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On the continuity of bending

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Abstract We examine the dependence of the deformation obtained by bending quasi-Fuchsian structures on the bending lamination. We show that when we consider bending quasi-Fuchsian structures on a closed surface, the conditions obtained by Epstein and Marden to relate weak convergence of arbitrary laminations to the convergence of bending cocycles are not necessary. Bending may not be continuous on the set of all measured laminations. However we show that if we restrict our attention to laminations with non negative real and imaginary parts then the deformation depends continuously on the lamination.

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The deformation of hyperbolic structures by bending along totally geodesic submanifolds of codimension one was introduced by Thurston in his lectures on *The Geometry and Topology of 3–manifolds.* The geometric and algebraic properties of the deformation were studied in [4] and [3]. Epstein and Marden [2] introduced the notion of a bending cocycle and used it to describe bending a hyperbolic surface along a measured geodesic lamination. The same notion was used in [5] to extend bending to a holomorphic family of local biholomorphic homeomorphisms of quasi-Fuchsian space $Q(S)$.

Epstein and Marden [2] give a careful analysis of the dependence of the bending cocycle on the measured lamination. They consider the set of measured laminations on $\mathcal{H}^2$ consisting of geodesics that intersect a compact subset $K \subset \mathcal{H}^2$. This is a subset of the space of measures on the space $G(K)$ of geodesics in $\mathcal{H}^2$ intersecting K, with the topology of weak convergence of measures. In this topology, the bending cocycle does not depend continuously on the lamination. One reason for this is the behaviour of the laminations near the endpoints of the segment over which we evaluate the cocycle. For example, consider the geodesic segment $[e^{i\theta}, i]$ in $\mathcal{H}^2$, for suitable θ in $[0, \pi/2]$, and the measured laminations μ_n, with weight 1 on the geodesic $(1/n, n)$ and weight -1 on the geodesic $(-1/n, -n)$. Then $\{\mu_n\}$ converges weakly to the zero lamination, but the cocycle of μ_n relative to $[e^{i\theta}, i]$ is approximately a hyperbolic isometry of translation length 1. Epstein and Marden find conditions under which a sequence of measured laminations gives a convergent sequence of cocycles relative to a given pair of points.

In this article we show that when the lamination is invariant by a discrete group and we only consider cocycles relative to points in the orbit of a suitable point $x \in \mathcal{H}^2$, any sequence of measured laminations $\{\mu_n\}$ which converges weakly gives rise to cocycles which converge up to conjugation. We show further that the same conjugating elements can be used for the cocycles for μ_n corresponding to the different generators of the group. Hence the laminations μ_n determine bending homomorphisms which, after conjugation by suitable isometries, converge to the bending homomorphism determined by μ_0. This implies that the deformations converge in $Q(S)$.

Theorem 1 *Let S be a closed hyperbolic surface and $Q(S)$ its space of quasi-Fuchsian structures. Let $\{\mu_n\}$ be a sequence of complex measured geodesic laminations, converging weakly to a lamination μ_0. Then the bending deformations*

$$B_{\mu_n} \colon \mathcal{D}_{\mu_n} \to Q(S)$$

converge to the deformation B_{μ_0}, uniformly on compact subsets of $\mathcal{D} = \mathcal{D}_{\mu_0} \cap (\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \mathcal{D}_{\mu_n})$.

We also state an infinitesimal version of the Theorem.

Theorem 2 *Let S be a closed hyperbolic surface and $Q(S)$ its space of quasi-Fuchsian structures. Let $\{\mu_n\}$ be a sequence of complex measured geodesic laminations, converging weakly to a lamination μ_0. Then the holomorphic bending vector fields T_{μ_n} on $Q(S)$ converge to T_{μ_0}, uniformly on compact subsets of $Q(S)$.*

These results do not necessarily imply the continuous dependence of the deformation on the bending lamination, because the space of measured laminations is not first countable. If however we restrict our attention to the subset of measured laminations with non negative real and imaginary parts, then we can apply results in [6] to obtain the following Theorem.

Theorem 3 *The mapping $\mathcal{ML}^{++}(S) \times Q(S) \to T(Q(S)) \colon (\mu, [\rho]) \mapsto T_\mu([\rho])$ is continuous, and holomorphic in $[\rho]$.*

The proof of Theorem 1 is based on the observation that, when the lamination is invariant by a discrete group and we are considering cocycles with respect to points x and $g(x)$, for some g in the group, the effect of a lamination near the endpoints of the segment $[x, g(x)]$ is controlled by its effect near x, provided that the lamination does not contain geodesics very close to the geodesic carrying $[x, g(x)]$. This last condition can be achieved by choosing x to be a point not on the axis of a conjugate of g (see Corollary 2.12).

In Section 1 we describe the space of measured laminations and we recall the definition of bending. In the beginning of Section 2 we recall or modify certain results from [2] and [5] which provide bounds for the effect of bending along nearby geodesics. Lemma 2.11 and the results following it examine the consequences of the above condition on the choice of x.

The proof of Theorems 1, 2 and 3 is given in Section 3. The laminations μ_n are replaced by finite approximations. The main result is Lemma 3.1, which gives the basic estimate for the difference between the bending homomorphism of μ_0 and a conjugate of the bending homomorphism of μ_n. Then a diagonal argument is used to obtain the convergence of bending.

1 The setting

We consider a closed surface S of genus greater than 1. We fix a hyperbolic structure on S, and let $\rho_0\colon \pi_1(S) \to PSL(2,\mathbb{R})$ be an injective homomorphism with discrete image $\Gamma_0 = \rho_0(\pi_1(S))$, such that S is isometric to $\mathcal{H}^2/\Gamma_0$.

We consider the space R of injective homomorphisms $\rho\colon \Gamma_0 \to PSL(2,\mathbb{C})$ obtained by conjugation with a quasiconformal homeomorphism ϕ of $\widehat{\mathbb{C}}$: if $g \in \Gamma_0$, acting on $\widehat{\mathbb{C}}$ as Möbius transformations, then $\rho(g) = \phi \circ g \circ \phi^{-1}$.

$PSL(2,\mathbb{C})$ acts on the left on R by inner automorphisms. The quotient of R by this action is the *space $Q(S)$ of quasi-Fuchsian structures* on S, or *quasi-Fuchsian space* of S. We denote the equivalence class in $Q(S)$ of a homomorphism $\rho \in R$ by $[\rho]$. Then $[\rho]$ is a *Fuchsian point* if there is a circle in $\widehat{\mathbb{C}}$ left invariant by $\rho(\Gamma_0)$, so that $\rho(\Gamma_0)$ is conjugate to a Fuchsian group of the first kind. The subset of Fuchsian points in $Q(S)$ is the *Teichmüller* space of S, $T(S)$.

