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Divergent sequences of Kleinian groups

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Abstract One of the basic problems in studying topological structures of deformation spaces for Kleinian groups is to find a criterion to distinguish convergent sequences from divergent sequences. In this paper, we shall give a sufficient condition for sequences of Kleinian groups isomorphic to surface groups to diverge in the deformation spaces.

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Dedicated to Prof David Epstein on the occasion of his 60th birthday

1 Introduction

The deformation space of a Kleinian group Γ is the space of faithful discrete representations of Γ into $PSL_2\mathbf{C}$ preserving parabolicity modulo conjugacy. It is one of the important aspects of Kleinian group theory to study the structures of deformation spaces. The first thing that was studied among the structures of deformation spaces was that of subspaces called quasi-conformal deformation spaces. By works of Ahlfors, Bers, Kra, Marden and Sullivan among others, the topological types and the parametrization of quasi-conformal deformation spaces are completely determined using the theory of quasi-conformal mappings and the ergodic theory on the sphere ([2], [5], [14], [24]). On the other hand, the total deformation spaces are less understood. A recent work of Minsky [16] makes it possible to determine the topological structure of the total deformation space completely in the case of once-puncture torus groups. The other cases are far from complete understanding. Although very rough topological structures, for instance the connected components of deformation spaces can be understood by virtue of recent works of Anderson–Canary and Anderson–Canary–McCullough, more detailed structures like the frontier of quasi-conformal deformation spaces are not yet known even in the case of surface groups with genus greater than 1.

A first step to understand the topological structure of the deformation space of a Kleinian group Γ is to give a criterion for a sequence $\{\Gamma_i\}$ in the deformation space to converge or diverge. In this paper, we shall consider the simplest case when the group Γ is isomorphic to a hyperbolic surface group $\pi_1(S)$ and has no accidental parabolic elements. In this case, Γ_i is either quasi-Fuchsian or a totally degenerate b–group, or a totally doubly degenerate group. Hence by taking a subsequence, we have only to consider the following three cases: all of the $\{\Gamma_i\}$ are quasi-Fuchsian, or totally degenerate b–groups, or totally doubly degenerate groups. For such groups, some conditions for sequences to converge are given for example in Bers [5], Thurston [28] and Ohshika [18]. Thurston's convergence theorem is called the double limit theorem. The purpose of this paper is to give a sufficient condition for sequences to diverge in the deformation space, which is in some sense complementary to the condition of the double limit theorem.

Before explaining the content of our main theorem, let us recall that a Kleinian group isomorphic to a hyperbolic surface group without accidental parabolic elements has two pieces of information describing the structures near ends as follows. When such a Kleinian group Γ is quasi-Fuchsian, by the Ahlfors–Bers theory, we get a pair of points in the Teichmuüller space $\mathcal{T}(S)$ corresponding to the group. In the case when Γ is a totally degenerate b–group, as there is one end of the non-cuspidal part $(\mathbf{H}^3/\Gamma)_0$ which is geometrically finite, we have a point in the Teichmüller space. In addition, the geometrically infinite end of $(\mathbf{H}^3/\Gamma)_0$ determines an ending lamination which is defined uniquely up to changes of transverse measures. Finally in the case when Γ is a totally doubly degenerate group, $(\mathbf{H}^3/\Gamma)_0$ have two geometrically infinite ends, and we have a pair of measured laminations which are ending laminations of the two ends. We shall define an end invariant of such a group Γ to be a pair (χ, v) where each factor is either a point of the Teichmüller space or a projective lamination represented by an ending lamination, which gives the information on one of the ends.

The statement of our main theorem is as follows. Suppose that we are given a sequence of Kleinian groups (Γ_i, ϕ_i) in the parabolicity-preserving deformation space $AH_p(S)$ of Kleinian groups isomorphic to $\pi_1(S)$ for a hyperbolic surface S. Suppose moreover that the end invariants (χ_i, v_i) have the following property: Either in the Thurston compactification or in the projective lamination space, $\{\chi_i\}$ and $\{v_i\}$ converge to maximal and connected projective laminations with the same support. Then the sequence $\{(\Gamma_i, \phi_i)\}$ does not converge in $AH_p(S)$.

