ISSN 1464-8997

Geometry & Topology Monographs
Volume 1: The Epstein Birthday Schrift
Pages 23–49

A brief survey of the deformation theory of Kleinian groups

James W Anderson

Abstract We give a brief overview of the current state of the study of the deformation theory of Kleinian groups. The topics covered include the definition of the deformation space of a Kleinian group and of several important subspaces; a discussion of the parametrization by topological data of the components of the closure of the deformation space; the relationship between algebraic and geometric limits of sequences of Kleinian groups; and the behavior of several geometrically and analytically interesting functions on the deformation space.

AMS Classification 30F40; 57M50

Keywords Kleinian group, deformation space, hyperbolic manifold, algebraic limits, geometric limits, strong limits

Dedicated to David Epstein on the occasion of his 60th birthday

1 Introduction

Kleinian groups, which are the discrete groups of orientation preserving isometries of hyperbolic space, have been studied for a number of years, and have been of particular interest since the work of Thurston in the late 1970s on the geometrization of compact 3–manifolds. A Kleinian group can be viewed either as an isolated, single group, or as one of a member of a family or continuum of groups.

In this note, we concentrate our attention on the latter scenario, which is the deformation theory of the title, and attempt to give a description of various of the more common families of Kleinian groups which are considered when doing deformation theory. No proofs are given, though it is hoped that reasonable coverage of the current state of the subject is given, and that ample references have been given for the interested reader to venture boldly forth into the literature.

It is possible to consider the questions raised here in much more general settings, for example for Kleinian groups in n–dimensions for general n, but that is beyond the scope of what is attempted here. Some material on this aspect of the question can be found in Bowditch [23] and the references contained therein.

The author would like to thank Dick Canary, Ed Taylor, and Brian Bowditch for useful conversations during the preparation of this work, as well as the referee for useful comments.

2 The deformation spaces

We begin by giving a few basic definitions of the objects considered in this note, namely Kleinian groups. We go on to define and describe the basic structure of the deformation spaces we are considering herein.

A *Kleinian group* is a discrete subgroup of $\mathrm{PSL}_2(\mathbf{C}) = \mathrm{SL}_2(\mathbf{C})/\{\pm \mathrm{I}\}$, which we view as acting both on the Riemann sphere $\overline{\mathbf{C}}$ by Möbius transformations and on real hyperbolic 3–space $\mathbf{H}^3$ by isometries, where the two actions are linked by the Poincaré extension.

The action of an infinite Kleinian group Γ partitions $\overline{\mathbf{C}}$ into two sets, the *domain of discontinuity* $\Omega(\Gamma)$, which is the largest open subset of $\overline{\mathbf{C}}$ on which Γ acts discontinuously, and the *limit set* $\Lambda(\Gamma)$. If $\Lambda(\Gamma)$ contains two or fewer points, Γ is *elementary*, otherwise Γ is *non-elementary*. For a non-elementary Kleinian group Γ, the limit set $\Lambda(\Gamma)$ can also be described as the smallest non-empty closed subset of $\overline{\mathbf{C}}$ invariant under Γ. We refer the reader to Maskit [68] or Matsuzaki and Taniguchi [71] as a reference for the basics of Kleinian groups.

An isomorphism $\varphi\colon \Gamma \to \Phi$ between Kleinian groups Γ and Φ is *type-preserving* if, for $\gamma \in \Gamma$, we have that γ is parabolic if and only if $\varphi(\gamma)$ is parabolic.

A Kleinian group is *convex cocompact* if its convex core is compact; recall that the *convex core* associated to a Kleinian group Γ is the minimal convex submanifold of $\mathbf{H}^3/\Gamma$ whose inclusion is a homotopy equivalence. More generally, a Kleinian group is *geometrically finite* if it is finitely generated and if its convex core has finite volume. This is one of several equivalent definitions of geometric finiteness; the interested reader is referred to Bowditch [22] for a complete discussion.

A Kleinian group Γ is *topologically tame* if its corresponding quotient 3–manifold $\mathbf{H}^3/\Gamma$ is homeomorphic to the interior of a compact 3–manifold. Geometrically finite Kleinian groups are topologically tame. It was conjectured by Marden [64] that all finitely generated Kleinian groups are topologically tame.

A compact 3–manifold M is *hyperbolizable* if there exists a hyperbolic 3–manifold $N = \mathbf{H}^3/\Gamma$ homeomorphic to the interior of M. Note that a hyperbolizable 3–manifold M is necessarily orientable; *irreducible*, in that every embedded 2–sphere in M bounds a 3–ball in M; and *atoroidal*, in that every embedded torus T in M is homotopic into ∂M. Also, since the universal cover $\mathbf{H}^3$ of N is contractible, the fundamental group of M is isomorphic to Γ. For a discussion of the basic theory of 3–manifolds, we refer the reader to Hempel [48] and Jaco [49].

Keeping to our viewpoint of a Kleinian group as a member of a family of groups, throughout this survey we view a Kleinian group as the image $\rho(G)$ of a representation ρ of a group G into $\mathrm{PSL}_2(\mathbf{C})$. Unless explicitly stated otherwise, we assume that G is finitely generated, torsion-free, and non-abelian, so that in particular $\rho(G)$ is non-elementary.

2.1 The representation varieties $\mathcal{HOM}(G)$ and $\mathrm{R}(G) = \mathcal{HOM}(G)/\mathrm{PSL}_2(\mathbf{C})$

The most basic of the deformation spaces is the *representation variety* $\mathcal{HOM}(G)$ which is the space of all representations of G into $\mathrm{PSL}_2(\mathbf{C})$ with the following topology. Given a set of generators $\{g_1, \ldots, g_k\}$ for G, we may naturally view $\mathcal{HOM}(G)$ as a subset of $\mathrm{PSL}_2(\mathbf{C})^k$, where a representation $\rho \in \mathcal{HOM}(G)$ corresponds to the k–tuple $(\rho(g_1), \ldots, \rho(g_k))$ in $\mathrm{PSL}_2(\mathbf{C})^k$. The defining polynomials of this variety are determined by the relations in G. In particular, if G is free, then $\mathcal{HOM}(G) = \mathrm{PSL}_2(\mathbf{C})^k$. It is easy to see that $\mathcal{HOM}(G)$ is a closed subset of $\mathrm{PSL}_2(\mathbf{C})^k$.

The representations in $\mathcal{HOM}(G)$ are *unnormalized*, in the sense that there is a natural free action of $\mathrm{PSL}_2(\mathbf{C})$ on $\mathcal{HOM}(G)$ by conjugation. Depending on the particular question being addressed, it is sometimes preferable to remove the ambiguity of this action and form the quotient space $\mathrm{R}(G) = \mathcal{HOM}(G)/\mathrm{PSL}_2(\mathbf{C})$.

Though a detailed description is beyond the scope of this survey, we pause here to mention work of Culler and Shalen [40], [41], in which a slight variant of the representation variety as defined above plays a fundamental role, and which has inspired further work of Morgan and Shalen [78], [79], [80] and Culler, Gordon, Luecke, and Shalen [39]. The basic object here is not the space $\mathrm{R}(G)$ of all representations of G into $\mathrm{PSL}_2(\mathbf{C})$ as defined above, but instead the related space $\mathrm{X}(G)$ of all representations of G into $\mathrm{SL}_2(\mathbf{C})$, modulo the action of $\mathrm{SL}_2(\mathbf{C})$. The introduction of this space $\mathrm{X}(G)$ does beg the question of when a representation of G into $\mathrm{PSL}_2(\mathbf{C})$ can be lifted to a representation of G into $\mathrm{SL}_2(\mathbf{C})$. We note in passing that this question of lifting representations has been considered by a number of authors, including Culler, Kra, and Thurston, to name but a few; we refer the reader to the article by Kra [61] for exact statements and a review of the history, including references.

By considering the global structure of the variety $\mathrm{X}(G)$ in the case that G is the fundamental group of a compact, hyperbolizable 3–manifold M, and in particular the ideal points of its compactification, Culler and Shalen [40] are able to analyze the actions of G on trees, which in turn has connections with the existence of essential incompressible surfaces in M, finite group actions on M, and has particular consequences in the case that M is the complement of a knot in $\mathbf{S}^3$. We refer the reader to the excellent survey article by Shalen [94], as well as to the papers cited above.

2.2 The spaces $\mathcal{HOM}_{\mathrm{T}}(G)$ and $\mathrm{R}_{\mathrm{T}}(G) = \mathcal{HOM}_{\mathrm{T}}(G)/\mathrm{PSL}_2(\mathbf{C})$ of the minimally parabolic representations

Let $\mathcal{HOM}_{\mathrm{T}}(G)$ denote the subspace of $\mathcal{HOM}(G)$ consisting of those representations ρ for which $\rho(g)$ is parabolic if and only if g lies in a rank two free abelian subgroup of G. We refer to $\mathcal{HOM}_{\mathrm{T}}(G)$ as the space of *minimally parabolic* representations of G. In particular, if G contains no $\mathbf{Z}\oplus\mathbf{Z}$ subgroups, then the image $\rho(G)$ of every ρ in $\mathcal{HOM}_{\mathrm{T}}(G)$ is *purely loxodromic*, in that every non-trivial element of $\rho(G)$ is loxodromic. Set $\mathrm{R}_{\mathrm{T}}(G) = \mathcal{HOM}_{\mathrm{T}}(G)/\mathrm{PSL}_2(\mathbf{C})$.