We fix a point $[\rho] \in Q(S)$, represented by the homomorphism $\rho\colon \Gamma_0 \to PSL(2,\mathbb{C})$ obtained by conjugation with the quasiconformal homeomorphism $\phi\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. We denote the image of ρ by Γ. The limit set of Γ_0 is $\widehat{\mathbb{R}}$. Then

$\phi(\widehat{\mathbb{R}})$ is the limit set of Γ. If γ is a geodesic in $\mathcal{H}^2$ with endpoints $u, v \in \widehat{\mathbb{R}}$, we denote by $\phi_*(\gamma)$ the geodesic in $\mathcal{H}^3$ with endpoints $\phi(u), \phi(v)$ in $\phi(\widehat{\mathbb{R}})$. In this way, geodesics on the surface $S \cong \mathcal{H}^2/\Gamma_0$ are associated to geodesics in the hyperbolic 3–manifold $\mathcal{H}^3/\Gamma$.

We want to study the deformation of quasi-Fuchsian structures by *bending*, [4], [2], [5]. Bending is determined by a geodesic lamination on S with a complex valued transverse measure.

A measured geodesic lamination on S lifts to a measured geodesic lamination on $\mathcal{H}^2$. The space $G(\mathcal{H}^2)$ of unoriented geodesics in $\mathcal{H}^2$ is homeomorphic to a Möbius strip without boundary. Let K be a compact subset of $\mathcal{H}^2$, projecting onto $\mathcal{H}^2/\Gamma_0$. The set $G(K)$ of geodesics in $\mathcal{H}^2$ intersecting K is a compact metrizable space.

A measured geodesic lamination on $\mathcal{H}^2$ determines a complex valued Borel measure μ on $G(K)$, with the property that if γ_1 and γ_2 are distinct geodesics in the support of μ, then they are disjoint. The set of measured geodesic laminations on S can be considered as a subset of $\mathcal{M}(G(K))$, the set of complex valued Borel measures on $G(K)$. The set $\mathcal{M}(G(K))$ has a norm, defined by

$$\|\mu\| = \sup\left\{\left|\int f\mu\right|, f \text{ continuous complex valued function on } G(K), |f| \leq 1\right\}$$

We shall use the weak* topology on $\mathcal{M}(G(K))$, with basis the sets of the form

$$U(\mu, \varepsilon, f_1, \dots, f_m) = \left\{\nu \in \mathcal{M}(G(K)) : \left|\int f_i\mu - \int f_i\nu\right| < \varepsilon, i = 1, \dots, m\right\}$$

where $\mu \in \mathcal{M}(G(K))$, f_i, $i = 1, \dots, m$ are continuous functions on $G(K)$, and ε is a positive number.

A measured geodesic lamination μ on S is called *finite* if it is supported on a finite set of simple closed geodesics in S. Then, for any compact subset K of $\mathcal{H}^2$, the measure on $G(K)$ determined by the lift of μ to $\mathcal{H}^2$ has finite support.

Given a finite measured geodesic lamination μ on S, we define bending the quasi-Fuchsian structure $[\rho]$ on S as follows.

Let $g_1, \dots, g_k$ be a set of generators of Γ_0. Choose a point x on $\mathcal{H}^2$ and, for each g_j, consider the geodesic segment $[x, g_j(x)]$. Let $\gamma_1, \dots, \gamma_m$ be the geodesics in the support of μ intersecting $[x, g_j(x)]$, and let $z_1, \dots, z_m$ be the corresponding measures. If γ_1 (or γ_m) go through x (or $g_j(x)$ respectively), we replace z_1 (or z_m) by $\frac{1}{2}z_1$ (or $\frac{1}{2}z_m$).

If γ is an oriented geodesic in $\mathcal{H}^3$ and $z \in \mathbb{C}$, we denote by $A(\gamma, z)$ the element of $PSL(2, \mathbb{C})$ with axis γ and complex displacement z. We will use the same notation for one of the matrices in $SL(2,\mathbb{C})$ corresponding to $A(\gamma,z)$. In such cases either the choice of the lift will not matter, or there will be an obvious choice.

We orient the geodesics $\gamma_1,\dots,\gamma_m$ so that they cross the segment $[x,g_j(x)]$ from right to left, and define the isometry

$$C_{t\mu}(x,g_j(x)) = A(\phi_*(\gamma_1),tz_1)\cdots A(\phi_*(\gamma_m),tz_m).$$

For each generator g_j, $j=1,\dots,k$, define

$$\rho_{t\mu}(g_j) = C_{t\mu}(x,g_j(x))\ \rho(g_j).$$

For t in an open neighbourhood of 0 in $\mathbb{C}$, the representation $[\rho_{t\mu}]$ is quasi-Fuchsian, [4].

Any measured geodesic lamination μ on S can be approximated by finite laminations so that the corresponding bending deformations converge, [2], [5]. In this way, we obtain for any measured geodesic lamination on S a deformation B_μ defined on an open set $\mathcal{D}_\mu \subset Q(S)\times\mathbb{C}$,

$$B_\mu\colon\ \mathcal{D}_\mu \to Q(S)\colon\ ([\rho],t)\mapsto[\rho_{t\mu}].$$

B_μ is a holomorphic mapping.

2 The lemmata

In the vector space $\mathbb{C}^2$ we introduce the norm

$$\|(z_1,z_2)\| = \max\{|z_1|,|z_2|\}.$$

A complex matrix $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ acts on $\mathbb{C}^2$ and has norm

$$\|A\| = \max\{|a|+|b|,|c|+|d|\}.$$

We will use this norm on $SL(2,\mathbb{C})$.

Lemma 2.1 ([2], 3.3.1) *Let X be a set of matrices in $SL(2,\mathbb{C})$ and $c=(0,0,1)\in\mathcal{H}^3$. Then the following are equivalent.*

i) *The closure of X is compact.*
ii) *There is a positive number M such that if $A\in X$ then $||A||\le M$.*
iii) *There is a positive number M such that if $A\in X$ then $||A||\le M$ and $||A^{-1}||\le M$.*

iv) *There is a positive number R such that if $A \in X$ then $d(c, A(c)) \leq R$.* $\square$

Let Λ be a maximal geodesic lamination on S, and $\psi\colon S \to \mathcal{H}^3/\Gamma$ the pleated surface representing the lamination Λ [1]. Let $\tilde{\psi}\colon \mathcal{H}^2 \to \mathcal{H}^3$ be the lift of ψ.

