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Rohlin's invariant and gauge theory III. Homology 4–tori

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Abstract

This is the third in our series of papers relating gauge theoretic invariants of certain 4–manifolds with invariants of 3–manifolds derived from Rohlin's theorem. Such relations are well-known in dimension three, starting with Casson's integral lift of the Rohlin invariant of a homology sphere. We consider two invariants of a spin 4–manifold that has the integral homology of a 4–torus. The first is a degree zero Donaldson invariant, counting flat connections on a certain $SO(3)$–bundle. The second, which depends on the choice of a 1–dimensional cohomology class, is a combination of Rohlin invariants of a 3–manifold carrying the dual homology class. We prove that these invariants, suitably normalized, agree modulo 2, by showing that they coincide with the quadruple cup product of 1–dimensional cohomology classes.

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1 Introduction

Let X be a closed smooth spin 4–manifold which has the integral homology of $T^4 = S^1 \times S^1 \times S^1 \times S^1$. Suppose that there exists a primitive cohomology class $\alpha \in H^1(X;\mathbb{Z})$ such that the infinite cyclic covering $\tilde{X}_\alpha$ corresponding to α has the integral homology of the 3–torus. Then X is called a $\mathbb{Z}[\mathbb{Z}]$*–homology 4–torus*. The intersection form on the second cohomology of a $\mathbb{Z}[\mathbb{Z}]$–homology 4–torus X is always isomorphic to the sum of three copies of the hyperbolic 2–form, but the cup-product on the first cohomology of X may vary.

In this paper, we discuss two invariants of $\mathbb{Z}[\mathbb{Z}]$–homology 4-tori. The first one is a Rohlin–type invariant $\bar\rho(X,\alpha)$, which *a priori* depends on the choice of a primitive class $\alpha \in H^1(X;\mathbb{Z})$ for which $\tilde{X}_\alpha$ has the integral homology of the 3–torus. It is defined in terms of an oriented 3–manifold M embedded in X that is Poincaré dual to the class α. Each spin structure on X induces one on M thus giving rise to a corresponding Rohlin invariant. The sum of these Rohlin invariants over all the spin structures on M induced from X is called $\bar\rho(X,\alpha)$, see Section 3.3. We show that this is a well defined invariant with values in $\mathbb{Z}/2\mathbb{Z}$. It extends the Rohlin invariant of $\mathbb{Z}[\mathbb{Z}]$–homology $S^1 \times S^3$, described in [9] and [21].

The other invariant, $\bar\lambda(X,P)$, comes from gauge theory. It equals one quarter times the signed count of projectively flat (or equivalently, projectively anti-self-dual) connections on an admissible $U(2)$–bundle P over X, see Section 4.5. This invariant can be viewed as either a degree-zero Donaldson polynomial of X, or a four–dimensional analogue of the Casson invariant. Yet another way to view $\bar\lambda(X,P)$ is as an extension of the Furuta–Ohta invariant defined for $\mathbb{Z}[\mathbb{Z}]$–homology $S^1 \times S^3$, see [9].

Theorem 1.1 *Let X be a $\mathbb{Z}[\mathbb{Z}]$–homology 4–torus. For any choice of a primitive cohomology class $\alpha \in H^1(X;\mathbb{Z})$ such that $\tilde{X}_\alpha$ has the integral homology of the 3–torus, and for any admissible $U(2)$–bundle P,*

$$\bar\rho(X,\alpha) \equiv \bar\lambda(X,P) \equiv \det X \pmod 2$$

where $\det X$ is defined as $(a_1 \cup a_2 \cup a_3 \cup a_4)\,[X] \pmod 2$, for any choice of basis $a_1,\ a_2,\ a_3,\ a_4 \in H^1(X;\mathbb{Z}_2)$.

Admissible bundles exist over all $\mathbb{Z}[\mathbb{Z}]$–homology 4–tori, see Section 4.1. This fact and Theorem 1.1 together imply that neither $\bar\rho(X,\alpha)$ nor $\bar\lambda(X,P) \pmod 2$ depend on choices made in their definition (primitive class α and admissible bundle P, respectively). The integer invariant $\bar\lambda(X,P)$ does depend on the choice of P, as we show by examples in Section 10.

Our proof of Theorem 1.1 takes advantage of a natural action of $H^1(X;\mathbb{Z}_2)=(\mathbb{Z}_2)^4$ on the moduli space of projectively flat connections. We identify the latter with the space of projective representations of $\pi_1 Y$ in $SU(2)$, and use this identification to show that the orbits with four elements are always non–degenerate and that the number of such orbits equals $\det X \pmod 2$. In the non–degenerate situation, this completes the proof because there are no orbits with one or two elements, and the orbits with eight and sixteen elements contribute zero to $\bar\lambda(X,P) \pmod 2$. The general case is handled similarly after one finds a generic perturbation which is $H^1(X;\mathbb{Z}_2)$–equivariant.

Because of the equivariance condition, constructing such a perturbation requires a rather delicate argument, which is carried out in Sections 6, 7, and 8. The different orbits (of size 4, 8, or 16) are handled separately, with no perturbation required at the 4-orbits. The hardest case is that of the 8-orbits, in which connections have a $\mathbb{Z}_2$ stabilizer inside $H^1(X;\mathbb{Z}_2)$. We consider first a perturbation that affects the deformation complex tangential to an orbit, and for this we perturb the ASD equations using holonomy functions, as in [7, 27, 5, 11, 14].

After this perturbation, the orbit is 0-dimensional, but the points in it may not be smooth points in the moduli space as there is potentially some nontrivial cohomology in the deformation complex normal to the orbit. If so, we make a further perturbation using the abstract perturbations found in [4] and [8]. Having achieved non-degeneracy at the 8–orbits using equivariant perturbations, we move on to the non-degeneracy at the 16–orbits in Section 8.

In [22], we studied a three–dimensional analogue of the invariant $\bar\lambda(X,P)$, denoted $\lambda'''(Y,w)$. It is obtained by counting projectively flat connections on a $U(2)$–bundle P over a homology 3–torus Y with $c_1(P)=w\neq 0 \pmod 2$. We showed that $\lambda'''(Y,w)\equiv \det Y \pmod 2$. Conjecturally, the invariant $\lambda'''(Y,w)$ coincides with the Lescop invariant [17] obtained via a surgery approach. Lescop's invariant is independent of the choice of w and in fact equals $(\det Y)^2$. It is natural to wonder if similar results hold for $\bar\lambda(X,P)$; at the end of the paper we present a family of examples that show that $\bar\lambda(X,P)$ is not determined by $\det X$, and that in fact it depends on the choice of bundle P.

Theorem 1.1 is part of a broader program for relating gauge–theoretic invariants of some simple 4–manifolds to Rohlin-type invariants [22, 23]. A survey of this work appears in [24]. In the spirit of Witten's conjecture [30] relating Donaldson and Seiberg–Witten invariants, Theorem 1.1 is also consistent with the result of the first author and S. Strle [25] on the (mod 2) evaluation of the Seiberg-Witten invariant for homology 4-tori.

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2 Algebraic topology of homology tori

The discussion of both the Rohlin and Donaldson invariants requires some information about algebraic topology of homology tori; relevant results are collected in this section.

An *n-dimensional homology torus* X is a manifold of dimension n such that $H_*(X;\mathbb{Z}) = H_*(T^n;\mathbb{Z})$ where $T^n = S^1 \times \ldots \times S^1$ (n times). Let $a_1,\ldots,a_n$ a basis in $H^1(X;\mathbb{Z}_2)$ then $\det X = (a_1 \cup \ldots \cup a_n)\,[X] \pmod 2$ is independent of the choice of $a_1,\ldots,a_n$ and is called the *determinant* of X. A homology torus X is called *odd* if $\det X = 1 \pmod 2$; otherwise, X is called *even*.

Let R be a commutative ring with unity; at various points we will use the rings $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_2$, and the ring $\mathbb{Z}_{(2)}$ of integers localized at 2, i.e. with all odd primes inverted. An *n-dimensional R-cohomology torus* is a manifold X such that $H^*(X;R)$ and $H^*(T^n;R)$ are isomorphic as rings.

Lemma 2.1 *Suppose that X is an n-dimensional odd homology torus. Then X is an n-dimensional $\mathbb{Z}_2$-cohomology torus and a $\mathbb{Q}$-cohomology torus.*

Proof By the universal coefficient theorem, the dimension of $H^k(X;\mathbb{Z}_2)$ is the same as the rank of $H^k(X;\mathbb{Z})$, and a basis $a_1,\ldots,a_n$ for $H^1(X;\mathbb{Z})$ gives rise to a basis for $H^1(X;\mathbb{Z}_2)$. It is straightforward to show that the $\binom{n}{k}$ distinct cup products of the a's are linearly independent in $H^k(X;\mathbb{Z}_2)$, and thus give a basis. The proof with $\mathbb{Q}$ coefficients is the same. □

We remind the reader that, for any Y, the cup-square $H^2(Y;\mathbb{Z}_2) \to H^4(Y;\mathbb{Z}_2)$ actually lifts to a map $H^2(Y;\mathbb{Z}_2) \to H^4(Y;\mathbb{Z}_4)$ known as the Pontrjagin square [20]. In our situation, where every $\mathbb{Z}_2$ cohomology class lifts to an integral class, the Pontrjagin square can be computed by lifting to an integral class, and reducing the cup product mod 4. We will write this with the usual notation for cup product.

Corollary 2.2 *Let X be an n–dimensional odd homology torus then the cup-product*

$$\cup : \Lambda^2 H^1(X;\mathbb{Z}_2) \to H^2(X;\mathbb{Z}_2) \tag{1}$$

gives rise to a bijective correspondence between decomposable elements in $\Lambda^2 H^1(X;\mathbb{Z}_2)$ and elements $w \in H^2(X;\mathbb{Z}_2)$ such that $w \cup w = 0 \pmod 4$.

Proof This is true for T^n hence also for X, since the $\mathbb{Z}_2$–cohomology rings of X and T^n are isomorphic. □

The natural map $X \to K(\pi_1 X, 1)$ induces a monomorphism (see [2])

$$\iota : H^2(\pi_1 X;\mathbb{Z}_2) \to H^2(X;\mathbb{Z}_2). \tag{2}$$

Corollary 2.3 *Let X be an odd homology torus then the map ι is an isomorphism.*

Proof This follows from the commutative diagram

$$\begin{tikzcd}
\Lambda^2 H^1(\pi_1 X;\mathbb{Z}_2) \arrow[r, "\cong"] \arrow[d, "\cup"'] & \Lambda^2 H^1(X;\mathbb{Z}_2) \arrow[d, "\cup"'] \\
H^2(\pi_1 X;\mathbb{Z}_2) \arrow[r, "\iota"] & H^2(X;\mathbb{Z}_2)
\end{tikzcd}$$

whose upper arrow is an isomorphism because $H^1(\pi_1 X;\mathbb{Z}_2) = H^1(X;\mathbb{Z}_2)$, and the right arrow is an isomorphism by Lemma 2.1. Since ι is injective, the remaining two arrows in the diagram are also isomorphisms. □

Lemma 2.4 *A manifold X is a $\mathbb{Z}_{(2)}$-cohomology torus if and only if it is a $\mathbb{Z}_2$-cohomology and $\mathbb{Q}$-cohomology torus.*

Proof The statement about the cohomology groups is a simple application of the universal coefficient theorem and Poincaré duality. To understand the ring structure, note that $H^1(X;\mathbb{Z}_{(2)}) \cong H^1(X;\mathbb{Z}) \otimes \mathbb{Z}_{(2)}$. Hence the n-fold cup product on $H^1(X;\mathbb{Z}_{(2)})$ is odd if and only if the n-fold cup products on $H^1(X;\mathbb{Q})$ and $H^1(X;\mathbb{Z}_2)$ are nontrivial. □

In this paper, we will be concerned with the following situation: we are given a homology 4–torus X, together with a specific double covering. This double covering corresponds to a cohomology class $\alpha \in H^1(X;\mathbb{Z}_2)$, and will be denoted $X_\alpha \to X$.

Proposition 2.5 *Suppose that X is a $\mathbb{Z}_{(2)}$-cohomology torus, and that $\pi : X_\alpha \to X$ is a nontrivial double covering corresponding to $\alpha \in H^1(X;\mathbb{Z}_2)$. Then X_α is a $\mathbb{Z}_{(2)}$-cohomology torus.*

Proof Associated to the covering $\pi : X_\alpha \to X$ is the Gysin sequence in cohomology (where the coefficients are understood to be $\mathbb{Z}_2$)

$$\begin{array}{ccccccccc}
\to H^{k-1}(X) & \xrightarrow{\cup\alpha} & H^k(X) & \xrightarrow{\pi^*} & H^k(X_\alpha) & \longrightarrow & H^k(X) & \xrightarrow{\cup\alpha} & H^{k+1}(X) \\
\cong\downarrow & & \cong\downarrow & & =\downarrow & & \cong\downarrow & & \cong\downarrow \\
\to \mathbb{Z}_2^{\binom{n}{k-1}} & \xrightarrow{\cup\alpha} & \mathbb{Z}_2^{\binom{n}{k}} & \xrightarrow{\pi^*} & H^k(X_\alpha) & \longrightarrow & \mathbb{Z}_2^{\binom{n}{k}} & \xrightarrow{\cup\alpha} & \mathbb{Z}_2^{\binom{n}{k+1}}
\end{array}$$

By Lemma 2.4, the $\mathbb{Z}_2$-cohomology ring of X is an exterior algebra over $\mathbb{Z}_2$. If we choose a generating set $\alpha = \alpha_1,\ \alpha_2,\dots,\alpha_n$, then the image of the map $\cup\,\alpha$ in $H^k(X;\mathbb{Z}_2)$ is the $\binom{n-1}{k-1}$ dimensional subspace spanned by products of α with monomials not involving α. Likewise, the kernel of $\cup\,\alpha : H^k(X;\mathbb{Z}_2) \to H^{k+1}(X;\mathbb{Z}_2)$ has dimension $\binom{n-1}{k-1}$. From exactness, the dimension of $H^k(X_\alpha;\mathbb{Z}_2)$ is $\binom{n}{k}$. It is straightforward to check, by similar arguments, that the n-fold cup product on $H^1(X_\alpha;\mathbb{Z}_2)$ is non-trivial. □

Finally, we record for later use some facts about the relation between the cohomology of a $\mathbb{Z}[\mathbb{Z}]$–homology 4–torus X and that of its infinite cyclic cover $\pi : \tilde{X}_\alpha \to X$ classified by $\alpha \in H^1(X;\mathbb{Z})$.

Lemma 2.6 *Let X be a $\mathbb{Z}[\mathbb{Z}]$–homology 4–torus whose infinite cyclic cover $\tilde{X}_\alpha$ has the integral homology of the 3–torus. Then*

(1) *The cup-product pairing $H^1(\tilde{X}_\alpha;\mathbb{Z}_2) \otimes H^2(\tilde{X}_\alpha;\mathbb{Z}_2) \to H^3(\tilde{X}_\alpha;\mathbb{Z}_2) \cong \mathbb{Z}_2$ is non–degenerate.*

(2) *$\pi^* : H^i(X;\mathbb{Z}_2) \to H^i(\tilde{X}_\alpha;\mathbb{Z}_2)$ is a surjection for all i.*

(3) *$\ker \pi^* : H^1(X;\mathbb{Z}_2) \to H^1(\tilde{X}_\alpha;\mathbb{Z}_2)$ is the subgroup generated by α.*

Proof The universal coefficient theorem implies that the $\mathbb{Z}_2$ homology of $\tilde{X}_\alpha$ is finitely generated, because the integral homology is finitely generated. Hence the first item follows from Milnor's duality theorem [19]. The other two parts are proved using the long exact sequence (where the coefficients are understood to be $\mathbb{Z}_2$)

$$\longrightarrow H^i(X) \xrightarrow{\pi^*} H^i(\tilde{X}_\alpha) \xrightarrow{t^*-1} H^i(\tilde{X}_\alpha) \xrightarrow{\delta} H^{i+1}(X) \xrightarrow{\pi^*} \tag{3}$$

derived in [19, proof of assertion 5]. Here t represents the covering translation. Looking at the beginning of this sequence, we see that $t^* : H^0(\tilde{X}_\alpha;\mathbb{Z}_2) \to H^0(\tilde{X}_\alpha;\mathbb{Z}_2)$ is the identity, since $\tilde{X}_\alpha$ is connected. Thus the image of $\delta : H^0(\tilde{X}_\alpha;\mathbb{Z}_2) \to H^1(X;\mathbb{Z}_2)$ is one–dimensional. The rest follows by counting dimensions and using exactness. □

We remark that the same proof would work with any field coefficients, from which we can deduce that the conclusions of Lemma 2.6 hold with integral coefficients as well. In the proof of Corollary 2.8 below, we need the following claim.

Lemma 2.7 *In the exact sequence* (3) *with integral coefficients, we have* $(\delta c)^2 = 0$ *for any* $c \in H^1(\tilde{X}_\alpha;\mathbb{Z})$.

Proof Consider the following commutative square:

$$\begin{array}{ccc} H^1(\tilde{X}_\alpha;\mathbb{Z}) & \xrightarrow{\delta} & H^2(X;\mathbb{Z}) \\ \Big\downarrow{\scriptstyle \cap[M]} & & \Big\downarrow{\scriptstyle \cap[X]} \\ H_2(\tilde{X}_\alpha;\mathbb{Z}) & \xrightarrow{\pi_*} & H_2(X;\mathbb{Z}) \end{array}$$

where the right-hand vertical arrow is Poincaré duality, and M^3 is a lift to $\tilde{X}_\alpha$ of the Poincaré dual of α. Now $\langle(\delta c)^2,[X]\rangle = (\delta c \cap [X])^2$ by the correspondence between intersections and cup products. Since $(\delta c \cap [X])^2 = (\pi_*(c \cap [M]))^2$, it suffices to show the vanishing of intersection numbers between classes in $H_2(X;\mathbb{Z})$ lying in the image of π_*. But the map $M \to \tilde{X}_\alpha$ has degree one, so the induced map $H_2(M;\mathbb{Z}) \to H_2(\tilde{X}_\alpha;\mathbb{Z})$ is onto. So if a, $b \in H_2(\tilde{X}_\alpha;\mathbb{Z})$, we can represent both classes by cycles in M, and so $\pi_*(a)$ and $\pi_*(b)$ are represented by cycles lying in $M \subset X$. It follows readily that $\pi_*(a)\cdot\pi_*(b) = 0$. □

Using this, we get the following corollary, which will be used in Section 4.1 to construct admissible bundles over $\mathbb{Z}[\mathbb{Z}]$–homology 4-tori.

Corollary 2.8 *Let* X *be a* $\mathbb{Z}[\mathbb{Z}]$*–homology* 4*–torus. Then there are classes* $w \in H^2(X;\mathbb{Z}_2)$ *and* $\xi \in H^1(X;\mathbb{Z}_2)$ *with* $w \cup w \equiv 0 \pmod 4$ *and* $w \cup \xi \neq 0$.

Proof We will show that there is a class $w \in H^2(X;\mathbb{Z}_2)$ with $w \cup w \equiv 0 \pmod 4$ such that $\pi^*(w) \neq 0 \in H^2(\tilde{X}_\alpha;\mathbb{Z}_2)$. By Milnor duality on $\tilde{X}_\alpha$, see Lemma 2.6, there is a class $\xi' \in H^1(\tilde{X}_\alpha;\mathbb{Z}_2)$ with $\pi^*(w) \cup \xi' \neq 0$. But $\xi' = \pi^*\xi$ for some $\xi \in H^1(X;\mathbb{Z}_2)$, see Lemma 2.6, and therefore $w \cup \xi \neq 0$ by naturality of the cup product.