2.3 The spaces $\mathcal{D}(G)$ and $\mathrm{AH}(G) = \mathcal{D}(G)/\mathrm{PSL}_2(\mathbf{C})$ of discrete, faithful representations

Let $\mathcal{D}(G)$ denote the subspace of $\mathcal{HOM}(G)$ consisting of the discrete, faithful representations of G, that is, the injective homomorphisms of G into $\mathrm{PSL}_2(\mathbf{C})$ with discrete image. For the purposes of this note, the space $\mathcal{D}(G)$ is our universe, as it is the space of all Kleinian groups isomorphic to G. Set $\mathrm{AH}(G) = \mathcal{D}(G)/\mathrm{PSL}_2(\mathbf{C})$.

We note that there exists an equivalent formulation of $\mathrm{AH}(G)$ in terms of manifolds. Given a hyperbolic 3–manifold N, let $\mathrm{H}(N)$ denote the set of all pairs (f, K), where K is a hyperbolic 3–manifold and $f\colon N \to K$ is a homotopy equivalence, modulo the equivalence relation $(f, K) \sim (g, L)$ if there exists an orientation preserving isometry $\alpha\colon K \to L$ so that $\alpha \circ f$ is homotopic to g. The topology on $\mathrm{H}(N)$ is given by noting that, if we let $\Gamma \subset \mathrm{PSL}_2(\mathbf{C})$ be a choice of conjugacy class of the fundamental group of N, then each element (f, K) in $\mathrm{H}(N)$ gives rise to a discrete, faithful representation $\varphi = f_*$ of Γ into $\mathrm{PSL}_2(\mathbf{C})$, with equivalent points in $\mathrm{H}(N)$ giving rise to conjugate representations into $\mathrm{PSL}_2(\mathbf{C})$. Hence, equipping $\mathrm{H}(N)$ with this topology once again gives rise to $\mathrm{AH}(G)$ with $G = \pi_1(N)$.

The following theorem, due to Jørgensen, describes the fundamental property of $\mathcal{D}(G)$, namely that the limit of a sequence of elements of $\mathcal{D}(G)$ is again an element of $\mathcal{D}(G)$.

Theorem 2.1 (Jørgensen [53]) *$\mathcal{D}(G)$ is a closed subset of $\mathcal{HOM}(G)$.*

There is one notable case in which $\mathrm{AH}(G)$ is completely understood, namely in the case that G is the fundamental group of a compact, hyperbolizable 3–manifold M whose boundary is the union of a (possibly empty) collection of tori. In this case, the hyperbolic structure on the interior of M is unique, by the classical Rigidity Theorem of Mostow, for closed manifolds, and Prasad, for manifolds with non-empty toroidal boundary. Rephrasing this statement as a statement about deformation spaces yields the following.

Theorem 2.2 (Mostow [81] and Prasad [91]) *Suppose that G is the fundamental group of a compact, orientable 3–manifold M whose boundary is the union of a (possibly empty) collection of tori. Then, $\mathrm{AH}(G)$ either is empty or consists of a single point.*

Given this result, it will cause us no loss of generality to assume that henceforth all Kleinian groups have infinite volume quotients.

2.4 The spaces $\mathcal{P}(G)$ and $\mathrm{MP}(G) = \mathcal{P}(G)/\mathrm{PSL}_2(\mathbf{C})$ of geometrically finite, minimally parabolic representations

Let $\mathcal{P}(G)$ denote the subset of $\mathcal{D}(G)$ consisting of those representations ρ with geometrically finite, minimally parabolic image $\rho(G)$. In particular, if G contains no $\mathbf{Z} \oplus \mathbf{Z}$ subgroups, then the image $\rho(G)$ of every representation $\rho \in \mathcal{P}(G)$ is convex cocompact. Set $\mathrm{MP}(G) = \mathcal{P}(G)/\mathrm{PSL}_2(\mathbf{C})$, and note that since $\mathrm{PSL}_2(\mathbf{C})$ is connected, the quotient map gives a one-to-one correspondence between the connected components of $\mathcal{P}(G)$ and those of $\mathrm{MP}(G)$.

It is an immediate consequence of the Core Theorem of Scott [93] and the Hyperbolization Theorem of Thurston that if $\mathcal{D}(G)$ is non-empty, then $\mathcal{P}(G)$ is non-empty. For a discussion of the Hyperbolization Theorem, see Morgan [77], Otal and Paulin [90], or Otal [89] for the fibered case.

We note here that, if there exists a geometrically finite, minimally parabolic representation of G into $\mathrm{PSL}_2(\mathbf{C})$, then in general there exist many geometrically finite representations which are not minimally parabolic, which can be constructed as limits of the geometrically finite, minimally parabolic representations. This construction has been explored in detail for a number of cases by Maskit [69] and Ohshika [85].

In the case that G is itself a geometrically finite, minimally parabolic Kleinian group, the structure of $\mathrm{MP}(G)$ is fairly well understood, both as a subset of $\mathrm{AH}(G)$ and in terms of how the components of $\mathrm{MP}(G)$ are parametrized by topological data. We spend the remainder of this section making these statements precise.

We begin with the Quasiconformal Stability Theorem of Marden [64].

Theorem 2.3 (Marden [64]) *If G is a geometrically finite, minimally parabolic Kleinian group, then $\mathrm{MP}(G)$ is an open subset of $\mathrm{R}(G)$.*

As a converse to this, we have the Structural Stability Theorem of Sullivan [97]. We note here that the versions of the Theorems of Marden and Sullivan given here are not the strongest, but are adapted to the point of view taken in this paper. The general statements holds valid in slices of $\mathrm{AH}(G)$ in which a certain collection of elements of G are required to have parabolic image, not just those which belong to $\mathbf{Z}\oplus\mathbf{Z}$ subgroups.

Theorem 2.4 (Sullivan [97]) *Let G be a finitely generated, torsion-free, non-elementary Kleinian group. If there exists an open neighborhood of the identity representation in $\mathrm{R}(G)$ which lies in $\mathrm{AH}(G)$, then G is geometrically finite and minimally parabolic.*

Combining these, we see that $\mathrm{MP}(G)$ is the interior of $\mathrm{AH}(G)$. A natural question which arises from this is whether there are points of $\mathrm{AH}(G)$ which do not lie in the closure of $\mathrm{MP}(G)$.

Conjecture 2.5 (Density conjecture) *$\mathrm{AH}(G)$ is the closure of $\mathrm{MP}(G)$.*

This Conjecture is due originally to Bers in the case that G is the fundamental group of a surface, see Bers [14], and extended by Thurston to general G.

There has been a good deal of work in the past couple of years on the global structure of $\mathrm{MP}(G)$ and its closure. We begin with an example to show that there exist groups G for which $\mathrm{MP}(G)$ is disconnected; the example we give here, in which $\mathrm{MP}(G)$ has finitely many components, comes from the discussion in Anderson and Canary [6].

Let T be a solid torus and for large k, let $A_1, \ldots, A_k$ be disjoint embedded annuli in ∂T whose inclusion into T induces an isomorphism of fundamental groups. For each $1 \leq j \leq k$, let S_j be a compact, orientable surface of genus j with a single boundary component, and let $Y_j = S_j \times I$, where I is a closed interval. Construct a compact 3–manifold M by attaching the annulus $\partial S_j \times I$ in ∂Y_j to the annulus A_j in ∂T. The resulting 3–manifold M is compact and hyperbolizable 3–manifold and has fundamental group G. This 3–manifold is an example of a *book of I–bundles*. Let ρ be an element of $\mathrm{MP}(G)$ for which the interior of M is homeomorphic to $\mathbf{H}^3/\rho(G)$.

Let τ be a permutation of $\{1, \ldots, k\}$, and consider now the manifold M_τ obtained by attaching the annulus $\partial S_j \times I$ in ∂Y_j to the annulus $A_{\tau(j)}$ in ∂T. By construction, M_τ is compact and hyperbolizable, and has fundamental group G; let ρ_τ be an element of $\mathrm{MP}(G)$ for which the interior of M_τ is homeomorphic to $\mathbf{H}^3/\rho_\tau(G)$. Since M and M_τ have isomorphic fundamental groups, they are homotopy equivalent. However, in the case that τ is not some power of the cycle $(12\cdots k)$, then there does not exist an orientation preserving homeomorphism between M and M_τ, and hence ρ and ρ_τ lie in different components of $\mathrm{MP}(G)$.

In the general case that G is finitely generated and does not split as a free product, there exists a characterization of the components of both $\mathrm{MP}(G)$ and its closure $\overline{\mathrm{MP}(G)}$ in terms of the topology of a compact, hyperbolizable 3–manifold M with fundamental group G. This characterization combines work of Canary and McCullough [33] and of Anderson, Canary, and McCullough [10]. We need to develop a bit of topological machinery before discussing this characterization.

For a compact, oriented, hyperbolizable 3–manifold M with non-empty, incompressible boundary, let $\mathcal{A}(M)$ denote the set of *marked homeomorphism types of compact, oriented 3–manifolds homotopy equivalent to* M. Explicitly, $\mathcal{A}(M)$ is the set of equivalence classes of pairs (M', h'), where M' is a compact, oriented, irreducible 3–manifold and $h'\colon M \to M'$ is a homotopy equivalence, and where two pairs (M_1, h_1) and (M_2, h_2) are equivalent if there exists an orientation preserving homeomorphism $j\colon M_1 \to M_2$ such that $j \circ h_1$ is homotopic to h_2. Denote the class of (M', h') in $\mathcal{A}(M)$ by $[(M', h')]$.