Lemma 2.2 ([5], 2.5) *Let K be a compact disc of radius R about $c = (0,0,1) \in \mathcal{H}^3$, and M a positive number. There is a positive number N with the following property. If $[x,y]$ is a geodesic segment in $\mathcal{H}^2$ such that $\tilde{\psi}([x,y]) \subset K$ and $\{\gamma_i, z_i\}$, $i = 1, \dots, m$ is a finite measured lamination with support contained in Λ, whose leaves all intersect $[x,y]$ and are numbered in order from x to y, and such that $\sum_{i=1}^{m} |\mathrm{Re}\, z_i| < M$, then*

$$\|A(\gamma_1, z_1) \cdots A(\gamma_m, z_m)\| \leq N. \qquad \square$$

Lemma 2.3 ([2], 3.4.1, [5], 2.4) *Let K be a compact subset of $SL(2,\mathbb{C})$, M a positive number, and let γ be the geodesic $(0,\infty)$. Then there is a positive number N with the following property. For any $B, C \in K$, and $z \in \mathbb{C}$ with $|z| \leq M$, we have*

$$\left\|BA(\gamma,z)B^{-1} - CA(\gamma,z)C^{-1}\right\| \leq N \left\|B - C\right\| |z|. \qquad \square$$

In order to examine the effect of bending along nearby geodesics, in Lemma 2.5 and 2.6, we shall use the notion of a solid cylinder in hyperbolic space. A *solid cylinder C over a disk D* in $\mathcal{H}^n$ is the union of all geodesics orthogonal to a $(n-1)$–dimensional hyperbolic disc D in $\mathcal{H}^n$. The *radius* of the cylinder is the hyperbolic radius of the disc D. If x is the centre of D, we say that C is a solid cylinder *based* at x. The boundary of C at infinity consists of two discs D_1 and D_2 in $\partial\mathcal{H}^n$. We say that the solid cylinder C is *supported* by D_1 and D_2. The geodesic orthogonal to D through its centre is the *core* of the solid cylinder C. We shall denote the cylinder with core γ, basepoint $x \in \gamma$ and radius r by $C(\gamma, x, r)$.

Lemma 2.4 ([5], 2.6) *Let L be a compact set in $\mathcal{H}^3$. Then there exists a positive number M with the following property. If D is a disc of radius r, contained in L, and α, β are two geodesics contained in the solid cylinder over D, then there is an element $A \in SL(2,\mathbb{C})$ such that $A(\alpha) = \beta$ and $||A - I|| \leq Mr$.* $\square$

If C is a solid cylinder supported on the discs D_1 and D_2, with $D_1 \cap D_2 = \emptyset$, and γ_1, γ_2 are two geodesics, each having one end point in D_1 and one in D_2, we say that γ_1 and γ_2 are *concurrently oriented* in C if their origins lie in the same component of $D_1 \cup D_2$.

Lemma 2.5 *Let m be a positive number and L a compact subset of $\mathcal{H}^3$. Then there are positive numbers M_1 and M_2 with the following property. If γ_1, γ_2 are concurrently oriented geodesics contained in a cylinder of radius r, based at a point in L, and z_1, z_2 are complex numbers such that $|z_i| \leq m$, then there are lifts of $A(\gamma_i, z_i)$ to $SL(2, \mathbb{C})$ such that*

$$\|A(\gamma_1, z_1) - A(\gamma_2, z_2)\| \leq M_1 r \min\{|z_1|, |z_2|\} + M_2 |z_1 - z_2|.$$

Proof We assume that $|z_1| \leq |z_2|$. We have

$$\|A(\gamma_1, z_1) - A(\gamma_2, z_2)\| \leq \|A(\gamma_1, z_1) - A(\gamma_2, z_1)\| + \|A(\gamma_2, z_1) - A(\gamma_2, z_2)\|.$$

Let $B \in SL(2, \mathbb{C})$ be an element mapping the geodesic $(0, \infty)$ to γ_2, and mapping the point $c = (0, 0, 1)$ to a point in L. Then, by Lemma 2.1, there is a constant K_1 depending only on L, such that $||B|| \leq K_1$. By Lemma 2.4 there is an element $C \in SL(2, \mathbb{C})$ such that $C(\gamma_2) = \gamma_1$, and $||C - I|| \leq K_2 r$ for some constant K_2 depending only on L.

By Lemma 2.3 there is a constant K_3 such that

$$\|A(\gamma_1, z_1) - A(\gamma_2, z_1)\| \leq K_3 \|CB - B\| \, |z_1| \leq K_1 K_2 K_3 r |z_1|.$$

On the other hand,

$$\|A(\gamma_2, z_1) - A(\gamma_2, z_2)\| \leq \|B\| \, \|A((0, \infty), z_1 - z_2) - I\| \, \|B^{-1}\| \, \|A((0, \infty), z_2)\|.$$

By Lemma 2.1 and the fact that the entries of $A((0, \infty), z_1 - z_2)$ depend analytically on $z_1 - z_2$, there is a constant K_4, depending on L and m such that

$$\|A(\gamma_2, z_1) - A(\gamma_2, z_2)\| \leq K_4 |z_1 - z_2|. \qquad \square$$

Lemma 2.6 ([5], 2.7) *Let m be a positive number and L a compact subset of $\mathcal{H}^3$. Then there is a positive number M with the following property. Let C be a solid cylinder of radius r based at a point in L. Let $\gamma_1, \ldots, \gamma_k$ be geodesics in C and $z_1, \ldots, z_k$ complex numbers with $\sum_{i=1}^k |\mathrm{Re}\,(z_i)| \leq m$. Then*

$$\left\| A(\gamma_1, z_1) \cdots A(\gamma_k, z_k) - A\left(\gamma_1, \sum_{i=1}^k z_i\right) \right\| \leq M r \sum_{i=1}^k |z_i|. \qquad \square$$

We want to show that if two geodesics on S are sufficiently close, then the corresponding geodesics in $\mathcal{H}^3/\Gamma$ will also be close, (Lemma 2.10).

Lemma 2.7 *Let K be a compact subset of $\mathcal{H}^2$, and $\phi\colon \partial\mathcal{H}^2 \to \partial\mathcal{H}^3$ a homeomorphism onto its image. Then there is a compact subset L of $\mathcal{H}^3$ such that if γ is a geodesic of $\mathcal{H}^2$ intersecting K, then $\phi_*(\gamma)$ intersects L, i.e. $\phi_*(G(K)) \subset G(L)$.*

Proof We consider the Poincaré disk model of hyperbolic space. There, it is clear that if K is a compact subset of B^2, then there is a positive number m such that if γ is a geodesic in $G(K)$ with end-points u, v, then $|u - v| \geq m$. Since ϕ^{-1} is uniformly continuous, there is a positive number M such that $|\phi(u) - \phi(v)| \geq M$, and hence there is a compact subset of B^3 intersecting $\phi_*(\gamma)$. □

Lemma 2.8 ([5], 2.2) *Let ε and η be two positive numbers. Then there is a positive number δ with the following property. If D_1 and D_2 are discs in S^2, with spherical radius $\leq \delta$, and the spherical distance between D_1 and D_2 is $\geq \eta$, then the solid cylinder supported by D_1 and D_2 has hyperbolic radius $r \leq \varepsilon$.* □