To show the existence of such a class w, consider the exact sequence (3), which shows that $H^2(X;\mathbb{Z}_2) \cong \operatorname{im}(\delta) \oplus V$ where both summands are 3–dimensional vector spaces over $\mathbb{Z}_2$ and $\pi^* : V \to H^2(\tilde{X}_\alpha;\mathbb{Z}_2)$ is an isomorphism. Suppose that there is no element $0 \neq a \in V$ with $a \cup a \equiv 0 \pmod 4$. Since X is spin, we then have that $a \cup a = 2 \pmod 4$, so the formula

$$(a+b)^2 \equiv a^2 + 2\,a \cup b + b^2 \pmod 4 \tag{4}$$

implies that $a \cup b \equiv 1 \pmod 2$ for all $a \neq b \in V$. It is easy to see (say, by considering a basis for V) that this makes the cup-product form on V singular. By non-singularity of the cup product on X, there is a non-zero cup product $a \cup b$ where $a \in V$ and $b \in \operatorname{im}(\delta)$. By the previous lemma, $b^2 \equiv 0 \pmod 4$, since b lifts to an integral class whose square is 0. Applying equation (4) once more, we conclude that $(a+b)^2 \equiv 0 \pmod 4$; by construction $\pi^*(a+b) = \pi^*(a) \neq 0$. To finish the proof, we set $w = a + b$. □

3 The Rohlin invariants

After describing a Poincaré duality theorem for smooth periodic-end manifolds, we introduce a Rohlin-type invariant for homology 4-tori, and then show that a certain combination of such Rohlin invariants is determined by the cup-product structure.

3.1 Periodic-end manifolds

A periodic-end manifold W_∞, in the sense of Taubes [26], has the following structure. The model for the end is an infinite cyclic covering $\pi : \tilde{X} \to X$ classified by a cohomology class $\alpha \in H^1(X;\mathbb{Z})$. This choice implies a specific generator t of the covering translations and thereby picks out one end of $\tilde{X}$ as lying in the positive direction. We denote this end by $\tilde{X}_+$. Choose a codimension-one submanifold $M \subset X$ dual to α, and a lift M_0 of M to $\tilde{X}$, giving copies $M_k = t^k(M_0)$. Thus $\tilde{X}$ is decomposed into copies $X_k = t^k(X_0)$

of a fundamental domain X_0, which is a manifold with boundary $-M_0 \cup M_1$. Write

$$N_k = \bigcup_{m \geq k} X_m.$$

Finally, suppose we have a compact manifold W with boundary M. The end-periodic manifold W_∞ is given by $W_\infty = W \cup_{M_0} N_0$. In other words, W_∞ is a non-compact manifold, with a single end that coincides with the positive end of an infinite cyclic cover.

The duality theorem is essentially a combination of duality theorems of Milnor [19] and Laitinen [16]; we briefly review these results. For a non-compact manifold Y, define the *end cohomology* $H^j_e(Y)$ to be the direct limit of the groups $H^i(Y - K; \mathbb{Z})$ where K runs over an exhausting set of compact sets. It is not hard to show that $H^j_e(Y)$ splits as a direct product of groups $H^j_e(Y_\epsilon)$ where Y_ϵ runs over the set of ends of Y. Dually, Laitinen defines the end homology in terms of (the inverse limit of) the algebraic mapping cone of $C_*(Y - K) \to C_*(Y)$. Again, it is not hard to define the end homology of a single end. It is evident from the definition that the end cohomology depends only on the end, or in other words that two manifolds that coincide off a compact set have the same end cohomology. The same is true about the end homology, although this requires more work. Laitinen proved that cap product with an appropriate fundamental class gives Poincaré-Lefschetz duality; in particular the end cohomology in dimension k is isomorphic to the end homology in dimension $n - k - 1$, where $n = \dim X$.

Recall that Milnor [19] showed that if X is oriented and $\tilde{X}$ has finitely generated homology with coefficients F (for F a field or the integers), then $\tilde{X}$ satisfies Poincaré duality:

$$\cap [M] : H^j(\tilde{X}; F) \stackrel{\cong}{\to} H_{n-j-1}(\tilde{X}; F).$$

In other words, $\tilde{X}$ resembles a compact manifold of dimension $n - 1$. Our duality theorem is a relative version of this.

Theorem 3.1 *Let W_∞ be an oriented periodic end-manifold, as described above. Assume that the homology of the infinite cyclic cover $\tilde{X}$, with coefficients in F = a field or the integers, is finitely generated. Then the groups $H^i(W_\infty; F)$ and $H_i(W_\infty; F)$ are finitely generated, and there are long exact sequences related by Poincaré duality isomorphisms*

$$\begin{tikzcd}
\to H^j_c(W_\infty;F) \arrow[r] \arrow[d, "\cap[W_\infty]"] & H^j(W_\infty;F) \arrow[r] \arrow[d, "\cap[W_\infty]"] & H^j(\tilde X;F) \arrow[r] \arrow[d, "\cap[M]"] & H^{j+1}_c(W_\infty;F) \to \arrow[d, "\cap[W_\infty]"] \\
\to H_{n-j}(W_\infty;F) \arrow[r] & H^{lf}_{n-j}(W_\infty;F) \arrow[r] & H_{n-j-1}(\tilde X;F) \arrow[r] & H_{n-j-1}(W_\infty;F) \to
\end{tikzcd} \tag{5}$$

where H^*_c stands for cohomology with compact support, and H^{lf}_* for locally finite homology.

In other words, W_∞ behaves like an n–manifold with boundary, where the homological 'boundary' $\tilde X$ looks like an $(n-1)$–manifold.

Proof Laitinen noted that there is an exact sequence like (5), with $H^j_e(Y)$ in place of $H^j(\tilde X)$, and proved that the end homology and cohomology groups satisfy Poincaré duality of dimension $n-1$. The end homology sits in an exact sequence involving the ordinary homology and the locally finite homology of Y. There are related definitions in Milnor [19, page 124 and footnote] of the homology/cohomology of a space relative to a single end.

To prove the theorem, we show that the finitely-generated hypothesis implies that the end homology and cohomology of W_∞ coincide with the ordinary homology and cohomology of $\tilde X$. Note that the end cohomology and homology of W_∞ are the same as the end cohomology and homology of the positive end $\tilde X_+$ of $\tilde X$. Let N_k be a neighborhood of the positive end of $\tilde X$, so there is a long exact sequence

$$\cdots \to H^j(\tilde X, N_k) \to H^j(\tilde X) \to H^j(N_k) \to \cdots \tag{6}$$

Passing to the limit as $k \to \infty$, we get a long exact sequence

$$\cdots \to H^j(\tilde X, \tilde X_+) \to H^j(\tilde X) \to H^j(\tilde X_+) \to \cdots \tag{7}$$

But [19, Assertion 8] says that under our hypotheses, the relative term vanishes, and so we get the desired isomorphism.

The statement about homology ends is proved similarly, using the analogue of the exact sequence [16, (2.6)] for the homology of a single end of $\tilde X$. □

For the rest of this subsection, assume that W_∞ is an oriented end-periodic 4–manifold, such that the end cohomology with rational coefficients is finitely generated. Note that the cup product $H^2_c(W_\infty;\mathbb{Q}) \otimes H^2_c(W_\infty;\mathbb{Q}) \to H^4_c(W_\infty;\mathbb{Q}) \cong \mathbb{Q}$ is well-defined, and hence has a signature which we denote by $\operatorname{sign}(W_\infty)$.

Now $W_\infty = W \cup_{M_0} N_0$, and it is easy to see that N_0 also has a signature. Any of these signatures can be computed using intersections rather than cup products.

Lemma 3.2 *If the end cohomology is finitely generated, then* $\mathrm{sign}(W_\infty) = \mathrm{sign}(W)$.

Proof The usual proof of Novikov additivity applies to show that $\mathrm{sign}(W_\infty) = \mathrm{sign}(W) + \mathrm{sign}(N_0)$, so it suffices to show that $\mathrm{sign}(N_0) = 0$. Now M_0 carries the fundamental class of $\tilde{X}$, so that the map $H_2(M_0;\mathbb{Q}) \to H_2(\tilde{X};\mathbb{Q})$ is surjective. Therefore, there are no non-trivial intersections between cycles in $\tilde{X}$, and so the same is true for N_0. It follows that $\mathrm{sign}(N_0) = 0$. □

3.2 The Rohlin invariants

For any manifold Y, denote by $\mathrm{Spin}(Y)$ the set of Spin structures on Y. Let X be a homology 4–torus, that is, a smooth spin 4–manifold with $H_*(X;\mathbb{Z}) = H_*(T^4;\mathbb{Z})$. Choose a spin structure $\sigma \in \mathrm{Spin}(X)$ and a primitive cohomology class $\alpha \in H^1(X;\mathbb{Z})$. These ingredients are enough to define an invariant $\rho(X,\alpha,\sigma) \in \mathbb{Q}/2\mathbb{Z}$ if X is odd. On the other hand, proving well-definedness of $\rho(X,\alpha,\sigma)$ for even X requires an additional assumption about the infinite cyclic covering $\tilde{X}_\alpha \to X$ corresponding to α. Here is the definition; the fact that it is independent of choices is the main result of this subsection.

Definition 3.3 Let (X,α,σ) be as above, and let M be a smooth oriented 3–manifold embedded in X that is Poincaré dual to α. Then σ induces in a canonical way a spin structure on M which we again call σ, and we define

$$\rho(X,\alpha,\sigma) = \rho(M,\sigma) \in \mathbb{Q}/2\mathbb{Z}$$

where $\rho(M,\sigma)$ is the usual Rohlin invariant of M. Our convention is that $\rho(M,\sigma) = 1/8\ \mathrm{sign}(V)$, where V is any smooth compact spin manifold with boundary M.

Theorem 3.4 *If X is an odd homology 4–torus, then $\rho(X,\alpha,\sigma)$ is independent of the choice of submanifold M. The same is true if X is an even homology 4–torus, and the rational homology of $\tilde{X}_\alpha$ is finitely generated. In either case, if the integral homology of $\tilde{X}_\alpha$ is the same as the homology of the 3–torus, then $\rho(X,\alpha,\sigma) \in \mathbb{Z}/2\mathbb{Z}$.*

Proof Suppose first that X is a homology torus for which the rational homology of $\tilde{X}_\alpha$ is finitely generated. It follows from Milnor's duality theorem [19] that $\tilde{X}_\alpha$ satisfies Poincaré duality over $\mathbb{Q}$ as if it were a 3–manifold. Note that any $M \subset X$ dual to α lifts to X_α, and gives rise to a fundamental class for $\tilde{X}_\alpha$. Thus the inclusion $M \to \tilde{X}_\alpha$ is a map of non-zero degree, and thus is surjective in rational homology. In particular, any classes $x, y \in H_2(\tilde{X}_\alpha; \mathbb{Q})$ have trivial intersection number.

Now suppose that M, M' are two submanifolds dual to α. It is easy to show that they have disjoint lifts to $\tilde{X}_\alpha$, and so cobound a spin submanifold $V \subset \tilde{X}_\alpha$. The intersection form on V is trivial, by the previous paragraph, so its signature is trivial. Hence the Rohlin invariants of M and M' must agree.

Now suppose that $\tilde{X}_\alpha$ has the integral homology of the 3–torus, and let M^3 be an oriented submanifold of X dual to α with a spin structure σ induced from X. Choose a spin 4–manifold W with boundary M and a lift M_0 of M to $\tilde{X}_\alpha$. Form, as in Section 3.1, a periodic end-manifold W_∞ whose end is modeled on the positive end of $\tilde{X}_\alpha$. By definition, $\rho(M, \sigma)$ equals one eighth of $\operatorname{sign}(W)$. The latter signature coincides with the signature of W_∞ by Lemma 3.2. Recall that for a compact 4–manifold whose boundary has no torsion in H_1, the intersection form splits as a unimodular form plus the radical. This follows from Poincaré duality, including the relation between duality on the manifold and on its boundary (as expressed in the exact sequence (5)). As we showed in Theorem 3.1, the same duality holds for a periodic end manifold. Hence with the hypothesis on $\tilde{X}_\alpha$, the intersection form on W_∞ is a sum of a unimodular form and the trivial form. But (cf. [12]) the signature of an even unimodular form is always divisible by 8, and so the Rohlin invariant lies in $\mathbb{Z}/2\mathbb{Z}$.

Finally, we verify that the Rohlin invariant of an odd homology torus is well-defined, without any additional hypothesis on the infinite cyclic cover. The main observation is the following. Suppose that X is a $\mathbb{Z}_{(2)}$ cohomology torus, and that M is dual to a non-trivial class $\alpha \in H^1(X; \mathbb{Z})$. Then the intersection form on $X - M$ is trivial. To see this, suppose that Σ is a surface in X, disjoint from M, with $\Sigma \cdot \Sigma \neq 0$. Setting $\beta = \mathrm{PD}(\Sigma) \in H^2(X; \mathbb{Z})$, it follows that $\alpha \cup \beta = 0$. This implies that $\beta = \alpha \cup \alpha'$ for some $\alpha' \in H^1(X; \mathbb{Z})$. But such a β satisfies $\beta \cup \beta = 0$, contradicting the fact that $\beta \cup \beta = \Sigma \cdot \Sigma \neq 0$.

To make use of this observation, let M and M' be submanifolds of X dual to α. We would like M and M' to be disjoint, so that they will cobound a submanifold with trivial intersection form. Consider the 2-fold covering space X_α dual to α. By Proposition 2.5, we again have a $\mathbb{Z}_{(2)}$ cohomology torus, and we can iterate, getting a 2^k fold covering $X^k \to X$. For some k, there are disjoint lifts of M and M' to X^k, and so we get a spin cobordism from M to M' inside X^k, with trivial intersection form. Hence the Rohlin invariants of M and M' must coincide. □

3.3 Rohlin invariants and the cup product

A homology 4–torus X has an assortment of Rohlin invariants $\rho(X,\alpha,\sigma)$, associated to different elements $\alpha \in H^1(X;\mathbb{Z})$ and different spin structures on X. In this subsection, we show that a certain combination of these Rohlin invariants is related to the parity of the cup product form. This is, in effect, half of the main theorem of the paper. The other half will be established in Section 9 by relating the Donaldson invariants of X to the same cup product.

Denote by $\sigma \to \sigma + x$ the action of $x \in H^1(X;\mathbb{Z}_2)$ on $\sigma \in \operatorname{Spin}(X)$.

Definition 3.5 Let X be an oriented homology 4–torus, and let $\alpha \in H^1(X;\mathbb{Z})$ be a primitive element such that $\tilde{X}_\alpha$ has finitely generated rational homology. Let $x_1, x_2, x_3 \in H^1(X;\mathbb{Z}_2)$ be such that $\{\alpha, x_1, x_2, x_3\}$ is a basis for $H^1(X;\mathbb{Z}_2)$. Choose a spin structure σ on X, and define

$$\bar\rho(X,\alpha) = \sum_{x\in \operatorname{Span}\{x_1,x_2,x_3\}} \rho(X,\alpha,\sigma+x). \tag{8}$$

Lemma 3.6 *The invariant $\bar\rho(X,\alpha)$ does not depend on the choice of spin structure σ, or the specific choice of $\{x_1,x_2,x_3\}$.*

Proof The main point in both statements is the following equality:

$$\rho(X,\alpha,\sigma) = \rho(X,\alpha,\sigma+\alpha).$$

This is clear from the definitions, because α restricts trivially to M and so changing σ to $\sigma+\alpha$ does not affect the spin structure on M (and *a fortiori* its Rohlin invariant).

Given another choice of basis $\{\alpha, x_1', x_2', x_3'\}$ for $H^1(X;\mathbb{Z}_2)$, the projection along α gives rise to a bijective correspondence between $\operatorname{Span}\{x_1,x_2,x_3\}$ and $\operatorname{Span}\{x_1',x_2',x_3'\}$. This proves the independence of $\bar\rho(X,\alpha)$ of the choice of $\{x_1,x_2,x_3\}$. To see that $\bar\rho(X,\alpha)$ does not depend on σ, notice that for any two choices, σ and σ', of spin structures on X,

$$\begin{aligned}\sum_{x\in \operatorname{Span}\{x_1,x_2,x_3\}} \rho(X,\alpha,\sigma'+x) &= \sum_{x\in \operatorname{Span}\{x_1,x_2,x_3\}} \rho(X,\alpha,\sigma+(\sigma'-\sigma)+x)\\ &= \sum_{x'\in \operatorname{Span}\{x_1',x_2',x_3'\}} \rho(X,\alpha,\sigma+x'),\end{aligned}$$

where $\mathrm{Span}\{x_1', x_2', x_3'\}$ is the bijective image of $(\sigma' - \sigma) + \mathrm{Span}\{x_1, x_2, x_3\}$ projected along α. $\square$

Remark 3.7 On the other hand, it is not so clear that $\bar\rho(X,\alpha)$ is independent of α. In many cases, this follows from the main theorem, but we do not know how to deduce this from first principles.

The relation between the invariant $\bar\rho$ and the cup-product is established via the following result of Turaev [28], extending a result of S. Kaplan [13].

Theorem 3.8 *Let (M^3,σ) be a spin manifold, and let $x, y, z \in H^1(M;\mathbb{Z}_2)$ be the mod 2 reductions of integral classes. Then*

$$\begin{aligned}\rho(M,\sigma) - \rho(M,\sigma+x) - \rho(M,\sigma+y) - \rho(M,\sigma+z)&\\ + \rho(M,\sigma+x+y) + \rho(M,\sigma+x+z) + \rho(M,\sigma+y+z)&\\ - \rho(M,\sigma+x+y+z) = (x\cup y\cup z)\,[M] \pmod 2 \quad (9)&\end{aligned}$$

Corollary 3.9 *Let X be an oriented homology 4–torus, and let $\alpha \in H^1(X;\mathbb{Z})$ be a primitive element such that $\tilde X_\alpha$ is a homology 3–torus. Then $\bar\rho(X,\alpha) \equiv \det X \pmod 2$.*

Proof This is immediate from Theorem 3.8 after we notice that all the Rohlin invariants in (9) take values in $\mathbb{Z}/2\mathbb{Z}$ (by Theorem 3.4) and that, for any choice of basis $\{\alpha, x_1, x_2, x_3\}$ in $H^1(X;\mathbb{Z}_2)$, we have $(x_1\cup x_2\cup x_3)\,[M] = (\alpha\cup x_1\cup x_2\cup x_3)\,[X] = \det X \pmod 2$. $\square$

4 The Donaldson invariants

In this section we introduce the $\bar\lambda$–invariants of $\mathbb{Z}[\mathbb{Z}]$–homology 4–tori via counting projectively flat connections on admissible $U(2)$–bundles. The definition is similar to that of the Donaldson invariants, but with two important differences. First, the metric perturbations that are usually used in the definition of Donaldson invariants have no effect here, so we make use of holonomy perturbations as employed in [7, 27, 5, 14] to obtain zero dimensional perturbed moduli spaces whose points may be counted (with signs). We make use of the standard cobordism resulting from a homotopy of perturbations to show that the invariants $\bar\lambda$ are independent of the perturbations. Second, we make essential use in our counting arguments of an action of $H^1(X;\mathbb{Z}_2)$ on the space of connections, and must take care that the perturbations we use are equivariant with respect to this action. A subtle point here is that we do *not* need to use the stronger claim that the action of $H^1(X;\mathbb{Z}_2)$ extends to the cobordism constructed from a homotopy of perturbations, compare with Remark 4.9. This stronger statement would be interesting to establish, but is beyond the methods of the current paper.

