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The mean curvature integral is invariant under bending

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Abstract Suppose $\mathcal{M}_t$ is a smooth family of compact connected two dimensional submanifolds of Euclidean space E^3 without boundary varying isometrically in their induced Riemannian metrics. Then we show that the mean curvature integrals

$$\int_{\mathcal{M}_t} H_t\, d\mathcal{H}^2$$

are constant. It is unknown whether there are nontrivial such bendings $\mathcal{M}_t$. The estimates also hold for periodic manifolds for which there are nontrivial bendings. In addition, our methods work essentially without change to show the similar results for submanifolds of H^n and S^n, to wit, if $\mathcal{M}_t = \partial X_t$

$$d\int_{\mathcal{M}_t} H_t\, d\mathcal{H}^2 = -kn - 1dV(X_t),$$

where $k = -1$ for H^3 and $k = 1$ for S^3. The Euclidean case can be viewed as a special case where $k = 0$. The rigidity of the mean curvature integral can be used to show new rigidity results for isometric embeddings and provide new proofs of some well-known results. This, together with far-reaching extensions of the results of the present note is done in the preprint [6]. Our result should be compared with the well-known formula of Herglotz (see [5], also [8] and [2]).

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1 Introduction

The underlying idea of this note is the following. Suppose $\mathcal{N}_t$ is a smoothly varying family of polyhedral solids having edges $\{E_t(k)\}_k$, and associated (signed) dihedral angles $\{\theta_t(k)\}_k$. According to a theorem of Schlafli [7]

$$\sum_k |E_t(k)| \frac{d}{dt}\theta_t(k) = 0.$$

In case edge length is preserved in the family, ie

$$\frac{d}{dt}\big|E_t(k)\big| = 0$$

for each time t and each k, then also (product rule)

$$\frac{d}{dt}\sum_k \big|E_t(k)\big|\,\theta_t(k) = 0.$$

Should the $\partial\mathcal{N}_t$'s be polyhedral approximations to submanifolds $\mathcal{M}_t$ varying isometrically, one might regard

$$\sum_k \big|E_t(k)\big|\,\theta_t(k)$$

as a reasonable approximation to the mean curvature integrals

$$\int_{\mathcal{M}_t} H_t\, d\mathcal{H}^2$$

and expect

$$\frac{d}{dt}\big|E_t(k)\big|$$

to be small. Hence it is plausible that the mean curvature integrals of the $\mathcal{M}_t$'s might be constant. In this note we show that that is indeed the case.

Examples such as the isometry pictured on page 306 of volume 5 of [8] show that the mean curvature integral is not preserved under discrete isometries.

Two comments are in order. The first is that it is very likely that there are *no* isometric bendings of hypersurfaces. One reason for the existence of the current work is to produce a tool for resolving this conjecture (as Herglotz' mean curvature variation formula can be used to give a simple proof of Cohn–Vossen's theorem on rigidity of convex hypersurfaces). Secondly, the main theorem can be viewed as a sort of dual bellows theorem (when the hypersurface in question lies in H^n or S^n): as the surface is isometrically deformed, the volume of the *polar dual* stays constant. This should be contrasted with the usual bellows theorem recently proved by Sabitov, Connelly and Walz [4].

2 Terminology and basic facts

Our object in this section is to set up terminology for a family of manifolds varying smoothly through isometries. We consider triangulations of increasing fineness varying with the manifolds. To make possible our mean curvature analysis we associate integral varifolds with both the manifolds and the polyhedral surfaces determined by the triangulations. The mean curvature integral of interest is identified with (minus two times) the varifold first variation associated with the unit normal initial velocity vector field.

2.1 Terminology and facts for a static manifold $\mathcal{M}$

2.1.1 We suppose that $\mathcal{M} \subset \mathbb{R}^3$ is a compact connected smooth two dimensional submanifold of $\mathbb{R}^3$ without boundary oriented by a smooth Gauss mapping $\mathbf{n}\colon \mathcal{M} \to \mathbb{S}^2$ of unit normal vectors.

2.1.2 $H\colon \mathcal{M} \to \mathbb{R}$ denotes half the sum of principal curvatures in direction $\mathbf{n}$ at points in $\mathcal{M}$ so that $H\mathbf{n}$ is the mean curvature vector field of $\mathcal{M}$.

2.1.3 We denote by U a suitable neighborhood of $\mathcal{M}$ in $\mathbb{R}^3$ in which a smooth nearest point retraction mapping $\rho\colon U \to \mathcal{M}$ is well defined. The smooth signed distance function $\sigma\colon U \to \mathbb{R}$ is defined by requiring $p = \rho(p) + \sigma(p)\,\mathbf{n}(\rho(p))$ for each p. We set

$$g = \nabla\sigma\colon U \to \mathbb{R}^3$$

(so that $g|\mathcal{M} = \mathbf{n}$); the vector field g is the initial velocity vector field of the deformation

$$G_t\colon U \to \mathbb{R}^3, \quad G_t(p) = p + t\,g(p) \quad \text{for } p \in U.$$

2.1.4 We denote by

$$V = \mathbf{v}(\mathcal{M})$$

the *integral varifold* associated with $\mathcal{M}$ [1, 3.5]. The first variation distribution of V [1, 4.1, 4.2] is representable by integration [1, 4.3] and can be written

$$\delta V = \mathcal{H}^2 \llcorner \mathcal{M} \wedge (-2H)\mathbf{n}$$

[1, 4.3.5] so that

$$\delta V(g) = \frac{d}{dt}\mathcal{H}^2\big(G_t(\mathcal{M})\big)\Big|_{t=0} = -2\int_{\mathcal{M}} g \cdot H\,\mathbf{n}\,d\mathcal{H}^2 = -2\int_{\mathcal{M}} H\,d\mathcal{H}^2;$$

here $\mathcal{H}^2$ denotes two dimensional Hausdorff measure in $\mathbb{R}^3$.

2.1.5 By a **vertex** p in $\mathcal{M}$ we mean any point p in $\mathcal{M}$. By an **edge** $\langle pq\rangle$ in $\mathcal{M}$ we mean any (unordered) pair of distinct vertexes p, q in $\mathcal{M}$ which are close enough together that there is a unique length minimizing geodesic arc $[\![pq]\!]$ in $\mathcal{M}$ joining them; in particular $\langle pq\rangle = \langle qp\rangle$. For each edge $\langle pq\rangle$ we write $\partial\langle pq\rangle = \{p, q\}$ and call p a vertex of edge $\langle pq\rangle$, etc. We also denote by $\overline{pq}$ the straight line segment in $\mathbb{R}^3$ between p and q, ie the convex hull of p and q. By a *facet* $\langle pqr\rangle$ in $\mathcal{M}$ we mean any (unordered) triple of distinct vertexes p, q, r which are not collinear in $\mathbb{R}^3$ such that $\langle pq\rangle$, $\langle qr\rangle$, $\langle rp\rangle$ are edges in $\mathcal{M}$; in particular, $\langle pqr\rangle = \langle qpr\rangle = \langle rpq\rangle$, etc. For each facet $\langle pqr\rangle$ we write

$\partial\langle pqr\rangle = \{\langle pq\rangle, \langle qr\rangle, \langle rp\rangle\}$ and call $\langle pq\rangle$ an edge of facet $\langle pqr\rangle$ and also denote by $\overline{pqr}$ the convex hull of p, q, r in $\mathbb{R}^3$.