There exists a natural map $\Theta\colon \mathrm{AH}(\pi_1(M)) \to \mathcal{A}(M)$, defined as follows. For $\rho \in \mathrm{AH}(\pi_1(M))$, let M_ρ be a compact core for $N_\rho = \mathbf{H}^3/\rho(\pi_1(M))$ and let $r_\rho\colon M \to M_\rho$ be a homotopy equivalence such that $(r_\rho)_*\colon \pi_1(M) \to \pi_1(M_\rho)$ is equal to ρ. Set $\Theta(\rho) = [(M_\rho, h_\rho)]$. It is known that the restriction of Θ to $\mathrm{MP}(\pi_1(M))$ is surjective, and that two elements ρ_1 and ρ_2 of $\mathrm{MP}(\pi_1(M))$ lie in the same component of $\mathrm{MP}(\pi_1(M))$ if and only if $\Theta(\rho_1) = \Theta(\rho_2)$. Hence, Θ induces a one-to-one correspondence between the components of $\mathrm{MP}(\pi_1(M))$ and the elements of $\mathcal{A}(M)$; the reader is directed to Canary and McCullough [33] for complete details.

Given a pair M_1 and M_2 of compact, hyperbolizable 3–manifolds with non-empty, incompressible boundary, say that a homotopy equivalence $h\colon M_1 \to M_2$ is a *primitive shuffle* if there exists a finite collection $\mathcal{V}_1$ of primitive solid torus components of the characteristic submanifold $\Sigma(M_1)$ and a finite collection $\mathcal{V}_2$ of solid torus components of $\Sigma(M_2)$, so that $h^{-1}(\mathcal{V}_2) = \mathcal{V}_1$ and so that h restricts to an orientation preserving homeomorphism from $\overline{M_1 - \mathcal{V}_1}$ to $\overline{M_2 - \mathcal{V}_2}$; we do not define the *characteristic submanifold* here, but instead refer the reader to Canary and McCullough [33], Jaco and Shalen [50], or Johannson [51].

Let $[(M_1, h_1)]$ and $[(M_2, h_2)]$ be two elements of $\mathcal{A}(M)$. Say that $[(M_2, h_2)]$ is *primitive shuffle equivalent* to $[(M_1, h_1)]$ if there exists a primitive shuffle $\varphi\colon M_1 \to M_2$ such that $[(M_2, h_2)] = [(M_2, \varphi \circ h_1)]$. We note that when M is hyperbolizable, this gives an equivalence relation on $\mathcal{A}(M)$, where each equivalence class contains finitely many elements of $\mathcal{A}(M)$; let $\widehat{\mathcal{A}}(M)$ denote the set of equivalence classes. By considering the composition $\widehat{\Theta} = q \circ \Theta$ of Θ with the quotient map $q\colon \mathcal{A}(M) \to \widehat{\mathcal{A}}(M)$, we obtain the following complete enumeration of the components of $\overline{\mathrm{MP}(\pi_1(M))}$.

Theorem 2.6 (Anderson, Canary, and McCullough [10]) *Let M be a compact, hyperbolizable 3–manifold with non-empty, incompressible boundary, and let $[(M_1, h_1)]$ and $[(M_2, h_2)]$ be two elements of $\mathcal{A}(M)$. The associated components of $\mathrm{MP}(\pi_1(M))$ have intersecting closures if and only if $[(M_2, h_2)]$ is primitive shuffle equivalent to $[(M_1, h_1)]$. In particular, $\widehat{\Theta}$ gives a one-to-one correspondence between the components of $\overline{\mathrm{MP}(\pi_1(M))}$ and the elements of $\widehat{\mathcal{A}}$.*

Before closing this section, we highlight two consequences of the analysis involved in the proof of Theorem 2.6. The first involves the accumulation, or, more precisely, the lack thereof, of components of $\mathrm{MP}(\pi_1(M))$.

Proposition 2.7 (Anderson, Canary, and McCullough [10]) *Let M be a compact, hyperbolizable 3–manifold with non-empty, incompressible boundary. Then, the components of* $\mathrm{MP}(\pi_1(M))$ *cannot accumulate in* $\mathrm{AH}(\pi_1(M))$*. In particular, the closure* $\overline{\mathrm{MP}(\pi_1(M))}$ *of* $\mathrm{MP}(\pi_1(M))$ *is the union of the closures of the components of* $\mathrm{MP}(\pi_1(M))$.

The second involves giving a complete characterization, in terms of the topology of M, as to precisely when $\overline{\mathrm{MP}(\pi_1(M)))}$ has infinitely many components. Recall that a compact, hyperbolizable 3–manifold M with non-empty, incompressible boundary has *double trouble* if there exists a toroidal component T of ∂M and homotopically non-trivial simple closed curves C_1 in T and C_2 and C_3 in $\partial M - T$ such that C_2 and C_3 are not homotopic in ∂M, but C_1, C_2 and C_3 are homotopic in M.

Theorem 2.8 (Anderson, Canary, and McCullough [10]) *Let M be a compact, hyperbolizable 3–manifold with non-empty, incompressible boundary. Then,* $\overline{\mathrm{MP}(\pi_1(M))}$ *has infinitely many components if and only if M has double trouble. Moreover, if M has double trouble, then* $\mathrm{AH}(\pi_1(M))$ *has infinitely many components.*

2.5 The spaces $\mathcal{QC}(G)$ and $\mathrm{QC}(G) = \mathcal{QC}(G)/\mathrm{PSL}_2(\mathbf{C})$ of quasiconformal deformations

In the case that G is itself a finitely generated Kleinian group, the classical deformation theory of G consists largely of the study of the space of *quasiconformal deformations* of G, which consists of those representations of G into $\mathrm{PSL}_2(\mathbf{C})$ which are induced by a quasiconformal homeomorphism of the Riemann sphere $\overline{\mathbf{C}}$.

We do not give a precise definition here, but roughly, a *quasiconformal homeomorphism* ω of $\overline{\mathbf{C}}$ is a homeomorphism which distorts the standard complex structure on $\overline{\mathbf{C}}$ by a bounded amount; the interested reader is referred to Ahlfors [2] or to Lehto and Virtanen [63] for a thorough discussion of quasiconformality. We do note that a quasiconformal homeomorphism $\omega\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ is completely determined (up to post-composition by a Möbius transformation) by the measurable function $\mu = \omega_{\overline{z}}/\omega_z$, and that to every measurable function μ on $\overline{\mathbf{C}}$ with $\| \mu \|_\infty < 1$ there exists a quasiconformal homeomorphism ω of $\overline{\mathbf{C}}$ which solves the Beltrami equation $\mu\omega_z = \omega_{\overline{z}}$.

Set $\mathcal{QC}(G)$ to be the space of those representations ρ of G into $\mathrm{PSL}_2(\mathbf{C})$ which are induced by a quasiconformal homeomorphism of $\overline{\mathbf{C}}$, so that $\rho \in \mathcal{QC}(G)$ if there exists a quasiconformal homeomorphism ω of $\overline{\mathbf{C}}$ so that $\rho(g) = \omega \circ g \circ \omega^{-1}$ for all $g \in G$. By definition, we have that $\mathcal{QC}(G)$ is contained in $\mathcal{D}(G)$. Set $\mathrm{QC}(G) = \mathcal{QC}(G)/\mathrm{PSL}_2(\mathbf{C})$.

It is known that $\mathrm{QC}(G)$ is a complex manifold, and is actually the quotient of the Teichmüller space of the (possibly disconnected) quotient Riemann surface $\Omega(G)/G$ by a properly discontinuous group of biholomorphic automorphims. This result, in its full generality, follows from the work of a number of authors, including Maskit [70], Kra [62], Bers [16], and Sullivan [95].

We note here, in the case that G is a geometrically finite, minimally parabolic Kleinian group, that it follows from the Isomorphism Theorem of Marden [64] that $\mathrm{QC}(G)$ is the component of $\mathrm{MP}(G)$ containing the identity representation.

Sullivan [95] has shown, for a finitely generated Kleinian group G, if there exists a quasiconformal homeomorphism ω of $\overline{\mathbf{C}}$ which conjugates G to a Kleinian group and which is conformal on $\Omega(G)$, then ω is necessarily a Möbius transformation. In other words, if ω conjugates G to subgroup of $\mathrm{PSL}_2(\mathbf{C})$, then $\mu = \omega_{\overline{z}}/\omega_z$ is equal to 0 on $\Lambda(G)$.

In particular, if $\Omega(G)$ is empty, then $\mathrm{QC}(G)$ consists of a single point, namely the identity representation. This can be viewed as a generalization of Theorem 2.2, as Sullivan's result also holds for an infinite volume hyperbolic 3–manifold N whose uniformizing Kleinian group G has limit set the whole Riemann sphere.

We note here that the study of quasiconformal deformations of finitely Kleinian groups is the origin of the Ahlfors Measure Conjecture. In [3], Ahlfors raises the question of whether the limit set of a finitely generated Kleinian group with non-empty domain of discontinuity necessarily has zero area. If this conjecture is true, then it would be impossible for a quasiconformal deformation of a finitely generated Kleinian group G to be supported on the limit set of G. The result of Sullivan mentioned above implies that no such deformation exists, though without solving the Measure Conjecture, which has not yet been completely resolved. It is known that the Measure Conjecture holds in a large number of cases, in particular it holds for all topologically tame groups. For a discussion of this connection, we refer the reader to Canary [30] and the references contained therein.