4.1 Admissible bundles

Let X be a homology 4–torus and consider a principal $U(2)$–bundle P over X and its associated $SO(3)=PU(2)$–bundle $\bar{P}$. The characteristic classes of P and $\bar{P}$ are related by the formulas

$$w_2(\bar{P}) = c_1(P) \pmod 2 \quad \text{and} \quad p_1(\bar{P}) = c_1(P)^2 - 4c_2(P).$$

Since $H^2(X;\mathbb{Z})$ is torsion free and $p_1(\bar{P}) = w_2(\bar{P})^2 \pmod 4$, every $SO(3)$–bundle over X arises as $\bar{P}$ for some $U(2)$–bundle P.

A $U(2)$–bundle P over a homology 4–torus X is called *admissible* if $p_1(\bar{P})=0$, $w_2(\bar{P})\neq 0$, and, in case X is even, there exists a $\xi\in H^1(X;\mathbb{Z}_2)$ such that $w_2(\bar{P})\cup\xi\neq 0$. Note that, for an admissible bundle, $w_2(\bar{P})\cup w_2(\bar{P}) = p_1(\bar{P}) = 0 \pmod 4$. Also note that according to Corollary 2.8, admissible bundles exist over any $\mathbb{Z}[\mathbb{Z}]$–homology 4–torus.

The existence of $\xi\in H^1(X;\mathbb{Z}_2)$ such that $w_2(\bar{P})\cup\xi\neq 0$ is automatic for odd homology tori (by Poincaré duality), therefore, any $U(2)$–bundle P over an odd homology torus such that $p_1(\bar{P})=0$ and $w_2(\bar{P})\neq 0$ is admissible. This is not the case for even homology tori, as can be seen from the following example.

Consider an even homology 3–torus, for example $M=\#_3\,(S^1\times S^2)$ or the result of 0–surgery on a band-sum of the Borromean rings as in [25, §5]. Note that the cup-product $\Lambda^2 H^1(M;\mathbb{Z}_2)\to H^2(M;\mathbb{Z}_2)$ is trivial and that $X = M\times S^1$ is an even $\mathbb{Z}[\mathbb{Z}]$–homology 4–torus. Choose a nonzero class $\alpha\in H^1(M;\mathbb{Z}_2)$, and let δ be the class pulled back from the S^1 factor. Now let $\bar{P}$ be the bundle with $p_1(\bar{P})=0$ and $w_2(\bar{P})=\alpha\cup\delta$. No $U(2)$–bundle P lifting $\bar{P}$ is admissible because any class $\xi\in H^1(X;\mathbb{Z}_2)$ has trivial cup product with $w_2(\bar{P})$.

4.2 $U(2)$ vs $SO(3)$ connections

Let X be a $\mathbb{Z}[\mathbb{Z}]$–homology 4–torus and fix a Riemannian metric on X. Every connection on a $U(2)$–bundle P over X induces connections on $\bar{P}$ and on the line bundle $\det P$, according to the splitting $\mathfrak{u}(2) = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$. If A is a $\mathfrak{u}(2)$–valued 1–form representing the connection in a local trivialization, this corresponds to the decomposition

$$A = \left(A - \frac{1}{2}\,\operatorname{tr} A \cdot \mathrm{Id}\right) + \frac{1}{2}\,\operatorname{tr} A \cdot \mathrm{Id}\,.$$

(In our discussion, we will denote by A either a connection, or its local representation as a 1-form; most calculations will involve the local representation.) The induced connection on $\bar{P}$ is the image of the first summand under the isomorphism $\mathrm{ad} : \mathfrak{su}(2) \to \mathfrak{so}(3)$ given by $\mathrm{ad}(\xi)(\eta) = [\xi, \eta]$, and the induced connection on $\det P$ is $\operatorname{tr} A$. Conversely, any two connections on $\bar{P}$ and $\det P$ determine a unique connection on P.

Fix a connection C on $\det P$ and let $\mathcal{A}(P)$ be the space of connections on P of Sobolev class L^2_l with $l \geq 3$, compatible with C. The assumption $l \geq 3$ ensures that $L^2_l \subset C^0$. The gauge group $\mathcal{G}(P)$ consisting of unitary automorphisms of P of Sobolev class L^2_{l+1} having determinant one preserves C and hence acts on $\mathcal{A}(P)$ with the quotient space $\mathcal{B}(P) = \mathcal{A}(P)/\mathcal{G}(P)$. Let $\mathcal{A}(\bar{P})$ be the affine space of connections on $\bar{P}$ and $\mathcal{G}(\bar{P})$ the full $SO(3)$ gauge group. Denote $\mathcal{B}(\bar{P}) = \mathcal{A}(\bar{P})/\mathcal{G}(\bar{P})$. The natural projection $\pi : \mathcal{A}(P) \to \mathcal{A}(\bar{P})$ commutes with the above gauge group actions and hence defines a projection

$$\pi : \mathcal{B}(P) \to \mathcal{B}(\bar{P}). \tag{10}$$

The space $\mathcal{B}(P)$ admits an $H^1(X;\mathbb{Z}_2)$ action whose orbit map is the map (10). The rest of this subsection is devoted to a detailed description of this action, compare with [6, Section 5.6].

Let $\mathcal{O}_2 : \mathcal{G}(\bar{P}) \to H^1(X;\mathbb{Z}_2)$ be the obstruction to lifting an automorphism $\bar{g} : \bar{P} \to \bar{P}$ to an $SU(2)$ gauge transformation. Under the standard identification of $H^1(X;\mathbb{Z}_2)$ as $\mathrm{Hom}(\pi_1 X, \mathbb{Z}_2)$, the evaluation of $\mathcal{O}_2(\bar{g})$ on a loop ψ may be calculated as follows. View $\bar{g} \in \mathcal{G}(\bar{P})$ as a section of the bundle $\mathrm{Ad}\,\bar{P} = \bar{P} \times_{\mathrm{Ad}} SO(3)$. The bundle $\mathrm{Ad}\,\bar{P}$ is trivial over ψ so the restriction of $\bar{g}$ on ψ can be viewed as simply a map to $SO(3)$. Then $\mathcal{O}_2(\bar{g})(\psi) = \bar{g}_*(\psi) \in \pi_1(SO(3)) = \mathbb{Z}_2$. Obstruction theory can be used to show that $\mathcal{O}_2 : \mathcal{G}(\bar{P}) \to H^1(X;\mathbb{Z}_2)$ is surjective.

Lemma 4.1 *Every gauge transformation $\bar{g} : \bar{P} \to \bar{P}$ lifts to a gauge transformation $g : P \to P$ of the $U(2)$–bundle P.*

Proof Observe that the adjoint representation $U(2) \to SO(3)$ gives rise to a $U(1)$ bundle $\pi : P \to \bar{P}$, and view gauge transformations $g : P \to P$ as sections of the bundle $\mathrm{Ad}\,P = P \times_{\mathrm{Ad}} U(2)$. The result will follow after we calculate the obstruction to lifting a section $\bar{g} : X \to \mathrm{Ad}\,\bar{P}$ to a section $g : X \to \mathrm{Ad}\,P$, and then show that this obstruction vanishes.

Since the fiber of $\pi : \mathrm{Ad}\,P \to \mathrm{Ad}\,\bar{P}$ is $U(1)$, there is a single obstruction $o(\bar{g}) \in H^2(X;\mathbb{Z})$ to the existence of a lift. By a standard argument, this obstruction is the pullback, via $\bar{g} : X \to \mathrm{Ad}\,\bar{P}$, of the obstruction $\mathcal{O} \in H^2(\mathrm{Ad}\,\bar{P};\mathbb{Z})$ to the existence of a section of π.

In order to evaluate $\mathcal{O}$, we apply the Leray–Serre spectral sequence to the $SO(3)$–bundle $\mathrm{Ad}\,\bar{P} \to X$ to calculate the second cohomology of $\mathrm{Ad}\,\bar{P}$. The only relevant non–trivial differentials are those at the E_3 level, and these would be $d_3 : E_3^{0,2} \to E_3^{3,0}$. But the former group is $H^2(SO(3);\mathbb{Z}) = \mathbb{Z}_2$ and the latter is $H^3(X;\mathbb{Z})$, which is torsion free. Thus we have an exact sequence

$$0 \longrightarrow H^2(X;\mathbb{Z}) \xrightarrow{p^*} H^2(\mathrm{Ad}\,\bar{P};\mathbb{Z}) \xrightarrow{i^*} H^2(SO(3);\mathbb{Z}) \longrightarrow 0,$$

where $i : SO(3) \to \mathrm{Ad}\,\bar{P}$ is the inclusion of $SO(3)$ as the fiber. This sequence is split because there is a section of the bundle $\mathrm{Ad}\,\bar{P} \to X$, which is the identity gauge transformation $\mathrm{Id}_{\bar{P}}$. Thus $H^2(\mathrm{Ad}\,\bar{P};\mathbb{Z}) \cong H^2(X;\mathbb{Z}) \oplus H^2(SO(3);\mathbb{Z})$. In particular, since the cohomology of X is torsion free, there is a unique 2–torsion class $t \in H^2(\mathrm{Ad}\,\bar{P};\mathbb{Z})$, and this class restricts to $H^2(SO(3);\mathbb{Z})$ as the generator.

We claim that $\mathcal{O} = t$. Since the above exact sequence splits, it suffices to show that $\mathrm{Id}_{\bar{P}}^*(\mathcal{O}) = 0$ but that $\mathcal{O}$ is nontrivial. Note that the identity gauge transformation $\mathrm{Id}_{\bar{P}}$ lifts to the identity transformation of P, hence $\mathrm{Id}_{\bar{P}}^*(\mathcal{O})$, which is the obstruction to this lifting, must vanish. On the other hand, we have that $i^*\mathcal{O} \neq 0 \in H^2(SO(3);\mathbb{Z})$ because this pullback represents the (complete) obstruction to a section of the adjoint representation $U(2) \to SO(3)$. But there certainly is no such section.

Finally, $o(\bar{g}) = \bar{g}^*(\mathcal{O})$ is a 2–torsion element of $H^2(X;\mathbb{Z})$ (since $\mathcal{O}$ is 2–torsion). Since $H^2(X;\mathbb{Z})$ is torsion free, $o(\bar{g}) = 0$ and $\bar{g}$ admits a lift. □

Given $\chi \in H^1(X;\mathbb{Z}_2)$, choose $\bar{g} \in \mathcal{G}(\bar{P})$ so that $\mathcal{O}_2(\bar{g}) = \chi$. Any gauge transformation $g : P \to P$ lifting $\bar{g}$ will be said to *realize* χ.

Lemma 4.2 *Let $\chi \in H^1(X;\mathbb{Z}_2)$ be realized by $g : P \to P$ and let $h = \det g : X \to S^1$. Then h induces a homomorphism $h_* \in \mathrm{Hom}\,(\pi_1 X, \mathbb{Z}) = H^1(X;\mathbb{Z})$ whose modulo 2 reduction is χ.*

Proof The result follows by a direct calculation with loops $\psi \in \pi_1 X$ using the observations that $\det : U(2) \to U(1)$ induces an isomorphism of the fundamental groups $\pi_1 U(2) = \pi_1 U(1) = \mathbb{Z}$ and that the homomorphism $\pi_1 U(2) \to \pi_1 SO(3)$ induced by the adjoint representation $U(2) \to SO(3)$ is the mod 2 reduction $\mathbb{Z} \to \mathbb{Z}_2$. $\square$

Given $\chi \in H^1(X;\mathbb{Z}_2)$, realize it by a gauge transformation $g : P \to P$ as above and define the action of χ on $\mathcal{B}(P)$ by the formula

$$\chi[A] = [\, g^* A - 1/2\, h\, dh^{-1}\,\mathrm{Id}\,] \tag{11}$$

where $h = \det g$ and, in local trivializations, $g^* A = g\, dg^{-1} + gAg^{-1}$. Since $\mathrm{tr}\,(g^* A - 1/2\, h\, dh^{-1}\,\mathrm{Id}) = \mathrm{tr}\, A + \mathrm{tr}\,(g\, dg^{-1}) - h\, dh^{-1} = \mathrm{tr}\, A$, the determinant connection is preserved and hence $\chi[A]$ belongs to $\mathcal{B}(P)$.

Any two gauge transformations $g_1, g_2 : P \to P$ lifting $\bar g : \bar P \to \bar P$ differ by an automorphism $\gamma = c\,\mathrm{Id}$, where $c \in U(1)$, so that $g_2 = \gamma g_1$. A straightforward calculation then shows that

$$\begin{aligned}
g_2^* A - 1/2\, h_2\, dh_2^{-1}\,\mathrm{Id} &= (\gamma g_1)^* A - 1/2\; c^2 h_1\, d(c^{-2} h_1^{-1})\;\mathrm{Id} \\
&= g_1^* A + \gamma d\gamma^{-1} - 1/2\, c^2 h_1 (2c^{-1} dc^{-1} h_1^{-1} + c^{-2} dh_1^{-1})\,\mathrm{Id} \\
&= g_1^* A + \gamma\, d\gamma^{-1} - c\, dc^{-1}\;\mathrm{Id} - 1/2\, h_1\, dh_1^{-1}\,\mathrm{Id} \\
&= g_1^* A - 1/2\, h_1\, dh_1^{-1}\,\mathrm{Id},
\end{aligned} \tag{12}$$

which implies that (11) is independent of the choice of a lift of $\bar g : \bar P \to \bar P$. Furthermore, any two choices of $\bar g$ realizing the same χ differ by an $SU(2)$ gauge transformation, and so do their respective lifts $g_1, g_2 : P \to P$. But then the connections $g_2^* A - 1/2\, h_2\, dh_2^{-1}\,\mathrm{Id}$ and $g_1^* A - 1/2\, h_1\, dh_1^{-1}\,\mathrm{Id}$ differ by an $SU(2)$ gauge transformation and thus define the same element in $\mathcal{B}(P)$. Therefore, (11) is independent of the choices made in its definition.

Given χ_1 and $\chi_2 \in H^1(X;\mathbb{Z}_2)$ realized by their respective lifts g_1 and $g_2 : P \to P$, the product $\chi_1\chi_2$ can be realized by the composition $g_1 g_2 : P \to P$. In particular,

$$\begin{aligned}
\chi_1(\chi_2[A]) &= [\, g_1^*(g_2^* A - 1/2\, h_2\, dh_2^{-1}\,\mathrm{Id}) - 1/2\, h_1\, dh_1^{-1}\,\mathrm{Id}\,] \\
&= [\,(g_1 g_2)^* - 1/2\, h_1 h_2\, d(h_1 h_2)^{-1}\,\mathrm{Id}\,] = (\chi_1\chi_2)[A]
\end{aligned}$$

since $h_1 = \det g_1$ and $h_2 = \det g_2$ commute. Moreover, $\chi = 0$ can be realized by $\mathrm{Id} : P \to P$ and hence induces an identity map on $\mathcal{B}(P)$. This completes the definition of the $H^1(X;\mathbb{Z}_2)$ action on $\mathcal{B}(P)$.

4.3 Projectively ASD connections

A connection A on an admissible bundle P is called *projectively ASD* if the connection $\bar{A}$ it induces on the bundle $\bar{P}$ is ASD. The latter means that $F_+(\bar{A}) = 0$ where, as usual, $F_+(\bar{A}) = 1/2\cdot(F(\bar{A}) + *\,F(\bar{A}))$ is the self dual part of the curvature. Equivalently, a connection A on an admissible bundle P is projectively ASD if $F_+(A)$ is central. Once a connection C on $\det P$ is fixed, the space of all projectively ASD connections on P compatible with C, modulo gauge group $\mathcal{G}(P)$, will be denoted by $\mathcal{M}(P)$. It is called the *moduli space of projectively ASD connections*; different choices of C give equivalent moduli spaces. The moduli space $\mathcal{M}(P)$ (and its perturbed version, $\mathcal{M}_\sigma(P)$, defined below) depends on the choice of Riemannian metric on X. When we want to indicate this dependence, we will use the notation $\mathcal{M}_g(P)$ (or $\mathcal{M}_{g,\sigma}(P)$).

The ASD equation is equivariant with respect to the $H^1(X;\mathbb{Z}_2)$ action in the following sense. The group $\mathcal{G}(P)$ acts on $\Omega^2_+(X;\operatorname{ad}\bar{P})$ completed in Sobolev L^2_{l-1} norm by pull back of differential forms, an element $g\in\mathcal{G}(P)$ sending ω to $(\operatorname{ad}\bar{g})^*\omega$. Let

$$\mathcal{E}(P) = \mathcal{A}(P)\times_{\mathcal{G}(P)}\Omega^2_+(X;\operatorname{ad}\bar{P}) \tag{13}$$

be the vector bundle over $\mathcal{B}(P)$ associated with the principal $\mathcal{G}(P)$ bundle $\mathcal{A}(P)\to\mathcal{B}(P)$. One can easily check that the formula $F_+[A] = [A, F_+(\bar{A})]$ defines a smooth section $F_+:\mathcal{B}(P)\to\mathcal{E}(P)$ of the above bundle.