2.1.6 Suppose $0 < \tau < 1$ and $0 < \lambda < 1$. By a τ, λ *regular triangulation* $\mathcal{T}$ *of* $\mathcal{M}$ *of maximum edge length* L we mean

(i) a family $\mathcal{T}_2$ of facets in $\mathcal{M}$, together with

(ii) the family $\mathcal{T}_1$ of all edges of facets in $\mathcal{T}_2$ together with

(iii) the family $\mathcal{T}_0$ of all vertexes of edges in $\mathcal{T}_1$

such that

(iv) $\overline{pqr} \subset U$ for each facet $\langle pqr\rangle$ in $\mathcal{T}_2$

(v) $\mathcal{M}$ is partitioned by the family of subsets

$$\left\{\rho\big(\overline{pqr} \sim (\overline{pq} \cup \overline{qr} \cup \overline{rq})\big) : \langle pqr\rangle \in \mathcal{T}_2\right\} \cup \left\{\rho(\overline{pq}) \sim \{p, q\} : \langle pq\rangle \in \mathcal{T}_1\right\} \\ \cup \left\{\{p\} : p \in \mathcal{T}_0\right\}$$

(vi) for facets $\langle pqr\rangle \in \mathcal{T}_2$ we have the uniform nondegeneracy condition: if we set $u = q - p$ and $v = r - p$ then

$$\left| v - \left(\frac{u}{|u|} \cdot v\right) \frac{u}{|u|} \right| \geq \tau |v|$$

(vii) $L = \sup\{|p - q| : \langle pq\rangle \in \mathcal{T}_1\}$

(viii) for edges in $\mathcal{T}_1$ we have the uniform control on the ratio of lengths:

$$\inf\{|p - q| : \langle pq\rangle \in \mathcal{T}_1\} \geq \lambda L.$$

2.1.7 Fact [3] It is a standard fact about the geometry of smooth submanifolds that there are $0 < \tau < 1$ and $0 < \lambda < 1$ such that for arbitrarily small maximum edge lengths L there are τ, λ regular triangulations of $\mathcal{M}$ of maximum edge length L. We fix such τ and λ. We hereafter consider only τ, λ regular triangulations $\mathcal{T}$ with very small maximum edge length L. Once L is small the triangles $\overline{pqr}$ associated with $\langle pqr\rangle$ in $\mathcal{T}_2$ are very nearly parallel with the tangent plane to $\mathcal{M}$ at p.

2.1.8 Associated with each facet $\langle pqr\rangle$ in $\mathcal{T}_2$ is the *unit normal vector* $\mathbf{n}(pqr)$ to $\overline{pqr}$ having positive inner product with the normal $\mathbf{n}(p)$ to $\mathcal{M}$ at p.

2.1.9 Associated with each edge $\langle pq \rangle$ in $\mathcal{T}_1$ are exactly two distinct facets $\langle pqr \rangle$ and $\langle pqs \rangle$ in $\mathcal{T}_2$. We denote by

$$\mathbf{n}(pq) = \frac{\mathbf{n}(pqr) + \mathbf{n}(pqs)}{\left|\mathbf{n}(pqr) + \mathbf{n}(pqs)\right|}$$

the *average normal vector* at $\overline{pq}$.

For each $\langle pq \rangle$ we further denote by $\theta(pq)$ the *signed dihedral angle* at $\overline{pq}$ between the oriented plane directions of $\overline{pqr}$ and $\overline{pqs}$ which is characterized by the condition

$$2\sin\left(\frac{\theta(pq)}{2}\right)\ \mathbf{n}(pq) = V + W$$

where

- V is the unit exterior normal vector to $\overline{pqr}$ along edge $\overline{pq}$, so that, in particular,

$$V \cdot (p-q) = V \cdot \mathbf{n}(pqr) = 0;$$

- W is the unit exterior normal vector to $\overline{pqs}$ along edge $\overline{pq}$.

One checks that

$$\cos\theta(pq) = \mathbf{n}(pqr) \cdot \mathbf{n}(pqs).$$

Finally for each $\langle pq \rangle$ we denote by

$$g(pq) = |p-q|^{-1} \int_{\overline{pq}} g \, d\mathcal{H}^1 \in \mathbb{R}^3$$

the $\overline{pq}$ *average of* g; here $\mathcal{H}^1$ is one dimensional Hausdorff measure in $\mathbb{R}^3$.

2.1.10 Associated with our triangulation $\mathcal{T}$ of $\mathcal{M}$ is the *polyhedral approximation*

$$\mathcal{N}[\mathcal{T}] = \cup\left\{\overline{pqr} : \langle pqr \rangle \in \mathcal{T}_2\right\}$$

and the integral varifold

$$V[\mathcal{T}] = \sum_{\langle pqr \rangle \in \mathcal{T}_2} \mathbf{v}\left(\overline{pqr}\right) = \mathbf{v}\left(\mathcal{N}(\mathcal{T})\right)$$

whose first variation distribution is representable by integration

$$\delta V[\mathcal{T}] = \sum_{\langle pq \rangle \in \mathcal{T}_1} \mathcal{H}^1 \llcorner \overline{pq} \wedge \left[2\ \sin\left(\frac{\theta(pq)}{2}\right)\right]\ \mathbf{n}(pq)$$

[1, 4.3.5] so that

$$\delta V[\mathcal{T}](g) = \sum_{\langle pq \rangle \in \mathcal{T}_1} \left[|p-q|\right] \left[2\ \sin\left(\frac{\theta(pq)}{2}\right)\right] \left[\mathbf{n}(pq) \cdot g(pq)\right].$$

2.2 Terminology and facts for a flow of manifolds $\mathcal{M}_t$

2.2.1 As in 2.1.1 we suppose that $\mathcal{M} \subset \mathbb{R}^3$ is a compact connected smooth two dimensional submanifold of $\mathbb{R}^3$ without boundary oriented by a smooth Gauss mapping $\mathbf{n}\colon \mathcal{M} \to \mathbb{S}^2$ of unit normal vectors. We suppose additionally that $\varphi\colon (-1,\, 1) \times \mathcal{M} \to \mathbb{R}^3$ is a smooth mapping with $\varphi(0,\, p) = p$ for each $p \in \mathcal{M}$. For each t we set

$$\varphi[t] = \varphi(t,\, \cdot)\colon \mathcal{M} \to \mathbb{R}^3 \quad \text{and} \quad \mathcal{M}_t = \varphi[t](\mathcal{M}).$$

Our principal assumption is that, for each t, the mapping $\varphi[t]\colon \mathcal{M} \to \mathcal{M}_t$ *is an orientation preserving isometric imbedding (of Riemannian manifolds).* In particular, each $\mathcal{M}_t \subset \mathbb{R}^3$ is a compact connected smooth two dimensional submanifold of $\mathbb{R}^3$ without boundary oriented by a smooth Gauss mapping $\mathbf{n_t}\colon \mathcal{M}_t \to \mathbb{S}^2$ of unit normal vectors.

