*(\alpha)f \tag{7.3}$$

and so any orbit of the (right) $\pi_1(X)$–action is contained in an orbit of the $\pi_1(Y)$–action.

Lemma 7.1 *The $\pi_1(Y)$–action on $[X,Y]^\bullet$ and the $\pi_1(X)$–action on $[X,Y]^\bullet$ commute.*

Proof Choose $\alpha\in\pi_1(X),\beta\in\pi_1(Y)$ and consider the composition

$$X\xrightarrow{\alpha(1_X)}X\xrightarrow{f}Y\xrightarrow{\beta(1_Y)}Y.$$

Then, in view of (7.2), we have

$$(\beta(f))\alpha=(\beta(1_Y)\circ f)\circ\alpha(1_X)=\beta(1_Y)\circ(f\circ\alpha(1_X))=\beta((f)\alpha). \qquad \square$$

Now, consider two pointed spaces X_1 and X_2 and the actions

$$[X_1,Y]^\bullet\times\pi_1(X_1)\to[X_1,Y]^\bullet \quad\text{and}\quad [X_2,Y]^\bullet\times\pi_1(X_2)\to[X_2,Y]^\bullet.$$

Together these actions yield the action

$$([X_1,Y]^\bullet \times [X_2,Y]^\bullet) \times (\pi_1(X_1) \times \pi_1(X_2)) \to [X_1,Y]^\bullet \times [X_2,Y]^\bullet. \qquad (7.4)$$

Because of Lemma 7.1, this action commutes with the diagonal $\pi_1(Y)$–action on $[X_1,Y]^\bullet \times [X_2,Y]^\bullet$.

Remark 7.2 Since $[X_1,Y]^\bullet \times [X_2,Y]^\bullet = [X_1 \vee X_2, Y]^\bullet$, we have the action of the group $\pi_1(X_1 \vee X_2) = \pi_1(X_1) * \pi_1(X_2)$ on $[X_1,Y]^\bullet \times [X_2,Y]^\bullet$. Let $i\colon \pi_1(X_1) \subset \pi_1(X_1) * \pi_1(X_2)$ be the standard inclusion. We emphasize that the $i(\pi_1(X_1))$–action on $[X_1,Y]^\bullet \times [X_2,Y]^\bullet$ does *not* coincide with the action coming from (7.4).

Given $f \in [X,Y]^\bullet$, let

$$S_f = \{\alpha \in \pi_1(Y) \mid \alpha f = f\}. \qquad (7.5)$$

Proposition 7.3 *$S_f \subset Z(f_*(\pi_1(X))$, where ZG denotes the centralizer of G in $\pi_1(Y)$.*

Proof Let H be the set of homomorphisms $\pi_1(X) \to \pi_1(Y)$ induced by pointed maps $X \to Y$. The $\pi_1(Y)$–action on $[X,Y]^\bullet$ induces the $\pi_1(Y)$–action on H. Here for $\alpha \in \pi_1(Y)$ we have

$$(\alpha(f_*))(u) = \alpha(f_*(u))\alpha^{-1},$$

where $u \in \pi_1(X)$ and $f\colon X \to Y$. Thus, if $\alpha \in S_f$ then $(\alpha(f_*))(u) = \alpha(f_*(u))\alpha^{-1} = f_*(u)$, ie $\alpha \in Z(f_*(\pi_1(X))$. □

Choose base points in N_1, N_2 and M and let $\mathcal{N}_i^\bullet, i = 1,2$, be the space of all pointed maps $f\colon N_i \to M$ such that the unpointed map f belongs to $\mathcal{N}_i$. Clearly, the subset $\pi_0(\mathcal{N}_i^\bullet)$ of $[N_i,M]^\bullet$ is invariant with respect to $\pi_1(N_i)$– and $\pi_1(M)$–actions. Moreover, because of (7.1), $\pi_0(\mathcal{N}_i^\bullet)$ is the orbit of $\pi_1(M)$–action on $[N_i,M]^\bullet$.

Clearly, the bijection $[N_1 \vee N_2, M]^\bullet = [N_1,M]^\bullet \times [N_2,M]^\bullet$ converts the $\pi_1(M)$–action on $[N_1 \vee N_2, M]$ into the diagonal action on $[N_1,M]^\bullet \times [N_2,M]^\bullet$.