Proposition 4.3 *There is a natural action of $H^1(X;\mathbb{Z}_2)$ on the bundle $\mathcal{E}(P)\to\mathcal{B}(P)$ extending the action on $\mathcal{B}(P)$ such that the section F_+ is equivariant.*

Proof Given a $\chi\in H^1(X;\mathbb{Z}_2)$ and a gauge transformation $g:P\to P$ realizing it, define $\chi:\mathcal{E}(P)\to\mathcal{E}(P)$ by the formula $\chi[A,\omega] = [g^*A - 1/2\,h\,dh^{-1}\,\mathrm{Id}, (\operatorname{ad}\bar{g})^*\omega]$ where $h=\det g$. That this definition is independent of the arbitrary choices follows by essentially the same argument as well–definedness of the action (11) on $\mathcal{B}(P)$. The equivariance of F_+ is immediate from the observation that $h\,dh^{-1}$ is a closed 1–form and hence $F(g^*A - 1/2\,h\,dh^{-1}\,\mathrm{Id}) = (\operatorname{ad} g)^*F(A)$. $\square$

Corollary 4.4 *The moduli space $\mathcal{M}(P)$ is the zero set of the equivariant section $F_+:\mathcal{B}(P)\to\mathcal{E}(P)$ and as such it is acted upon by $H^1(X;\mathbb{Z}_2)$. The quotient space of this action is the moduli space $\mathcal{M}(\bar{P})$ of ASD connections on $\bar{P}$ modulo gauge group $\mathcal{G}(\bar{P})$, compare with (10).*

Let $A \in \mathcal{A}(P)$ then the map $\mathcal{A}(P) \to \Omega^1(X;\operatorname{ad}\bar{P})$ sending B to $\bar{B} - \bar{A}$ establishes an isomorphism between $T_A\mathcal{A}(P)$ and $\Omega^1(X;\operatorname{ad}\bar{P})$ completed in Sobolev L^2_l norm. Under this isomorphism, the slice through A of the $\mathcal{G}(P)$ action on $\mathcal{A}(P)$ is isomorphic to $\ker d^*_{\bar{A}} \subset \Omega^1(X;\operatorname{ad}\bar{P})$. The map $F_+ : \mathcal{B}(P) \to \mathcal{E}(P)$ linearizes at $[A]$ to $d^+_{\bar{A}} : \ker d^*_{\bar{A}} \to \Omega^2_+(X;\operatorname{ad}\bar{P})$ so the local structure of $\mathcal{M}(P)$ near $[A]$ is described by the deformation complex

$$\Omega^0(X;\operatorname{ad}\bar{P}) \xrightarrow{d_{\bar{A}}} \Omega^1(X;\operatorname{ad}\bar{P}) \xrightarrow{d^+_{\bar{A}}} \Omega^2_+(X;\operatorname{ad}\bar{P}) \tag{14}$$

with cohomology $H^0(X;\operatorname{ad}A)$, $H^1(X;\operatorname{ad}A)$, and $H^2_+(X;\operatorname{ad}A)$. Note for future use that the deformation complexes at $[A]$ and $\chi[A] \in \mathcal{M}(P)$ are isomorphic to each other via $(\operatorname{ad}\bar{g})^*$ where g realizes χ:

$$\begin{array}{ccccc}
\Omega^0(X;\operatorname{ad}\bar{P}) & \xrightarrow{d_{\bar{g}^*\bar{A}}} & \Omega^1(X;\operatorname{ad}\bar{P}) & \xrightarrow{d^+_{\bar{g}^*\bar{A}}} & \Omega^2_+(X;\operatorname{ad}\bar{P}) \\
\Big\downarrow{\scriptstyle (\operatorname{ad}\bar{g})^*} & & \Big\downarrow{\scriptstyle (\operatorname{ad}\bar{g})^*} & & \Big\downarrow{\scriptstyle (\operatorname{ad}\bar{g})^*} \\
\Omega^0(X;\operatorname{ad}\bar{P}) & \xrightarrow{d_{\bar{A}}} & \Omega^1(X;\operatorname{ad}\bar{P}) & \xrightarrow{d^+_{\bar{A}}} & \Omega^2_+(X;\operatorname{ad}\bar{P}).
\end{array} \tag{15}$$

The above allows for a computation of the formal dimensions of both $\mathcal{M}(P)$ and $\mathcal{M}(\bar{P})$ using the index theorem: since X is a $\mathbb{Z}[\mathbb{Z}]$–homology 4–torus, we have $b^+_2(X) = 3$ and therefore

$$\dim \mathcal{M}(P) = \dim \mathcal{M}(\bar{P}) = -2p_1(\bar{P}) - 3(1 - b_1 + b^+_2)(X) = 0.$$

A straightforward application of the Chern–Weil theory to the bundle $\bar{P}$ with $p_1(\bar{P}) = 0$ shows that projectively ASD connections on P are in fact *projectively flat*, which means that they induce flat connections on $\bar{P}$. Therefore, the moduli spaces $\mathcal{M}(P)$ and $\mathcal{M}(\bar{P})$ can be viewed as moduli spaces of projectively flat (respectively, flat) connections. The holonomy correspondence then identifies $\mathcal{M}(\bar{P})$ with a compact subset of the $SO(3)$–character variety of $\pi_1 X$. Since $\mathcal{M}(P)$ is a finite sheeted covering of $\mathcal{M}(\bar{P})$, see Corollary 4.4, it is also compact. Proposition 5.3 below will provide an explicit description of $\mathcal{M}(P)$ in terms of (projective) $SU(2)$–representations of $\pi_1 X$.

We say that $\mathcal{M}(P)$ is *non-degenerate* at $[A] \in \mathcal{M}(P)$ if $H^2_+(X;\operatorname{ad}A) = 0$. According to (15), $\mathcal{M}(P)$ is non–degenerate at $\chi[A]$ if and only if it is non–degenerate at $[A]$. We say that $\mathcal{M}(P)$ is *non–degenerate* if it is non–degenerate at all $[A] \in \mathcal{M}(P)$. The above discussion implies that, if $\mathcal{M}(P)$ is non–degenerate then it is a smooth compact manifold of dimension zero. It is canonically oriented once a homology orientation is fixed, see [5, Section 3], and the action of $H^1(X;\mathbb{Z}_2)$ on $\mathcal{M}(P)$ is then orientation preserving, see [5, Corollary 3.27].

If $\mathcal{M}(P)$ fails to be non–degenerate, the same holds true after $\mathcal{M}(P)$ is properly perturbed.

4.4 Admissible perturbations

Let S^1 be the interval $[0,1]$ with identified ends, and consider an embedding $\psi : S^1 \times D^3 \to X$ so that, for each point $z \in D^3$, we have a loop $\psi_z = \psi(S^1 \times \{z\})$. The loop ψ_0 will also be called ψ. We will say that ψ is *mod-2 trivial* if $0 = [\psi] \in H_1(X;\mathbb{Z}_2)$.

Given $A \in \mathcal{A}(P)$, denote by $\mathrm{hol}_A(\psi)$ the function on $\psi(S^1 \times D^3)$ whose value at $\psi(s,z)$ equals the $SU(2)$ holonomy of A around the loop ψ_z starting at the point $\psi(s,z)$. Recall from [22, Section 3.2] that, once a connection C on $\det P$ is fixed, a choice of square roots of hol_C on a set of representative loops makes $\mathrm{hol}_A(\psi)$ into a well defined $SU(2)$ valued function, with different choices leading to equivalent theories. Let $\Pi : SU(2) \to \mathfrak{su}(2)$ be the projection given by

$$\Pi(u) = u - \frac{1}{2}\,\mathrm{tr}(u)\cdot \mathrm{Id}\,.$$

It is equivariant with respect to the adjoint action of $SU(2)$ on both $SU(2)$ and $\mathfrak{su}(2)$, therefore, $\Pi\,\mathrm{hol}_A(\psi)$ is a well defined section of $\mathrm{ad}\,\bar{P}$ over $\psi(S^1 \times D^3)$ after we identify $\mathfrak{su}(2)$ with $\mathfrak{so}(3)$. For any $\nu \in \Omega^2_+(X)$ supported in $\psi(S^1 \times D^3)$, the formula

$$\sigma(\nu,\psi)[A] = [\,A, \Pi\,\mathrm{hol}_A(\psi)\,\otimes\,\nu\,] \tag{16}$$

defines a section $\sigma(\nu,\psi) : \mathcal{B}(P) \to \mathcal{E}(P)$ of the bundle (13). That $\sigma(\nu,\psi)$ is well defined follows from the fact that $\Pi\,\mathrm{hol}_{g^*A}(\psi) = \mathrm{ad}\,\bar{g}\cdot\Pi\,\mathrm{hol}_A(\psi)$ for any $g \in \mathcal{G}(P)$.

Proposition 4.5 *If ψ is mod-2 trivial then the section (16) is equivariant with respect to the $H^1(X;\mathbb{Z}_2)$ action on the bundle $\mathcal{E}(P) \to \mathcal{B}(P)$, compare with Proposition 4.3.*

Proof Given $\chi \in H^1(X;\mathbb{Z}_2)$, realize it by a gauge transformation $g : P \to P$. Let $h = \det g : X \to S^1$ and consider the flat connection $1/2\,h\,dh^{-1}$ on a trivial line bundle $L \to X$. We wish first to compute the holonomy of this connection.

For an arbitrary loop $\psi : S^1 \to X$, the restriction of h to ψ is a map $S^1 \to S^1$ of the form $t \to \exp(if(t))$ where $f(2\pi) = f(0) + 2\pi k$ and $k = h_*(\psi) \in \mathbb{Z}$. Then $1/2\,h\,dh^{-1} = i/2\,f'(t)\,dt$, and the holonomy of $1/2\,h\,dh^{-1}$ around ψ equals

$$\exp\left(\frac{i}{2}\int_0^{2\pi} f'(t)\,dt\right) = \exp\left(\frac{i}{2}\left(f(2\pi)-f(0)\right)\right) = \exp(i\pi k) = \chi(\psi),$$

where χ is viewed as a homomorphism from $\pi_1 X$ to $\mathbb{Z}_2 = \{\pm 1\}$, compare with Lemma 7.3.

Let ψ be a mod-2 trivial loop. For any $A \in \mathcal{A}(P)$, the connection $g^*A - 1/2\,h\,dh^{-1}\,\mathrm{Id} = g^*\left(A - 1/2\,h\,dh^{-1}\,\mathrm{Id}\right)$ can be viewed as the pull back via $g : P \to P$ of the connection on $P\otimes L = P$ induced by A and $-1/2\,h\,dh^{-1}$. Then

$$\begin{aligned}\Pi\,\mathrm{hol}_{g^*A-1/2\,h\,dh^{-1}\,\mathrm{Id}}(\psi) &= \mathrm{ad}\,\bar g\cdot\Pi(\mathrm{hol}_A(\psi)\cdot\mathrm{hol}_{-1/2\,h\,dh^{-1}}(\psi))\\ &= \mathrm{ad}\,\bar g\cdot\Pi\left(\mathrm{hol}_A(\psi)\cdot\chi(\psi)\right) = \mathrm{ad}\,\bar g\cdot\Pi\,\mathrm{hol}_A(\psi),\end{aligned}$$

after we notice that $\chi(\psi)=1$ because $\chi:\pi_1X\to\mathbb{Z}_2$ factors through $H_1(X;\mathbb{Z}_2)$ and ψ is mod-2 trivial. Therefore,

$$\Pi\,\mathrm{hol}_{g^*A-1/2\,h\,dh^{-1}\,\mathrm{Id}}(\psi)\otimes\nu = (\mathrm{ad}\,\bar g)^*(\Pi\,\mathrm{hol}_A(\psi)\otimes\nu),$$

which implies equivariance of the section $\sigma(\nu,\psi)$. □

An *admissible perturbation* is a smooth section σ of the bundle $\mathcal{E}(P)\to\mathcal{B}(P)$ of the form

$$\sigma = \sum_{k=1}^{N}\varepsilon_k\cdot\sigma(\nu_k,\psi_k),\quad \varepsilon_k\in\mathbb{R}, \tag{17}$$

where $\sigma(\nu_k,\psi_k)$ are sections (16) built from a family of loops $\psi_k : S^1\times D^3\to X$ with disjoint images, and $\nu_k\in\Omega^2_+(X)$ are self–dual 2–forms, each supported in its respective $\psi_k(S^1\times D^3)$. An admissible perturbation is *equivariant* if it is equivariant with respect to the action of $H^1(X;\mathbb{Z}_2)$ on the bundle $\mathcal{E}(P)\to\mathcal{B}(P)$. According to Proposition 4.5, all admissible perturbations built from mod-2 trivial loops ψ_k are equivariant.

Let $\sigma:\mathcal{B}(P)\to\mathcal{E}(P)$ be an admissible perturbation and define $\zeta_\sigma:\mathcal{B}(P)\to\mathcal{E}(P)$ by the formula $\zeta_\sigma = F_+ + \sigma$. We call a connection $A\in\mathcal{A}(P)$ *perturbed projectively ASD* if $\zeta_\sigma[A]=0$, and denote by $\mathcal{M}_\sigma(P)=\zeta_\sigma^{-1}(0)$ the moduli space of perturbed projectively ASD connections. If $\sigma=0$ then $\mathcal{M}_\sigma(P)$ is the moduli space $\mathcal{M}(P)$ of projectively ASD (equivalently, projectively flat) connections.

Any admissible perturbation $\sigma:\mathcal{B}(P)\to\mathcal{E}(P)$ factors through the embedding $\mathcal{E}'(P)\to\mathcal{E}(P)$ where $\mathcal{E}'(P)$ is the bundle (13) with fiber $\Omega^2_+(X;\mathrm{ad}\,\bar P)$ completed in Sobolev L^2_l norm, see [14, Proposition 7]. Since the inclusion $L^2_l\to L^2_{l-1}$ is compact, the derivatives of σ are compact operators and hence the perturbed section ζ_σ is Fredholm of index zero.

Proposition 4.6 *For every admissible perturbation σ, the perturbed moduli space $\mathcal{M}_\sigma(P)$ is compact.*

Proof The key to proving compactness is the observation that the forms $\Pi\,\mathrm{hol}_A(\psi)\otimes\nu$ admit a uniform L^∞ bound independent of A. Given a sequence $[A_n]\in\mathcal{M}_\sigma(P)$, the perturbed projectively ASD equations then imply that the $F_+(\bar{A}_n)$ are uniformly bounded in L^∞, and $F(\bar{A}_n)$ are uniformly bounded in L^2 because of the Chern–Weil formula. Using Uhlenbeck's compactness theorem [29] in the absence of bubbling, we conclude that (after passing to a subsequence and gauge equivalent connections, if necessary) the sequence A_n converges in L^p_1 for all $p\geq 2$ to a connection A such that $[A]\in\mathcal{M}_\sigma(P)$, compare with [14, Proposition 11]. Finally, bootstrapping leads to the C^∞ convergence. □

The local structure of $\mathcal{M}_\sigma(P)$ near a point $[A]$ is described by the deformation complex

$$\Omega^0(X,\mathrm{ad}\,\bar{P}) \xrightarrow{d_{\bar{A}}} \Omega^1(X,\mathrm{ad}\,\bar{P}) \xrightarrow{d^+_{\bar{A}}+D_A\sigma} \Omega^2_+(X,\mathrm{ad}\,\bar{P}). \tag{18}$$

Here, assuming that σ is given by the formula (17), we have $D_A\sigma=\sum\varepsilon_k\cdot D_A\sigma(\nu_k,\psi_k)$ where $D_A\sigma(\nu_k,\psi_k)$ is the differential at A of the function $\mathcal{A}(P)\to\Omega^2_+(X;\mathrm{ad}\,\bar{P})$ mapping A to $\Pi\,\mathrm{hol}_A(\psi_k)\otimes\nu_k$.

We call $\mathcal{M}_\sigma(P)$ *non-degenerate* at $[A]\in\mathcal{M}_\sigma(P)$ if the second cohomology of the complex (18) vanishes, that is, $\operatorname{coker}(d^+_{\bar{A}}+D_A\sigma)=0$. We say that $\mathcal{M}_\sigma(P)$ is *non-degenerate* if it is non–degenerate at all $[A]\in\mathcal{M}_\sigma(P)$. A non–degenerate $\mathcal{M}_\sigma(P)$ is a compact zero–dimensional manifold, canonically oriented by a choice of homological orientation. If σ is equivariant then, according to Proposition 4.5, $\mathcal{M}_\sigma(P)$ is acted upon by $H^1(X;\mathbb{Z}_2)$, and this action is orientation preserving.

Our immediate goal will be to show that $\mathcal{M}_\sigma(P)$ is non–degenerate for a generic (not necessarily equivariant) admissible perturbation σ. This will allow us to define the Donaldson invariant in Section 4.5; later in the paper, we will prove a more refined result that $\mathcal{M}_\sigma(P)$ can be made non–degenerate using a generic *equivariant* admissible perturbation σ.

Recall that, after making certain choices as in [22, Section 3.2], the holonomy of a connection $A\in\mathcal{A}(P)$ can be viewed as a map $\mathrm{hol}_A:\Omega(X,x)\to SU(2)$ on the monoid of loops based at $x\in X$ such that $\mathrm{hol}_A(\psi_1\psi_2)=\pm\,\mathrm{hol}_A(\psi_1)\,\mathrm{hol}_A(\psi_2)$. We will call A *reducible* if the image of hol_A is contained in a copy of $U(1)\subset SU(2)$, and *irreducible* otherwise. Among reducible connections, those with the image of hol_A in the center of $SU(2)$ will be called *central*.

Lemma 4.7 *If A is a projectively flat irreducible connection on P then there are finitely many loops $\psi_k : S^1 \times D^3 \to X$ with disjoint images and self–dual 2–forms ν_k, each supported in a small ball inside $\psi_k(S^1 \times D^3)$, such that the sections $\Pi \operatorname{hol}_A(\psi_k) \otimes \nu_k \in \Omega^2_+(X; \operatorname{ad} \bar{P})$ generate the vector space $H^2_+(X; \operatorname{ad} A)$.*

Proof This is essentially Lemma 2.5 from Donaldson [5]. □

The moduli space $\mathcal{M}(P)$ consists of the gauge equivalence classes of projectively flat connections all of which are irreducible because of the assumption that $w_2(\bar{P}) \neq 0$. Since $\mathcal{M}(P)$ is compact, we can suppose that the loops ψ_k and the forms ν_k can be chosen so that Lemma 4.7 holds for all the points $[A] \in \mathcal{M}(P)$ simultaneously. We fix a choice of ψ_k and ν_k, and define σ by the formula (17).

Proposition 4.8 *There exists a real number $r > 0$ such that, with the choice of ψ_k and ν_k as above, the moduli space $\mathcal{M}_\sigma(P)$ is non–degenerate for a generic $\bar{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_N)$ inside the ball $\|\bar{\varepsilon}\| < r$ in $\mathbb{R}^N$.*

Proof Let us pull the bundle $\mathcal{E}(P)$ back to a bundle $\pi^*\mathcal{E}(P) \to \mathcal{B}(P) \times \mathbb{R}^N$ via the projection $\pi : \mathcal{B}(P) \times \mathbb{R}^N \to \mathcal{B}(P)$, and consider the "universal section"

$$\Psi([A], \bar{\varepsilon}) = [\, A, \bar{\varepsilon}, F_+(\bar{A}) + \sum \varepsilon_k \cdot \Pi \operatorname{hol}_A(\psi_k) \otimes \nu_k \,]. \tag{19}$$

Let $[A] \in \mathcal{M}(P)$ and choose $a \in T_{[A]}\mathcal{A}(P) = \ker d^*_{\bar{A}}$, $\bar{\eta} = (\eta_1, \dots, \eta_N) \in \mathbb{R}^N$, and a small $t \in \mathbb{R}$. Then, up to higher order terms in t, we have

$$\begin{aligned} \Psi([A + ta], t\bar{\eta}) &= [\, A + ta, t\bar{\eta}, F_+(\bar{A} + ta) + \sum t\eta_k \cdot \Pi \operatorname{hol}_{A+ta}(\psi_k) \otimes \nu_k \,] \\ &= \Psi([A], 0) + t\, [\, a,\, \bar{\eta},\, d^+_{\bar{A}} a + \sum \eta_k \cdot \Pi \operatorname{hol}_A(\psi_k) \otimes \nu_k \,] + \dots \end{aligned}$$

Since sections $\Pi \operatorname{hol}_A(\psi_k) \otimes \nu_k$ generate $H^2_+(X; \operatorname{ad} A) = \operatorname{coker} d^+_{\bar{A}}$, we conclude that Ψ is transversal at $\mathcal{M}(P) \times \{0\}$ to the zero section of $\pi^*\mathcal{E}(P) \to \mathcal{B}(P) \times \mathbb{R}^N$. Therefore, there exist an open neighborhood $\mathcal{U}$ of $\mathcal{M}(P)$ in $\mathcal{B}(P)$ and a ball $\mathcal{V} \subset \mathbb{R}^N$ centered at the origin such that the intersection of $\Psi^{-1}(0)$ with the product $\mathcal{U} \times \mathcal{V} \subset \mathcal{B}(P) \times \mathbb{R}^N$ is a smooth manifold of dimension N.