2.2.2 As in 2.1.2, for each t, we denote by $H_t\mathbf{n}_t$ the mean curvature vector field of $\mathcal{M}_t$.

2.2.3 As in 2.1.3, for each t we denote by U_t a suitable neighborhood of $\mathcal{M}_t$ in $\mathbb{R}^3$ in which a smooth nearest point retraction mapping $\rho_t\colon U_t \to \mathcal{M}_t$ is well defined together with smooth signed distance function $\sigma_t\colon U_t \to \mathbb{R}$; also we set $g[t] = \nabla\sigma_t\colon U_t \to \mathbb{R}^3$ as an initial velocity vector field.

2.2.4 By a convenient abuse of notation we assume that we can define a smooth map

$$\varphi\colon (-1,\, 1) \times U_0 \to \mathbb{R}^3,$$

$$\varphi(t,\, p) = \varphi\big(t,\, \rho_0(p) + \sigma_0(p)\mathbf{n}_0(\rho(p))\big) = \varphi\big(t,\, \rho_0(p)\big) + \sigma_0(p)\mathbf{n}_t(\rho_0(p))$$

for each t and p. With $\varphi[t] = \varphi(t,\, \cdot)$ we have $\varphi[0] = \mathbf{1}_{U_0}$ and, additionally, $\sigma_0(p) = \sigma_t\big(\varphi[t](p)\big)$. We further assume that

$$U_t = \varphi[t]\, U_0$$

for each t.

2.2.5 Fact If we replace our initial $\varphi[t]\colon \mathcal{M} \to \mathbb{R}^3$'s by $\varphi[\mu t]$ for large enough μ (equivalently, restrict times t to $-1/\mu < t < 1/\mu$) and decrease the size of U_0 then the extended $\varphi[t]\colon U_0 \to \mathbb{R}^3$'s will exist. Such restrictions do not matter in the proof of our main assertion, since it is local in time and requires only small neighborhoods of the $\mathcal{M}_t$'s.

2.1.6 As in 2.1.4, for each t we denote by

$$V_t = \mathbf{v}(\mathcal{M}_t)$$

the integral varifold associated with $\mathcal{M}_t$.

2.2.7 We fix $0 < \tau < 1/2$ and $0 < \lambda < 1/2$ as in 2.1.7 and fix 2τ, 2λ regular triangulations $\mathcal{T}(1)$, $\mathcal{T}(2)$, $\mathcal{T}(3)$, $\ldots$ of $\mathcal{M}$ having maximum edge lengths $L(1)$, $L(2)$, $L(3)$ $\ldots$ respectively with $\lim_{j\to\infty} L(j) = 0$. For each j, the vertexes of $\mathcal{T}(j)$ are denoted $\mathcal{T}_0(j)$, the edges are denoted $\mathcal{T}_1(j)$, and the facets are denoted $\mathcal{T}_2(j)$. For all large j and each t we have triangulations $\mathcal{T}(1, t)$, $\mathcal{T}(2, t)$, $\mathcal{T}(3, t)$, $\ldots$ of $\mathcal{M}_t$ as follows. With notation similar to that above we specify, for each j and t,

$$\mathcal{T}_0(j, t) = \Big\{\varphi[t](p) : p \in \mathcal{T}_0(j)\Big\}, \quad \mathcal{T}_1(j, t) = \Big\{\big\langle\varphi[t](p)\,\varphi[t](q)\big\rangle : \langle pq\rangle \in \mathcal{T}_1(j)\Big\},$$

$$\mathcal{T}_2(j, t) = \Big\{\big\langle\varphi[t](p)\,\varphi[t](q)\,\varphi[t](r)\big\rangle : \langle pqr\rangle \in \mathcal{T}_2(j)\Big\}.$$

2.2.8 Fact If we replace $\varphi[t]$ by $\varphi[\mu t]$ for large enough μ (equivalently, restrict times t to $-1/\mu < t < 1/\mu$) then $\mathcal{T}(1, t)$, $\mathcal{T}(2, t)$, $\mathcal{T}(3, t)$, $\ldots$ will a sequence of τ, λ regular triangulations of $\mathcal{M}$ with maximum edge lengths $L(j, t)$ converging to 0 uniformly in time t as $j \to \infty$. Such restrictions do not matter in the proof of our main assertion, since it is local in time. We assume this has been done, if necessary, and that each of the triangulations $\mathcal{T}(j, t)$ is τ, λ regular with maximum edge lengths $L(j, t)$ converging to 0 as indicated.

2.2.9 As in 2.1.8 we associate with each j, t, and $\langle pqr\rangle \in \mathcal{T}_2(j)$ a unit normal vector $\mathbf{n}[t, j](pqr)$ to $\overline{\varphi[t](p)\,\varphi[t](q)\,\varphi[t](r)}$. As in 2.1.9 we associate with each j, t, and $\langle pq\rangle \in \mathcal{T}_1(j)$ an average normal vector $\mathbf{n}[t, j](pq)$ at $\overline{\varphi[t](p)\,\varphi[t](q)}$ and a signed dihedral angle $\theta[t, j](pq)$ at $\overline{\varphi[t](p)\,\varphi[t](q)}$ and the $\overline{\varphi[t](p)\,\varphi[t](q)}$ average $g[t, j](pq)$ of $g[t]$.

2.2.10 As in 2.1.10 we associate with each triangulation $\mathcal{T}(j, t)$ of $\mathcal{M}_t$ a *polyhedral approximation* $\mathcal{N}[\mathcal{T}(j, t)]$ and an integral varifold

$$V[\mathcal{T}(j, t)] = \mathbf{v}\big(\mathcal{N}[\mathcal{T}(j, t)]\big) = \sum_{\langle pqr\rangle \in \mathcal{T}_1(j)} \mathbf{v}\left(\overline{\varphi[t](p)\,\varphi[t](q)\,\varphi[t](r)}\right)$$

with first variation distribution

$$\delta V[\mathcal{T}(j, t)] = \sum_{\langle pq\rangle \in \mathcal{T}_1(j)} \mathcal{H}^1 \llcorner \left[\overline{\varphi[t])p)\,\varphi[t](q)}\right] \wedge \left[2\,\sin\left(\frac{\theta[t, j](pq)}{2}\right)\right] \mathbf{n}[t, j](pq).$$

so that

$$\delta V[\mathcal{T}(j,\,t)]\big(g[t]\big)$$
$$= \sum_{\langle pq\rangle\in\mathcal{T}_1(j)} \Big[\big|\varphi[t](p)-\varphi[t](q)\big|\Big]\left[2\,\sin\left(\frac{\theta[t,\,j](pq)}{2}\right)\right]\Big[\mathbf{n}[t,\,j](pq)\cdot g[t,\,j](pq)\Big].$$

2.2.11 The quantity we wish to show is constant in time is

$$\int_{\mathcal{M}_t} H_t\,d\mathcal{H}^2 = -\left(\frac{1}{2}\right)\delta V_t\big(g[t]\big).$$

Since, for each time t,

$$V_t = \lim_{j\to\infty} V[\mathcal{T}(j,\,t)] \qquad \text{(as varifolds)}$$

we know, for each t,

$$\delta V_t\big(g[t]\big) = \lim_{j\to\infty} \delta V[\mathcal{T}(j,\,t)]\big(g[t]\big).$$

We are thus led to seek to estimate

$$\frac{d}{dt}\delta V[\mathcal{T}(j,\,t)]\big(g[t]\big)$$

using the formula in 2.2.10. A key equality it provided by Schlafli's theorem mentioned above which, in the present terminology, asserts for each j and t,

$$\sum_{\langle pq\rangle\in\mathcal{T}_1(j)} \Big[\big|\varphi[t](p)-\varphi[t](q)\big|\Big]\frac{d}{dt}\Big[\theta[t,\,j](pq)\Big] = 0.$$