Theorem 7.4

$$\pi_0(\mathcal{B}) = (\pi_1(N_1) \times \pi_1(N_2)) \backslash (\pi_0(\mathcal{N}_1{}^\bullet) \times \pi_0(\mathcal{N}_2{}^\bullet)) / \pi_1(M).$$

Proof Consider the inclusion $j\colon \mathcal{N}_1^\bullet \times \mathcal{N}_2^\bullet \to \mathcal{B}$. Since M and N_i are connected, the map $j_*\colon \pi_0(\mathcal{N}_1^\bullet \times \mathcal{N}_2^\bullet) \to \pi_0(\mathcal{B})$ is surjective. Now, consider two points $b, b' \in \mathcal{N}_1^\bullet \times \mathcal{N}_2^\bullet$ and set $j(b) = (\rho_1, \rho_2, \phi_1, \phi_2)$ and $j(b') = (\rho_1', \rho_2', \phi_1', \phi_2') \in \mathcal{B}$. Then $\phi_1\rho_1 = \phi_1'\rho_1'$. Suppose that there is a path from b to b' in $\mathcal{B}$. Consider this path as a quadruple $(\rho_1(t), \rho_2(t), \phi_1(t), \phi_2(t)), t \in I$. Now, the loop $\rho_i(t)$ gives us the loop (homotopy class) $\alpha \in \pi_1(N_i), i = 1, 2$, while the loop

$$(\phi_1(t))(\rho_1(t)) = (\phi_2(t))(\rho_2(t))$$

gives us an element $\beta \in \pi_1(M)$. Clearly, $\beta([b])(\alpha_1, \alpha_2) = [b']$ where $[b]$ and $[b']$ are components of $\mathcal{N}_1^\bullet \times \mathcal{N}_2^\bullet$ containing b and b', respectively. □

Corollary 7.5 *The mapping* $\operatorname{alk}\colon \pi_0(\mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma) \to \mathbb{A}$ *of Theorem* 3.9 *is surjective.*

Proof Take an arbitrary point $b \in \mathcal{B}$ and consider the class $[b] \in \Omega_0(\mathcal{B})$. Clearly it suffices to show that, given $(f_1, f_2) \in \mathcal{N}_1 \times \mathcal{N}_2 \setminus \Sigma$, there exist generic paths $\gamma_i\colon I \to \mathcal{N}_1 \times \mathcal{N}_2, i = 1, 2$, starting at (f_1, f_2) such that $\gamma_i(t)$ intersects Σ_0 for exactly one value $t_i \in (0, 1), i = 1, 2$, and $[\gamma_1(t_1)] = [b], [\gamma_2(t_2)] = -[b]$. Clearly there exists $b' = [\phi_1, \phi_2, \rho_1, \rho_2] \in \mathcal{B}$ such that $\pm[b]' = [b]$ and that one of the two possible resolutions of the double point between ϕ_1 and ϕ_2 is isotopic to (f_1, f_2). Let (ϕ_1', ϕ_2') be this resolution and let $\tilde\gamma$ be the isotopy path that connects (f_1, f_2) to (ϕ_1', ϕ_2').

It is easy to construct (see Figure 1) a local deformation $\tilde\gamma_+$ (respectively, $\tilde\gamma_-$) of (ϕ_1', ϕ_2') that involves only one passage through a double point and such that the sign of the corresponding crossing of Σ_0 is positive (respectively, negative) and the associated element of $\pi_0(\mathcal{B})$ is the same as the one for b'. The desired paths γ_i are the paths $\tilde\gamma_+\tilde\gamma$ and $\tilde\gamma_-\tilde\gamma$. □

Corollary 7.6 *Assume that N_1 and N_2 are simply connected and that $\pi_1(M)$ is infinite. Also, assume that, for both $i = 1$ and $i = 2$, the $\pi_1(M)$–action on $\pi_0(\mathcal{N}_i^\bullet)$ is free. Then $\pi_0(B)$ is an infinite set.* □

Example 7.7 Let $N_1 = N_2 = S^n$, and let M be a manifold which has the $(n+2)$–type of the wedge $S^1 \vee S^n$. Fix a map $f\colon S^n \to M$ which induces an isomorphism in n–dimensional homology. Let $\mathcal{N} = \mathcal{N}_1 = \mathcal{N}_2$ be the path component of f in the space of smooth maps $S^n \to M$. Then $\pi_0(B)$ is an infinite set because of Corollary 7.6.