Let us consider the projection $p : \Psi^{-1}(0) \cap (\mathcal{U} \times \mathcal{V}) \to \mathcal{V}$. For every $\bar{\varepsilon} \in \mathcal{V}$ we have $p^{-1}(\bar{\varepsilon}) = \mathcal{M}_\sigma(P) \cap \mathcal{U}$ where σ is given by the formula (17). By Uhlenbeck's compactness theorem, if $\bar{\varepsilon}$ is small enough, then the entire moduli space $\mathcal{M}_\sigma(P)$ is contained in $\mathcal{U}$: otherwise, there would exist a sequence A_n of perturbed projectively ASD connections with $\|F(\bar{A}_n)\|_{L^2} \to 0$ and $\|F_+(\bar{A}_n)\|_{L^p} \to$

0 but with no subsequence converging in L^p_1 to a flat connection. Thus there exists an $r > 0$ such that $p^{-1}(\bar\varepsilon) = \mathcal{M}_\sigma(P)$ as long as $\|\bar\varepsilon\| < r$. The Sard–Smale Theorem now implies that the projection p is a submersion for an open dense subset of the ball $\|\bar\varepsilon\| < r$, that is, that $\operatorname{coker}(d^+_{\tilde A} + D_A\sigma) = 0$ for all $[A] \in \mathcal{M}_\sigma(P)$. □

4.5 Definition of the invariants

Let P be an admissible $U(2)$–bundle over a $\mathbb{Z}[\mathbb{Z}]$–homology 4–torus X. After perturbation, if necessary, the moduli space $\mathcal{M}_\sigma(P)$ consists of finitely many points, and it is canonically oriented up to a choice of homology orientation. We define the invariant $\bar\lambda(X,P)$ as one quarter times signed count of points in $\mathcal{M}_\sigma(P)$,

$$\bar\lambda(X,P) = \frac{1}{4}\,\#\mathcal{M}_\sigma(P).$$

In what follows, we will only be interested in the modulo 2 reduction of $\bar\lambda(X,P)$, therefore, a particular choice of homology orientation, which is implicit in the above definition, will not matter.

The invariant $\bar\lambda(X,P)$ is well defined because, for any generic path of Riemannian metrics $g(t)$ and small perturbations $\sigma(t)$, $t \in [0,1]$, the moduli spaces $\mathcal{M}_{g(0),\sigma(0)}(P)$ and $\mathcal{M}_{g(1),\sigma(1)}(P)$ are cobordant via the compact oriented cobordism

$$\overline{\mathcal{M}}(P) = \bigcup_{t\in[0,1]} \mathcal{M}_{g(t),\sigma(t)}(P) \times \{t\}.$$

It is not hard to show that $\bar\lambda(X,P)$ is independent of the choice of determinant connection C.

Remark 4.9 It should be noted that the definition of $\bar\lambda(X,P)$ does not require that σ be equivariant. On the other hand, Sections 7 and 8 provide us with an equivariant admissible perturbation σ such that $\mathcal{M}_\sigma(P)$ is non–degenerate *and* is acted upon by $H^1(X;\mathbb{Z}_2)$. Since this action is orientation preserving, all points in the orbit of a (perturbed) projectively flat connection are counted in $\bar\lambda(X,P)$ with the same sign. It would be tempting to define the invariant $\bar\lambda(X,P)$ by counting points in $\mathcal{M}(\bar P)$ instead of those in $\mathcal{M}(P)$. However, such an approach comes against two problems: one is that the points of $\mathcal{M}(\bar P)$ would need to be counted with weights given by the order of the orbits of their respective lifts to $\mathcal{M}(P)$. A more serious problem would be proving the well definedness of such an invariant: one would have to deal with a generic path of *equivariant* perturbations, which might not be an easy task.

Remark 4.10 The class of admissible perturbations used in Sections 7 and 8 is larger than one defined in Section 4.4. However, the above argument showing well–definedness of $\bar{\lambda}(X,P)$ remains valid for these more general perturbations given their compactness properties, as described in Section 7.4.

5 Projective representations

The holonomy map embeds the moduli space $\mathcal{M}(\bar{P})$ of flat connections on $\bar{P}$ into the $SO(3)$–representation variety of $\pi_1(X)$. An attempt to extend this correspondence to the moduli space $\mathcal{M}(P)$ of projectively flat connections leads to the concept of a projective representation. We briefly describe the construction and refer the reader to [22] for a detailed treatment.

5.1 Algebraic background

Let G be a finitely presented group and view $\mathbb{Z}_2=\{\pm 1\}$ as the center of $SU(2)$. A map $\rho: G\to SU(2)$ is called a *projective representation* if $\rho(gh)\rho(h)^{-1}\rho(g)^{-1}$ belongs to $\mathbb{Z}_2$ for all $g,h\in G$. Given a projective representation ρ, the function $c: G\times G\to\mathbb{Z}_2$ defined as $c(g,h)=\rho(gh)\rho(h)^{-1}\rho(g)^{-1}$ is a 2–cocycle, that is, $c(gh,k)c(g,h)=c(g,hk)c(h,k)$. We will refer to c as the *cocycle associated with* ρ.

Let us fix a cocycle $c: G\times G\to\mathbb{Z}_2$ and denote by $\mathcal{PR}_c(G;SU(2))$ the set of conjugacy classes of projective representations $\rho: G\to SU(2)$ whose associated cocycle is c. Up to an isomorphism, this set only depends on the cohomology class $[c]\in H^2(G;\mathbb{Z}_2)$.

The group $H^1(G;\mathbb{Z}_2)$ acts on $\mathcal{PR}_c(G;SU(2))$ by the formula $\rho\mapsto\rho^{\chi}$ where $\rho^{\chi}(g)=\chi(g)\rho(g)$. The quotient space of this action is the space $\mathcal{R}_{[c]}(G;SO(3))$ consisting of (the conjugacy classes of) $SO(3)$–representations whose second Stiefel–Whitney class equals $[c]$. The projection map $\pi:\mathcal{PR}_c(G;SU(2))\to\mathcal{R}_{[c]}(G;SO(3))$ can be given explicitly by composing $\rho: G\to SU(2)$ with the adjoint representation $\operatorname{ad}: SU(2)\to SO(3)$.

A projective representation $\rho: G\to SU(2)$ is called *irreducible* if the centralizer of its image equals the center of $SU(2)$. All projective representations whose associated 2–cocycle is not homologous to zero are irreducible, assuming that $H_1(G;\mathbb{Z})$ has no 2–torsion.

5.2 The holonomy correspondence

In this section we establish a correspondence between projectively flat connections over a $\mathbb{Z}[\mathbb{Z}]$–homology 4–torus X and projective representations of its fundamental group. We first deal with the discrepancy arising from the fact that $H^2(\pi_1 X;\mathbb{Z}_2)$ and $H^2(X;\mathbb{Z}_2)$ need not be isomorphic.

Lemma 5.1 *Let X be a $\mathbb{Z}[\mathbb{Z}]$–homology torus and P an admissible $U(2)$–bundle over X. If $w_2(\bar P)$ does not belong to the image of $\iota : H^2(\pi_1 X;\mathbb{Z}_2) \to H^2(X;\mathbb{Z}_2)$, see (2), then the moduli space $\mathcal{M}(\bar P)$ is empty.*

Proof The Hopf exact sequence $\pi_2 X \to H_2(X;\mathbb{Z}) \to H_2(\pi_1 X;\mathbb{Z}) \to 0$, see [2], implies that, if $w_2(\bar P)$ does not belong to the image of ι, it evaluates non-trivially on a 2–sphere in X. Such a bundle $\bar P$ cannot support any flat connections, for a flat connection on $\bar P$ would pull back to a flat connection on the 2–sphere, whose holonomy would trivialize the bundle. □

Corollary 5.2 *The main theorem holds for all $\bar\lambda(X,P)$ such that $w_2(\bar P)$ is not in the image of $\iota : H^2(\pi_1 X;\mathbb{Z}_2) \to H^2(X;\mathbb{Z}_2)$.*

Proof Let P be an admissible $U(2)$–bundle such that $w_2(\bar P)$ is not in the image of ι. Then, according to Lemma 5.1, the moduli space $\mathcal{M}(\bar P)$ is empty. On the other hand, this situation is only possible if X is an even homology torus, see Corollary 2.3. □

From now on, we will concentrate on admissible $U(2)$–bundles P such that $w_2(\bar P)$ is in the image of $H^2(\pi_1 X;\mathbb{Z}_2)$, and will identify $H^2(\pi_1 X;\mathbb{Z}_2)$ with its (monomorphic) image in $H^2(X;\mathbb{Z}_2)$.

It is a well known fact that the holonomy defines a bijection $\bar\varphi : \mathcal{M}(\bar P) \to \mathcal{R}_w(X;SO(3))$ where $w = w_2(\bar P)$ (from now on we abbreviate $\mathcal{R}_w(\pi_1 X;SO(3))$ to $\mathcal{R}_w(X;SO(3))$ etc). The following result lifts this correspondence to the level of projectively flat connections. It is proved in [22].

Proposition 5.3 *Let P be an admissible $U(2)$–bundle over a homology 4–torus X, and c a 2–cocycle representing $w = w_2(\bar P)$. Then the holonomy correspondence defines a map $\varphi : \mathcal{M}(P) \to \mathcal{PR}_c(X;SU(2))$ which is an $H^1(X;\mathbb{Z}_2)$–equivariant bijection.*

In particular, we have the following commutative diagram whose horizontal arrows are bijections

$$\begin{array}{ccc}
\mathcal{M}(P) & \xrightarrow{\varphi} & \mathcal{PR}_c(X;SU(2)) \\
\Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\pi} \\
\mathcal{M}(\bar P) & \xrightarrow{\bar\varphi} & \mathcal{R}_w(X;SO(3))
\end{array}$$

Here, $\pi : \mathcal{PR}_c(X;SU(2)) \to \mathcal{R}_w(X;SO(3))$ is the natural projection map, see Section 5.1.

An argument similar to that for representation varieties shows that Zariski tangent space to $\mathcal{PR}_c(X;SU(2))$ at a projective representation $\rho : \pi_1(X) \to SU(2)$ equals $H^1(X;\operatorname{ad}\rho)$ where $\operatorname{ad}\rho : \pi_1(X) \to SU(2) \to SO(3)$ is a representation. It is identified as usual with the tangent space to $\mathcal{M}(P)$ at the corresponding projectively flat connection.

A point $\rho \in \mathcal{PR}_c(X;SU(2))$ is called *non–degenerate* if $H^1(X;\operatorname{ad}\rho) = 0$; the space $\mathcal{PR}_c(X;SU(2))$ is called *non–degenerate* if all of its points are non–degenerate. Since $p_1(\bar P) = 0$ the formal dimension of $\mathcal{M}(P)$ equals zero, and so $H^1(X;\operatorname{ad}\rho)$ and $H^2_+(X;\operatorname{ad}\rho)$ have the same dimension. In particular, $\mathcal{M}(P)$ is non–degenerate if and only if $\mathcal{PR}_c(X;SU(2))$ is, compare with Section 4.3. Because of the identification of Proposition 5.3, the invariant $\bar\lambda(X,P)$ in this situation can be computed as signed count of points in $\mathcal{PR}_c(X;SU(2))$ with $[c] = w_2(\bar P)$.

6 The four–orbits

Let X be a homology 4–torus. According to the action of $H^1(X;\mathbb{Z}_2) = (\mathbb{Z}_2)^4$ the space $\mathcal{PR}_c(X;SU(2))$ splits into orbits of possible orders one, two, four, eight, and sixteen. In this section we study the *four–orbits* (orbits with four elements, or those with stabilizer $\mathbb{Z}_2 \oplus \mathbb{Z}_2$). The results of this section are quite parallel to those in Section 4 of [22], and the proofs are only sketched.

6.1 The four–orbits and invariant $\bar\lambda$

Consider a subgroup of $SO(3)$ that is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Such a subgroup is generated by π–rotations about two perpendicular axes in $\mathbb{R}^3$, and any two such subgroups are conjugate to each other in $SO(3)$. Hence the following definition makes sense. Given $w \in H^2(\pi_1 X;\mathbb{Z}_2)$, define $\mathcal{R}_w(X;\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ to be the (finite) subspace of $\mathcal{R}_w(X;SO(3))$ consisting of the $SO(3)$ conjugacy classes of representations $\alpha:\pi_1(X)\to SO(3)$ which factor through $\mathbb{Z}_2\oplus\mathbb{Z}_2\subset SO(3)$, compare with Section 5.1.

Proposition 6.1 *Let $[c]=w$ be a non-trivial class in $H^2(\pi_1 X;\mathbb{Z}_2)$. Then the map $\pi:\mathcal{PR}_c(X;SU(2))\to\mathcal{R}_w(X;SO(3))$ establishes a bijective correspondence between the set of four–orbits in $\mathcal{PR}_c(X;SU(2))$ and the set $\mathcal{R}_w(X;\mathbb{Z}_2\oplus\mathbb{Z}_2)$.*

Proof Suppose that the conjugacy class of a projective representation $\rho:\pi_1(X)\to SU(2)$ is fixed by a subgroup $\mathbb{Z}_2\oplus\mathbb{Z}_2$ of $H^1(X;\mathbb{Z}_2)$ generated by homomorphisms α, $\beta:\pi_1(X)\to\mathbb{Z}_2$. As in [22, Proposition 4.1], ρ factors through a copy of the quaternion 8-group $Q=\{\pm1,\pm i,\pm j,\pm k\}$ and therefore its associated $SO(3)$ representation $\operatorname{ad}\rho$ factors through a copy of $\mathbb{Z}_2\oplus\mathbb{Z}_2\subset SO(3)$.

To complete the proof, we only need to show that the orbit of ρ consists of exactly four points. Let γ, δ be vectors in $H^1(X;\mathbb{Z}_2)$ completing α, β to a basis. Then ρ, ρ^γ, ρ^δ and $\rho^{\gamma\delta}$ lie in the same $H^1(X;\mathbb{Z}_2)$–orbit but are not conjugate. The latter can be seen as follows: suppose there exists an element $w\in SU(2)$ such that $\gamma(x)\rho(x)=w\,\rho\,w^{-1}$ then $w=\pm k$ and $\alpha(x)\beta(x)\gamma(x)\rho(x)=(ijk)\,\rho(x)\,(ijk)^{-1}=\rho(x)$ for all x, a contradiction. □

Remark 6.2 The above proof also shows that no point of $\mathcal{PR}_c(X;SU(2))$ with $[c]\neq0$ is fixed by more than two basis elements in $H^1(X;\mathbb{Z}_2)$ so that $\mathcal{PR}_c(X;SU(2))$ has no orbits of orders one or two.

6.2 The number of four–orbits

Our next goal is to find a formula for the number of points in $\mathcal{R}_w(X;\mathbb{Z}_2\oplus\mathbb{Z}_2)$ modulo 2.

Proposition 6.3 *If X is an odd homology 4–torus then for any non-zero $w\in H^2(\pi_1X;\mathbb{Z}_2)$ such that $w\cup w\equiv 0\mod 4$ we have $\#\,\mathcal{R}_w(X;\mathbb{Z}_2\oplus\mathbb{Z}_2)\equiv1$ (mod 2).*

Proof Recall that any two subgroups of $SO(3)$ that are isomorphic to $\mathbb{Z}_2\oplus\mathbb{Z}_2$ are conjugate, and that moreover any automorphism of such a subgroup is realized by conjugation by an element of $SO(3)$. Let us fix a subgroup $\mathbb{Z}_2\oplus\mathbb{Z}_2$ and a basis in it.

Since $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is abelian, every $\alpha \in \mathcal{R}_w(X;\mathbb{Z}_2\oplus\mathbb{Z}_2)$ factors through a homomorphism $H_1(X;\mathbb{Z}) \to \mathbb{Z}_2\oplus\mathbb{Z}_2$. The two components of this homomorphism determine elements $\beta,\gamma \in \operatorname{Hom}(H_1(X;\mathbb{Z});\mathbb{Z}_2) \cong H^1(X;\mathbb{Z}_2)$. It is straightforward to see that the $SO(3)$ representation α may be recovered from β and γ via the formula $\alpha \cong \beta\oplus\gamma\oplus\det(\beta\oplus\gamma)$. This establishes a bijective correspondence

$$\mathcal{R}(X;\mathbb{Z}_2\oplus\mathbb{Z}_2) \to \Lambda^2_0 H^1(X;\mathbb{Z}_2), \tag{20}$$

where $\mathcal{R}(X;\mathbb{Z}_2\oplus\mathbb{Z}_2)$ is union of $\mathcal{R}_w(X;\mathbb{Z}_2\oplus\mathbb{Z}_2)$ over all possible w and $\Lambda^2_0 H^1(X;\mathbb{Z}_2)$ is the subset of decomposable elements of $\Lambda^2 H^1(X;\mathbb{Z}_2)$. Since $H_1(X;\mathbb{Z})$ is torsion free, any element in $H^1(X;\mathbb{Z}_2)$ is the mod 2 reduction of a class in $H^1(X;\mathbb{Z})$. It follows that the cup product of any element $a\in H^1(X;\mathbb{Z}_2)$ with itself is 0. We compute

$$\begin{aligned} w_2(\alpha) &= w_1(\beta)\cup w_1(\gamma) + w_1(\beta)\cup w_1(\det(\beta\oplus\gamma)) + w_1(\gamma)\cup w_1(\det(\beta\oplus\gamma)) \\ &= w_1(\beta)\cup w_1(\gamma) + w_1(\beta)\cup(w_1(\beta)+w_1(\gamma)) + w_1(\gamma)\cup(w_1(\beta)+w_1(\gamma)) \\ &= w_1(\beta)\cup w_1(\gamma). \end{aligned} \tag{21}$$

Since $w_1(\beta)=\beta$ and $w_1(\gamma)=\gamma$, this shows that $w_2(\alpha)$ belongs to the image of $\Lambda^2_0 H^1(X;\mathbb{Z}_2)$ under the map (1).

The result now follows by combining (1) and (20): Corollary 2.2 implies that the space $\mathcal{R}_w(X;\mathbb{Z}_2\oplus\mathbb{Z}_2)$ consists of exactly one point for every choice of $0\neq w\in H^2(\pi_1 X;\mathbb{Z}_2)$ such that $w\cup w = 0 \pmod 4$. □

Proposition 6.4 *If X is an even homology 4–torus then for any non-zero $w\in H^2(\pi_1X;\mathbb{Z}_2)$ such that $w\cup w\equiv 0 \pmod 4$ and $w\cup\xi\neq 0$ for some $\xi\in H^1(X;\mathbb{Z}_2)$ the space $\mathcal{R}_w(X;\mathbb{Z}_2\oplus\mathbb{Z}_2)$ is empty.*

Proof This follows by combining (1) and (20) and using the following observation. Suppose that $w\in H^2(X;\mathbb{Z}_2)$ is of the form $w=\beta\cup\gamma$ then $\omega\cup\xi=\beta\cup\gamma\cup\xi = 0$ for all $\xi\in H^1(X;\mathbb{Z}_2)$: if not, then by Poincaré duality, there exists $\eta\in H^1(X;\mathbb{Z}_2)$ such that $\beta\cup\gamma\cup\xi\cup\eta\neq 0 \pmod 2$, which contradicts the fact that X is an even homology torus. □

Corollary 6.5 *Let P be an admissible $U(2)$–bundle over a homology 4–torus X and let $w = w_2(\bar P)$ be in the image of (2). Then $\#\,\mathcal{R}_w(X;\mathbb{Z}_2\oplus\mathbb{Z}_2)\equiv \det X$* $\pmod 2$.

6.3 Non–degeneracy of the four–orbits

We wish to use Corollary 6.5 to calculate the contribution of the four–orbits into $\bar{\lambda}(X, P) \pmod 2$. In order to do that, we need to check the non–degeneracy condition for such orbits in the case when X is an odd homology 4–torus. This is similar to the non–degeneracy statement about two–orbits in [22].