2.2.12 Fact Since, for each $\langle ppq\rangle$ in $\mathcal{T}_2(j)$, $\partial\langle pqr\rangle$ consists of exactly three edges, and, for each $\langle pq\rangle$ in $\mathcal{T}_1(j)$, there are exactly two distinct facets $\langle pqr\rangle$ in $\mathcal{T}_2(j)$ for which $\langle pq\rangle\in\partial\langle pqr\rangle$ we infer that, for each j,

$$card\big[\mathcal{T}_1(j)\big] = \frac{3}{2}card\big[\mathcal{T}_2(j)\big].$$

We then use the τ,λ regularity of the the $\mathcal{T}(j)$'s to check that that, for each time t and each $\langle ppq\rangle$ in $\mathcal{T}_2(j)$ the following four numbers have bounded ratios (independent of j, t, and $\langle ppq\rangle$) with each other

$$\mathcal{H}^2\left(\overline{\varphi[t](p)\,\varphi[t](q)\,\varphi[t](r)}\right), \qquad \big|\varphi[t](p)-\varphi[t](q)\big|^2, \qquad L(j,\,t)^2, \qquad L(j)^2.$$

Since

$$\lim_{j\to\infty}\mathcal{H}^2\big(\mathcal{N}[j,\,t]\big) = \mathcal{H}^2\big(\mathcal{M}_t\big) = \mathcal{H}^2\big(\mathcal{M}\big),$$

we infer

$$\sup_j \sum_{\langle pq\rangle\in\mathcal{T}_1(j)} L(j)^2 < \infty, \qquad \lim_{j\to\infty}\sum_{\langle pq\rangle\in\mathcal{T}_1(j)} L(j)^3 = 0.$$

3 Modifications of the flow

3.1 Justification for computing with modified flows

As indicated in 2.2, we wish to estimate the time derivatives of

$$\delta V[\mathcal{T}(j,\,t)]\big(g[t]\big)$$
$$= \sum_{\langle pq\rangle\in\mathcal{T}_1(j)} \Big[\big|\varphi[t](p)-\varphi[t](q)\big|\Big]\left[2\,\sin\left(\frac{\theta[t,\,j](pq)}{2}\right)\right]\Big[\mathbf{n}[t,\,j](pq)\cdot g[t,\,j](pq)\Big].$$

In each of the $\langle pq\rangle$ summands, each of the three factors

$$\Big[\big|\varphi[t](p)-\varphi[t](q)\big|\Big],\quad \left[2\,\sin\left(\frac{\theta[t,\,j](pq)}{2}\right)\right],\quad \Big[\mathbf{n}[t,\,j](pq)\cdot g[t,\,j](pq)\Big]$$

is an intrinsic geometric quantity (at each time) whose value does not change under isometries of the ambient $\mathbb{R}^3$. With $\langle pqr\rangle$ and $\langle pqs\rangle$ denoting the two facets sharing edge $\langle pq\rangle$, we infer that each of the factors depends at most on the relative positions of $\varphi[t](p)$, $\varphi[t](q)$, $\varphi[t](r)$, $\varphi[t](s)$ and $\varphi[t]\mathcal{M}$. Suppose $\psi\colon(-1,\,1)\times\mathbb{R}^3\to\mathbb{R}^3$ is continuously differentiable, and for each t, the function $\psi[t]=\psi(t,\,\cdot)\colon\mathbb{R}^3\to\mathbb{R}^3$ is an isometry. Suppose further, we set

$$\varphi^*(t,\,p)=\psi\big(t,\,\varphi(t,\,p)\big),\quad \varphi^*[t]=\varphi^*(t,\,\cdot)$$

for each t and p so that $\varphi^*[t]=\psi[t]\circ\varphi[t]$. If we replace $\mathcal{M}$ by $\mathcal{M}^*=\psi[0]\mathcal{M}$ and φ by φ^* then we could follow the procedures of 2.1 and 2.2 to construct triangulations and polyhedral approximations $\mathcal{T}^*[j,\,t]$ and varifolds V^*, etc. with

$$\delta V[\mathcal{T}(j,\,t)]\big(g[t]\big)=\delta V^*[\mathcal{T}^*(j,\,t)]\big(g^*[t]\big).$$

Not only do we have equality in the sum, but, for each $\langle pq\rangle$ the corresponding summands are identical numerically. Hence, in evaluating $\delta V[\mathcal{T}(j,\,t)]\big(g[t]\big)$ we are free to (and will) use a different ψ and φ^* for each summand.

3.2 Conventions for derivatives

Suppose W is an open subset of $\mathbb{R}^M$ and $f=\big(f^1,\,f^2,\,\ldots,\,f^N\big)\colon W\to\mathbb{R}^N$ is K times continuously differentiable. We denote by

$$|||D^K f|||$$

the supremum of the partial derivatives

$$\frac{\partial^k f^K}{\partial x_{i(1)}\partial x_{i(2)}\ldots\partial x_{i(K)}}(p)$$

corresponding to all points $p\in W$, all $\big\{i(1),\,i(2),\,\ldots,\,i(K)\big\}\subset\big\{1,\,\ldots,\,M\big\}$ and $k=1,\,\ldots,N$, all choices of orthonormal coordinates $(x_1,\,\ldots,\,x_M)$ for $\mathbb{R}^M$ and all choices of orthonormal coordinates $(y_1,\,\ldots,\,y_N)$ for $\mathbb{R}^N$.

3.3 Conventions for inequalities

In making various estimates we will use use the largest edge length of the jth triangulation, typically called L, and a general purpose constant C. The constant C will have different values in different contexts (even in the same formula). What is implied is that, with $\mathcal{M}$ and φ fixed, the constants C can be chosen independent of the level of triangulation (once it is fine enough) and independent of time t and independent of the various modifications of our flow which are used in obtaining our estimates. As a representative example of our terminology, the expression

$$A = B \pm CL^2$$

means

$$-CL^2 \leq A - B \leq CL^2.$$

3.4 Fixing a vertex at the origin

Suppose p is a vertex in $\mathcal{M}$ and

$$\varphi_*(-1,\,1) \times U_0 \to \mathbb{R}^3, \qquad \varphi_*(t,\,q) = \varphi(t,\,q) - \varphi(t,\,p) \quad \text{for each } q.$$

Then $\varphi^*(t,\,p) = (0,\,0,\,0)$ for each t. One checks, for $K = 0,\,1,\,2,\,3$ that

$$|||\,D^K\varphi^*\,||| \leq 2|||\,D^K\varphi\,|||, \qquad |||\,D^K\varphi^*[t]\,||| = |||\,D^K\varphi[t]\,|||$$

for each t.