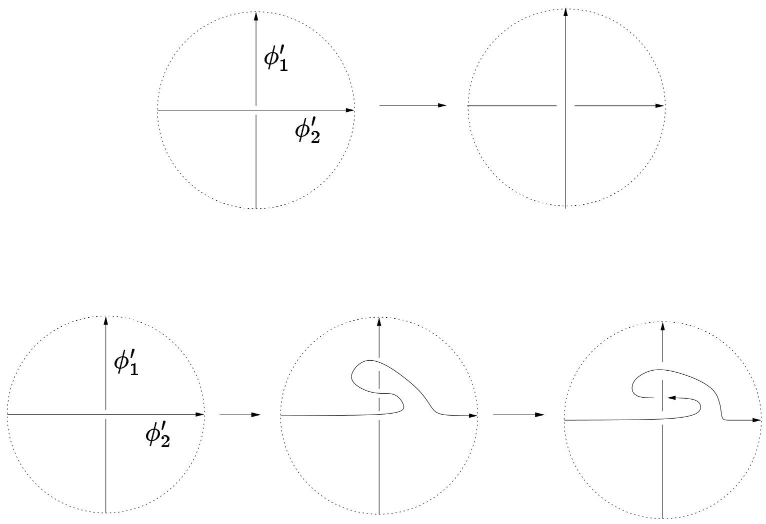

Figure 1: The paths $\tilde{\gamma}_+$ and $\tilde{\gamma}_-$ in the 1–dimensional case

Another series of cases with $\pi_0(\mathcal{B})$ infinite appears as follows.

Take $f_i \in \pi_0(\mathcal{N}_i^\bullet), i = 1, 2$, and let G_i be the subgroup of $\pi_1(M)$ generated by S_{f_i} and $f_{i*}(\pi_1(N_i))$, where S_{f_i} is the stabilizer of f_i as in (7.5).

Theorem 7.8 *If the Reidemeister set of double cosets*

$$G_2 \backslash \pi_1(M) / G_1$$

is infinite, then $\pi_0(\mathcal{B})$ is an infinite set and $\Omega_0(\mathcal{B})$ is an infinitely generated abelian group.

Proof Take any $u, v \in \pi_1(M)$ such that the elements (uf_1, f_2) and (vf_1, f_2) of $\pi_0(\mathcal{N}_1{}^\bullet) \times \pi_0(\mathcal{N}_2{}^\bullet)$ give us the same element of $\pi_0(\mathcal{B})$.

It follows from Theorem 7.4 that there exists $(\alpha, \beta, \gamma) \in \pi_1(N_1) \times \pi_1(N_2) \times \pi_1(M)$ such that $\gamma(uf_1, f_2)(\alpha, \beta) = (vf_1, f_2)$ or

$$\gamma u f_1 \alpha = v f_1 \text{ in } \pi_0(\mathcal{N}_1{}^\bullet) \quad \text{and} \quad \gamma f_2 \beta = f_2 \text{ in } \pi_0(\mathcal{N}_2{}^\bullet).$$

Because of Lemma 7.1 and (7.3), $\gamma f_2 \beta = \gamma f_{2*}(\beta) f_2$. So, $\gamma f_{2*}(\beta) \in S_{f_2}$, and therefore $\gamma \in G_2$.

Similarly, $v^{-1}\gamma u f_1 \alpha = f_1$, ie $v^{-1}\gamma u f_{1*}(\alpha) f_1 = f_1$, and hence $v^{-1}\gamma u f_{1*}(\alpha) \in S_{f_1}$. Thus $v^{-1}\gamma u \in G_1$, ie $\gamma u \in vG_1$, and $G_2 u \cap vG_1 \neq \varnothing$.