Lemma 2.9 *Let K be a compact subset of B^n, and d a positive number. Then there is a positive number δ with the following property. If C is a solid cylinder in B^n, over a disc with radius $r \leq \delta$ and centre at a point in K, then the spherical radius of each of the discs supporting C is $\leq d$.*

Proof The radii of the supporting discs are given by continuous functions of the core geodesic, the base point and the radius of the cylinder. For a fixed base point, they tend to zero with the radius of the cylinder. The result follows by compactness. □

Lemma 2.10 *Let $[\rho]$ be a quasi-Fuchsian structure on S, K a compact subset of $\mathcal{H}^2$, and L a compact subset of $\mathcal{H}^3$ such that $\phi_*(G(K)) \subset G(L)$. Let r be a positive number. Then there is a positive number δ with the following property. If $\gamma \in G(K)$, $x \in \gamma \cap K$ and $0 \leq r_1 \leq \delta$, then there is some point $x' \in L$ such that for any geodesic α contained in the solid cylinder $C(\gamma, x, r_1)$, the geodesic $\phi_*(\alpha)$ is contained in the solid cylinder $C(\phi_*(\gamma), x', r) \subset \mathcal{H}^3$.*

Proof We work in the Poincaré disc model of the hyperbolic plane and space, B^2 and B^3. Since L is a compact subset of B^3, there is a number $\eta_2 > 0$ such that if u and v are the endpoints of any geodesic in B^3 intersecting L, then the spherical distance between u and v is $\geq \eta_2$. Then, by Lemma 2.8, there is a positive number δ_2, such that any solid cylinder with core a geodesic $\gamma \in G(L)$ and supported on discs of spherical radius $\leq \delta_2$, has hyperbolic radius $\leq r$.

Since $\phi\colon S^1 \to S^2$ is uniformly continuous, there is a positive number δ_1, such that any arc in S^1 of length $\leq \delta_1$ is mapped into a disc in S^2, of radius $\leq \delta_2$. Then, by Lemma 2.9, there is a positive number δ such that any solid cylinder of radius $\leq \delta$ and based at a point in K, is supported on two arcs of length $\leq \delta_1$. □

Recall that, if X is a subset of $\mathcal{H}^2$, we denote by $G(X)$ the set of geodesics in $\mathcal{H}^2$ which intersect X. To simplify notation, we will write $G(x)$ for the set of geodesics through the point $x \in \mathcal{H}^2$, and $G(x,y)$ for the set of geodesics intersecting the open geodesic segment (x,y).

If Γ is a group of isometries of $\mathcal{H}^2$, we denote by G'_Γ the set of geodesics in $\mathcal{H}^2$ which do not intersect any of their translates by Γ:

$$G'_\Gamma = \{\gamma \in G(\mathcal{H}^2) : \forall g \in \Gamma, g(\gamma) \cap \gamma = \emptyset \text{ or } g(\gamma) = \gamma\}.$$

In the following Lemma we consider the angle between unoriented geodesics to lie in the interval $[0, \frac{\pi}{2}]$.

Lemma 2.11 *Let ℓ and θ be positive numbers. Then there is a positive number ζ with the following property. Let $x, y \in \mathcal{H}^2$, γ the geodesic carrying the segment $[x,y]$, $g \in PSL(2,\mathbb{R})$ and $\gamma' \in G'_{\langle g \rangle}$, such that:*

i) *The hyperbolic distance $d(x,y) \leq \ell$.*
ii) *The geodesic segments $[x,y]$ and $[g(x), g(y)]$ intersect, and the angle between γ and $g(\gamma)$ is $\alpha \geq \theta$.*
iii) *γ' intersects the segment $[x,y]$ and the angle between γ and γ' is β.*

Then $\beta \geq \zeta$.

Proof Without loss of generality, we may asume that $x = i \in \mathcal{H}^2$ and $y = ti$. The angle of intersection between the geodesics δ and $g(\delta)$ is a continuous function of δ. Hence there is a neighbourhood U of $\gamma \in G(\mathcal{H}^2)$ disjoint from $G'_{\langle g \rangle}$, that is consisting of geodesics δ such that $g(\delta)$ intersects δ.

There is a positive number r such that the (two dimensional) solid cylinder $C(\gamma, i\sqrt{t}, r)$ has the property: if $\delta \subset C(\gamma, i\sqrt{t}, r)$ then $\delta \in U$. Then it is easy to show, using hyperbolic trigonometry, that there is a positive number ζ such that any geodesic δ intersecting $[x,y]$ at an angle $\leq \zeta$ is contained in $C(\gamma, i\sqrt{t}, r)$, and hence $\delta \notin G'_{\langle g \rangle}$. □

Corollary 2.12 *If g is a hyperbolic isometry of $\mathcal{H}^2$ and $x \in \mathcal{H}^2$ does not lie on the axis of g, then there is a positive number ζ with the following property. If μ is any geodesic lamination invariant by g, then no leaf of the lamination intersects the geodesic segment $[x, g(x)]$ at an angle smaller than ζ.* □

Lemma 2.13 *Let ℓ, θ and ε be positive numbers. Then there is a positive number r with the following property. Let $x, y \in \mathcal{H}^2$ with $d(x,y) \leq \ell$, and let γ be the geodesic carrying the segment $[x,y]$. Let $g \in PSL(2,\mathbb{R})$ be such that $[x,y]$ intersects $[g(x), g(y)]$ at the point x_0, and at an angle $\alpha \geq \theta$. If $\delta \in G'_{\langle g \rangle} \cap G(D(x_0, r))$, then δ intersects both γ and $g(\gamma)$, and the points of intersection lie in $D(x_0, \varepsilon)$.*

Proof Since $g^{-1}(x_0) \in [x,y]$, we have $d(g^{-1}(x_0), x_0) \leq \ell$. We consider the geodesic segment $[x', y']$ of length 3ℓ on the geodesic γ, centred at x_0.

Let U be a neighbourhood of $\gamma \in G(\mathcal{H}^2)$ disjoint from $G'_{\langle g \rangle}$. There is r_1 such that any geodesic which intersects $D(x_0, r_1)$ and does not intersect $[x', y']$, lies in U, and hence it is not in $G'_{\langle g \rangle}$. So, if $\delta \in G'_{\langle g \rangle} \cap G(D(x_0, r_1))$, δ intersects the segment $[x', y']$. Similarly, there is r_2 such that if $\delta \in G'_{\langle g \rangle} \cap G(D(x_0, r_2))$, δ intersects the segment $[g(x'), g(y')]$.

By Lemma 2.11, the angle at the points of intersection is greater than a constant ζ. If r satisfies $0 < r < \min(r_1, r_2)$ and $\sinh r < \sin\zeta \sinh\varepsilon$, then it has the required property. □

The following Lemma shows that, under certain conditions, taking integrals along geodesic segments describes weak convergence of measures.