There are several classes of Kleinian groups for which $\mathrm{QC}(G)$ has been extensively studied, which we discuss here.

A *Schottky group* is a finitely generated, purely loxodromic Kleinian group G which is free on g generators and whose domain of discontinuity is non-empty;

the number of generators is sometimes referred to as the *genus* of the Schottky group. This is not the original definition, but is equivalent to the usual definition by a theorem of Maskit [67]. In particular, a Schottky group is necessarily convex cocompact. Chuckrow [37] shows that any two Schottky groups of the same rank are quasiconformally conjugate, so that $\mathrm{QC}(G)$ is in fact equal to the space $\mathrm{MP}(G)$ of all convex cocompact representations of a group G which is free on g generators into $\mathrm{PSL}_2(\mathbf{C})$.

In the same paper [37], Chuckrow also engages in an analysis of the closure of $\mathrm{QC}(G)$ in $\mathrm{R}(G)$ for a Schottky group G of genus g. In particular, she shows that every point ρ in $\partial\mathrm{QC}(G)$ has the property that $\rho(G)$ is free on g generators, and contains no elliptic elements of infinite order. However, this in itself is not enough to show that $\rho(G)$ is discrete, as Greenberg [46] has constructed free, purely loxodromic subgroups of $\mathrm{PSL}_2(\mathbf{C})$ which are not discrete.

More generally, Chuckrow also shows that the limit ρ of a convergent sequence $\{\rho_n\}$ of type-preserving faithful representations in $\mathcal{HOM}(G)$ is again a faithful representation of G, and that $\rho(G)$ contains no elliptic elements of infinite order.

Jørgensen [53] credits his desire to generalize the results of Chuckrow [37] to leading him to what is now commonly referred to as Jørgensen's inequality, which states that if γ and φ are elements of $\mathrm{PSL}_2(\mathbf{C})$ which generate a non-elementary Kleinian group, then $|\mathrm{tr}^2(\gamma)-4|+|\mathrm{tr}([\gamma,\varphi])-2|\geq 1$, where $\mathrm{tr}(\gamma)$ is the trace of a matrix representative of γ in $\mathrm{SL}_2(\mathbf{C})$. The proof of Theorem 2.1 is a direct application of this inequality.

For a Schottky group G, it is known that $\mathrm{AH}(G)$ is not compact. There is work of Canary [27] and Otal [88] on a conjecture of Thurston which gives conditions under which sequences in $\mathrm{QC}(G)$ have convergent subsequences; we do not give details here, instead referring the interested reader to the papers cited above.

We also mention here the work of Keen and Series [59] on the Riley slice of the space of 2–generator Schottky groups, in which they introduce coordinates on the Riley slice and study the cusp points on the boundary of the Riley slice.

A *quasifuchsian* group is a finitely generated Kleinian group whose limit set is a Jordan curve and which contains no element interchanging the two components of its domain of discontinuity. Consequently, every quasifuchsian group is isomorphic to the fundamental group of a surface. It is known that any two isomorphic purely loxodromic quasifuchsian groups are quasiconformally conjugate, by work of Maskit [66], and hence for a purely loxodromic quasifuchsian group G we have that $\mathrm{MP}(G)=\mathrm{QC}(G)$.

This equality does not hold for quasifuchsian groups uniformizing punctured surfaces, for several reasons. First, the quasifuchsian groups uniformizing the three-times punctured sphere and the once-punctured torus are isomorphic, namely the free group of rank two, but cannot be quasiconformally conjugate, as the surfaces are not homeomorphic. Second, as every quasifuchsian group isomorphic to the free group G of rank two contains parabolic elements, no quasifuchsian group isomorphic to G lies in $\mathrm{MP}(G)$.

It is known that $\mathrm{QC}(G)$ is biholomorphically equivalent to the product of Teichmüller spaces $\mathcal{T}(S)\times\mathcal{T}(\overline{S})$, where S is one of the components of $\Omega(G)/G$ and $\overline{S}$ is its complex conjugate.

A *Bers slice* of $\mathrm{QC}(G)$ for a quasifuchsian group G is a subspace of $\mathrm{QC}(G)$ of the form $B(s_0)=\mathcal{T}(S)\times\{s_0\}$. The structure of the closure of $B(s_0)$ in $\mathrm{AH}(G)$ has been studied by a number of authors, including Bers [14], Kerckhoff and Thurston [60], Maskit [66], McMullen [74], and Minsky [76]. In particular, Bers [14] showed that the closure $\overline{B(s_0)}$ of $B(s_0)$ is compact, and Kerckhoff and Thurston [60] have shown that the compactification $\overline{B(s_0)}$ depends on the basepoint s_0, and so there are actually uncountably many such compactifications. Among other major results, Minsky [76] has shown that every punctured torus group lies in the boundary of $\mathrm{QC}(G)$, where G is a quasifuchsian group uniformizing a punctured torus and where a *punctured torus group* is a Kleinian group generated by two elements with parabolic commutator. In particular, this shows that the relative version of the Density Conjecture holds for punctured torus groups.

There are other slices of $\mathrm{QC}(G)$ which have been extensively studied. There is the extensive work of Keen and Series, see for instance [56], [57], and [58], inspired in part by unpublished work of Wright [103], on the Maskit slice of the Teichmüller space of a punctured torus in terms of *pleating coordinates*, which are natural and geometrically interesting coordinates on the Teichmüller space of the punctured torus which are given in terms of the geometry of the corresponding hyperbolic 3–manifolds.

In the case that G is a Kleinian group for which the corresponding 3–manifold $M=(\mathbf{H}^3\cup\Omega(G))/G$ is a compact, acylindrical 3–manifold with non-empty, incompressible boundary, then every representation in $\mathrm{MP}(G)$ in fact lies in $\mathrm{QC}(G)$; this follows from work of Johannson [51]. In addition, Thurston [99] has shown that $\mathrm{AH}(G)$ is compact for such G; another proof is given by Morgan and Shalen [80].

2.6 The spaces of $\mathcal{TT}(G)$ and $\mathrm{TT}(G) = \mathcal{TT}(G)/\mathrm{PSL}_2(\mathbf{C})$ of topologically tame representations

There is one last class of deformations which we need to define, before beginning our discussion of the relationships between these spaces. We begin with a topological definition. A compact submanifold M of a hyperbolic 3–manifold N is a *compact core* if the inclusion of M into N is a homotopy equivalence. The Core Theorem of Scott [93] implies that every hyperbolic 3–manifold with finitely generated fundamental group has a compact core. Marden [64] asked whether every hyperbolic 3–manifold N with finitely generated fundamental group is necessarily *topologically tame*, in that N is homeomorphic to the interior of its compact core.

Set $\mathcal{TT}(G)$ to be the subspace of $\mathcal{D}(G)$ consisting of the representations ρ with minimally parabolic, topologically tame image $\rho(G)$.
Set $\mathrm{TT}(G) = \mathcal{TT}(G)/\mathrm{PSL}_2(\mathbf{C})$.

There is a notion related to topological tameness, namely *geometric tameness*, first defined by Thurston [102]. We do not discuss geometric tameness here; the interested reader should consult Thurston [102], Bonahon [18], or Canary [29]. Thurston [102] showed that geometrically tame hyperbolic 3–manifolds with freely indecomposible fundamental group are topologically tame and satisfy the Ahlfors Measure Conjecture. Bonahon [18] showed that if every non-trivial free product splitting of a finitely generated Kleinian group Γ has the property that there exists a parabolic element of Γ not conjugate into one of the free factors, then Γ is geometrically tame. Canary [29] extended the definition of geometrically tame to all hyperbolic 3–manifolds, proved that topologically tame hyperbolic 3–manifolds are geometrically tame, and proved that topological tameness has a number of geometric and analytic consequences; in particular, he established that the Ahlfors Measure Conjecture holds for topologically tame Kleinian groups.

3 Geometric limits

There is a second notion of convergence for Kleinian groups which is distinct from the topology described above, which is equally important in the study of deformations spaces.

A sequence $\{\Gamma_n\}$ of Kleinian groups converges *geometrically* to a Kleinian group $\widehat{\Gamma}$ if two conditions are met, namely that every element of $\widehat{\Gamma}$ is the limit of a sequence of elements $\{\gamma_n \in \Gamma_n\}$ and that every accumulation point of every sequence $\{\gamma_n \in \Gamma_n\}$ lies in $\widehat{\Gamma}$. Note that, unlike the topology of algebraic convergence described above, the geometric limit of a sequence of isomorphic Kleinian groups need not be isomorphic to the groups in the sequence, and indeed need not be finitely generated. However, it is known that the geometric limit of a sequence of non-elementary, torsion-free Kleinian groups is again torsion-free.

We note here that it is possible to phrase the definition of geometric convergence in terms of the quotient hyperbolic 3–manifolds. Setting notation, let 0 denote a choice of basepoint for $\mathbf{H}^3$, and let $p_j\colon \mathbf{H}^3 \to N_j = \mathbf{H}^3/\rho_j(G)$ and $p\colon \mathbf{H}^3 \to \widehat{N} = \mathbf{H}^3/\widehat{\Gamma}$ be the covering maps. Let $B_R(0) \subset \mathbf{H}^3$ be a ball of radius R centered at the basepoint 0.