This means that the images of u and v in $G_2\backslash\pi_1(M)/G_1$ coincide. This completes the proof. □

Corollary 7.9 *Let $h\colon \pi_1(M) \to H_1(M)$ be the Hurewicz homomorphism. Then $\pi_0(B)$ is infinite whenever the group $H_1(M)/(h(G_1)+h(G_2))$ is infinite.*

Corollary 7.10 *Assume that $\pi_1(M)$ is a Preissman group. Let $\mathcal{N}_1 \times \mathcal{N}_2$ be such that for $(f_1, f_2) \in \mathcal{N}_1 \times \mathcal{N}_2$ the subgroups $f_{1*}(\pi_1(N_1)), f_{2*}(\pi_1(N_2)) \subset \pi_1(M)$ are both nontrivial and such that $H_1(M)/\big(f_{1*}(H_1(N_1)) + f_{2*}(H_1(N_2))\big)$ is infinite. Then $\pi_0(\mathcal{B})$ is infinite and hence $\Omega_0(\mathcal{B})$ is an infinitely generated abelian group.*

Proof It suffices to prove that each of the groups $f_{i*}(\pi_1(\mathcal{N}_i)), i = 1, 2$, is a subgroup of finite index in G_i. For sake of simplicity, we fix i and denote G_i and $f_{i*}(\pi_1(\mathcal{N}_i))$ by G and H, respectively, and prove that the index $[G : H]$ is finite.

Case 1 Assume that the group H is abelian. Then since $\pi_1(M)$ is a Preissman group, we conclude that the centralizer ZH of H is a cyclic subgroup which contains H. Since $H \neq \{e\}$, H is a subgroup of finite index in ZH. Thus, we are done since, by Proposition 7.3, $S_f \subset ZH$ and so $G \subset ZH$.

Case 2 Assume that the group H is non-abelian. Then, by Proposition 6.3, $ZH = \{e\}$. Thus, by Proposition 7.3, $S_f = \{e\}$, ie $G = H$. □

8 The pairing $\mu_{i,j}$ as an obstruction for double point elimination and as a splitting of the intersection pairing

In this section we do not assume that $\dim N_1^{n_1} + \dim N_2^{n_2} = \dim M^m$.

A classical question is: given two C^∞ maps $f_1\colon N_1 \to M$ and $f_2\colon N_2 \to M$, can we make their images disjoint by homotopies of f_1 and f_2? The necessary condition for the existence of such homotopies is that the intersection of the homology classes $f_{1*}([N_1])$ and $f_{2*}([N_2])$ is zero in $H_{m-n_1-n_2}(M)$. It is well-known that this condition is not sufficient in general.

From now on we assume that $n_1 + n_2 = m$ and consider the pairing

$$\mu = \mu_{0,0}\colon\ \Omega_0(\mathcal{N}_1) \otimes \Omega_0(\mathcal{N}_2) \to \Omega_{n_1+n_2+0+0-m}(\mathcal{B}) = \Omega_0(\mathcal{B}).$$

The mapping $e\colon \mathcal{B} \to M$ that maps $(\phi_1, \phi_2, \rho_1, \rho_2)$ to $\phi_1 \circ \rho_1(\mathrm{pt}) = \phi_2 \circ \rho_2(\mathrm{pt}) \in M$ induces the (augmentation) map $\varepsilon\colon \Omega_*(\mathcal{B}) \to \Omega_*(M)$. Clearly $\tau \circ \varepsilon(\mu([f_1],[f_2]))$ is equal to the intersection of $f_{1*}([N_1])$ and $f_{2*}([N_2])$ in $H_{m-n_1-n_2}(M)$, where τ is the Steenrod–Thom homomorphism.

We observe that the necessary condition for the existence of $f_1 \in \mathcal{N}_1, f_2 \in \mathcal{N}_2$ with $\mathrm{Im}(f_1) \cap \mathrm{Im}(f_2) = \emptyset$ is that

$$\mu([\bar{f}_1],[\bar{f}_2]) = 0 \in \Omega_0(\mathcal{B}),$$

for some (and therefore for all) $\bar{f}_1 \in \mathcal{N}_1, \bar{f}_2 \in \mathcal{N}_2$. Clearly this condition is much stronger than the vanishing of intersection condition. (Related results for intersections of mappings of S^1 into an oriented 2–dimensional surface were previously obtained by Turaev and Viro [33, 34].)

In fact, the following Theorem 8.1 describes a big class of manifolds M where the condition

$$\mu([\bar{f}_1],[\bar{f}_2]) = 0 \in \Omega_0(\mathcal{B})$$

is necessary and sufficient for the existence of $f_1 \in \mathcal{N}_1$ and $f_2 \in \mathcal{N}_2$ with disjoint images.