Lemma 2.14 *Let $\{\mu_n\}$ be a sequence of measured geodesic laminations on $\mathcal{H}^2$, invariant by $g \in PSL(2,\mathbb{R})$, and assume that μ_n converge weakly to a measured lamination μ. Let γ be a geodesic in $\mathcal{H}^2$, such that γ and $g(\gamma)$ intersect at one point. Then, for every geodesic segment $[u,v]$ on γ and for every continuous function $f\colon [u,v] \to [0,1]$, with $f(u) = f(v) = 0$, the sequence $\int_{[u,v]} f\mu_n$ converges to $\int_{[u,v]} f\mu$.*

Proof Since γ intersects $g(\gamma)$ at one point, there is a neighbourhood U of γ in $G(\mathcal{H}^2)$ which is disjoint from $G'_{\langle g \rangle}$. We define a continuous function $\tilde{f}\colon G(\mathcal{H}^2) \to [0,1]$ by letting $\tilde{f}(\delta) = f(y)$ if $y \in [u,v]$ and $\delta \in G(y) - U$, and extending continuously to the rest of $G(\mathcal{H}^2)$. Then, for any measured geodesic lamination ν invariant by g,

$$\tilde{f}\nu(G(u,v)) = \int_{[u,v]} f\nu.$$ □

3 The theorems

We fix a reference point $[\rho_0] \in T(S)$, and we consider a point $[\rho] \in Q(S)$. Let $g_1, \ldots, g_k \in PSL(2, \mathbb{R})$ be a set of generators for $\Gamma_0 = \rho_0(\pi_1(S))$. Let $x \in \mathcal{H}^2$ be a point which does not lie on the axis of any conjugate of the generators g_j.

Let θ be the minimum of the angles between the geodesics carrying the segments $[g_j^{-1}(x), x]$ and $[x, g_j(x)]$, for $j = 1, \ldots, k$. Let d and d' be the maximum and the minimum, respectively, of the distances between x and $g_j(x)$, for $j = 1, \ldots, k$.

Let K be a compact disc in $\mathcal{H}^2$ containing in its interior the points x, $g_j(x)$, $g_j^{-1}(x)$, for $j = 1, \ldots, k$, and projecting onto $S_0 = \mathcal{H}^2/\Gamma_0$. Let L be a compact disc in $\mathcal{H}^3$ such that $\phi_*(G(K)) \subset G(L)$.

We consider a positive integer m, and a positive number $r(m)$ such that d/m is less than the number $\delta(K, L, r(m))$ given by Lemma 2.10.

Let μ be a complex measured geodesic lamination on $\mathcal{H}^2$, invariant by the group Γ_0, with $||\mu|| < M_0$. We consider one of the generators g_j, $j = 1, \ldots, k$, and to simplify notation we drop the suffix j for the time being. Let γ denote the geodesic carrying the segment $[x, g(x)]$. We divide the segment $[x, g(x)]$ into m equal subsegments, by the points

$$x = x_0, x_1, \ldots, x_{m-1}, x_m = g(x).$$

If $[x, y]$ is a geodesic segment in $\mathcal{H}^2$ and ν is a measure on a set of geodesics in $\mathcal{H}^2$, we introduce the notation

$$\int'_{[x,y]} \nu = \frac{1}{2}\nu\left(G\left(x\right)\right) + \nu\left(G\left(x, y\right)\right) + \frac{1}{2}\nu\left(G\left(y\right)\right)$$

We define two new measures on the set $G(\mathcal{H}^2)$ of geodesics in $\mathcal{H}^2$ in the following way. For every $i = 1, \ldots, m$, let $\tilde{\gamma}_i$ be a geodesic in $\operatorname{supp}\mu$, intersecting γ in $[x_{i-1}, x_i]$. We define, for $i = 1, \ldots, m$,

$$\tilde{\mu}(\tilde{\gamma}_i) = \int'_{[x_{i-1}, x_i]} \mu.$$

For every $i = 1, \ldots, m-1$, let γ'_i be the geodesic in $\operatorname{supp}\mu$ intersecting the open segment (x_{i-1}, x_{i+1}) as near as possible to x_i. Let $\lambda_i \colon [x_0, x_m] \to [0, 1]$, $i = 1, \ldots, m-1$, be continuous functions satisfying

(1) $\operatorname{supp}(\lambda_i) \subset [x_{i-1}, x_{i+1}]$ and

(2) $\sum_{i=1}^{m-1} \lambda_i(x) = 1$ for all $x \in [x_0, x_m]$.

Then, in particular, $[x_0, x_1] \subset \lambda_i^{-1}(1)$ and $[x_{m-1}, x_m] \subset \lambda_{m-1}^{-1}(1)$. We define, for $i = 1, \dots, m-1$,

$$\mu'(\gamma_i') = \int_{[x_{i-1}, x_{i+1}]} \lambda_i \mu$$

Now we define

$$C_i = A(\phi_*(\tilde{\gamma}_i), \tilde{\mu}(\tilde{\gamma}_i)) \qquad \text{for } i = 1, \dots, m$$

and

$$D_i = A(\phi_*(\gamma_i'), \mu'(\gamma_i')) \qquad \text{for } i = 1, \dots, m-1.$$

We want to bound the norm $||C_1 C_2 \cdots C_m - D_1 D_2 \cdots D_{m-1}||$.

We put $a_i = \int'_{[x_{i-1}, x_i]} \lambda_i \mu$ and $b_i = \int'_{[x_i, x_{i+1}]} \lambda_i \mu$. Then $\mu'(\gamma_i') = a_i + b_i$, for $i = 1, \dots, m-1$, and $\tilde{\mu}(\tilde{\gamma}_1) = a_1$, $\tilde{\mu}(\tilde{\gamma}_m) = b_{m-1}$, and for $i = 2, \dots, m-1$, $\tilde{\mu}(\tilde{\gamma}_i) = b_{i-1} + a_i$.