3.5 Mapping a frame to the basis vectors

Suppose $(0,\,0,\,0) \in \mathcal{M}$ and that $\mathbf{e}_1$ and $\mathbf{e}_2$ are tangent to $\mathcal{M}$ at $(0,\,0,\,0)$. Suppose also $\varphi(t,\,0,\,0,\,0) = (0,\,0,\,0)$ for each t. Then the mapping φ^* given by setting

$$\varphi^*[t] = \begin{pmatrix} \frac{\partial\varphi^1}{\partial x_1}(t,\,0,\,0,\,0) & \frac{\partial\varphi^2}{\partial x_1}(t,\,0,\,0,\,0) & \frac{\partial\varphi^3}{\partial x_1}(t,\,0,\,0,\,0) \\ \frac{\partial\varphi^1}{\partial x_2}(t,\,0,\,0,\,0) & \frac{\partial\varphi^2}{\partial x_2}(t,\,0,\,0,\,0) & \frac{\partial\varphi^3}{\partial x_2}(t,\,0,\,0,\,0) \\ \frac{\partial\varphi^1}{\partial x_3}(t,\,0,\,0,\,0) & \frac{\partial\varphi^2}{\partial x_3}(t,\,0,\,0,\,0) & \frac{\partial\varphi^3}{\partial x_3}(t,\,0,\,0,\,0) \end{pmatrix} \circ \varphi[t]$$

satisfies

$$\varphi^*[t](0,\,0,\,0) = (0,\,0,\,0), \quad D\varphi^*[t](0,\,0,\,0) = \mathbf{1}_{\mathbf{R}^3}$$

with

$$|||D^K\varphi^*[t]||| = |||D^K\varphi[t]|||$$

for each $K = 1,\,2,\,3$ and each t, and

$$\left|\left|\left|\frac{\partial\varphi^*}{\partial t}(t,\,\cdot)\right|\right|\right| \leq 3\Big(|||D^0\varphi|||\cdot|||D^2\varphi||| + |||D^1\varphi[t]|||^2\Big).$$

3.6 Theorem *There is $C < \infty$ such that the following is true for all sufficiently small $\delta > 0$. Suppose $\gamma_0\colon [0, \delta] \to \mathcal{M}$ is an arc length parametrization of a length minimizing geodesic in $\mathcal{M}$ and set*

$$\gamma(s, t) = \varphi[t]\big(\gamma_0(s)\big) \quad \text{for each } s \text{ and } t$$

so that $s \to \gamma(s, t)$ is an arc length parametrization of a geodesic in $\mathcal{M}_t$. We also set

$$r(s, t) = \big|\gamma(0, t) - \gamma(s, t)\big| \quad \text{for each } s \text{ and } t$$

and, for (fixed) $0 < R < \delta$, consider

$$r(R, t) = \big|\gamma(0, t) - \gamma(R, t)\big| \quad \text{for each } t.$$

Then

$$\frac{d}{dt} r(R, t) = \pm C R^2$$

and

$$\lim_{R \downarrow 0} R^{-1} \frac{d}{dt} r(R, t) = 0.$$

Proof We will show

$$\left.\frac{d}{dt} r(R, t)\right|_{t=0} = \pm C R^2.$$

Step 1 Replacing $\varphi(t, p)$ by $\varphi^*(t, p) = \varphi(t, p) - \varphi(t, \gamma_0(0))$ as in 3.4 if necessary we assume without loss of generality that $\gamma(0, t) = (0, 0, 0)$ for each t.

Step 2 Rotating coordinates if necessary we assume without loss of generality that $\mathbf{e}_1$ and $\mathbf{e}_2$ are tangent to $\mathcal{M}_0$ at $(0, 0, 0)$ and that $\gamma_0'(0) = \mathbf{e}_1$

Step 3 Rotating coordinates as time changes as in 3.5 if necessary we assume without loss of generality that $D\varphi[t](0, 0, 0) = \mathbf{1}_{\mathbf{R}^3}$ for each t.

Step 4 We define

$$X(s, t) = \gamma(s, t) \cdot \mathbf{e}_1, \quad Y(s, t) = \gamma(s, t) \cdot \mathbf{e}_2, \quad Z(s, t) = \gamma(s, t) \cdot \mathbf{e}_3$$

so that

$$\gamma(s, t) = \big(X(s, t), Y(s, t), Z(s, t)\big)$$

and estimate for each s and t:

(a) $X(0, t) = Y(0, t) = Z(0, t) = 0$ (by step 1)

(b) $X_t(0,\,0) = Y_t(0,\,0) = Z_t(0,\,0) = 0$

(c) $X_s(s,\,t)^2 + Y_s(s,\,t)^2 + Z_s(s,\,t)^2 = 1$

(d) $X_s(s,\,t) = \pm 1,\ Y_s(s,\,t) = \pm 1,\ Z_s(s,\,t) = \pm 1$

(e) $1/2 \le r(s,\,t)/|s| \le 1$ (since δ is small)

(f) $X(s,\,0) = \pm Cs,\ Y(s,\,0) = \pm Cs,\ Z(s,\,0) = \pm Cs$

(g) $X_s(0,\,t) = X_s(0,\,0),\ Y_s(0,\,t) = Y_s(0,\,0),\ Z_s(0,\,t) = Z_s(0,\,0)$ (by step 3)

(h) $X_{st}(0,\,0) = Y_{st}(0,\,0) = Z_{st}(0,\,0) = 0$

(*i*)
$$X_{st}(s,\,0) = X_{st}(0,\,0) + \int_0^s X_{sst}(\eta,\,0)\,d\eta = 0 \pm s \sup\left|X_{sst}\right| = \pm Cs,$$
$$Y_{st}(s,\,0) = \pm Cs, \qquad Z_{st}(s,\,0) = \pm Cs$$

(j)
$$X_t(s,\,0) = X_t(0,\,0) + \int_0^s X_{st}(\eta,\,0)\,d\eta = 0 \pm Cs^2,$$
$$Y_t(s,\,0) = \pm Cs^2, \qquad Z_t(s,\,0) = \pm Cs^2$$

(k) $r^2 = X^2 + Y^2 + Z^2$

(ℓ)
$$rr_s = XX_s + YY_s + ZZ_s, \quad r_s = \frac{1}{r}\big(XX_s + YY_s + ZZ_s\big)$$

(m)
$$rr_t = XX_t + YY_t + ZZ_t, \quad r_t = \frac{1}{r}\big(XX_t + YY_t + ZZ_t\big)$$

(n) $r_s r_t + rr_{st} = X_s X_t + XX_{st} + Y_s Y_t + YY_{st} + Z_s Z_t + ZZ_{st}$

(o) evaluating (n) at $t = 0,\ r > 0$ we see
$$\frac{1}{r(s,\,0)^2}\big((\pm Cs)(\pm 1)\big)\big((\pm Cs)(\pm Cs^2)\big) + r(s,\,0)r_{st}(s,\,0)$$
$$= (\pm 1)(\pm Cs^2) + (\pm Cs)(\pm Cs)$$

(p) $r_{st}(s,\,0) = \pm Cs$

(q)
$$r_t(R,\,0) = r_t(0,\,0) + \int_0^R r_{st}(s,\,0)\,ds = 0 + \int_0^R \pm Cs\,ds = \pm CR^2.$$
□

3.7 Corollary *Suppose triangulation $\mathcal{T}(j)$ has maximum edge length $L = L(j)$ and $\langle pq \rangle$ is an edge in $\mathcal{T}_1(j)$. Then, for each t,*

$$\Big|\varphi[t](p) - \varphi[t](q)\Big| = \pm CL \quad \text{and} \quad \frac{d}{dt}\Big|\varphi[t](p) - \varphi[t](q)\Big| = \pm CL^2.$$

3.8 Stabilizing the facets of an edge

Suppose $\mathcal{T}(j)$ is a triangulation with maximum edge length $L = L(j)$ and that $\langle ABC \rangle$, $\langle ACD \rangle$ are facets in $\mathcal{T}_2(j)$ as illustrated

$$\begin{array}{ccccc} & & D = (e, f, 0) & & \\ & \swarrow & & \nwarrow & \\ (0,0,0) = A & & \longleftrightarrow & & C = (d, 0, 0)\,. \\ & \searrow & & \nearrow & \\ & & B = (a, b, c) & & \end{array}$$

Interchanging B and D if necessary we assume without loss of generality the the average normal $\mathbf{n}[0, AC]$ to $\mathcal{M}_0$ at A has positive inner product with $(C - A) \times (D - A)$.