When N_1 and N_2 are simply connected and $n_1+n_2=m$ such $f_1 \in \mathcal{N}_1, f_2 \in \mathcal{N}_2$ with disjoint images do exist if and only if the $\mathbb{Z}[\pi_1(M)]$–valued intersection index of the lifts of the mappings $\bar{f}_1$ and $\bar{f}_2$ to the universal covering $\widetilde{M} \to M$ vanishes (see Kervaire [17], for example). In the conditions of Theorem 8.1 N_1 and N_2 are not assumed to be simply connected. Thus the mappings $\bar{f}_1$ and $\bar{f}_2$ can not be lifted to the universal covering and Kervaire's approach to this classical problem does not work.

Observe also that if $\mathcal{N}_1, \mathcal{N}_2$ consist of mappings homotopic to immersions, then the statement of Theorem 8.1 follows immediately from Theorem 2.2 of the work [15] of Hatcher and Quinn (see Section 2.2).

Theorem 8.1 *Let $N_1^{n_1}, N_2^{n_2}$, $n_1, n_2 > 2$, be closed manifolds and let M^m be a (not necessarily closed) manifold such that $n_1+n_2=m$, $\pi_1(M)$ is Preissman (see Definition 6.2) and $\pi_i(M)=0$ for $i=2,\dots,\max(n_1,n_2)+1$. Let $\mathcal{N}_1, \mathcal{N}_2$ be connected components of the space of mappings of N_1 and of N_2 into M. Then $\mu\colon \Omega_0(\mathcal{N}_1) \otimes \Omega_0(\mathcal{N}_2) \to \Omega_0(\mathcal{B}) = \mathbb{Z}$ is the zero pairing if and only there exists $f_1 \in \mathcal{N}_1$, $f_2 \in \mathcal{N}_2$ with $\mathrm{Im}\, f_1 \cap \mathrm{Im}\, f_2 = \emptyset$.*

Remark 8.2 In particular, by the Preissman and Hadamard Theorems, see Theorem 6.1 and [11], $\pi_1(M)$ is Preissman and $\pi_j(M)=0$, $j>1$ for all closed M that admit a Riemannian metric of negative sectional curvature; and any such M satisfies all the conditions of Theorem 8.1.

Example 8.3 Here we give an example where neither Kervaire non Hatcher–Quinn approaches work while Theorem 8.1 implies that f_1 and f_2 can be made disjoint via a homotopy. Let $N_1 = S^1 \times \mathbb{RP}^9$, let N_2 be a 4–dimensional manifold and let M^{14} be a hyperbolic manifold. Then one can prove directly that any pair (f_1, f_2) of maps $f_1\colon N_1 \to M^{14}$ and $f_2\colon N_2 \to M^{14}$ is homotopic to a pair with disjoint images, and so $\mu(f_1, f_2) = 0$. On the other hand, the Kervaire approach is not applicable if we assume that f_1 is non-trivial on π_1. The Hatcher–Quinn [15] Theorem 2.2 is not applicable, since $S^1 \times \mathbb{RP}^9$ is not immersible into M^{14}. (Since $\overline{w}_6(\mathbb{R}P^9) \neq 0$, neither $\mathbb{RP}^9$ nor $S^1 \times \mathbb{RP}^9$ are immersible into the universal cover $\mathbb{R}^{14}$ of M^{14}.)

Proof It is clear that if $f_1 \in \mathcal{N}_1$ and $f_2 \in \mathcal{N}_2$ with $\operatorname{Im} f_1 \cap \operatorname{Im} f_2 = \emptyset$ do exist, then μ is the zero pairing.

Let us show that if μ is the zero pairing, then such f_1, f_2 do exist. Take $\bar f_1 \in \mathcal{N}_1$, $\bar f_2 \in \mathcal{N}_2$ so that they are transversal to each other, and hence $\operatorname{Im} \bar f_1$ and $\operatorname{Im} \bar f_2$ intersect at isolated transverse double points. For a double point $d \in \operatorname{Im} \bar f_1 \cap \operatorname{Im} \bar f_2$ we denote by $[d] \in \Omega_0(\mathcal{B})$ the input of this double point into $\mu([\bar f_1], [\bar f_2])$. The proof is via induction on the number of double points of $\operatorname{Im} \bar f_1 \cap \operatorname{Im} \bar f_2$.