To understand the meaning of this theorem, let us contrast it with Thurston's double limit theorem. For simplicity, we only consider the case when Γ_i is a quasi-Fuchsian group for the time being. By Ahlfors–Bers theory, a sequence of quasi-Fuchsian groups $\{(\Gamma_i, \phi_i)\}$ corresponds to a sequence of pairs of marked hyperbolic structures $\{(m_i, n_i)\}$ on S. Consider the case when both m_i and n_i diverge in the Teichmüller space and their limits in the Thurston compactification are projective laminations $[\mu]$ and $[\nu]$ respectively. The double limit theorem asserts that if μ and ν fill up S, viz., any measured lamination has non-zero intersection number with either μ or ν, then the sequence $\{(\Gamma_i, \phi_i)\}$ converges in the deformation space passing through a subsequence if necessary. The situation of our theorem is at the opposite pole to that of the double limit theorem. We assume in our theorem that μ and ν are equal except for the transverse measures and that they are maximal and connected.

We can see the assumption of maximality is essential by taking look at an example of Anderson–Canary [3]. They constructed an example of quasi-Fuchsian groups converging in $AH_p(S)$ which correspond to pairs of marked hyperbolic structures (m_i, n_i) such that $\{m_i\}$ and $\{n_i\}$ converge to the same point in $\mathcal{PL}(S)$. In this example, the support of the limit projective lamination is a simple closed curve, far from being maximal.

The proof of our theorem is based on an argument sketched in Thurston [26] which was used to prove his theorem stating that sequences of Kleinian groups isomorphic to surface groups which converge algebraically to Kleinian groups without accidental parabolic elements converge strongly. We shall give a detailed proof of this theorem in the last section as an application of our theorem.

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2 Preliminaries

Kleinian groups are discrete subgroups of the Lie group $PSL_2\mathbf{C}$ which is the group of conformal automorphisms of the 2–sphere S^2 and the orientation preserving isometry group of the hyperbolic 3–space $\mathbf{H}^3$. A Kleinian group acts conformally on S^2 and discontinuously on $\mathbf{H}^3$ by isometries. In this paper, we always assume that Kleinian groups are torsion free. For a torsion-free Kleinian group Γ, the quotient $\mathbf{H}^3/\Gamma$ is a complete hyperbolic 3–manifold.

Let Γ be a Kleinian group, which is regarded as acting on S^2. The subset of S^2 which is the closure of the set consisting of the fixed points of non-trivial elements in Γ, is called the limit set of Γ, and denoted by Λ_Γ. The limit set Λ_Γ is invariant under the action of Γ. The complement of Λ_Γ is called the region of discontinuity of Γ and denoted by Ω_Γ. The group Γ acts on Ω_Γ properly discontinuously. If Γ is finitely generated, the quotient Ω_Γ/Γ is a Riemann surface of finite type (ie a disjoint union of finitely many connected Riemann surfaces of finite genus with finitely many punctures) by Ahlfors' finiteness theorem [1].

A homeomorphism $\omega\colon S^2 \to S^2$ is said to be quasi-conformal if it has an L^2–distributional derivative and there exists a function $\mu\colon S^2 \to \mathbf{C}$ called a Beltrami coefficient whose essential norm is strictly less than 1, such that $\omega_{\overline{z}} = \mu\omega_z$. If the Beltrami coefficient μ for ω satisfies the condition $\mu \circ \gamma(z)\overline{\gamma}'(z)/\gamma'(z) = \mu(z)$ for every $\gamma \in \Gamma$, then the conjugate $\omega\Gamma\omega^{-1}$ is again a Kleinian group. A Kleinian group obtained by such a fashion from Γ is called a quasi-conformal deformation of Γ. By identifying two quasi-conformal deformations which are conformally conjugate, and giving the topology induced from the representation space, we obtain the quasi-conformal deformation space of Γ, which we shall denote by $QH(\Gamma)$. A quasi-conformal deformation of Ω_Γ/Γ can be extended to that of Γ. This gives rise to a continuous map $\rho\colon \mathcal{T}(\Omega_\Gamma/\Gamma) \to QH(\Gamma)$. By the works of Ahlfors, Bers, Kra, Marden and Sullivan among others, it is known that when Γ is finitely generated, ρ is a covering map, and that especially if Γ is isomorphic to a surface group (or more generally if Γ satisfies the condition $(*)$ introduced by Bonahon [7]), then ρ is a homeomorphism. The inverse of ρ is denoted by Q.