Lemma 3.1 *A sequence of torsion-free Kleinian groups $\{\Gamma_n\}$ converges geometrically to a torsion-free Kleinian group $\widehat{\Gamma}$ if and only if there exists a sequence $\{(R_n, K_n)\}$ and a sequence of orientation preserving maps $\tilde{f}_n\colon B_{R_n}(0) \to \mathbf{H}^3$ such that the following hold:*

1) *$R_n \to \infty$ and $K_n \to 1$ as $i \to \infty$;*
2) *the map $\tilde{f}_n$ is a K_n–bilipschitz diffeomorphism onto its image, $\tilde{f}_n(0) = 0$, and $\{\tilde{f}_n|_A\}$ converges to the identity for any compact set A; and*
3) *$\tilde{f}_n$ descends to a map $f_n\colon Z_n \to \widehat{N}$, where $Z_n = B_{R_n}(0)/\Gamma_n$ is a submanifold of N_n; moreover, f_n is also an orientation preserving K_n–biLipschitz diffeomorphism onto its image.*

For a proof of this Lemma, see Theorem 3.2.9 of Canary, Epstein, and Green [32], and Theorem E.1.13 and Remark E.1.19 of Benedetti and Petronio [13].

A fundamental example of the difference between algebraic and geometric convergence of Kleinian groups is given by the following explicit example of Jørgensen and Marden [55]; earlier examples are given in Jørgensen [52]. Choose ω_1 and ω_2 in $\mathbf{C} - \{0\}$ which are linearly independent over $\mathbf{R}$, and for each $n \geq 1$ set $\omega_{1n} = \omega_1 + n\omega_2$, $\omega_{2n} = \omega_2$, and $\tau_n = \omega_{2n}/\omega_{1n}$. Consider the loxodromic elements $L_n(z) = \exp(-2\pi\mathbf{i}\tau_n)z + \omega_2$. Then, as $n \to \infty$, L_n converges to $L(z) = z + \omega_2$, and so $\langle L_n \rangle$ converges algebraically to $\langle L \rangle$. However, note that $L_n^{-n}(z)$ converges to $K(z) = z + \omega_1$ as $n \to \infty$. Hence, $\langle L_n \rangle$ converges geometrically to $\langle L, K \rangle = \mathbf{Z} \oplus \mathbf{Z}$.

This example of the geometric convergence of loxodromic cyclic groups to rank two parabolic groups underlies much of the algebra of the operation of *Dehn surgery*, which we describe here.

Let M be a compact, hyperbolizable 3–manifold, let T be a torus component of ∂M, and choose a meridian–longitude system (α,β) on T. Let P be a solid torus and let c be a simple closed curve on ∂P bounding a disc in P. For each pair (m,n) of relatively prime integers, let $M(m,n)$ be the 3–manifold by attaching ∂P to T by an orientation-reversing homeomorphism which identifies c with $m\alpha+n\beta$; we refer to $M(m,n)$ as the result of (m,n) Dehn surgery along T. The following Theorem describes the basic properties of this operation; the version we state is due to Comar [38].

Theorem 3.2 (Comar [38]) *Let M be a compact, hyperbolizable 3–manifold and let $T=\{T_1,\ldots,T_k\}$ be a non-empty collection of tori in ∂M. Let $\hat{N}=\mathbf{H}^3/\Gamma$ be a geometrically finite hyperbolic 3–manifold and let $\psi\colon \mathrm{int}(M)\to N$ be an orientation preserving homeomorphism. Further assume that every parabolic element of Γ lies in a rank two parabolic subgroup. Let (m_i,l_i) be a meridian–longitude basis for T_i. Let $\{(\mathbf{p}_n,\mathbf{q}_n)=((p_n^1,q_n^1),\ldots,(p_n^k,q_n^k))\}$ be a sequence of k–tuples of pairs of relatively prime integers such that, for each i, $\{(p_n^i,q_n^i)\}$ converges to ∞ as $n\to\infty$.*

Then, for all sufficiently large n, there exists a representation $\beta_n\colon\Gamma\to \mathrm{PSL}_2(\mathbf{C})$ with discrete image such that

1) *$\beta_n(\Gamma)$ is geometrically finite, uniformizes $M(\mathbf{p}_n,\mathbf{q}_n)$, and every parabolic element of $\beta_n(\Gamma)$ lies in a rank two parabolic subgroup;*
2) *the kernel of $\beta_n\circ\psi_*$ is normally generated by $\{m_1^{p_n^1}l_1^{q_n^1},\ldots,m_k^{p_n^k}l_k^{q_n^k}\}$; and*
3) *$\{\beta_n\}$ converges to the identity representation of Γ.*

The idea of Theorem 3.2 is due to Thurston [102] in the case that the hyperbolic 3–manifold N has finite volume, so that ∂M consists purely of tori. In this case, it is also known that $\mathrm{volume}(\mathbf{H}^3/\beta_n(\Gamma)) < \mathrm{volume}(\mathbf{H}^3/\Gamma)$ for each n, and that $\mathrm{volume}(\mathbf{H}^3/\beta_n(\Gamma)) \to \mathrm{volume}(\mathbf{H}^3/\Gamma)$ as $n\to\infty$. For a more detailed discussion of this phenomenon, we refer the reader to Gromov [47] and Benedetti and Petronio [13]. The generalization to the case that N has infinite volume is due independently to Bonahon and Otal [21] and Comar [38]. Note that the $\beta_n(\Gamma)$ are not isomorphic, and hence there is no notion of algebraic convergence for these groups.

In the case that we have a sequence of representations in $\mathcal{D}(G)$, the following result of Jørgensen and Marden is extremely useful.

Proposition 3.3 (Jørgensen and Marden [55]) *Let $\{\rho_n\}$ be a sequence in $\mathrm{AH}(G)$ converging to ρ; then, there is a subsequence of $\{\rho_n\}$, again called*

$\{\rho_n\}$, so that $\{\rho_n(G)\}$ converges geometrically to a Kleinian group $\widehat{\Gamma}$ containing $\rho(G)$.

A sequence $\{\rho_n\}$ in $\mathcal{D}(G)$ converges *strongly* to ρ if $\{\rho_n\}$ converges algebraically to ρ and if $\{\rho_n(G)\}$ converges geometrically to $\rho(G)$. Note that we may consider $\mathcal{D}(G)$, and $\mathrm{AH}(G)$, to be endowed with topology of strong convergence, instead of the topology of algebraic convergence. We also refer the reader to the recent article of McMullen [75], in which a variant of the notion of strong convergence is explored in a somewhat more general setting.

Generalizing the behavior of the sequence of loxodromic cyclic groups described above, examples of sequences $\{\rho_n\}$ in $\mathcal{D}(G)$ which converge algebraically to ρ and for which $\{\rho_n(G)\}$ converges geometrically to a Kleinian group Γ properly containing $\rho(G)$ have been constructed by a number of authors, including Thurston [102], [100], Kerckhoff and Thurston [60], Anderson and Canary [6], Ohshika [84], and Brock [26], [25], among others.

Jørgensen and Marden [55] carry out a very detailed study of the relationship between the algebraic limit and the geometric limit in the case when the geometric limit is assumed to be geometrically finite. In general, not much is known about the relationship between the algebraic and geometric limits of a sequence of isomorphic Kleinian groups. We spend the remainder of this section discussing this question.

A fundamental point in understanding how algebraic limits sit inside geometric limits is the following algebraic fact, which is an easy application of Jørgensen's inequality.

Proposition 3.4 (Anderson, Canary, Culler, and Shalen [9]) *Let $\{\rho_n\}$ be a sequence in $\mathcal{D}(G)$ which converges to ρ and for which $\{\rho_n(G)\}$ converges geometrically to a Kleinian group $\widehat{\Gamma}$ containing $\rho(G)$. Then, for each $\gamma \in \widehat{\Gamma} - \rho(G)$, the intersection $\gamma\rho(G)\gamma^{-1}$ is either trivial or parabolic cyclic.*

One of the first applications of this result, also in [9], was to show, when the algebraic limit is a maximal cusp, that the convex hull of the quotient 3–manifold corresponding to the algebraic limit embeds in the quotient 3–manifold corresponding to the geometric limit. This was part of a more general attempt to understand the relationship between the volume and the rank of homology for a finite volume hyperbolic 3–manifold.

Another application was given by Anderson and Canary [7]. Before stating the generalization, we need to give a definition. Given a Kleinian group Γ, consider its associated 3–manifold $M = (\mathbf{H}^3 \cup \Omega(\Gamma))/\Gamma$, where $\Omega(\Gamma)$ is the domain of discontinuity of Γ. Then, Γ has connected limit set and no accidental parabolics if and only if every closed curve γ in ∂M which is homotopic to a curve of arbitrarily small length in the interior of M with the hyperbolic metric, is homotopic to a curve of arbitrarily small length in ∂M, with its induced metric.

Theorem 3.5 (Anderson and Canary [7]) *Let G be a finitely generated, torsion-free, non-abelian group, let $\{\rho_n\}$ be a sequence in $\mathcal{D}(G)$ converging to ρ, and suppose that $\{\rho_n(G)\}$ converges geometrically to $\widehat{\Gamma}$. Let $N = \mathbf{H}^3/\rho(G)$ and $\widehat{N} = \mathbf{H}^3/\widehat{\Gamma}$, and let $\pi\colon N \to \widehat{N}$ be the covering map. If $\rho(G)$ has non-empty domain of discontinuity, connected limit set, and contains no accidental parabolics, then there exists a compact core M for N such that π is an embedding restricted to M.*

One can apply Theorem 3.5 to show that certain algebraically convergent sequences are actually strongly convergent. This is of interest, as it is generally much more difficult to determine strong convergence of a sequence of representations than to determine algebraic convergence.