Take a double point $p \in \operatorname{Im} \bar f_1 \cap \operatorname{Im} \bar f_2$. Since $\mu([\bar f_1], [\bar f_2]) = 0$, there exists a transversal double point $q \in \operatorname{Im} \bar f_1 \cap \operatorname{Im} \bar f_2$ such that $[p] = -[q] \in \Omega_0(\mathcal{B})$. In particular we get that the double points p and q correspond to the same path connected component of $\mathcal{B}$. Put p_i and q_i (for $i = 1, 2$) to be, respectively, the preimages of the double points p and q on N_i (for $i = 1, 2$).

Theorem 7.4 and the definition of $\mathcal{B}$ imply that there exist paths $\alpha_i\colon [0,1] \to N_i$ with $\alpha_i(0) = p_i$ and $\alpha_i(1) = q_i$ for $i = 1, 2$; and a path $\beta\colon [1,2] \to M$ with $\beta(1) = q$, $\beta(2) = p$ such that the loop $\bar f_i(\alpha_i)\beta$ acts trivially on the pointed homotopy class of the mapping $(N_i, p_i) \to (M, p)$ for $i = 1, 2$. Thus $\bar f_1(\alpha_1)\beta \in \pi_1(M, p)$ commutes with all the elements of $\operatorname{Im}(\bar f_{1*}\colon \pi_1(N_1) \to \pi_1(M))$.

If $\operatorname{Im} \bar f_1 \subset \pi_1(M)$ is an infinite cyclic group or a trivial group, then, since $\pi_j(M) = 0$ for $j = 2, \dots, \max(n_1, n_2) + 1$, the elementary obstruction theory implies that $\bar f_1$ is homotopic to a mapping that passes through a mapping $S^1 \to M$. Since $n_1 + n_2 = m$ and $n_1 > 2$ we get that after the small perturbation the mapping $S^1 \to M$ does not pass through $\operatorname{Im} \bar f_2$ and thus $\bar f_1$ is homotopic to a mapping that has disjoint image with $\bar f_2$.

If $\operatorname{Im} \bar f_{1*} \subset \pi_1(M)$ is a group which is not trivial and is not infinite cyclic, then, since $\pi_1(M)$ is Preissman (see Definition 6.2), $\bar f_1(\alpha_1)\beta = 1 \in \pi_1(M, p)$.

Similarly, we get that either $\bar{f}_2$ is homotopic to mapping passing through a mapping $S^1 \to M$, and hence it is homotopic to a mapping with the image disjoint from $\operatorname{Im} \bar{f}_1$; or $\bar{f}_2(\alpha_2)\beta$ is also trivial in $\pi_1(M)$. Thus the paths $\bar{f}_1(\alpha_1)$ and $\bar{f}_2(\alpha_2)$ can be assumed to be homotopic. Thus $\bar{f}_1(\alpha_1)\big(\bar{f}_2(\alpha_2)\big)^{-1} = 1 \in \pi_1(M)$ and this loop bounds a disk $D \subset M$.

For dimension reasons D can be assumed to be embedded and $\operatorname{Int} D$ can be assumed to be disjoint from $\operatorname{Im} \bar{f}_1$ and $\operatorname{Im} \bar{f}_2$. Since $[p] = -[q] \in \Omega_0(\mathcal{B})$, we get that the signs of the intersection points p and q of $\operatorname{Im} \bar{f}_1 \cap \operatorname{Im} \bar{f}_2$ are opposite. Thus we can apply the Whitney trick to the points p and q and the disk D and cancel the double points. Since D was assumed to be embedded and $\operatorname{Int} D$ was assumed to be disjoint from $\operatorname{Im} \bar{f}_1$ and $\operatorname{Im} \bar{f}_2$, no new double points of $\operatorname{Im} \bar{f}_1 \cap \operatorname{Im} \bar{f}_2$ will appear during the Whitney trick. Thus we managed to decrease the number of double points of $\operatorname{Im} \bar{f}_1 \cap \operatorname{Im} \bar{f}_2$ and we proceed by induction. □

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