Let Γ be a finitely generated Kleinian group. We shall define the deformation space of Γ. An element γ of $PSL_2\mathbf{C}$ is said to be parabolic if it is conjugate to a parabolic element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The deformation space of Γ, denoted by $AH_p(\Gamma)$, is the space of faithful discrete representations of Γ into $PSL_2\mathbf{C}$ preserving the parabolicity modulo conjugacy with the quotient topology induced from the representation space. We shall often denote an element (ie an equivalence class of groups) in $AH_p(\Gamma)$ in a form (G, ϕ) where ϕ is a faithful discrete representation with the image G which represents the equivalence class. The quasi-conformal deformation space $QH(\Gamma)$ is regarded as a subspace of $AH_p(\Gamma)$.

Let $C(\Lambda_\Gamma)$ be the intersection of $\mathbf{H}^3$ and the convex hull of the limit set Λ_Γ in the Poincaré ball $\mathbf{H}^3 \cup S^2_\infty$. As $C(\Lambda_\Gamma)$ is Γ–invariant, $C(\Lambda_\Gamma)$ can be taken quotient by Γ and gives rise to a closed convex set $C(\Lambda_\Gamma)/\Gamma$ in $\mathbf{H}^3/\Gamma$, which is called the convex core of $\mathbf{H}^3/\Gamma$. The convex core is the minimal closed convex set of $\mathbf{H}^3/\Gamma$ which is a deformation retract. A Kleinian group Γ is said to be geometrically finite if it is finitely generated and if the convex core of $\mathbf{H}^3/\Gamma$ has finite volume, otherwise it is geometrically infinite. When Γ is geometrically finite, $QH(\Gamma)$ is an open subset of $AH_p(\Gamma)$.

For a sequence $\{\Gamma_i\}$ of Kleinian groups, its geometric limit is defined as follows.

Definition 2.1 A Kleinian group H is called the geometric limit of $\{\Gamma_i\}$ if every element of H is the limit of a sequence $\{\gamma_i\}$ for $\gamma_i \in \Gamma_i$, and the limit of any convergent sequence $\{\gamma_{i_j} \in \Gamma_{i_j}\}$ for a subsequence $\{\Gamma_{i_j}\} \subset \{\Gamma_i\}$ is contained in H.

The geometric limit of non-elementary Kleinian groups is also a Kleinian group. We call a limit in the deformation space an algebraic limit to distinguish it from a geometric limit. We also call the first factor of a limit in the deformation space, ie the Kleinian group which is the image of the limit representation, an algebraic limit. Suppose that $\{(\Gamma_i, \phi_i)\}$ converges in $AH_p(\Gamma)$ to (Γ', ϕ). Then there is a subsequence of $\{\Gamma_i\}$ converging to a Kleinian group H geometrically. Moreover, the algebraic limit Γ' is contained in the geometric limit H. (Refer to Jørgensen–Marden [13] for the proofs of these facts.) When the algebraic limit Γ' coincides with the geometric limit H, we say that the sequence $\{(\Gamma_i, \phi_i)\}$ converges to (Γ', ϕ) strongly.

When $\{\Gamma_i\}$ converges geometrically to H, there exists a framed (K_i, r_i)–approximate isometry defined below between $\mathbf{H}^3/\Gamma_i$ and $\mathbf{H}^3/H$ with base-frames which are the projections of a base-frame on a point in $\mathbf{H}^3$ where $K_i \to 1$ and $r_i \to \infty$ as $i \to \infty$. (See Canary–Epstein–Green [9]).)

Definition 2.2 Let (M_1, e_1) and (M_2, e_2) be two Riemannian 3–manifolds with base-frame whose base-frames are based at $x_1 \in M_1$, and $x_2 \in M_2$ respectively. A (K, r)–approximate isometry between (M_1, e_1) and (M_2, e_2) is a diffeomorphism from (X_1, x_1) to (X_2, x_2) for subsets X_1, X_2 of M_1, M_2 containing the r–balls centred at x_1, x_2 such that $df(e_1) = e_2$ and

$$d_{M_1}(x,y)/K \le d_{M_2}(f(x), f(y)) \le K d_{M_1}(x,y)$$

for any $x, y \in X_1$.