1) Fixing A at the origin Modifying φ if necessary as in 3.4 if necessary we can assume without loss of generality that $\varphi[t](A) = (0,\, 0,\, 0)$ for each t. As indicated there, various derivative bounds are increased by, at most, a controlled amount.

2) Convenient rotations We set $u(t) = \varphi[t](C)$, $v(t) = \varphi[t](D)$ and use the Gramm–Schmidt orthonormalization process to construct

$$U(t) = \frac{u(t)}{|u(t)|}, \quad V(t) = \frac{v(t) - v(t)\cdot U(t)\,U(t)}{|v(t) - v(t)\cdot U(t)\,U(t)|}, \quad W(t) = U(t) \times V(t).$$

One uses the mean value theorem in checking

$$|||D^K U(t)||| \le C \left(\sum_{j=0}^{K+1} |||D^j \varphi||| \right), \quad \text{etc}$$

for each $K = 0,\, 1,\, 2$. We denote by $Q(t)$ the orthogonal matrices having columns equal to $U(t)$, $V(t)$, $W(t)$ respectively (which is the inverse matrix to its transpose). Replacing φ_t by $Q(t) \circ \varphi_t$ if necessary, we assume without loss of generality that there are functions $a(t)$, $b(t)$, $c(t)$, $d(t)$, $e(t)$, $f(t)$, such that

$$\varphi[t](A) = (0,\, 0,\, 0), \quad \varphi[t](B) = (a(t),\, b(t),\, c(t)),$$
$$\varphi[t](C) = (d(t),\, 0,\, 0), \quad \varphi[t](D) = (e(t),\, f(t),\, 0).$$

We assume without loss of generality the existence of functions $F[t]\big(x,\, y\big)$ defined for $(x,\, y)$ near $(0,\, 0)$ such that, near $(0,\, 0,\, 0)$ our manifold $\mathcal{M}_t$ is the graph of $F[t]$. In particular,

$$c(t) = F[t]\big(a(t),\, b(t)\big).$$

We assert that if $|p| \leq CL$, then

$$|F[t](p)| \leq CL^2, \qquad |\nabla F[t](p)| \leq CL. \tag{3.8.1}$$

To see this, first we note that $F[t](A) = F[t](C) = F[t](D) = 0$. Next we invoke Rolle's theorem to conclude the existence of c_1 on segment AD and c_2 on segment CD such

$$\left\langle \frac{D-A}{|D-A|},\, DF[t](c_1) \right\rangle = 0 = \left\langle \frac{D-C}{|D-C|},\, DF[t](c_2) \right\rangle .$$

Since $|p| \leq CL$ we infer

$$\left\langle \frac{D-A}{|D-A|},\, DF[t](p) \right\rangle = \pm CL, \qquad \left\langle \frac{D-C}{|D-C|},\, DF[t](p) \right\rangle = \pm CL.$$

In view of 2.1.6(vi)(vii)(viii) and 2.2.7 we infer that $\mathbf{e}_1$ and $\mathbf{e}_2$ are bounded linear combinations of $(D-A)/|D-A|$ and $(D-C)/|D-C|$ from which we conclude that $|\nabla F[t](p)| \leq CL$. This in turn implies that $|F[t](p)| \leq CL^2$ as asserted.

Since

$$\frac{\partial}{\partial t} F[t](0,\, 0) = 0$$

we infer

$$\frac{\partial}{\partial t} F[t](p) = \pm CL \tag{3.8.2}$$

and since

$$\frac{\partial}{\partial t}\, (\varphi[t](A) \cdot \mathbf{e}_3) = 0$$

we infer

$$c'(t) = \frac{\partial}{\partial t} F[t](a(t),\, b(t)) = \frac{\partial}{\partial t}\, (\varphi[t](B) \cdot \mathbf{e}_3) = \pm CL. \tag{3.8.3}$$

3.9 Proposition *Let L, A, B, C, D, a, b, c, d, e, f be as in 3.8. Then*

(1) $a'(t) = \pm CL^2$

(2) $b'(t) = \pm CL^2$

(3) $c'(t) = \pm CL$

(4) $d'(t) = \pm CL^2$

(5) $e'(t) = \pm CL^2$

(6) $f'(t) = \pm CL^2$.

Proof According to 3.7, if $r(t)$ denotes the distance between the endpoints of an edge of arc length L at time t, then

$$r'(t) = \pm CL^2.$$

(i) We invoke 3.7 directly to infer (4) above.

(ii) We apply 3.7 to the distance between $(0, 0, 0)$ and $(e, f, 0)$ to infer

$$\frac{d}{dt}\left(e^2+f^2\right)^{\frac{1}{2}} = \frac{\left(ee'+ff'\right)}{\left(e^2+f^2\right)^{\frac{1}{2}}} = \pm CL^2, \qquad ee'+ff' = \pm CL^3.$$

(iii) We apply 3.7 to the distance between $(d, 0, 0)$ and $(e, f, 0)$ to infer

$$\frac{d}{dt}\left((e-d)^2+f^2\right)^{\frac{1}{2}} = \frac{\left(e-d)(e'-d')+ff'\right)}{\left((e-d)^2+f^2\right)^{\frac{1}{2}}} = \pm CL^2,$$

$$(e{-}d)(e'-d')+ff' = \pm CL^3.$$

We subtract the first inequality from the second to infer

$$ed'-de'+dd' = \pm CL^3, \qquad de' \pm CL^3, \qquad e' = \pm CL^2.$$

Assertions (5) and (6) follow readily.

(iv) We apply 3.7 to the distance between $(0, 0, 0)$ and (a, b, c) to infer

$$\frac{d}{dt}\left(a^2+b^2+c^2\right)^{\frac{1}{2}} = \frac{\left(aa'+bb'+cc'\right)}{\left(a^2+b^2+c^2\right)^{\frac{1}{2}}} = \pm CL^2, \qquad aa'+bb'+cc' = \pm CL^3.$$

(v) We apply 3.7 to the distance between $(d, 0, 0)$ and (a, b, c) to infer

$$\frac{d}{dt}\left((a-d)^2+b^2+c^2\right)^{\frac{1}{2}} = \frac{\left((a-d)(a'-d')+bb'+cc'\right)}{\left((a-d)^2+b^2+c^2\right)^{\frac{1}{2}}} = \pm CL^2,$$

$$(a-d)(a'-d')+bb'+cc' = \pm CL^3.$$

We subtract the first inequality form the second to infer

$$ad'-da'+dd' = \pm CL^3, \qquad da' \pm CL^3, \qquad a' = \pm CL^2,$$

which gives assertion (1).