Theorem 3.6 (Anderson and Canary [7]) *Let G be a finitely generated, torsion-free, non-abelian group and let $\{\rho_n\}$ be a sequence in $\mathcal{D}(G)$ converging to ρ. Suppose that $\rho_n(G)$ is purely loxodromic for all n, and that $\rho(G)$ is purely loxodromic. If $\Omega(\rho(G))$ is non-empty, then $\{\rho_n(G)\}$ converges strongly to $\rho(G)$. Moreover, $\{\Lambda(\rho_n(G))\}$ converges to $\Lambda(\rho(G))$.*

Theorem 3.7 (Anderson and Canary [7]) *Let G be a finitely generated, torsion-free, non-abelian group and let $\{\rho_n\}$ be a sequence in $\mathcal{D}(G)$ converging to ρ. Suppose that $\rho_n(G)$ is purely loxodromic for all n, that $\rho(G)$ is purely loxodromic, and that G is not a non-trivial free product of (orientable) surface groups and cyclic groups, then $\{\rho_n(G)\}$ converges strongly to $\rho(G)$. Moreover, $\{\Lambda(\rho_n(G))\}$ converges to $\Lambda(\rho(G))$.*

Both Theorem 3.6 and Theorem 3.7 have been generalized by Anderson and Canary [8] to Kleinian groups containing parabolic elements, under the hypothesis that the sequences are type-preserving.

One reason that strong convergence is interesting is that strongly convergent sequences of isomorphic Kleinian groups tend to be extremely well behaved, as one has the geometric data coming from the convergence of the quotient

3–manifolds as well as the algebraic data coming from the convergence of the representations. For instance, there is the following Theorem of Canary and Minsky [34]. We note that a similar result is proven independently by Ohshika [86].

Theorem 3.8 (Canary and Minsky [34]) *Let M be a compact, irreducible 3–manifold and let $\{\rho_n\}$ be a sequence in $\mathrm{TT}(\pi_1(M))$ converging strongly to ρ, where each $\rho_n(\pi_1(M))$ and $\rho(\pi_1(M))$ are purely loxodromic. Then, $\rho(\pi_1(M))$ is topologically tame; moreover, for all sufficiently large n, there exists a homeomorphism $\varphi_n\colon \mathbf{H}^3/\rho_n(\pi_1(M)) \to \mathbf{H}^3/\rho(\pi_1(M))$ so that $(\varphi_n)_* = \rho \circ \rho_n^{-1}$.*

By combining the results of Anderson and Canary [7] and of Canary and Minsky [34] stated above, one may conclude that certain algebraic limits of sequences of isomorphic topologically tame Kleinian groups are again topologically tame.

There is also the following result of Taylor [98].

Theorem 3.9 (Taylor [98]) *Let G be a finitely generated, torsion-free, non-abelian group, and let $\{\rho_n\}$ be a sequence in $\mathcal{D}(G)$ converging strongly to ρ, where each $\rho_n(G)$ has infinite co-volume. If $\rho(G)$ is geometrically finite, then $\rho_n(G)$ is geometrically finite for n sufficiently large.*

The guiding Conjecture in the study of the relationship between algebraic and geometric limits, usually attributed to Jørgensen, is stated below.

Conjecture 3.10 (Jørgensen) *Let Γ be a finitely generated, torsion-free, non-elementary Kleinian group, let $\{\rho_n\}$ be a sequence in $\mathcal{D}(\Gamma)$ converging to ρ, and suppose that $\{\rho_n(\Gamma)\}$ converges geometrically to $\widehat{\Gamma}$. If ρ is type-preserving, then $\rho(\Gamma) = \widehat{\Gamma}$.*

As we have seen above, this conjecture has been shown to hold in a wide variety of cases, including the case in which the sequence $\{\rho_n\}$ is type-preserving and the limit group $\rho(\Gamma)$ either has non-empty domain of discontinuity or is not a non-trivial free product of cyclic groups and the fundamental groups of orientable surfaces.

4 Functions on deformation spaces

There are several numerical quantities associated to a Kleinian group Γ; one is the *Hausdorff dimension* $\mathrm{D}(\Gamma)$ of the limit set $\Lambda(\Gamma)$ of Γ, another is the smallest positive eigenvalue $\mathrm{L}(\Gamma)$ of the Laplacian on the corresponding hyperbolic 3–manifold $\mathbf{H}^3/\Gamma$. These two functions are closely related; namely, if Γ is topologically tame, then $\mathrm{L}(\Gamma) = \mathrm{D}(\Gamma)(2 - \mathrm{D}(\Gamma))$ when $\mathrm{D}(\Gamma) \geq 1$, and $\mathrm{L}(\Gamma) = 1$ when $\mathrm{D}(\Gamma) \leq 1$. The relationship between these two quantities has been studied by a number of authors, including Sullivan [96], Bishop and Jones [17], Canary [31], and Canary, Minsky, and Taylor [35] (from which the statement given above is taken). It is natural to consider how these functions behave on the spaces we have been discussing in this note.

We begin by giving a few topological definitions. A compact, hyperbolizable 3–manifold with incompressible boundary is a *generalized book of I–bundles* if there exists a disjoint collection A of essential annuli in M so that each component of the closure of the complement of A in M is either a solid torus, a thickened torus, or an I–bundle whose intersection with ∂M is the associated ∂I–bundle.

An *incompressible core* of a compact hyperbolizable 3–manifold is a compact submanifold P, possibly disconnected, with incompressible boundary so that M can be obtained from P by adding 1–handles.

We begin with a pair of results of Canary, Minsky, and Taylor [35] which relates the topology of M to the behavior of these functions on a well-defined subset of $\mathrm{AH}(\pi_1(M))$, and show that they are in a sense dual to one another.

Theorem 4.1 (Canary, Minsky, and Taylor [35]) *Let M be a compact, hyperbolizable 3–manifold. Then, $\sup L(\rho(\pi_1(M))) = 1$ if and only if every component of the incompressible core of M is a generalized book of I–bundles; otherwise, $\sup L(\rho(\pi_1(M))) < 1$. Here, the supremum is taken over all ρ in $\mathrm{AH}(\pi_1(M))$ for which $\mathbf{H}^3/\rho(\pi_1(M))$ is homeomorphic to the interior of M.*

Theorem 4.2 (Canary, Minsky, and Taylor [35]) *Let M be a compact, hyperbolizable 3–manifold which is not a handlebody or a thickened torus. Then, $\inf D(\rho(\pi_1(M))) = 1$ if and only if every component of the incompressible core of M is a geneneralized book of I–bundles; otherwise, $\inf D(\rho(\pi_1(M))) > 1$. Here, the infimum is taken over all ρ in $\mathrm{AH}(\pi_1(M))$ for which $\mathbf{H}^3/\rho(\pi_1(M))$ is homeomorphic to the interior of M.*

It is also possible to consider how these quantities behave under taking limits. We note that results similar to Theorem 4.3 have been obtained by McMullen [75], who also shows that the function D is not continuous on $\mathcal{D}(\pi_1(M))$ in the case that M is a handlebody.

Theorem 4.3 (Canary and Taylor [36]) *Let M be a compact, hyperbolizable 3–manifold which is not homeomorphic to a handlebody. Then $D(\rho)$ is continuous on $\mathcal{D}(\pi_1(M))$ endowed with the topology of strong convergence.*

Recently, Fan and Jorgenson [44] have made use of the heat kernel to prove the continuity of small eigenvalues and small eigenfunctions of the Laplacian for sequences of hyperbolic 3–manifolds converging to a geometrically finite limit manifold, where the convergence is the variant of strong convergence considered by McMullen [75].

There are several functions on $\mathrm{QC}(G)$ which have been studied by Bonahon. In order to keep definitions to a minimum, we state his results for geometrically finite G, though we note that they hold for a general finitely generated Kleinian group G. Given a representation ρ in $\mathrm{QC}(G)$, recall that the *convex core* C_ρ of $\mathbf{H}^3/\rho(G)$ is the smallest convex submanifold of $\mathbf{H}^3/\rho(G)$ whose inclusion is a homotopy equivalence. By restricting the hyperbolic metric on $\mathbf{H}^3/\rho(G)$ to ∂C_ρ, we obtain a map μ from $\mathrm{QC}(G)$ to the Teichmüller space $\mathcal{T}(\Omega(G)/G)$ of the Riemann surface $\Omega(G)/G$.

Theorem 4.4 (Bonahon [20]) *For a geometrically finite Kleinian group G, the map $\mu\colon \mathrm{QC}(G) \to \mathcal{T}(\Omega(G)/G)$ is continuously differentiable.*

Another function on $\mathrm{QC}(G)$ studied by Bonahon, by developing an analog of the Schläfli formula for the volume of a polyhedron in hyperbolic space, is the function $\mathrm{vol}\colon \mathrm{QC}(G) \to [0,\infty)$, which associates to $\rho \in \mathrm{QC}(G)$ the volume $\mathrm{vol}(\rho)$ of the convex core C_ρ of $\mathbf{H}^3/\rho(G)$.

Theorem 4.5 (Bonahon [19]) *Let G be a geometrically finite Kleinian group. If the boundary ∂C_ρ of the convex core C_ρ of $\mathbf{H}^3/\rho(G)$ is totally geodesic, then ρ is a local minimum of* $\mathrm{vol}\colon \mathrm{QC}(G) \to [0,\infty)$.