Proposition 6.6 *If X is an odd homology 4–torus then $H^1(X; \operatorname{ad}\rho) = 0$ for any projective representation $\rho : \pi_1(X) \to SU(2)$ such that $\operatorname{ad}\rho \in \mathcal{R}_w(X; \mathbb{Z}_2 \oplus \mathbb{Z}_2)$ with $w \neq 0$ and $w \cup w = 0 \pmod 4$.*

As in the proof of [22, Proposition 4.4], the cohomology group in question is related to the usual cohomology of X via a Gysin sequence. The non-triviality of the cup product determines enough about the maps to lead to the non-vanishing result. We refer the reader to [22] for all the details.

7 The eight–orbits

Let X be a $\mathbb{Z}[\mathbb{Z}]$–homology 4–torus and P an admissible $U(2)$–bundle over X. The moduli space $\mathcal{M}(P)$ need not be non–degenerate at the eight–orbits as it was at the four–orbits and therefore has to be perturbed. Doing so equivariantly requires a detailed knowledge of the geometry of $\mathcal{M}(P)$ at the eight–orbits, which is the topic of the first part of this section. We proceed by perturbing $\mathcal{M}(P)$ at the eight–orbits, first in the tangential and then in the normal direction.

7.1 Connections with stabilizer $\mathbb{Z}_2$

Let us fix $0 \neq \chi \in H^1(X; \mathbb{Z}_2)$ and consider the action $\chi : \mathcal{B}^*(P) \to \mathcal{B}^*(P)$ induced on the irreducible part of $\mathcal{B}(P)$ by (11). The fixed points of this action will be denoted by $\mathcal{B}^\chi(P)$, and we begin by describing $\mathcal{B}^\chi(P)$ as a moduli space in its own right. To this end we make the following definitions.

Let $u : P \to P$ be a $U(2)$ gauge transformation realizing $\chi \in H^1(X; \mathbb{Z}_2)$ and let $h = \det u$. Then we denote by $\mathcal{A}^u(P)$ the subset of $\mathcal{A}(P)$ consisting of connections A such that $u^* A - 1/2\, h^{-1} dh \operatorname{Id} = A$, and by $\mathcal{G}^u(P)$ the subgroup of $\mathcal{G}(P)$ consisting of gauge transformations $g \in \mathcal{G}(P)$ having the property that $gu = \pm\, ug$.

Proposition 7.1 *Suppose that $\mathcal{B}^\chi(P)$ is non–empty. If $u : P \to P$ realizes $\chi \in H^1(X;\mathbb{Z}_2)$ then $\mathcal{B}^\chi(P) = \mathcal{A}^u(P)/\,\mathcal{G}^u(P)$. Moreover, $\bar u^2 = \mathrm{Id}$.*

Proof Define a map $\mathcal{A}^u(P) \to \mathcal{B}^\chi(P)$ by $A \to [A]$. This map is surjective: given $[A] \in \mathcal{B}^\chi(P)$, there exists a lift g of χ and a gauge transformation $k \in \mathcal{G}(P)$ such that $g^*A - 1/2\,\det g \cdot d\,(\det g)^{-1}\,\mathrm{Id} = k^*A$. Let $v = k^{-1}g$ then $v^*A - 1/2\,\det v \cdot d\,(\det v)^{-1}\,\mathrm{Id} = A$ since $\det g = \det v$ and k commutes with $d(\det g^{-1})\,\mathrm{Id}$. Since both u and v lift χ, there exists an automorphism $\gamma = c\,\mathrm{Id}$ with $c \in U(1)$ such that $\gamma v = w^{-1} u\, w$ for some $w \in \mathcal{G}(P)$. A calculation similar to (12) shows that

$$\begin{aligned} v^*A - 1/2\,\det v \cdot d\,(\det v)^{-1}\,\mathrm{Id} &= (\gamma v)^*A - 1/2\,\det \gamma v \cdot d\,(\det \gamma v)^{-1}\,\mathrm{Id} \\ &= (w^{-1} u\, w)^*A - 1/2\,\det u \cdot d\,(\det u)^{-1}\,\mathrm{Id} = A. \end{aligned}$$

Therefore, $u^*w^*A - 1/2\,h\,dh^{-1}\,\mathrm{Id} = w^*A$, which implies that $[A] = [w^*A]$ with $w^*A \in \mathcal{A}^u(P)$.

The map $\mathcal{A}^u(P) \to \mathcal{B}^\chi(P)$ becomes injective after we factor $\mathcal{A}^u(P)$ out by the gauge group $\mathcal{G}^u(P)$. This can be seen as follows. Suppose that $A, B \in \mathcal{A}^u(P)$ are such that $[A] = [B]$, then there is $g \in \mathcal{G}(P)$ such that $B = g^*A$ and $u^*g^*A - 1/2\,h\,dh^{-1}\,\mathrm{Id} = g^*A$. Therefore, in addition to $u^*A - 1/2\,h\,dh^{-1}\,\mathrm{Id} = A$, we also have $(g^{-1}ug)^*A - 1/2\,h\,dh^{-1}\,\mathrm{Id} = A$. This implies that $(g^{-1}ug)^*A = u^*A$ and that $(u^{-1}g^{-1}ug)^*A = A$. Since $\det(u^{-1}g^{-1}ug) = 1$ and A is irreducible, we conclude that $u^{-1}g^{-1}ug = \pm 1$.

The equality $(u^2)^*A - h\,dh^{-1}\,\mathrm{Id} = A$ obtained by applying u^* to A twice is equivalent to $(h^{-1}u^2)^*A = A$ which is seen from the following calculation: $(h^{-1}u^2)^*A = h^{-1}dh\,\mathrm{Id} + (u^2)^*A = h^{-1}dh\,\mathrm{Id} + h\,dh^{-1}\,\mathrm{Id} + A = A$. Since $\det(h^{-1}u^2) = h^{-2}h^2 = 1$ and A is irreducible, we conclude that $h^{-1}u^2 = \pm 1$ and therefore $\bar u^2 = 1$. □

7.2 The local structure of $\mathcal{M}(P)$ near eight–orbits

It follows from Corollary 4.4 that the action $\chi : \mathcal{B}(P) \to \mathcal{B}(P)$ restricts to a well–defined action $\chi : \mathcal{M}(P) \to \mathcal{M}(P)$, whose fixed point set will be denoted by $\mathcal{M}^\chi(P) \subset \mathcal{B}^\chi(P)$. We wish to investigate the non–degeneracy condition for $\mathcal{M}(P)$ at points $[A] \in \mathcal{M}^\chi(P)$. If $\mathcal{M}(P)$ fails to be non–degenerate at $[A] \in \mathcal{M}^\chi(P)$, this may be due to the nonvanishing of the obstruction (lying in the second cohomology of the deformation complex) to infinitesimal deformations in the direction of $\mathcal{M}^\chi(P)$ (tangential deformations) and/or in the direction normal to $\mathcal{M}^\chi(P) \subset \mathcal{M}(P)$.

More precisely, according to Proposition 7.1, one can choose a gauge transformation $u: P \to P$ realizing χ so that every point in $\mathcal{M}^{\chi}(P)$ is the $\mathcal{G}^u(P)$ equivalence class of a connection A having the property $u^*A - 1/2\, h\, dh^{-1}\,\mathrm{Id} = A$ with $h = \det u$. The isomorphism (15) of deformation complexes at $[A]$ and $\chi[A] = [A]$ then becomes an order two automorphism $(\mathrm{ad}\,\bar{u})^*$ of the deformation complex at $[A]$, thus splitting it into a direct sum of two elliptic complexes according to the (± 1)–eigenvalues of the operator $(\mathrm{ad}\,\bar{u})^*$. The complex

$$\Omega^0(X;\mathrm{ad}\,\bar{P})^+ \xrightarrow{d_{\bar{A}}} \Omega^1(X;\mathrm{ad}\,\bar{P})^+ \xrightarrow{d^+_{\bar{A}}} \Omega^2_+(X;\mathrm{ad}\,\bar{P})^+ \tag{22}$$

describes tangential deformations, while normal deformations are described by the complex

$$\Omega^0(X;\mathrm{ad}\,\bar{P})^- \xrightarrow{d_{\bar{A}}} \Omega^1(X;\mathrm{ad}\,\bar{P})^- \xrightarrow{d^+_{\bar{A}}} \Omega^2_+(X;\mathrm{ad}\,\bar{P})^-. \tag{23}$$

The cohomology of the above two complexes will be denoted by $H^0(X;\mathrm{ad}\,A)^{\pm}$, $H^1(X;\mathrm{ad}\,A)^{\pm}$, and $H^2_+(X;\mathrm{ad}\,A)^{\pm}$, respectively.

Lemma 7.2 *Both complexes (22) and (23) have zero index.*

Proof Since the index of the deformation complex (14) is zero, it is sufficient to show that the complex (22) has zero index. Observe that the $(+1)$–eigenspace $\mathrm{ad}\,\bar{P}^+$ of the operator $\mathrm{ad}\,\bar{u} : \mathrm{ad}\,\bar{P} \to \mathrm{ad}\,\bar{P}$ is a $\mathbb{Z}_2$–bundle with fiber $\mathbb{R}$ and $w_1(\mathrm{ad}\,\bar{P}^+) = \chi$, and that $\Omega^k(X;\mathrm{ad}\,\bar{P})^+ = \Omega^k(X;\mathrm{ad}\,\bar{P}^+)$ for all k. Therefore, the complex (22) is isomorphic to the deformation complex of the moduli space of ASD connections on $\mathrm{ad}\,\bar{P}^+$, and the latter has index $-(1 - b_1 + b_2^+)\,(X) = 0$ by the index theorem. □

7.3 Tangential perturbations

Let $\sigma: \mathcal{B}(P) \to \mathcal{E}(P)$ be an equivariant admissible perturbation, and $\mathcal{M}_\sigma(P)$ the corresponding moduli space of perturbed projectively ASD connections. Every $0 \neq \chi \in H^1(X;\mathbb{Z}_2)$ defines a map $\chi: \mathcal{M}_\sigma(P) \to \mathcal{M}_\sigma(P)$ whose fixed point set will be denoted by $\mathcal{M}^{\chi}_\sigma(P)$. Let us choose a gauge transformation $u: P \to P$ as in Proposition 7.1 then the deformation complex of $\mathcal{M}^{\chi}_\sigma(P)$ at $[A]$,

$$\Omega^0(X;\mathrm{ad}\,\bar{P})^+ \xrightarrow{d_{\bar{A}}} \Omega^1(X;\mathrm{ad}\,\bar{P})^+ \xrightarrow{d^+_{\bar{A}} + D_A\sigma} \Omega^2_+(X;\mathrm{ad}\,\bar{P})^+, \tag{24}$$

is the invariant part of the complex (18) with respect to the action of $(\mathrm{ad}\,\bar{u})^*$. We say that $\mathcal{M}^{\chi}_\sigma(P)$ is *non-degenerate* at $[A]$ if the second cohomology of the complex (24) vanishes (observe that, in general, this condition may be weaker than the non–degeneracy condition for the full moduli space $\mathcal{M}_\sigma(P)$ at $[A]$). If $\sigma = 0$, this condition is equivalent to the vanishing of the second cohomology group $H^2_+(X;\operatorname{ad} A)^+$ of the complex (22). We call $\mathcal{M}^\chi_\sigma(P)$ *non–degenerate* if it is non–degenerate at all $[A] \in \mathcal{M}^\chi_\sigma(P)$. Our goal in this section will be to find enough equivariant admissible perturbations σ to ensure that $\mathcal{M}^\chi_\sigma(P)$ is non–degenerate for a generic σ.

We begin by restating the above perturbation problem in slightly different terms. Proposition 4.5 implies that, for every $0 \neq \chi \in H^1(X;\mathbb{Z}_2)$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{B}(P) & \xrightarrow{\ \sigma\ } & \mathcal{E}(P) \\ \Big\downarrow \chi & & \Big\downarrow \chi \\ \mathcal{B}(P) & \xrightarrow{\ \sigma\ } & \mathcal{E}(P). \end{array}$$

In particular, the restriction of σ onto $\mathcal{B}^\chi(P)$ defines a map $\sigma^\chi : \mathcal{B}^\chi(P) \to \mathcal{E}^\chi(P)$, where $\mathcal{E}^\chi(P)$ stands for the fixed point set of $\chi : \mathcal{E}(P) \to \mathcal{E}(P)$. For a choice of $u : P \to P$ as above, $\mathcal{E}^\chi(P)$ is isomorphic to the bundle

$$\mathcal{E}^\chi(P) = \mathcal{A}^u(P) \times_{\mathcal{G}^u(P)} \Omega^2_+(X;\operatorname{ad}\bar{P})^+$$

over $\mathcal{B}^\chi(P)$, and σ^χ is a section of this bundle. The restriction of the section $F_+ : \mathcal{B}(P) \to \mathcal{E}(P)$ onto $\mathcal{B}^\chi(P)$ defines a section $F^\chi_+ : \mathcal{B}^\chi(P) \to \mathcal{E}^\chi(P)$ in a similar fashion. The zero set of $\zeta^\chi_\sigma = F^\chi_+ + \sigma^\chi : \mathcal{B}^\chi(P) \to \mathcal{E}^\chi(P)$ is precisely the moduli space $\mathcal{M}^\chi_\sigma(P)$, only now viewed as obtained by perturbing $\mathcal{M}^\chi(P)$ using perturbation σ^χ.

This reformulation makes the tangential perturbation problem similar to the non–equivariant perturbation problem studied in Section 4.4. The key reduction step is provided by the following lemma.

Lemma 7.3 *Let $p : \tilde{X} \to X$ be the regular 16–fold covering of X corresponding to the mod 2 abelianization homomorphism $\pi_1 X \to H_1(X;\mathbb{Z}_2)$. For every projectively flat connection A on $P \to X$, denote by $\tilde{A}$ the pull back connection over $\tilde{X}$. Let $\operatorname{Stab}(A)$ denote the stabilizer of $[A]$ in $H^1(X;\mathbb{Z}_2)$ then*

(a) $\operatorname{Stab}(A) = 1$ *if and only if $\tilde{A}$ is irreducible,*

(b) $\operatorname{Stab}(A) = \mathbb{Z}_2$ *if and only if $\tilde{A}$ is reducible non–central, and*

(c) $\operatorname{Stab}(A) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *if and only if $\tilde{A}$ is central.*

No other stabilizers $\operatorname{Stab}(A)$ *may occur.*

Proof A version of this statement for the homology 3–tori is proved in [22, Lemma 5.6]; the proof works exactly the same in the 4–dimensional case. □

Lemma 7.4 *Let A be a projectively flat connection with stabilizer $\mathbb{Z}_2$ in $H^1(X;\mathbb{Z}_2)$. Then there exists a collection of disjoint mod-2 trivial loops ψ_k and a collection of self–dual 2–forms ν_k, each supported in its respective $\psi_k(S^1\times D^3)$, such that the forms $\Pi\,\mathrm{hol}_A(\psi_k)\otimes\nu_k$ generate the vector space $H^2_+(X;\mathrm{ad}\,A)^+$.*

Proof Fix a harmonic lift $H^2_+(X;\mathrm{ad}\,A)^+\subset\Omega^2_+(X;\mathrm{ad}\,\bar P)^+$ of $H^2_+(X;\mathrm{ad}\,A)^+$. It consists of 2–forms ω with coefficients in $\mathrm{ad}\,\bar P$ such that ω is self–dual, $d_{\bar A}\omega=0$, and $(\mathrm{ad}\,\bar u)^*\omega=\omega$. Observe that, for a mod-2 trivial loop ψ_k, the equality $u^*A-1/2\,h\,dh^{-1}\,\mathrm{Id}=A$ with $h=\det u$ implies that

$$\begin{aligned}\Pi\,\mathrm{hol}_A(\psi_k)\otimes\nu_k&=\Pi\,\mathrm{hol}_{u^*A-1/2\,h\,dh^{-1}\,\mathrm{Id}}(\psi_k)\otimes\nu_k\\&=(\mathrm{ad}\,\bar u)^*\,(\Pi\,\mathrm{hol}_A(\psi_k)\otimes\nu_k),\end{aligned}$$

as in the proof of Proposition 4.5, hence the forms $\Pi\,\mathrm{hol}_A(\psi_k)\otimes\nu_k$ belong to $\Omega^2_+(X;\mathrm{ad}\,\bar P)^+$. We need to construct sufficiently many of these forms so that the subspace of $\Omega^2_+(X;\mathrm{ad}\,\bar P)^+$ that they span projects orthogonally *onto* $H^2_+(X;\mathrm{ad}\,A)^+$.

There exist finitely many points $x_1,\dots,x_m$ in X such that the evaluation map

$$\mathrm{ev}:H^2_+(X;\mathrm{ad}\,A)^+\longrightarrow\bigoplus_{p=1}^m\Lambda^2_+(\mathrm{ad}\,\bar P)^+_{x_p}$$

given by $\mathrm{ev}(\omega)=(\omega(x_1),\dots,\omega(x_m))$ is injective. Here, $(\mathrm{ad}\,\bar P)^+_{x_p}$ is the $(+1)$–eigenspace of $(\mathrm{ad}\,\bar u)_{x_p}$ acting on $(\mathrm{ad}\,\bar P)_{x_p}$. Since the latter eigenspace is one–dimensional, each of the spaces $\Lambda^2_+(\mathrm{ad}\,\bar P)^+_{x_p}$ has real dimension three.

Let $\rho:\pi_1X\to SU(2)$ be the holonomy representation of A. For each of the points x_p, consider the possible holonomies $\rho(\psi)$ around mod-2 trivial loops ψ based at x_p. According to Lemma 7.3, the restriction of ρ to the subgroup $\pi_1\tilde X\subset\pi_1X$ generated by mod-2 trivial loops is non–central, so there exists a mod-2 trivial loop ψ_p such that $\Pi\,\mathrm{hol}_A(\psi_p)\in(\mathrm{ad}\,\bar P)^+_{x_p}$ is not zero.

Let $\omega_{i,p}$, $i=1,2,3$, be a basis for $(\Lambda^2_+)_{x_p}$ then the tensor products $\Pi\,\mathrm{hol}_A(\psi_p)\otimes\omega_{i,p}$ span the entire space

$$\bigoplus_{p=1}^m\Lambda^2_+(\mathrm{ad}\,\bar P)^+_{x_p}.$$

For each index p, choose a small ball around x_p over which the forms in $\Omega^2_+(X;\operatorname{ad}\bar{P})^+$ have small variations, and three distinct points inside it. Label the three points near x_p by $x_{i,p}$. Then we can choose mod-2 trivial loops $\psi_{i,p}$ based at $x_{i,p}$ whose holonomy is close to $\operatorname{hol}_A(\psi_p)$, and also 2–forms $\nu_{i,p}$ supported in small balls around $x_{i,p}$ and close to the respective multiples of $\omega_{i,p}$ in a local trivialization. The resulting sections $\Pi\operatorname{hol}_A(\psi_{i,p})\otimes\nu_{i,p}$ have disjoint supports and, if the approximations in the above construction are made sufficiently fine, no non-zero element of $H^2_+(X;\operatorname{ad}A)^+$ can be orthogonal to all of them. $\square$

Observe that, for every $0\neq\chi\in H^1(X;\mathbb{Z}_2)$, the moduli space $\mathcal{M}^\chi(P)$ is compact as it is the fixed point set of an involution acting on the compact space $\mathcal{M}(P)$. In particular, we can suppose that the mod-2 trivial loops ψ_k and forms ν_k can be chosen so that the conclusion of Lemma 7.4 holds for all $[A]\in\mathcal{M}^\chi(P)$ simultaneously. We fix such a choice and define σ by the formula (17).