(vi) We estimate from 3.8 that

$$c = F[t](a, b) = \pm CL^2, \quad c' = \frac{d}{dt}F[t](a, b) + \nabla F[t](a, b)\cdot(a', b') = \pm CL,$$

which gives (3) above. We have also $cc' = \pm CL^3$. We recall (iv) above and estimate

$$aa'+bb'+cc' = \pm CL^3, \quad bb' = \pm CL^3, \quad b' = \pm CL^2,$$

which is (2) above. □

3.10 Proposition *Suppose $\mathcal{T}(j)$ is a triangulation with maximum edge length $L = L(j)$ and $\langle pq\rangle$ is an edge in $\mathcal{T}_1(j)$. Abbreviate $\theta(t) = \theta[t, j](pq)$. Then, for each t,*

$$\theta(t) = \pm CL \tag{1}$$

$$2\sin\left(\frac{\theta(t)}{2}\right) = \pm CL \tag{2}$$

$$\theta'(t) = \pm C \tag{3}$$

$$\frac{d}{dt}\left[2\sin\left(\frac{\theta(t)}{2}\right)\right] = \pm C \tag{4}$$

$$\frac{d}{dt}\left[2\sin\left(\frac{\theta(t)}{2}\right) - \theta\right] = \pm CL^2. \tag{5}$$

Proof Making the modifications of 3.8 if necessary, we assume without loss of generality (in the terminology there) that $\varphi[t](p) = A = (0, 0, 0)$, $\varphi[t](q) = C = (d(t), 0, 0)$, and that there are $\langle pqB_*\rangle, \langle pqD_*\rangle \in \mathcal{T}_2(j)_0$ with $\varphi[t](B_*) = B = (a(t), b(t), c(t))$, $\varphi[t](D_*) = D = (e(t), f(t), 0)$.

The unit normal to $\overline{ACD}$ is $(0, 0, 1)$ while the unit normal to $\overline{ABC}$ is

$$\frac{(0, -c, b)}{(b^2 + c^2)^{\frac{1}{2}}}$$

so that $\qquad \cos\theta = \dfrac{b}{(b^2+c^2)^{\frac{1}{2}}},$

$$\sin\theta = \pm\left(1 - \cos^2\theta\right)^{\frac{1}{2}} = \pm\left(1 - \frac{b^2}{b^2+c^2}\right)^{\frac{1}{2}} = \pm\frac{c}{(b^2+c^2)^{\frac{1}{2}}} = \pm CL$$

in view of 3.8. Assertions (1) and (2) follow. We compute further

$$(\sin\theta)' = \cos\theta\,\theta' = \pm\frac{(b^2+c^2)^{\frac{1}{2}}c' - c\frac{bb'+cc'}{(b^2+c^2)^{\frac{1}{2}}}}{b^2+c^2} = \pm C$$

in view of 3.9(1)(2)(3) and 3.8. Assertion (3) and (4) follow. Assertion (5) follows from differentiation and assertions (1) and (3). ☐

3.11 Proposition *Suppose $\mathcal{T}(j)$ is a triangulation with maximum edge length $L = L(j)$ and $\langle pq\rangle$ is an edge in $\mathcal{T}_1(j)$. Then*

(1) $\mathbf{n}[t,\, j](pq) = \big(0,\, \pm CL,\, 1 \pm CL^4\big)$

(2) $(d/dt)\big(\mathbf{n}[t,\, j](pq)\big) = \big(0,\, \pm C,\, \pm CL\big) + \big(\pm CL,\, \pm CL,\, \pm CL\big)$

(3) $g[t,\, j](pq) = \big(\pm CL,\, \pm CL,\, 1 \pm CL^2\big)$

(4) $(d/dt)g[t,\, j](pq) = \big(\pm C,\, \pm C,\, 0\big) + \big(\pm CL,\, \pm CL,\, \pm CL\big)$

(5) $\mathbf{n}[t,\, j](pq) \cdot g[t,\, j](pq) = 1 \pm CL^2$

(6) $(d/dt)\Big(\mathbf{n}[t,\, j](pq) \cdot g[t,\, j](pq)\Big) = \pm CL$

(7) $1 - \mathbf{n}[t,\, j](pq) \cdot g[t,\, j](pq) = \pm CL^2$.

Proof We let $A,\, B,\, C,\, D,\, F[t],\, b(t),\, c(t),\, d(t)$ be as in 3.8. We abbreviate $\mathbf{n} = \mathbf{n}[t,\, j](pq)$ and estimate

$$\begin{aligned}\mathbf{n} &= \frac{(0,\,0,\,1) + (0,\,-c,\,b)/(b^2+c^2)^{\frac12}}{\big|(0,\,0,\,1) + (0,\,-c,\,b)/(b^2+c^2)^{\frac12}\big|} \\ &= \frac{\big(0,\,-c,\,b + (b^2+c^2)^{\frac12}\big)}{2^{\frac12}\big(b^2+c^2+b(b^2+c^2)^{\frac12}\big)^{\frac12}}.\end{aligned}$$

The first assertion follows from 3.8.1. We differentiate to conclude $\mathbf{n}' =$

$$\begin{aligned}&\frac{\pm CL\big(0,\,-c',\,b' \pm C(bb'+cc')/L - (L/L)\big(bb'+cc' \pm b'L + \pm C(b/L)(bb'+cc'\big)}{\pm L^2} \\ &\quad = \big(0,\,\pm C,\,\pm CL\big) + \big(\pm CL,\,\pm CL,\,\pm CL\big)\end{aligned}$$

in view of 3.9(2)(3). This is assertion (2).

We abbreviate $g = g[t,\, j](pq)$ and estimate

$$\begin{aligned}g &= \frac{1}{d(t)}\int_0^{d(t)} \frac{\big(-F[t]_x,\,-F[t]_y,\,1\big)}{\big|\big(-F[t]_x,\,-F[t]_y,\,1\big)\big|} \\ &= \frac{1}{d(t)}\int_0^{d(t)} \frac{\big(-F[t]_x,\,-F[t]_y,\,1\big)}{\Big(\big(F[t]_x^2 F[t]_y^2 + 1\Big)^{\frac12}}.\end{aligned}$$

The third assertion follows from 3.8.1. We differentiate to estimate that dg/dt equals

$$\frac{-d'}{d^2}\int_0^{d(t)}\frac{\big(-F[t]_x,\,-F[t]_y,\,1\big)}{\big(1+F[t]_x^2+F[t]_y^2\big)^{\frac{1}{2}}}+\frac{d'}{d}\frac{\big(-F[t]_x,\,-F[t]_y,\,1\big)}{\big(1+F[t]_x^2+F[t]_y^2\big)^{\frac{1}{2}}}$$
$$+\frac{1}{d}\int_0^{d}\frac{\pm CL\big(-F[t]_{tx},\,-F[t]_{ty},\,0\big)}{1+F[t]_x^2+F[t]_y^2}$$
$$-\frac{1}{d}\int_0^{d}\frac{\big(-F[t]_x,\,-F[t]_y,\,1\big)(\pm C/L)\big(F[t]_xF[t]_{tx}+F[t]_yF[t]_{ty}\big)}{1+F[t]_x^2+F[t]_y^2}=$$
$$L\big(\pm C,\,\pm C,\,\pm C\big)+L\big(\pm C,\,\pm C,\,\pm C\big)+\big(\pm C,\,\pm C,\,0\big)+L\big(\pm C,\,\pm C,\,\pm C\big)$$

which gives assertion (4). Assertion (5) follows from assertions (1) and (3). Assertion (6) follows from assertions (1), (2), (3), (4) and integration by parts. Assertion (7) follows from assertions (1) and (3). □