We put $D_i^l = A(\phi_*(\gamma_i'), a_i)$ and $D_i^r = A(\phi_*(\gamma_i'), b_i)$. With this notation we have

$$\begin{aligned}
&\|C_1 \cdots C_m - D_1 \cdots D_{m-1}\| \leq \\
&\qquad \|C_1 \cdots C_{m-1}\| \left\|C_m - D_{m-1}^r\right\| \\
&\qquad + \|C_1 \cdots C_{m-2}\| \left\|C_{m-1} - D_{m-2}^r D_{m-1}^l\right\| \left\|D_{m-1}^r\right\| \\
&\qquad + \cdots + \|C_1 \cdots C_{s-1}\| \left\|C_s - D_{s-1}^r D_s^l\right\| \|D_s^r D_{s+1} \cdots D_{m-1}\| \\
&\qquad + \cdots + \left\|C_1 - D_1^l\right\| \|D_1^r D_2 \cdots D_{m-1}\|.
\end{aligned}$$

Then, by Lemma 2.2, there is a positive number M_1, depending on L and M_0, which is an upper bound for the norm of the factors of the form $C_1 \cdots C_s$, $D_s^r D_{s+1} \cdots D_{m-1}$. By Lemma 2.6, there is a positive number M_2, depending on L and M_0, such that each factor of the form $C_s - D_{s-1}^r D_s^l$ has norm bounded by $M_2 r(m) \tilde{\mu}(\tilde{\gamma}_s)$. Then

$$\|C_1 \cdots C_m - D_1 \cdots D_{m-1}\| \leq M_0 M_1^2 M_2 r(m). \tag{1}$$

In the following we want to examine the behaviour of $D_1 \cdots D_{m-1}$ as $m \to \infty$ and as the lamination μ changes. For this we must consider more carefully the leaves of the lamination near x.

By Lemma 2.13, there is an open set $U \subset G(K)$, depending on d, θ and d'/m such that, if δ is any geodesic in $U \cap \operatorname{supp} \mu$, then δ intersects the geodesics

γ and $g(\gamma)$ at a distance less than d'/m from x. Let $\chi\colon G(K) \to [0,1]$ be a continuous function, with $\operatorname{supp}\chi \subset U$ and $\chi|_{G(x)} = 1$. We introduce the notation

$$a' = \int_{[x_0,x_1]} \chi\mu \qquad\qquad a'' = \int'_{[x_0,x_1]} (1-\chi)\mu$$

$$b' = \int_{[x_{m-1},x_m]} (\chi\circ g^{-1})\mu \qquad\qquad b'' = \int'_{[x_{m-1},x_m]} (1-\chi\circ g^{-1})\mu$$

$$P = A(\phi_*(\gamma'_1), a') \qquad\qquad Q = A(\phi_*(\gamma'_1), a'')$$

$$R = A(\phi_*(\gamma'_{m-1}), b'') \qquad\qquad S = A(\phi_*(\gamma'_{m-1}), b'),$$

and we have

$$D_1 = PQD_1^r \qquad\qquad D_{m-1} = D_{m-1}^l RS.$$

Let $\{\mu_n\}$ be a sequence of complex measured geodesic laminations on the surface S_0, converging weakly in $\mathcal{M}(G(K))$ to a measured lamination μ_0. Then, by the Uniform Boundedness Principle, there is a positive number M_0 such that $||\mu_n|| \leq M_0$ for all $n \geq 0$.

For each positive integer m, for each $i = 1, \ldots, m-1$, for each $j = 1, \ldots, k$ and for each measured lamination μ_n, $n \geq 0$, we define as above the points $x_{j,m,i}$, the geodesics $\gamma'_{n,j,m,i}$, the functions $\lambda_{j,m,i}$, the quantities $a_{n,,j,m,i}$, $b_{n,j,m,i}$, $a'_{n,j,m}$, $b'_{n,j,m}$ and the isometries $D_{n,j,m,i}$, $P_{n,j,m}$, $Q_{n,j,m}$, $R_{n,j,m}$, $S_{n,j,m}$.

Let $B_{n,j,m} = D_{n,j,m,1} \cdots D_{n,j,m,m-1}$. We want to find a bound for the norm of the difference between $B_{0,j,m}g_j$ and some conjugate of $B_{n,j,m}g_j$.

Lemma 3.1 *With the above notation, there exist positive numbers N_1, N_2 and functions $r\colon \mathbb{N} \to \mathbb{R}$, $\varepsilon\colon \mathbb{N}\times\mathbb{N} \to \mathbb{R}$ such that*

$$\lim_{m\to\infty} r(m) = 0, \qquad \lim_{n\to\infty} \varepsilon(m,n) = 0 \quad \textit{for each } m \in \mathbb{N}$$

and

$$\left\| P_{0,1,m}P_{n,1,m}^{-1}B_{n,j,m}g_jP_{n,1,m}P_{0,1,m}^{-1} - B_{0,j,m}g_j \right\| \leq N_1 r(m) + N_2\varepsilon(m,n).$$

Proof To simplify notation, we drop the index m for the time being, and write, for example, $D_{n,j;i}$ for $D_{n,j,m,i}$. We have

$$\begin{aligned}
\left\|P_{0,1}P_{n,1}^{-1}B_{n,j}g_jP_{n,1}P_{0,1}^{-1}-B_{0,j}g_j\right\| \leq \\
\left\|P_{0,1}P_{n,1}^{-1}B_{n,j}g_jP_{n,1}P_{0,1}^{-1}-P_{0,j}P_{n,j}^{-1}B_{n,j}g_jP_{n,j}P_{0,j}^{-1}\right\| \\
+\left\|P_{0,j}P_{n,j}^{-1}B_{n,j}g_jP_{n,j}P_{0,j}^{-1}g_j^{-1}-P_{0,j}P_{n,j}^{-1}B_{n,j}S_{n,j}^{-1}S_{0,j}\right\|\|g_j\| \\
+\left\|P_{0,j}P_{n,j}^{-1}B_{n,j}S_{n,j}^{-1}S_{0,j}-B_{0,j}\right\|\|g_j\|.
\end{aligned} \tag{2}$$

We will find upper bounds for the three terms of the right hand side of the above inequality.

The first term of (2) is bounded above by

$$\begin{aligned}
\left\|P_{0,1}P_{n,1}^{-1}-P_{0,j}P_{n,j}^{-1}\right\|\left\|B_{n,j}g_jP_{n,1}P_{0,1}^{-1}\right\| \\
+\left\|P_{0,j}P_{n,j}^{-1}B_{n,j}g_j\right\|\left\|P_{n,j}P_{0,j}^{-1}-P_{n,j}P_{0,j}^{-1}\right\|.
\end{aligned}$$

By Lemma 2.2, the factors containing g_j are bounded above by M_1. We consider the other factor in each term. Recall that $P_{n,j}=A(\phi_*(\gamma'_{n,j;1}),a'_{n,j})$. We have

$$\begin{aligned}
\left\|P_{0,j}P_{n,j}^{-1}-P_{0,1}P_{n,1}^{-1}\right\| \leq \\
\|P_{0,j}\|\left\|P_{n,j}^{-1}-A(\phi_*(\gamma'_{0,j;1}),-a'_{n,j})\right\| \\
+\left\|A(\phi_*(\gamma'_{0,j;1}),a'_{0,j}-a'_{n,j})-A(\phi_*(\gamma'_{0,1;1}),a'_{0,1}-a'_{n,1})\right\| \\
+\|P_{0,1}\|\left\|A(\phi_*(\gamma'_{0,1;1}),-a'_{n,1})-P_{n,1}^{-1}\right\|.
\end{aligned} \tag{3}$$

By Lemma 2.5, there is a positive constant M' such that the first and the third term of the right hand side of (3) are bounded by $M_0M_1M'r(m)$. To find a bound for the second term we consider two cases.