Proposition 7.5 *There exists a real number $r>0$ such that, with the choice of ψ_k and ν_k as above, the moduli space $\mathcal{M}^\chi_\sigma(P)$ is non–degenerate for a generic $\bar{\varepsilon}=(\varepsilon_1,\dots,\varepsilon_N)$ inside the ball $\|\bar{\varepsilon}\|<r$ in $\mathbb{R}^N$.*

Proof Let us pull the bundle $\mathcal{E}^\chi(P)$ back to a bundle $\pi^*\mathcal{E}^\chi(P)\to\mathcal{B}^\chi(P)\times\mathbb{R}^N$ and consider the "universal section" Ψ^χ obtained by restricting the section (19) to $\mathcal{B}^\chi(P)\times\mathbb{R}^N$. Since the forms $\Pi\operatorname{hol}_A(\psi_k)\otimes\nu_k$ generate $H^2_+(X;\operatorname{ad}A)^+$, we conclude that Ψ^χ is transversal at $\mathcal{M}^\chi(P)\times\{0\}$ to the zero section of the above bundle. The rest of the argument is identical to the proof of Proposition 4.8. $\square$

One can use the holonomy correspondence of Section 5.2 to conclude that $\mathcal{M}^\chi(P)$ for each $0\neq\chi\in H^1(X;\mathbb{Z}_2)$ contains at most one four–orbit. Since the four–orbits are non–degenerate, they are isolated in each of the $\mathcal{M}^\chi(P)$ and hence the entire moduli space of eight–orbits is compact. The above argument then shows that, after a generic equivariant admissible perturbation, the moduli space of eight–orbits is non–degenerate.

7.4 Normal perturbations

The non–degeneracy of the moduli space of eight–orbits achieved in the previous section by the means of a perturbation σ implies that the moduli space $\mathcal{M}_\sigma(P)$

has finitely many eight–orbits, and it is non–degenerate at each of them in the tangential direction. However, $\mathcal{M}_\sigma(P)$ may fail to be non–degenerate in the normal direction, in which case another perturbation is needed. Such a perturbation, if it is small enough, will not spoil the tangential non–degeneracy of $\mathcal{M}_\sigma(P)$. This observation, together with the fact that there are only finitely many eight–orbits in $\mathcal{M}_\sigma(P)$, allows us to perturb at each of them separately. Furthermore, to save some notation, we will assume that the *unperturbed* moduli space $\mathcal{M}^\chi(P)$ is non–degenerate; the argument below remains valid after a small equivariant perturbation.

First we wish to extend the class of admissible perturbations, following ideas of Donaldson [4] and Furuta [8]. Assume that $\mathcal{M}^\chi(P)$ is non–degenerate at $[A]$ and choose an open neighborhood $\mathcal{U}([A]) \subset \mathcal{B}(P)$ which does not contain any other points of $\mathcal{M}(P)$ with non–trivial stabilizer. The slice theorem then implies that, if $\mathcal{U}([A])$ is small enough, one can find a real number $r > 0$ such that every $B \in \mathcal{A}(P)$ with $[B] \in \mathcal{U}([A])$ can be written as $B = g^*(A + a)$ for a unique (up to sign) $g \in \mathcal{G}(P)$ and a unique $a \in \ker d^*_{\bar{A}}$ such that $\|a\| = \|a\|_{L^2_l} < r$.

Choose a bounded linear operator $\varphi : \ker d^*_{\bar{A}} \to \Omega^2_+(X; \operatorname{ad}\bar{P})$, where both spaces are completed in Sobolev L^2_l norms, and let $\beta : \mathbb{R}_+ \to [0,1]$ be a smooth cut off function such that $\beta(t) = 1$ for $t \le r/2$ and $\beta(t) = 0$ for $t \ge r$. The formula

$$\tau_{[A]}([B]) = [\,A + a,\ \beta(\|a\|)\,\varphi(a)\,], \quad [B] \in \mathcal{U}([A]), \tag{25}$$

extended to be zero on the complement of $\mathcal{U}([A])$, defines a section $\tau_{[A]} : \mathcal{B}(P) \to \mathcal{E}(P)$.

Next we address equivariance properties of the above section. Remember that $[A] \in \mathcal{M}^\chi(P)$ for some $0 \ne \chi \in H^1(X; \mathbb{Z}_2)$, so we first handle the equivariance of $\tau_{[A]} : \mathcal{B}(P) \to \mathcal{E}(P)$ with respect to the action of χ. Fix a representative A and use Proposition 7.1 to realize χ by a gauge transformation $u : P \to P$ such that $u^*A - 1/2\, h\, dh^{-1}\ \mathrm{Id} = A$.

Lemma 7.6 *The perturbation $\tau_{[A]}$ is equivariant with respect to the action of χ if and only if the following diagram commutes*

$$\begin{array}{ccc} \ker d^*_{\bar{A}} & \xrightarrow{\ \varphi\ } & \Omega^2_+(X; \operatorname{ad}\bar{P}) \\ \Big\downarrow{\scriptstyle (\operatorname{ad}\bar{u})^*} & & \Big\downarrow{\scriptstyle (\operatorname{ad}\bar{u})^*} \\ \ker d^*_{\bar{A}} & \xrightarrow{\ \varphi\ } & \Omega^2_+(X; \operatorname{ad}\bar{P}). \end{array} \tag{26}$$

Proof Observe that the operator $(\mathrm{ad}\,\bar{u})^*$ is an isometry on the Sobolev L^2_l completions of $\ker d^*_{\bar{A}}$ and $\Omega^2_+(X;\mathrm{ad}\,\bar{P})$. The statement now follows by a straightforward calculation using the formula (25). □

Thus, for any $[A] \in \mathcal{M}^\chi(P)$, the equivariance of $\tau_{[A]}$ with respect to the action of χ can be achieved by making a proper choice of φ in its definition, see Lemma 7.8 below. The residual action of $H^1(X;\mathbb{Z}_2)/\chi = (\mathbb{Z}_2)^3$ can then be used to spread the perturbation $\tau_{[A]}$ to a small neighborhood of the orbit of $[A]$. More explicitly, if $\eta \in H^1(X;\mathbb{Z}_2)/\chi$, then $\mathcal{U}(\eta[A]) = \eta\,\mathcal{U}([A])$ is a neighborhood of $\eta[A]$, and we can assume that all of the sets $\mathcal{U}(\eta[A])$ have disjoint closures. Then $\tau_{\eta[A]} = \eta\,\tau_{[A]}\,\eta^{-1}$ is a perturbation that is equivariant with respect to the stabilizer $\chi\eta^{-1}$ of $\eta[A]$. Hence

$$\tau = \sum \tau_{\eta[A]} : \mathcal{B}(P) \to \mathcal{E}(P) \tag{27}$$

is a section which is equivariant with respect to the action of the full group $H^1(X;\mathbb{Z}_2)$ and which is supported in a small neighborhood of the eight–orbit through $[A]$.

We define new admissible perturbations $\tau : \mathcal{B}(P) \to \mathcal{E}(P)$ as finite linear combinations of perturbations defined in (27). Each section τ is smooth and compact since it factors through the embedding $\mathcal{E}'(P) \to \mathcal{E}(P)$, where $\mathcal{E}'(P)$ is the bundle (13) whose fiber is completed in Sobolev L^2_l norm, and since the inclusion $L^2_l \to L^2_{l-1}$ is compact, compare with Section 4.4. Therefore, the section $\zeta_\tau = F_+ + \tau : \mathcal{B}(P) \to \mathcal{E}(P)$ is Fredholm, and the perturbed moduli space $\mathcal{M}_\tau(P) = (\zeta_\tau)^{-1}(0)$ has formal dimension zero.

Proposition 7.7 *For any choice of perturbation τ as above, the moduli space $\mathcal{M}_\tau(P)$ is compact.*

Proof The proof of Proposition 4.6 goes through as before to give the above statement because the curvatures $F_+(\bar{A})$ are uniformly bounded in L^∞ for all $[A] \in \mathcal{M}_\tau(P)$, see Furuta [8, Proposition 3.1]. □

Our next step is to show that the operators φ we use to define τ can be chosen so that $\mathcal{M}_\tau(P)$ is acted upon by $H^1(X;\mathbb{Z}_2)$ and is non–degenerate at the eight–orbits of this action.

Lemma 7.8 *For any $[A] \in \mathcal{M}^\chi(P)$, there exist finitely many bounded linear operators $\varphi_k : \ker d^*_{\bar{A}} \to \Omega^2_+(X;\mathrm{ad}\,\bar{P})$ which make the diagram (26) commute and have the following properties. Let τ_k be the perturbation (27) defined* *using φ_k and $D_A\tau_k : H^1(X;\operatorname{ad}A)^- \to H^2_+(X;\operatorname{ad}A)^-$ the linear map obtained by restricting the derivative of τ_k on $H^1(X;\operatorname{ad}A)^- \subset \Omega^1(X,\operatorname{ad}\bar{P})^-$ and projecting onto $H^2_+(X;\operatorname{ad}A)^- \subset \Omega^2_+(X;\operatorname{ad}\bar{P})^-$. Then the maps $D_A\tau_k$ span* $\operatorname{Hom}(H^1(X;\operatorname{ad}A)^-, H^2_+(X;\operatorname{ad}A)^-)$.

Proof The derivative of τ_k at A is the operator φ_k. Let us choose a basis ψ_k for $\operatorname{Hom}(H^1(X;\operatorname{ad}A)^-, H^2_+(X;\operatorname{ad}A)^-)$ and extend ψ_k by composing with the projection $\ker d^*_{\bar{A}} \to H^1(X;\operatorname{ad}A)^-$ to obtain a family of bounded linear operators $\varphi_k : \ker d^*_{\bar{A}} \to \Omega^2_+(X,\operatorname{ad}\bar{P})$. The action of $(\operatorname{ad}\bar{u})^*$ on $H^1(X;\operatorname{ad}A)^-$ and $H^2_+(X;\operatorname{ad}A)^-$ is by minus identity, hence commutativity of (26) is immediate from the linearity of φ_k. □

Remark 7.9 If we start with an already perturbed moduli space $\mathcal{M}_\sigma(P)$, where σ is any linear combination of equivariant perturbations of the types (17) and (27), the above lemma can easily be amended to provide perturbations τ_k whose derivatives span $\operatorname{Hom}(H^1_\sigma(X;\operatorname{ad}A)^-, H^2_{+,\sigma}(X;\operatorname{ad}A)^-)$, where $H^1_\sigma(X;\operatorname{ad}A)^-$ and $H^2_{+,\sigma}(X;\operatorname{ad}A)^-$ are the cohomology groups of the complex

$$\Omega^0(X;\operatorname{ad}\bar{P})^- \xrightarrow{d_{\bar{A}}} \Omega^1(X;\operatorname{ad}\bar{P})^- \xrightarrow{d^+_{\bar{A}}+D_A\sigma} \Omega^2_+(X;\operatorname{ad}\bar{P})^-.$$

Proposition 7.10 *There exist equivariant perturbations τ_k as in (27) such that, for any small generic $\varepsilon_k \in \mathbb{R}$ and $\tau = \sum \varepsilon_k \cdot \tau_k$, the moduli space $\mathcal{M}_\tau(P)$ is non–degenerate at each of its eight–orbits.*

Proof First observe that the argument with Uhlenbeck's compactness theorem as in the proof of Proposition 4.8 shows that, by choosing ε_k small enough, one can make $\mathcal{M}^\chi_\tau(P)$ belong to an arbitrarily small neighborhood of $\mathcal{M}^\chi(P)$.

Since $\mathcal{M}(P)$ is assumed to be non–degenerate in the tangential direction, its local structure at any $[A] \in \mathcal{M}^\chi(P)$ is given by the Kuranishi map

$$D_A\tau = \sum \varepsilon_k \cdot D_A\tau_k : H^1(X;\operatorname{ad}A)^- \to H^2_+(X;\operatorname{ad}A)^-.$$

According to Lemma 7.8, this map is an isomorphism for a generic choice of ε_k, hence its kernel vanishes at each point in the orbit of $[A]$. If the ε_k are chosen to be sufficiently small, this guarantees the non–degeneracy of $\mathcal{M}_\tau(P)$ at all nearby eight–orbits.

Repeating this argument for the rest of the (finitely many) $[A] \in \mathcal{M}^\chi(P)$ and keeping in mind Remark 7.9, one achieves the non–degeneracy of $\mathcal{M}_\tau(P)$ at each of the eight–orbits. □

8 The sixteen–orbits

Throughout this section, by an equivariant admissible perturbation σ we will always mean a linear combination of equivariant admissible perturbations of the types (17) and (27). Moreover, we will assume that $\mathcal{M}_\sigma(P)$ is non–degenerate at the eight–orbits and compact; that such σ exists is the main result of the previous section.

In this section, we will achieve the full non–degeneracy of $\mathcal{M}_\sigma(P)$ by further perturbing it at the 16–orbits. This task is simplified by the fact that the stratum of 16–orbits, $\mathcal{M}^0_\sigma(P) \subset \mathcal{M}_\sigma(P)$, is compact: both four– and eight–orbits in the compact $\mathcal{M}_\sigma(P)$ are non–degenerate and hence isolated. On the negative side, we will need to perturb the already perturbed 16–orbits. These no longer consist of projectively flat connections, hence the methods of Section 7.3 and specifically Lemma 7.3 need to be adapted to this new situation.

Lemma 8.1 *Let $p : \tilde{X} \to X$ be the regular 16–fold covering of X corresponding to the mod 2 abelianization homomorphism $\pi_1 X \to H_1(X;\mathbb{Z}_2)$. If A is a connection on P whose stabilizer in $H^1(X;\mathbb{Z}_2)$ is trivial then the pull back connection $\tilde{A}$ is irreducible.*

Proof Let $x \in X$ and $\tilde{x} \in \tilde{X}$ be points such that $p(\tilde{x}) = x$, then we have an exact sequence of monoids

$$1 \longrightarrow \Omega(\tilde{X},\tilde{x}) \xrightarrow{\;p_*\;} \Omega(X,x) \xrightarrow{\;q\;} H_1(X;\mathbb{Z}_2) \longrightarrow 1$$

where $q(\gamma) = [\gamma]$ is the homology class of γ. Write each of the sixteen elements of $H_1(X;\mathbb{Z}_2)$ in the form $q(\gamma_k)$ for some fixed loops $\gamma_k \in \Omega(X,x)$. Then every $h \in \operatorname{im}(\operatorname{hol}_A) \subset SU(2)$ can be written in the form

$$h = \operatorname{hol}_A(\gamma) = \operatorname{hol}_A(p_*(\tilde{\gamma}) \cdot \gamma_k) = \pm\, \operatorname{hol}_{\tilde{A}}(\tilde{\gamma}) \cdot \operatorname{hol}_A(\gamma_k),$$

where $\tilde{\gamma} \in \Omega(\tilde{X},\tilde{x})$. This means that $\operatorname{im}(\operatorname{hol}_{\tilde{A}})$ has finite index in $\operatorname{im}(\operatorname{hol}_A)$.

Suppose that $\tilde{A}$ is reducible so that $\operatorname{im}(\operatorname{hol}_{\tilde{A}})$ is contained in a copy of $U(1)$. Then $\operatorname{im}(\operatorname{hol}_A)$ is contained in a finite extension of $U(1)$, that is, a copy of binary dihedral group $U(1) \cup j\, U(1)$. Equivalently, if $\bar{A}$ is the connection on $\bar{P}$ adjoint to A then $\operatorname{im}(\operatorname{hol}_{\bar{A}}) \subset O(2)$ where the embedding $j : O(2) \to SO(3)$ is given by $j(a) = \det(a) \oplus a$. We wish to show that A has a non–trivial stabilizer in $H^1(X;\mathbb{Z}_2)$.

Let $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to U(2)$ be a gluing cocycle for P then $\bar{\varphi}_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(3)$ is a gluing cocycle for $\bar{P}$. Since $\bar{P}$ admits a connection $\bar{A}$ whose holonomy is

$O(2)$ we may assume that $\bar{\varphi}_{\alpha\beta}$ factors through the embedding $j : O(2) \to SO(3)$. We wish to find a family of functions $u_\alpha : U_\alpha \to U(2)$ such that

$$\begin{aligned} &u_\alpha \varphi_{\alpha\beta} = \varphi_{\alpha\beta} u_\beta \quad \text{on} \quad U_\alpha \cap U_\beta, \text{ and} \\ &u_\alpha du_\alpha^{-1} + u_\alpha A_\alpha u_\alpha^{-1} - 1/2\, h_\alpha\, dh_\alpha^{-1}\,\mathrm{Id} = A_\alpha \quad \text{over} \quad U_\alpha. \end{aligned} \tag{28}$$

Here, $h_\alpha = \det u_\alpha$.

The strategy is to make an initial choice $u_\alpha = i \in SU(2)$, and then to modify that choice to make conditions (28) hold. Since $\mathrm{ad}(i) = \mathrm{diag}(1,-1,-1)$ we have $\mathrm{ad}(i)\bar{\varphi}_{\alpha\beta} = \bar{\varphi}_{\alpha\beta}\,\mathrm{ad}(i)$ in $SO(3)$. The same equality holds in $U(2)$ modulo center, that is, $i\,\varphi_{\alpha\beta} = t_{\alpha\beta}\,\varphi_{\alpha\beta}\, i$ for some $t_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1)$. Observe that $t_{\alpha\beta}$ is in the commutator subgroup of $U(2)$, hence, being central, only takes values ± 1. Thus $t_{\alpha\beta}$ defines an obstruction $t \in H^1(X;\mathbb{Z}_2)$.

This obstruction need not vanish. On the other hand, we are allowed to redefine u_α to be $t_\alpha i$ for some $t_\alpha : U_\alpha \to U(1)$. Because of this extra freedom, we only need vanishing of the image of t in $H^1(X;S^1) = H^2(X;\mathbb{Z})$. This image equals $\beta(t) \in H^2(X;\mathbb{Z})$ where $\beta : H^1(X;\mathbb{Z}_2) \to H^2(X;\mathbb{Z})$ is the Bockstein operator. Since $H^2(X;\mathbb{Z})$ is torsion free, $\beta(t) = 0$.

This means that there exist $t_\alpha : U_\alpha \to S^1$ such that $u_\alpha = t_\alpha i$ satisfy the equation $u_\alpha \varphi_{\alpha\beta} = \varphi_{\alpha\beta} u_\beta$ on $U_\alpha \cap U_\beta$. Finally, to verify the equation $u_\alpha du_\alpha^{-1} + u_\alpha A_\alpha u_\alpha^{-1} - 1/2\, h_\alpha\, dh_\alpha^{-1}\,\mathrm{Id} = A_\alpha$, we notice that A is uniquely determined by $\bar{A}$ and the determinant connection, hence it is enough to check the above equality for connections $\bar{A}$ and $\mathrm{tr}\, A$. Both are immediate from the definition of u_α. □

Lemma 8.2 *For any $[A] \in \mathcal{M}^0_\sigma(P)$ there is a collection of mod-2 trivial disjoint loops ψ_k and a collection of self–dual 2–forms ν_k, each supported in its respective $\psi_k(S^1 \times D^3)$, such that the forms $\Pi\,\mathrm{hol}_A(\psi_k) \otimes \nu_k$ generate the second cohomology $H^2_{+,\sigma}(X;\mathrm{ad}\,A)$ of complex (18).*

Proof As in the proof of Lemma 7.4, we follow the proof of Lemma 2.5 in Donaldson [5].