It is known that the Hausdorff dimension of the limit set is a continuous function on $\mathrm{QC}(\Gamma)$, using estimates relating the Hausdorff dimension and quasiconformal dilitations due to Gehring and Väisälä [45]. In some cases, it is possible to obtain more analytic information.

Theorem 4.6 (Ruelle [92]) *Let Γ be a convex cocompact Kleinian group whose limit set supports an expanding Markov partition. Then, the Hausdorff dimension of the limit set is a real analytic function on* $\mathrm{QC}(\Gamma)$.

Earlier work of Bowen [24] shows that quasifuchsian and Schottky groups support such Markov partitions. The following Theorem follows by combining these results of Bowen and Ruelle with a condition which implies the existence of an expanding Markov partition, namely that there exists a fundamental polyhedron in $\mathbf{H}^3$ for the Kleinian group G which has the *even cornered property*, together with the Klein Combination Theorem.

Theorem 4.7 (Anderson and Rocha [11]) *Let G be a convex cocompact Kleinian group which is isomorphic to the free product of cyclic groups and fundamental groups of 2–orbifolds. Then, the Hausdorff dimension of the limit set is a real analytic function on* $\mathrm{QC}(G)$.

We note here that it is not yet established that all convex cocompact Kleinian groups support such Markov partitions.

Another function one can consider is the injectivity radius of the corresponding quotient hyperbolic 3–manifold. For a hyperbolic 3–manifold N, the *injectivity radius* $\mathrm{inj}_N(x)$ at a point $x \in N$ is one-half the length of the shortest homotopically non-trivial closed curve through x. The following Conjecture is due to McMullen.

Conjecture 4.8 *Let G be a finitely generated group with g generators. Then, there exists a constant $C = C(g)$ so that, if N is a hyperbolic 3–manifold with fundamental group isomorphic to G and if x lies in the convex core of N, then* $\mathrm{inj}_N(x) \leq C$.

Kerckhoff and Thurston [60] show that, if M is the product of a closed, orientable surface S of genus at least 2 with the interval, then there exists a constant $C = C(M)$ so that if N is a hyperbolic 3–manifold which is homeomorphic to the interior of M and if N has no cusps, then the injectivity radius on the convex core of N is bounded above by C. Fan [42] generalizes this to show that, if M is a compact, hyperbolizable 3–manifold which is either a book of I–bundles or is acylindrical, then there exists a constant $C = C(M)$ so that, if N is any hyperbolic 3–manifold homeomorphic to the interior of M, then the injectivity radius on the convex core of N is bounded above by C.

We close by mentioning recent work of Basmajian and Wolpert [12] concerning the persistance of intersecting closed geodesics. Say that a Kleinian group Γ has the *SPD property* if all the closed geodesics in $\mathbf{H}^3/\Gamma$ are simple and pairwise disjoint.

Theorem 4.9 (Basmajian and Wolpert [12]) *Let G be a torsion-free, convex co-compact Kleinian group, and let U be the component of* $\mathrm{MP}(G)$ *containing the identity representation. Then, either*

1) *there exists a subset V of U, which is the intersection of a countably many open dense sets, so that $\rho(G)$ has the SPD property for every $\rho \in V$, or*
2) *there exists a pair of loxodromic elements α and β of G so that the closed geodesics in $\mathbf{H}^3/\rho(G)$ corresponding to loxodromic elements $\rho(\alpha)$ and $\rho(\beta)$ intersect at an angle constant over all $\rho \in U$; in particular, there is* no *element $\rho \in U$ so that $\rho(G)$ has the SPD property.*

They also show that the first possibility holds in the case that G is a purely loxodromic Fuchsian group.

References

[1] **W Abikoff**, **B Maskit**, *Geometric decompositions of Kleinian groups*, Amer. J. Math. 99 (1977) 687–697

[2] **L V Ahlfors**, *Lectures on Quasiconformal Mappings*, Wadsworth and Brooks–Cole (1987)

[3] **L V Ahlfors**, *Finitely generated Kleinian groups*, Amer. J. Math. 86 (1964) 413–29

[4] **L V Ahlfors**, *The structure of a finitely generated Kleinian group*, Acta Math. 122 (1969) 1–17

[5] **L V Ahlfors**, **L Bers**, *Riemann's mapping theorem for variable metrics*, Annals of Math. 72 (1960) 385–404

[6] **J W Anderson**, **R D Canary**, *Algebraic limits of Kleinian groups which rearrange the pages of a book*, Invent. Math. 126 (1996) 205–214

[7] **J W Anderson**, **R D Canary**, *Cores of hyperbolic 3–manifolds and limits of Kleinian groups*, Amer. J. Math. 118 (1996) 745–779

[8] **J W Anderson**, **R D Canary**, *Cores of hyperbolic 3–manifolds and limits of Kleinian groups II*, Journal London Math. Soc. (to appear)

[9] **J W Anderson**, **R D Canary**, **M Culler**, **P Shalen**, *Free Kleinian groups and volumes of hyperbolic 3–manifolds*, J. Differential Geometry 43 (1996) 738–782

[10] **J W Anderson**, **R D Canary**, **D McCullough**, *On the topology of deformation spaces of Kleinian groups*, preprint

[11] **J W Anderson**, **A C Rocha**, *Analyticity of Hausdorff dimension of limit sets of Kleinian groups*, Ann. Acad. Sci. Fenn. 22 (1997) 349–364

[12] **A Basmajian**, **S A Wolpert**, *Hyperbolic 3–manifolds with non-intersecting closed geodesics*, in preparation

[13] **R Benedetti**, **C Petronio**, *Lectures on Hyperbolic Geometry*, Springer–Verlag Universitext, (1992)

[14] **L Bers**, *On boundaries of Teichmüller spaces and on Kleinian groups*, Annals of Math. 91 (1970) 570–600

[15] **L Bers**, *Simultaneous uniformization*, Bull. Amer. Math. Soc. 66 (1960) 94–97

[16] **L Bers**, *Spaces of Kleinian groups*, from: "Maryland Conference in Several Complex Variables I", Springer–Verlag Lecture Notes in Math. vol. 155 (1970) 9–34

[17] **C J Bishop**, **P W Jones**, *Hausdorff dimension and Kleinian groups*, Acta Math. 197 (1997) 1–39

[18] **F Bonahon**, *Bouts des variétés hyperboliques de dimension 3*, Annals of Math. 124 (1986) 71–158

[19] **F Bonahon**, *A Schläfli-type formula for convex cores of hyperbolic 3–manifolds*, preprint

[20] **F Bonahon**, *Variations of the boundary geometry of 3–dimensional hyperbolic convex cores*, preprint

[21] **F Bonahon**, **J P Otal**, *Variétés hyperboliques à géodésiques arbitrairement courtes*, Bull. London Math. Soc. 20 (1988) 255–261

[22] **B H Bowditch**, *Geometrical finiteness for hyperbolic groups*, J. Funct. Anal. 113 (1993) 245–317

[23] **B H Bowditch**, *Spaces of geometrically finite representations* Ann. Acad. Sci. Fenn. (to appear)

[24] **R Bowen**, *Hausdorff dimension of quasi-circles*, IHES Publ. Math. 50 (1979) 1–25

[25] **J F Brock**, *Iteration of mapping classes on a Bers' slice: examples of algebraic and geometric limits of hyperbolic 3–manifolds*, Proceedings of the Bers colloquium (1997) (to appear)

[26] **J F Brock**, *Iteration of mapping classes and limits of hyperbolic 3–manifolds*, PhD Thesis UC Berkeley (1997)

[27] **R D Canary**, *Algebraic convergence of Schottky groups*, Trans. Amer. Math. Soc. 337 (1993) 235–258

[28] **R D Canary**, *Covering theorems for hyperbolic 3–manifolds*, Proceedings of Low-Dimensional Topology, International Press (1994) 21–30

[29] **R D Canary**, *Ends of hyperbolic 3–manifolds*, Journal Amer. Math. Soc. 6 (1993) 1–35

[30] **R D Canary**, *Geometrically tame hyperbolic 3–manifolds*, Proceedings of Symposia in Pure Mathematics 54 (1993) 99–109

[31] **R D Canary**, *On the Laplacian and the geometry of hyperbolic 3–manifolds*, J. Differential Geometry 36 (1992) 349–367

[32] **R D Canary**, **D B A Epstein**, **P Green**, *Notes on notes of Thurston*, from: "Analytical and Geometrical Aspects of Hyperbolic Spaces", Cambridge University Press, (1987) 3–92

[33] **R D Canary**, **D McCullough**, *Homotopy equivalences of 3–manifolds and deformation theory of Kleinian groups*, in preparation

[34] **R D Canary**, **Y N Minsky**, *On limits of tame hyperbolic 3–manifolds*, J. Differential Geometry 43 (1996) 1–41

[35] **R D Canary**, **Y N Minsky**, **E C Taylor**, *Spectral theory, Hausdorff dimension and the topology of hyperbolic 3–manifolds*, J. Geom. Analysis (to appear)

[36] **R D Canary**, **E C Taylor**, *Hausdorff dimension and limits of Kleinian groups*, preprint

[37] **V Chuckrow**, *On Schottky groups with applications to Kleinian groups*, Annals of Math. 88 (1968) 47–61

[38] **T D Comar**, *Hyperbolic Dehn surgery and convergence of Kleinian groups*, PhD Thesis University of Michigan (1996)

[39] **M Culler**, **C McA Gordon**, **J Luecke**, **P B Shalen**, *Dehn surgery on knots*, Annals of Math. 125 (1987) 237–300