(1) The segment $[x_0,x_{j;1}]$ intersects the same geodesics in $\operatorname{supp}(\chi\mu_n)$ as does the segment $[x_0,x_{1;1}]$.

(2) The two segments intersect different sets of geodesics in $\operatorname{supp}(\chi\mu_n)$.

Let $z_{n,i}=\int_{[x_0,x_{i;1}]}\chi(\mu_0-\mu_n)=a'_{0,i}-a'_{n,i}$.

In case (1), $z_{n,j}=z_{n,1}$, and the geodesics $\gamma'_{0,j;1},\gamma'_{0,1;1}$ lie in a (2–dimensional) solid cylinder of radius d/m based at x_0. The segments $[x_0,x_{j;1}]$ and $[x_0,x_{1;1}]$

induce concurrent orientations on the geodesics $\gamma'_{0,j;1}$ and $\gamma'_{0,1;1}$ respectively. So, by Lemma 2.5,

$$\left\|A(\phi_*(\gamma'_{0,j;1}), z_{n,j}) - A(\phi_*(\gamma'_{0,1;1}), z_{n,1}\right\| \leq M_0 M' r(m).$$

Note that if μ_n satisfies the conditions of case (1) for large enough n, then μ_0 also satisfies these conditions.

In case (2), the orientations induced by the segments $[x_0, x_{j;1}]$ and $[x_0, x_{1,1}]$ on the geodesics $\gamma'_{0,j;1}$ and $\gamma'_{0,1;1}$ respectively, are not concurrent. Hence, by Lemma 2.5,

$$\left\|A(\phi_*(\gamma'_{0,j;1}), z_{n,j}) - A(\phi_*(\gamma'_{0,1;1}), z_{n,1)}\right\| \leq M_0 M' r(m) + M''|z_{n,j} + z_{n,1}|.$$

Note that, in this case,

$$a'_{0,j} + a'_{0,1} = \int_{[x_0, x_{j;1}]} \chi\mu_0 + \int_{[x_0, x_{1;1}]} \chi\mu_0 = \chi\mu_0(G)$$

and similarly for μ_n. Hence $z_{n,j} + z_{n,1} = \chi\mu_0(G) - \chi\mu_n(G)$. Let

$$\varepsilon_0(m,n) = \sup_{s \geq n} |\chi_m \mu_0(G) - \chi_m \mu_s(G)|.$$

Now we turn our attention to the second term of equation (2). This term involves only the generator g_j, so we drop the subscript j from the notation. We have

$$\begin{aligned} &\left\|P_0 P_n^{-1} B_n g P_n P_0^{-1} g^{-1} - P_0 P_n^{-1} B_n S_n^{-1} S_0\right\| \leq \\ &\qquad \left\|P_0 P_n^{-1} B_n\right\| \left\|S_n^{-1}\right\| \left\|S_n g P_n^{-1} g^{-1} - S_0 g P_0 g^{-1}\right\| \left\|g P_0^{-1} g^{-1}\right\|. \end{aligned}$$

We consider the term $S_n g P_n^{-1} g^{-1}$, which is equal to

$$A\left(\phi_*(\gamma'_{n;m-1}), \int_{[x_{;m-1}, x_{;m}]} (\chi \circ g^{-1})\mu_n\right) A\left(\phi_*(g(\gamma'_{n;1}), \int_{[x_0, x_{;1}]} \chi\mu_n\right).$$

Since μ_n is invariant by g, and $x_{;m} = g(x_0)$, we have

$$\int_{[x_{;m}, g(x_{;1})]} (\chi \circ g^{-1})\mu_n = \int_{[x_0, x_{;1}]} \chi\mu_n.$$

We have to consider two cases:

(1) The segments $[x_{;m-1}, x_{;m}]$ and $[x_{;m}, g(x_{;1})]$ intersect the same geodesics in $\operatorname{supp}((\chi \circ g^{-1})\mu_n)$.

(2) The segments $[x_{;m-1}, x_{;m}]$ and $[x_{;m}, g(x_{;1})]$ intersect different sets of geodesics in $\operatorname{supp}((\chi \circ g^{-1})\mu_n)$.

In case (1), we let $z_n = \int_{[x_{;m-1},x_{;m}]}(\chi\circ g^{-1})\mu_n = \int_{[x_{;m},g(x_{;1})]}(\chi\circ g^{-1})\mu_n$. The geodesics $\gamma'_{n;m-1}$ and $g(\gamma'_{n;1})$ lie in a solid cylinder of radius d/m, based at $x_{;m}$, and the orientations induced by the segments $[x_{;m-1},x_{;m}]$ and $[x_{;m},g(x_{;1})]$ are not concurrent. Hence, by Lemma 2.6, $\left\|S_ngP_ng^{-1}-I\right\| \le M_0M_2r(m)$. As before, if μ_n satisfies the conditions of case (1) for large enough n, then μ_0 also satisfies these conditions. Hence

$$\left\|S_ngP_ng^{-1}-S_0gP_0g^{-1}\right\| \le 2M_0M_2r(m).$$

In case (2), since μ_n is invariant by g, and $x_{;m}=g(x_0)$, we have

$$\int_{[x_{;m},g(x_{;1})]}(\chi\circ g^{-1})\mu_n + \int_{[x_{;m-1},x_{;m}]}(\chi\circ g^{-1})\mu_n = \chi\mu_n(G)$$

and if n is large enough, the same is true of μ_0. Then

$$\begin{aligned}\left\|S_ngP_ng^{-1}-S_0gP_0g^{-1}\right\| \le{}& \\ &\left\|S_ngP_ng^{-1}-A(\phi_*(\gamma'_{n;m-1}),\chi\mu_n(G))\right\| \\ &+\left\|A(\phi_*(\gamma'_{n;m-1}),\chi\mu_n(G))-A(\phi_*(\gamma'_{0;m-1}),\chi\mu_0(G))\right\| \\ &+\left\|A(\phi_*(\gamma'_{0;m-1}),\chi\mu_0(G))-S_0gP_0g^{-1}\right\|.\end{aligned}$$

By Lemma 2.5 and Lemma 2.6, this is bounded above by $M'r(m)+M''\varepsilon(m,n)$.