4 Constancy of the mean curvature integral

4.1 The derivative estimates

Suppose triangulation $\mathcal{T}(j)$ has maximum edge length $L = L(j)$. We recall from 2.2.10 that

$$\delta V[\mathcal{T}(j,\,t)]\big(g[t]\big)$$
$$=\sum_{\langle pq\rangle\in\mathcal{T}_1(j)}\Big[\big|\varphi[t](p)-\varphi[t](q)\big|\Big]\left[2\,\sin\left(\frac{\theta[t,\,j](pq)}{2}\right)\right]\Big[\mathbf{n}[t,\,j](pq)\cdot g[t,\,j](pq)\Big]$$

and we estimate, for each t that

$$\frac{d}{dt}\Big(\delta V[\mathcal{T}(j)_t]\big(g[t]\big)\Big)$$
$$=\sum_{\langle pq\rangle\in\mathcal{T}_1(j)}\Big[\big|\varphi[t](p)-\varphi[t](q)\big|\Big]'\left[2\,\sin\left(\frac{\theta[t,\,j](pq)}{2}\right)\right]\Big[\mathbf{n}[t,\,j](pq)\cdot g[t,\,j](pq)\Big]$$
$$+\sum_{\langle pq\rangle\in\mathcal{T}_1(j)}\Big[\big|\varphi[t](p)-\varphi[t](q)\big|\Big]\left[2\,\sin\left(\frac{\theta[t,\,j](pq)}{2}\right)\right]'\Big[\mathbf{n}[t,\,j](pq)\cdot g[t,\,j](pq)\Big]$$
$$+\sum_{\langle pq\rangle\in\mathcal{T}_1(j)}\Big[\big|\varphi[t](p)-\varphi[t](q)\big|\Big]\left[2\,\sin\left(\frac{\theta[t,\,j](pq)}{2}\right)\right]\Big[\mathbf{n}[t,\,j](pq)\cdot g[t,\,j](pq)\Big]'.$$

We assert that

$$\frac{d}{dt}\Big(\delta V[\mathcal{T}(j,\,t)]\big(g[t]\big)\Big) = \sum_{\langle pq\rangle\in\mathcal{T}_1(j)} \pm CL^3 = \sum_{\langle pq\rangle\in\mathcal{T}_1(j)} \pm CL(j)^3.$$

To see this we will estimate each of the three summands above.

First summand We use 3.7, 3.10(2), 3.11(5) to estimate for each pq,

$$\Big[\big|\varphi[t](p)-\varphi[t](q)\big|\Big]'\left[2\,\sin\left(\frac{\theta[t,\,j](pq)}{2}\right)\right]\Big[\mathbf{n}[t,\,j](pq)\cdot g[t,\,j](pq)\Big] = \big(CL^2\big)\big(CL\big)\big(1\pm CL^2\big).$$

Second summand We use 3.10(5), 3.11(7) to estimate for each pq,

$$\begin{aligned}
&\Big[\big|\varphi[t](p)-\varphi[t](q)\big|\Big]\left[2\,\sin\left(\frac{\theta[t,\,j](pq)}{2}\right)\right]'\Big[\mathbf{n}[t,\,j](pq)\cdot g[t,\,j](pq)\Big]\\
&= \Big[\big|\varphi[t](p)-\varphi[t](q)\big|\Big]\Big[\theta[t,\,j](pq)\Big]'\\
&\quad + \Big[\big|\varphi[t](p)-\varphi[t](q)\big|\Big]\left[2\,\sin\left(\frac{\theta[t,\,j](pq)}{2}\right)-\theta[t,\,j](pq)\right]'\\
&\quad + \Big[\big|\varphi[t](p)-\varphi[t](q)\big|\Big]\left[2\,\sin\left(\frac{\theta[t,\,j](pq)}{2}\right)\right]'\Big[\mathbf{n}[t,\,j](pq)\cdot g[t,\,j](pq)-1\Big]\\
&= \Big[\big|\varphi[t](p)-\varphi[t](q)\big|\Big]\Big[\theta[t,\,j](pq)\Big]' \pm \big(CL\big)\big(CL^2\big) \pm \big(CL\big)\big(C\big)\big(CL^2\big).
\end{aligned}$$

Third summand We use 3.10(2) and 3.11(6) to estimate

$$\Big[\big|\varphi[t](p)-\varphi[t](q)\big|\Big]\left[2\,\sin\left(\frac{\theta[t,\,j](pq)}{2}\right)\right]\Big[\mathbf{n}[t,\,j](pq)\cdot g[t,\,j](pq)\Big]' = \big(CL\big)\big(CL\big)\big(CL\big).$$

According to Schlafli's formula [7],

$$\sum_{\langle pq\rangle\in\mathcal{T}_1(j)} \Big[\big|\varphi[t](p)-\varphi[t](q)\big|\Big]\Big[\theta[t,\,j](pq)\Big]' = 0.$$

Our assertion follows.

4.2 Main Theorem

(1) *For each fixed time* t,

$$\lim_{j\to\infty} \delta V[\mathcal{T}(j,\, t)]\big(g[t]\big) = \delta V_t\big(g[t]\big).$$

(2) *For each fixed* j, $\delta V[\mathcal{T}(j)_t]\big(g[t]\big)$ *is a differentiable function of* t *and*

$$\lim_{j\to\infty} \frac{d}{dt}\Big(\delta V[\mathcal{T}(j)_t]\big(g[t]\big)\Big) = 0$$

uniformly in t.

(3) *For each* t

$$\int_{\mathcal{M}_t} H_t \, d\mathcal{H}^2 = \int_{\mathcal{M}} H \, d\mathcal{H}^2.$$

This is the main result of this note.

Proof To prove the first assertion, we check that

$$(\rho_t)_\sharp V[\mathcal{T}(j,\, t)] = V_t$$

for each t and all large j. Indeed, the τ regularity of our triangulations implies that the normal directions of the $\mathcal{N}[\mathcal{T}(j)_t]$ are very nearly equal to the normal directions of nearby points on $\mathcal{M}_t$ and that the restriction of $D\rho_t$ to the tangent planes of the $\mathcal{N}[\mathcal{T}(j)_t]$ is very nearly an orthogonal injection. The first assertion follows with use of the first variation formula given in [14.1, 4.2]. Assertion (2) follows from 4.1 since

$$\sum_{\langle pq\rangle \in \mathcal{T}_1(j)} L(j)^2$$

is dominated by the area of $\mathcal{M}$ (see 2.2.12) and $\lim_{j\to\infty} L(j) = 0$. Assertion (3) follows from assertions (1) and (2) and our observation in 2.1.4. □

Acknowledgements Fred Almgren tragically passed away shortly after this note was written. Since then, the main result for smooth surfaces has been reproved in an easier way and generalized to the setting of Einstein manifolds by J-M Schlenker together with the second author of the current paper [6]. Nonetheless, it seems clear that the methods used here can be used to extend these results in other directions.

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