The subspace $H^2_{+,\sigma}(X;\mathrm{ad}\,A) = \mathrm{coker}(d^*_{\bar{A}} + D_A\sigma)$ of $\Omega^2_+(X;\mathrm{ad}\,\bar{P})$ is finite dimensional because of the ellipticity of complex (18). Therefore, there exist finitely many points $x_1,\dots,x_m$ in X such that the evaluation map

$$\mathrm{ev} : H^2_{+,\sigma}(X;\mathrm{ad}\,A) \longrightarrow \bigoplus_{p=1}^{m} \Lambda^2_+(\mathrm{ad}\,\bar{P})_{x_p}$$

given by $\mathrm{ev}(\omega) = (\omega(x_1),\dots,\omega(x_m))$ is injective. Note that, since $(\mathrm{ad}\,\bar{P})_{x_p}$ is three–dimensional, each vector space $\Lambda^2_+(\mathrm{ad}\,\bar{P})_{x_p}$ has dimension nine.

For each of the points x_p, consider the possible holonomies $\operatorname{hol}_A(\psi)$ around mod-2 trivial loops ψ based at x_p. The restriction of hol_A onto mod-2 trivial loops coincides with the map $\operatorname{hol}_{\tilde{A}}$ where $\tilde{A}$ is the pull back connection as in Lemma 8.1. According to that lemma, $\tilde{A}$ is irreducible, hence one can find three mod-2 trivial loops ψ_i, $i = 1,2,3$, such that the vectors $\Pi \operatorname{hol}_A(\psi_i)$ span $(\operatorname{ad}\bar{P})_{x_p}$.

Let ν_j, $j = 1,2,3$, be self–dual 2–forms supported near x_p such that $\nu_j(x_p)$ span $(\Lambda^2_+)_{x_p}$. Then the nine vectors $(\Pi \operatorname{hol}_A(\psi_i) \otimes \nu_j)(x_p)$ span the vector space $\Lambda^2_+(\operatorname{ad}\bar{P})_{x_p}$. Repeating this construction for the rest of the points x_p results in a collection of loops and forms which we will call ψ_k and ν_k, respectively. One may assume that $\operatorname{supp}\nu_k \subset \psi_k(S^1 \times D^3)$ for all k and, after a small isotopy, $\psi_k(S^1 \times D^3) \cap \psi_l(S^1 \times D^3) = \emptyset$ for all $k \neq l$. The result now follows because the supports of ν_k can be chosen so small that no non–zero form in $H^2_{+,\sigma}(X; \operatorname{ad} A)$ can be orthogonal to all of the $\Pi \operatorname{hol}_A(\psi_k) \otimes \nu_k$. □

The moduli space $\mathcal{M}^0_\sigma(P)$ is compact therefore we can suppose that the mod-2 trivial loops ψ_k and forms ν_k can be chosen so that the conclusion of Lemma 8.2 holds for all $[A] \in \mathcal{M}^0_\sigma(P)$ simultaneously. We will fix such a choice.

Proposition 8.3 *There exists a real number $r > 0$ such that, with the choice of ψ_k and ν_k as above and perturbation σ' defined by the formula (17), the moduli space $\mathcal{M}^0_{\sigma+\sigma'}(P)$ is non–degenerate for a generic $\bar{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_N)$ inside the ball $\|\bar{\varepsilon}\| < r$ in $\mathbb{R}^N$.*

Proof Given Lemma 8.2, the proof is almost identical to the proof of Proposition 4.8. □

Since small enough perturbations σ' used in Proposition 8.3 to make $\mathcal{M}^0_\sigma(P)$ non–degenerate do not spoil the non–degeneracy at the four– and eight–orbits, we are finished.

9 Proof of Theorem 1.1

Let P be an admissible $U(2)$–bundle on X, and $\bar{P}$ its associated $SO(3)$–bundle. If $w_2(\bar{P})$ is not in the image of the map $\iota : H^2(\pi_1 X; \mathbb{Z}_2) \to H^2(X; \mathbb{Z}_2)$, see (2), then the theorem holds by Corollary 5.2.

If $\mathcal{M}(P)$ is non–degenerate, consider the action on $\mathcal{M}(P)$ of $H^1(X; \mathbb{Z}_2) = (\mathbb{Z}_2)^4$. First, there are no orbits of orders one or two, see Remark 6.2. According to Corollary 6.5, the contribution of the four–orbits to $\bar\lambda(X,P)$ equals $\det X \pmod 2$. The theorem now follows since the orbits consisting of eight and sixteen elements do not contribute to $\bar\lambda(X,P) \pmod 2$ at all.

In general, $\mathcal{M}(P)$ becomes non–degenerate after an admissible equivariant perturbation σ as above. Since the four–orbits are already non–degenerate, they will remain such if σ is sufficiently small. The perturbation σ will not create orbits with one or two elements, or new orbits with four elements. Since σ is equivariant, the above argument discarding the orbits with more than four elements can be applied again to show that $\bar\lambda(X,P) \equiv \det X \pmod 2$.

The statement of Theorem 1.1 about the Rohlin invariant has been verified in Corollary 3.9.

10 An example

Theorem 1.1 shows that, modulo 2, the invariant $\bar\lambda(X,P)$ equals the determinant of X. It is natural to wonder if a similar result holds over the integers. By analogy with the three–dimensional case, see [22], one would conjecture that $\bar\lambda(X,P) = (\det X)^2$. In this section, we present a simple example that demonstrates that this is not the case, and moreover, that $\bar\lambda(X,P)$ depends on the choice of P.

The example may be described succinctly by viewing the 4–torus as an elliptic fibration over T^2. Then we define a manifold $T^4(q,-q)$ to be the result of log transforms of orders q and $-q$ on two fibers. For any value of q, the manifold $T^4(q,-q)$ has the integral homology of T^4, and for q odd (which we assume to be the case henceforth) $T^4(q,-q)$ is an odd $\mathbb{Z}[\mathbb{Z}]$–homology torus (in particular, it is spin). We will find that the value of $\bar\lambda(T^4(q,-q),P)$ depends on the choice of bundle P; for some bundles it is given by $\pm q^2$ while for others it is equal to ± 1.

This result is similar to the calculation of degree zero Donaldson invariants of the log transform of an elliptic surface. In fact the manifold $T^4(q,-q)$ is the double branched cover (with exceptional curves blown down) of a log transform of a K3 surface. More generally, one could consider homology tori coming from the double branched cover of the Gompf-Mrowka non-complex homotopy K3 surfaces [10], which are log transforms of K3 surfaces along two or three non-homologous tori. The value of $\bar\lambda$ appears to be given by the analogue of [10, Theorem 3.3] but the calculations are not particularly revealing. That the two results are similar is perhaps not very surprising; compare [15].

10.1 Log transforms on a torus

Let us describe the construction in a little more detail. Write $T^4 = S^1 \times S^1 \times S^1 \times S^1$ and let u, v, x, y be the coordinates on the four circles. Regard this torus as an elliptic fibration via projection onto the first two coordinates, and call the base torus T and the fiber torus F. Choose disjoint disks D_a, D_b in T, and let α and β be their boundary curves, oriented so that $\alpha + \beta = 0$ in $H_1(T - (D_a \cup D_b))$. For all of these curves, we will use the same letter to denote the curve and a coordinate on the curve. Define

$$T^4(q, -q) = ((T - (D_a \cup D_b)) \times F) \cup_{\varphi_a} (D_a \times F) \cup_{\varphi_b} (D_b \times F)$$

where $\varphi_a : \partial D_a \times F \to \partial(T - (D_a \cup D_b)) \times F$ and $\varphi_b : \partial D_b \times F \to \partial(T - (D_a \cup D_b)) \times F$ are given by the formulas

$$\begin{aligned} \varphi_a(\alpha, x, y) &= (\alpha^q x^{q-1}, \alpha x, y) \quad \text{and} \\ \varphi_b(\beta, x, y) &= (\beta^{-q} x^{-q-1}, \beta x, y). \end{aligned}$$

It is not difficult to show that $T^4(q, -q)$ is a homology 4–torus and that its fundamental group $\pi_1(T^4(q, -q))$ has presentation (with the above-named curves representing fundamental group elements)

$$\langle x, y, u, v, \alpha, \beta \mid x, y \,\text{central},\ \alpha^q x = 1,\ \beta^{-q} x = 1,\ [u, v]\alpha\beta = 1 \rangle. \tag{29}$$

Note that projection onto the y-circle (the last S^1 factor) represents $T^4(q, -q)$ as $T^3(q, -q) \times S^1$, where $T^3(q, -q)$ is obtained by Dehn filling of $(T - (D_a \cup D_b)) \times S^1$.

The tori $u \times x$, $u \times y$, $v \times x$, $v \times y$, $x \times y$ give generators for the second homology of $T^4(q, -q)$ with $\mathbb{Z}_2$–coefficients (this is not true over the integers; as in [10, §2], the regular fiber $F = x \times y$ is q times the multiple fiber. But since we assume that q is odd, we may as well use the regular fiber). The remaining generator can be described as an immersed surface in $T^3(q, -q)$ as follows. Consider a disc D in the base torus T containing D_a and D_b. Start with q copies of the twice punctured torus $T - (D_a \cup D_b)$, whose boundary consists of q parallel copies of α on one side, and q copies of β on the other. In the solid torus that is glued in to create the multiple fiber (with meridian glued to $\alpha^q x$), there is a planar surface with boundary the q copies of α plus one copy of x; likewise in the other solid torus, except with inverted orientation on x. The two copies of x are parallel in $D \times S^1$, so one can fill in an annulus between them. The union of these is an immersed surface, say $\Sigma \subset T^3(q, -q)$, carrying the correct homology class.

10.2 The pull-back case

Let P be a $U(2)$–bundle over $T^4(q,-q)$ that pulls back from a bundle over $T^3(q,-q)$ (which we call again P) having $w_2(\bar{P}) \neq 0$. Since $T^4(q,-q)$ is odd, the bundle P is admissible. Let $\lambda'''(T^3(q,-q),w)$ with $w = w_2(P)$ be the Casson-type invariant of the homology 3–torus $T^3(q,-q)$ introduced in [22]; we refer to that paper for all the definitions. It is not difficult to show using techniques of [23] that the invariants $\bar{\lambda}(T^4(q,-q),P)$ and $\lambda'''(T^3(q,-q),w)$ coincide, at least when the $SO(3)$–representation variety of $\pi_1(T^3(q,-q))$ is non–degenerate.

The remainder of this subsection will be devoted to a direct calculation of the invariant $\lambda'''(T^3(q,-q),w)$ for two of the possible seven non–trivial values of $w \in H^2(T^3(q,-q);\mathbb{Z}_2)$ (a similar argument can be used to show that the other five choices of w give the same result, compare with Remark 10.1). To describe our two choices for w, we present the group $\pi_1(T^3(q,-q))$ as

$$\langle x,u,v,\alpha,\beta \mid x \text{ central},\ \alpha^q x = 1,\ \beta^{-q}x = 1,\ [u,v]\alpha\beta = 1\rangle,$$

compare with (29), and consider representations $\rho : \pi_1(T^3(q,-q)) \to SO(3)$ whose $w_2(\rho)$ is non–trivial on the generator $x \times u \in H_2(T^3(q,-q);\mathbb{Z}_2)$ and trivial on $x \times v$. This gives us the two possibilities for $w = w_2(\rho)$, depending on how it evaluates on Σ, the third generator of $H_2(T^3(q,-q);\mathbb{Z}_2)$.

It is a standard fact [1] that since $w_2(\rho)$ evaluates non-trivially on the torus $x \times u$, the restriction of ρ to $x \times u$ is a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ representation, as described in Section 6. Therefore, we may assume that $\rho(x)$ is a π–rotation about the x–axis and $\rho(u)$ is a π–rotation about the y–axis in $\mathbb{R}^3$. Note that since $\langle w_2(\rho), x \times v\rangle = 0$, then $\rho(v)$ is a rotation around the x-axis by an angle to be determined shortly.

It turns out that the representation ρ, and in particular the value of $\rho(v)$, are completely determined by $\rho(\alpha)$, $\rho(\beta)$ and the evaluation of $w_2(\rho)$ on Σ. This is immediate from the following recipe for computing $\langle w_2(\rho),\Sigma\rangle$. First, choose lifts $\tilde{\rho}(x)$, $\tilde{\rho}(u)$, $\tilde{\rho}(v) \in SU(2)$ of the specified $SO(3)$ elements $\rho(x)$, $\rho(u)$ and $\rho(v)$. Of course we know that $[\tilde{\rho}(x),\tilde{\rho}(u)] = -1$, because $w_2(\rho)$ evaluates non-trivially on the torus $x \times u$. Now choose lifts $\tilde{\rho}(\alpha)$ and $\tilde{\rho}(\beta)$ such that $[\tilde{\rho}(u),\tilde{\rho}(v)]\tilde{\rho}(\alpha)\tilde{\rho}(\beta) = 1$. This can be done by first choosing arbitrary lifts and then adjusting the value of (say) $\tilde{\rho}(\beta)$. Then $\langle w_2(\rho),\Sigma\rangle$ is trivial or non–trivial depending on whether $\tilde{\rho}(\alpha)^q\tilde{\rho}(\beta)^q$ equals 1 or -1.

To find $\rho(\alpha)$ and $\rho(\beta)$, we simply extract q-th roots of $\rho(x)$ so that $\rho(\alpha)$ is a rotation about the x–axis through $\pi k/q$ with odd $k = 1,\dots,2q-1$, and $\rho(\beta)$

is a rotation about the x axis through $\pi\ell/q$ with odd $\ell = 1, \dots, 2q-1$. One can conjugate the entire representation (using the π–rotation with respect to the y–axis, for instance) without changing $\rho(x)$ and $\rho(u)$ but switching between (k,ℓ) and $(2q-k, 2q-\ell)$. This leaves us with pairs $(\rho(\alpha),\rho(\beta))$ no two of which are conjugate to each other as long as $1 \le k \le q$. Among these, there is a special pair with $k = \ell = q$, and a total of $(q^2-1)/2$ other pairs.

The special pair gives a unique $SO(3)$ representation with image in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and given w_2. This can be easily seen from the last relation, which in this case says that $\rho(u)$ and $\rho(v)$ commute – therefore, $\rho(v)$ is a rotation through 0 or π about the x-axis. From the recipe, it is immediate that this rotation angle is 0 if and only if $\langle w_2(\rho), x \times v\rangle = 0$. Once w_2 is fixed, this $SO(3)$ representation lifts to the unique four–orbit of projective $SU(2)$ representations; in particular, it is non–degenerate, see Proposition 4.4 of [22].

For each of the remaining pairs $(\rho(\alpha),\rho(\beta))$, we use the last relation to find the rotation angle of $\rho(v)$. A little calculation shows that this equation determines two possible angles, θ and $\theta+\pi$. As before, the evaluation $\langle w_2(\rho), x \times v\rangle$ is 0 for exactly one of these choices. Therefore, for a given choice of w_2, there is a unique representation ρ for each of the pairs $(\rho(\alpha),\rho(\beta))$. It lifts to an eight–orbit of projective $SU(2)$ representations – this follows from the observation that the image of ρ belongs to $O(2) \subset SO(3)$ but not to a copy of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. All these representations are non–degenerate (proving this is an easy exercise in linear algebra) hence each of them is counted in $\lambda'''(T^3(q,-q),w)$ with multiplicity two.

Thus we have a total of $2\cdot(q^2-1)/2 + 1 = q^2$ representations (counted with multiplicities). All of them should be counted with the same sign – for instance, because they all give rise to holomorphic bundles so the orientations are the same at all points. Therefore, $\lambda'''(T^3(q,-q),w) = \pm\, q^2$.

Another way to prove the latter formula (without referring to holomorphic bundles) is by using surgery theory. A framed link describing $T^3(q,-q)$ can be obtained from the Borromean rings, with all three components framed by zero, by adding two meridians of one of the components with respective framings q and $-q$. Surgering out one component of the Borromean rings at a time and using Casson's surgery formula, one reduces the calculation to that of Casson's invariant of a certain plumbed manifold. The latter is given by an explicit formula. We leave details to the reader. The downside of this surgery approach is that it does not give any information about the non–degeneracy of the representation variety, which we use to identify $\bar\lambda(T^4(q,-q),P)$ with $\lambda'''(T^3(q,-q),w)$.

Remark 10.1 Conjecturally, the gauge-theoretic invariant $\lambda'''(Y,w)$ of a homology 3–torus Y is independent of w and coincides (up to an overall sign) with the combinatorially defined invariant of Lescop [17]. According to [22] this conjecture holds modulo 2, and one hopes that it can also be verified over the integers by either establishing a surgery formula in the gauge theory context, or by an argument analogous to Taubes' theorem [27, 18, 3]. Lescop shows that her invariant is given by $(\det Y)^2$, which is manifestly independent of w. In the case at hand, $Y = T^3(q,-q)$, the determinant equals q so the above calculation of $\lambda'''(T^3(q,-q),w) = \pm q^2$ weighs in for the conjecture.

10.3 The non-pull-back case

The case when the bundle P does not pull back from $T^3(q,-q)$ yields a different answer for $\bar\lambda(T^4(q,-q),P)$. Suppose that ρ is an SO(3) representation with $\langle w_2(\rho), x\times y\rangle \neq 0$, and assume that $\rho(x)$ is a π–rotation about the x-axis and $\rho(y)$ is a π–rotation about the y-axis. Note that, since any element that centralizes a $\mathbb{Z}_2\oplus\mathbb{Z}_2$ subgroup in $SO(3)$ is an element of that subgroup, we conclude that $\rho(u)$, $\rho(v)$, $\rho(\alpha)$ and $\rho(\beta)$ all belong to the same $\mathbb{Z}_2\oplus\mathbb{Z}_2$ subgroup as $\rho(x)$ and $\rho(y)$. Since $\rho(\alpha)$ is a q-th root of the π-rotation about the x-axis, it also is a π-rotation about the x-axis. Thus $\rho(\alpha)=\rho(x)$, and similarly, $\rho(\beta)=\rho(x)$.

Now, the relations in (29) hold for arbitrary choices of $\rho(u),\rho(v)\in\mathbb{Z}_2\oplus\mathbb{Z}_2$ – in fact, fixing $\rho(u)$ and $\rho(v)$ amounts to fixing a particular value for $w_2(\rho)$. For instance, if $w_2(\rho)$ vanishes on the 2–torus $x\times u$, then $\rho(u)$ must be either identity or a π–rotation about the x–axis. On the other hand, if $\langle w_2(\rho), x\times u\rangle \neq 0$, then $\rho(u)$ is a π–rotation around the y–axis or the z–axis. Thus our freedom in choosing $\rho(u)$ and $\rho(v)$ is only restricted by the requirement that $w_2(\rho)\cup w_2(\rho)$ be zero modulo 4.

Choose for instance $w_2(\rho) = x^*\cup y^*$ where $x,y\in H^1(T^4(q,-q);\mathbb{Z}_2)$ are dual to the curves x and y. Then obviously $w_2(\rho)\cup w_2(\rho) = x^*\cup y^*\cup x^*\cup y^* = 0 \pmod 4$, and one can easily see that this choice corresponds to having $\rho(u) = \rho(v) = 1$. For this particular choice of w_2, the above $\mathbb{Z}_2\oplus\mathbb{Z}_2$ representation ρ is the only point in the representation variety. By Proposition 6.6, such a representation is automatically non–degenerate (remember that since q is odd, $T^4(q,-q)$ is an odd homology torus), so the invariant equals ± 1.

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