[40] **M Culler**, **P B Shalen**, *Varieties of group representations and splittings of 3–manifolds*, Annals of Math. 117 (1983) 109–146

[41] **M Culler**, **P. B. Shalen**, *Bounded separating incompressible surfaces in knot manifolds*, Invent. Math. 75 (1984) 537–545

[42] **C E Fan**, *Injectivity radius bounds in hyperbolic convex cores*, PhD Thesis University of Michigan (1997)

[43] **C E Fan**, *Injectivity radius bounds in hyperbolic I-bundle convex cores*, preprint

[44] **C E Fan**, **J Jorgenson**, *Small eigenvalues and Hausdorff dimension of sequences of hyperbolic 3–manifolds*, preprint

[45] **F W Gehring**, **J Väisälä**, *Hausdorff dimension and quasiconformal mappings*, Journal London Math. Soc. 6 (1973) 504–512

[46] **L Greenberg**, *Discrete subgroups of the Lorentz group*, Math. Scand. 10 (1962) 85–107

[47] **M Gromov**, *Hyperbolic manifolds after Jørgensen and Thurston*, from: "Séminaire Bourbaki", Lecture Notes in Math. 842, Springer–Verlag (1981) 40–53

[48] **J Hempel**, *3–manifolds*, Annals of Math. Studies 86, Princeton University Press (1976)

[49] **W Jaco**, *Lectures on three-manifold topology*, Conference board of the mathematical sciences, regional conference series in mathematics no. 43, AMS (1980)

[50] **W Jaco**, **P B Shalen**, *Seifert fibered spaces in 3–manifolds*, Memoirs AMS 220 (1979)

[51] **K Johannson**, *Homotopy equivalences of 3–manifolds with boundaries*, Lecture notes in Math. vol. 761, Springer–Verlag (1979)

[52] **T Jørgensen**, *On cyclic groups of Möbius transformations*, Math. Scand. 33 (1973) 250–260

[53] **T Jørgensen**, *On discrete groups of Möbius transformations*, Amer. J. Math. 98 (1976) 739–749

[54] **T Jørgensen**, **P Klein**, *Algebraic convergence of finitely generated Kleinian groups*, Quart. J. Math. Oxford 33 (1982) 325–332

[55] **T Jørgensen**, **A Marden**, *Algebraic and geometric convergence of Kleinian groups*, Math. Scand. 66 (1990) 47–72

[56] **L Keen**, **C Series**, *Pleating coordinates for the Teichmüller space of a punctured torus*, Bull. Amer. Math. Soc. 26 (1992) 141–146

[57] **L Keen**, **C Series**, *Pleating coordinates for the Maskit embedding of the Teichmüller space of a punctured torus*, Topology 32 (1993) 719–749

[58] **L Keen**, **C Series**, *Pleating invariants for punctured torus groups*, preprint

[59] **L Keen**, **C Series**, *The Riley slice of Schottky space*, Proc. London Math. Soc. 69 (1994) 72–90

[60] **S P Kerckhoff**, **W P Thurston**, *Non-continuity of the action of the mapping class group at Bers' boundary of Teichmüller space*, Invent. Math. 100 (1990) 25–47

[61] **I Kra**, *On lifting Kleinian groups to* $\mathrm{SL}_2(\mathbf{C})$, from: "Differential Geometry and Complex Analysis", I Chavel and H Farkas (editors), Springer–Verlag (1985) 181–193

[62] **I Kra**, *On spaces of Kleinian groups*, Comm. Math. Helv. 47 (1972) 53–69

[63] **O Lehto**, **K I Virtanen**, *Quasiconformal mappings in the plane*, Springer–Verlag (1973)

[64] **A Marden**, *The geometry of finitely generated Kleinian groups*, Annals of Math. 99 (1974) 383–462

[65] **A Marden**, *Geometric relations between homeomorphic Riemann surfaces*, Bull. Amer. Math. Soc. 3 (1980) 1001–1017

[66] **B Maskit**, *On boundaries of Teichmüller spaces and on Kleinian groups II*, Annals of Math. 91 (1970) 607–639

[67] **B Maskit**, *A characterization of Schottky groups*, J. d'Analyse Math. 19 (1967) 227–230

[68] **B Maskit**, *Kleinian groups*, Springer–Verlag (1988)

[69] **B Maskit**, *Parabolic elements in Kleinian groups*, Annals of Math. 117 (1983) 659–668

[70] **B Maskit**, *Self-maps of Kleinian groups*, Amer. J. Math. 93 (1971) 840–856

[71] **K Matsuzaki**, **M Taniguchi**, *Hyperbolic Manifolds and Kleinian Groups*, Clarendon Press, (1998)

[72] **D McCullough**, *Compact submanifolds of 3–manifolds with boundary*, Quart. J. Math. Oxford 37 (1986) 299–307

[73] **D McCullough**, **A Miller**, **G A Swarup**, *Uniqueness of cores of non-compact 3–manifolds*, Journal London Math. Soc. 32 (1985) 548–556

[74] **C T McMullen**, *Cusps are dense*, Annals of Math. 133 (1991) 217–247

[75] **C T McMullen**, *Hausdorff dimension and conformal dynamics I: Strong convergence of Kleinian groups*, preprint

[76] **Y N Minsky**, *The classification of punctured-torus groups*, preprint

[77] **J W Morgan**, *On Thurston's uniformization theorem for three-dimensional manifolds*, from: "The Smith Conjecture", J Morgan and H Bass (editors), Academic Press (1984) 37–125

[78] **J Morgan**, **P B Shalen**, *Valuations, trees, and degenerations of hyperbolic structures I*, Annals of Math. 120 (1984) 401–476

[79] **J Morgan**, **P B Shalen**, *Degenerations of hyperbolic structures II: Measured laminations in 3-manifolds*, Annals of Math. 127 (1988) 403–456

[80] **J Morgan**, **P B Shalen**, *Degenerations of hyperbolic structures III: Actions of 3-manifold groups on trees and Thurston's compactification theorem*, Annals of Math. 127 (1988) 457–519

[81] **G D Mostow**, *Strong rigidity of locally symmetric spaces*, Annals of Math. Studies 78, Princeton University Press (1973)

[82] **K Ohshika**, *Divergent sequences of Kleinian groups*, preprint

[83] **K Ohshika**, *Ending laminations and boundaries for deformation spaces of Kleinian groups*, Journal London Math. Soc. 42 (1990) 111–121

[84] **K Ohshika**, *Geometric behaviour of Kleinian groups on boundaries for deformation spaces*, Quart. J. Math. Oxford 43 (1992) 97–111

[85] **K Ohshika**, *Geometrically finite Kleinian groups and parabolic elements*, preprint

[86] **K Ohshika**, *Kleinian groups which are limits of geometrically finite groups*, preprint

[87] **K Ohshika**, *Strong convergence of Kleinian groups and Caratheodory convergence of domains of discontinuity*, Math. Proc. Camb. Phil. Soc. 112 (1992) 297–307

[88] **J P Otal**, *Sur la degenerescence des groupes de Schottky*, Duke Math. J. 74 (1994) 777–792

[89] **J P Otal**, *Le Théoréme d'hyperbolisation pour les variétés fibrées de dimension 3*, Astérisque 235 (1996)

[90] **J P Otal**, **F Paulin**, *Géométrie hyperbolique et groupes kleiniens*, in preparation

[91] **G Prasad**, *Strong rigidity in* $\mathbf{Q}$*-rank* 1 *lattices*, Invent. Math. 21 (1973) 255–286

[92] **D Ruelle**, *Repellers for real analytic maps*, Ergodic Theory and Dynam. Sys. 2 (1982) 99–107

[93] **G P Scott**, *Compact submanifolds of 3-manifolds*, Journal London Math. Soc. 7 (1973) 246-250

[94] **P B Shalen**, *Representations of 3-manifold groups and applications in topology*, Proceedings of the International Congress of Mathematicians (1986) 607–614

[95] **D P Sullivan**, *On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions*, from: "Riemann surfaces and related topics", I Kra and B Maskit (editors), Annals of Math. Studies 97, Princeton University Press (1981) 465–496

[96] **D P Sullivan**, *Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups*, Acta Math. 153 (1984) 259–277

[97] **D P Sullivan**, *Quasiconformal homeomorphisms and dynamics II: Structural stability implies hyperbolicity of Kleinian groups*, Acta Math. 155 (1985) 243–260

[98] **E C Taylor**, *Geometric finiteness and the convergence of Kleinian groups*, Comm. Analysis and Geometry, (to appear)

[99] **W P Thurston**, *Hyperbolic structures on 3-manifolds I: Deformation of acylindrical manifolds*, Annals of Math. 124 (1986) 203–246

[100] **W P Thurston**, *Hyperbolic structures on 3-manifolds II: Surface groups and 3-manifolds which fiber over the circle*, preprint

[101] **W P Thurston**, *Three-dimensional manifolds, Kleinian groups, and hyperbolic geometry*, Bull. Amer. Math. Soc. 6 (1982) 357–381

[102] **W P Thurston**, *The geometry and topology of 3-manifolds*, Lecture notes Princeton University (1978)

[103] **D J Wright**, *The shape of the boundary of Maskit's embedding of the Teichmüller space of once punctured tori*, preprint

Faculty of Mathematical Studies
University of Southampton
Southampton, SO17 1BJ, England

Email: `jwa@maths.soton.ac.uk`

Received: 15 November 1997