The third term of equation (2) is bounded by

$$\|P_0\|\left\|P_n^{-1}B_nS_n^{-1}-P_0^{-1}B_0S_0^{-1}\right\|\|S_0\|\|g\|.$$

But

$$\begin{aligned}&\left\|P_n^{-1}B_nS_n^{-1}-P_0^{-1}B_0S_0^{-1}\right\| = \\ &\qquad\left\|Q_nD^r_{n;1}D_{n;2}\cdots D_{n;m-2}D^l_{n;m-1}R_n - Q_0D^r_{0;1}D_{0;2}\cdots D_{0;m-2}D^l_{0;m-1}R_0\right\|\end{aligned}$$

and by Lemma 2.2, this is bounded by

$$\begin{aligned}M_1^2\Big(\left\|D^l_{n;m-1}R_n-D^l_{0;m-1}R_0\right\| + \sum_{i=2}^{m-2}\|D_{n,i}-D_{0,i}\| + \\ +\left\|Q_nD^r_{n;1}-Q_0D^r_{0;1}\right\|\Big).\end{aligned} \tag{4}$$

Note that $Q_nD^r_{n;1} = A\left(\phi_*(\gamma'_{n;1}),\int_{[x_0,x_{;1}]}\lambda_{;1}(1-\chi)\mu_n\right)$ and hence

$$\left\|Q_nD^r_{n;1}-Q_0D^r_{0;1}\right\| \le M'r(m)+M''\varepsilon_1(m,n)$$

where $\varepsilon_1(m,n) = \sup_{s\ge n}\left|\int_{[x_0,x_{;1}]}\lambda_{;1}(1-\chi_m)(\mu_s-\mu_0)\right|$, and similarly for the other terms of (4), for suitable ε_i, $i=2,\dots,m-1$.

To complete the proof of Lemma 3.1 we must show that $r(m)$ and $\varepsilon(m,n) = \sum_{i=0}^{m-1} \varepsilon_i(m,n)$ have the required properties. It is clear that we can choose a sequence $r(m)$, with $\lim_{m\to\infty} r(m) = 0$, such that the pair $r = r(m)$, $\delta = d/m$ satisfy the conditions of Lemma 2.10. Lemma 2.14 implies that, for each m, $\lim_{n\to\infty} \varepsilon(m,n) = 0$. □

We let $E_{n,j,m} = C_{n,j,m,1} \cdots C_{n,j,m,m}$ and $H_{n,m} = P_{0,1,m} P_{n,1,m}^{-1}$. Then, combining the above result with (1), we have

$$\left\| H_{n,m} E_{n,j,m} g_j H_{n,m}^{-1} - E_{0,j,m} g_j \right\| \le M(r(m) + \varepsilon(m,n). \tag{5}$$

If $g_1, \ldots, g_k$ is a set of generators for Γ_0, the space R of homomorphisms $\rho\colon \Gamma_0 \to PSL(2,\mathbb{C})$ with quasi-Fuchsian image is a subspace of $PSL(2,\mathbb{C})^k$, and $Q(S)$ is a subspace of the quotient by the adjoint action on the left, $PSL(2,\mathbb{C})^k \big/ PSL(2,\mathbb{C})$. Let

$$\rho_{n,m} = \left(H_{n,m} E_{n,j,m} g_j H_{n,m}^{-1}, \quad j = 1, \ldots, k \right)$$

$$\rho_{n,m} = (E_{0,j,m} g_j, \quad j = 1, \ldots, k)$$

and let $[\rho_{n,m}]$ denote the equivalence class of $\rho_{n,m}$ in $PSL(2,\mathbb{C})^k \big/ PSL(2,\mathbb{C})$.

Let $n(m)$ be a sequence such that $n(m) \ge m$ and $\varepsilon(n(m), m) \le 1/m$. Then $\lim_{m\to\infty} \rho_{n(m),m} = \rho_{\mu_0}$. As $m \to \infty$, $[\rho_{n,m}]$ converge, uniformly in n, to the bending deformation $[\rho_{\mu_n}]$, [5]. Hence, $\lim_{m\to\infty} [\rho_{n(m),m}] = \lim_{m\to\infty} [\rho_{\mu_{n(m)}}] = \lim_{n\to\infty} [\rho_{\mu_n}]$, and we have

$$\lim_{n\to\infty} [\rho_{\mu_n}] = [\rho_{\mu_0}]. \tag{6}$$

To complete the proof of Theorem 1, it remains to show that the convergence is uniform in compact subsets of $\mathcal{D}$. If $([\rho], t) \in \mathcal{D}$, each bound used in the proof of (6) depends at most linearly on t, while it depends on ρ only in terms of the endpoints of a finite number of geodesics $\phi_*(\gamma)$. The endpoints of the geodesic $\phi_*(\gamma)$ are, for each γ, holomorphic functions of $[\rho]$. Hence each bound can be chosen uniformly on each compact subset of $\mathcal{D}$.

Note that $\mathcal{D}$ contains in its interior the set $Q(S) \times \{0\}$. If the laminations μ_n are real for all but a finite number of n, then $\mathcal{D}$ also contains the set $Q(S) \times \mathbf{R}$, but this is not true in the general case.

To prove Theorem 2 we recall that the bending vector field T_μ is defined by

$$T_\mu([\rho]) = \frac{\partial}{\partial t} B_\mu([\rho], t).$$

The vector fields T_{μ_n} are holomorphic, and $B_{\mu_n}([\rho],t)$ converge to $B_{\mu_0}([\rho],t)$ for $([\rho],t) \in \mathcal{D}$. It follows that T_{μ_n} converge to T_{μ_0}, uniformly on compact subsets of $Q(S)$.

We conclude with the proof of Theorem 3. We consider the subset of $\mathcal{ML}(S)$ consisting of measured laminations with non negative real and imaginary parts, and we denote it by $\mathcal{ML}^{++}(S)$. We identify $\mathcal{ML}^{++}(S)$ with a subset of the set of pairs of positive measured laminations $\mathcal{ML}^+_{\mathbb{R}}(S) \times \mathcal{ML}^+_{\mathbb{R}}(S)$. If $\nu \in \mathcal{ML}^{++}(S)$, then $\operatorname{Re}\nu$ and $\operatorname{Im}\nu$ are in $\mathcal{ML}^+_{\mathbb{R}}(S)$ and they satisfy the condition

$$\operatorname{supp}(\operatorname{Re}\nu) \cup \operatorname{supp}(\operatorname{Im}\nu) \text{ is a geodesic lamination.} \tag{7}$$

Conversely, any pair ν_1, ν_2 of positive measured laminations satisfying (7) define a measure $\nu = \nu_1 + i\nu_2 \in \mathcal{ML}^{++}(S)$. The mapping is a homeomorphism of $\mathcal{ML}^{++}(S)$ onto a subset of $\mathcal{ML}^+_{\mathbb{R}}(S) \times \mathcal{ML}^+_{\mathbb{R}}(S)$. But $\mathcal{ML}^+_{\mathbb{R}}(S)$ is homeomorphic to $\mathbb{R}^{6g-6}$, [6]. Thus $\mathcal{ML}^{++}(S)$ is first countable, and Theorem 2 implies that $\mu \mapsto T_\mu$ is continuous. Theorem 3 then follows by the continuity of the evaluation map.

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