Let $\{(M_i, v_i)\}$ be a sequence of hyperbolic 3–manifolds with base-frame. We say that (M_i, v_i) converges geometrically (in the sense of Gromov) to a hyperbolic 3–manifold with base-frame (N, w) when for any large r and $K > 1$ there exists an integer i_0 such that there exists a (K, r)–approximate isometry between

(M_i, v_i) and (N, w) for $i \geq i_0$. As described above, by choosing base-frames which are the images of a fixed base-frame in $\mathbf{H}^3$, the sequence of $\mathbf{H}^3/\Gamma_i$ with the base-frame converges geometrically to $\mathbf{H}^3/H$ with the base-frame when Γ_i converges to H geometrically.

Let $M = \mathbf{H}^3/\Gamma$ be a complete hyperbolic 3–manifold. A parabolic element of Γ is contained in a maximal parabolic subgroup, which is isomorphic to either $\mathbf{Z}$ or $\mathbf{Z} \times \mathbf{Z}$ and corresponds to a cusp of M. This is derived from Margulis' lemma. By deleting mutually disjoint neighbourhoods of the cusps of M, we obtain a non-cuspidal part of M, which we shall denote by M_0. We delete the cusp neighbourhoods where the injectivity radius is less than ϵ for some universal constant $\epsilon > 0$ so that this procedure of deleting cusp neighbourhoods is consistent among all the hyperbolic 3–manifolds. The non-cuspidal part M_0 is a 3–manifold whose boundary component is either a torus or an open annulus.

By theorems of Scott [22] and McCullough [15], there exists a submanifold $C(M)$ of M_0 such that $(C(M), C(M) \cap \partial M_0)$ is relatively homotopy equivalent to $(M_0, \partial M_0)$ by the inclusion, which is called a core of M. An end of M_0 is said to be geometrically finite if some neighbourhood of the end contains no closed geodesics, otherwise it is called geometrically infinite. A geometrically infinite end e is called geometrically infinite tame (or simply degenerate) if that end faces an incompressible frontier component S of a core and there exists a sequence of simple closed curves $\{\gamma_i\}$ on S such that the closed geodesic in M homotopic to γ_i tends to the end e as $i \to \infty$. (In this paper we use this term only when every component of the frontier of the core is incompressible.) A Kleinian group Γ is geometrically finite if and only if every end of $(\mathbf{H}^3/\Gamma)_0$ is geometrically finite.

In this paper, we shall consider sequences of Kleinian groups isomorphic to surface groups. Let S be a hyperbolic surface of finite area. We call punctures of S cusps. We denote by $AH_p(S)$ the space of Kleinian groups modulo conjugacy which are isomorphic to $\pi_1(S)$ by isomorphisms mapping elements represented by cusps to parabolic elements. We can also identify this space $AH_p(S)$ with the deformation space of a Fuchsian group G such that $\mathbf{H}^2/G = S$. Let (Γ, ϕ) be a class in $AH_p(S)$. We say that a parabolic element $\gamma \in \Gamma$ is accidental parabolic when $\phi^{-1}(\gamma)$ does not correspond to a cusp of S. Assume that (Γ, ϕ) in $AH_p(S)$ has no accidental parabolic element. Then the non-cuspidal part $(\mathbf{H}^3/\Gamma)_0$ has only two ends since one can see that a core is homeomorphic to $S \times I$ and has exactly two frontier components. Therefore in this case, Γ is either (1) a quasi-Fuchsian group, ie geometrically finite and the limit set Λ_Γ is homeomorphic to the circle or (2) a totally degenerate b–group, ie Ω_Γ is connected and simply connected, and $(\mathbf{H}^3/\Gamma)_0$ has one geometrically finite end and one geometrically infinite end, or (3) a totally doubly degenerate group, ie $\Omega_\Gamma = \emptyset$, and $(\mathbf{H}^3/\Gamma)_0$ has two geometrically infinite tame ends. Recall that a Kleinian group is called a b–group when its region of discontinuity has a unique invariant component, which is simply connected.

For a hyperbolic surface $S = \mathbf{H}^2/\Gamma$, we denote the quasi-conformal deformation space of Γ by $QF(S)$. This space consists of quasi-Fuchsian groups isomorphic to $\pi_1(S)$ by isomorphisms taking elements representing cusps to parabolic elements. By the Ahlfors–Bers theory, there is a homeomorphism $Q\colon QF(S) \to \mathcal{T}(S) \times \mathcal{T}(\overline{S})$, which we shall call the Ahlfors–Bers homeomorphism. Here $\mathcal{T}(\overline{S})$ denotes the Teichmüller space of the "complex conjugate" of S. This can be interpreted as the space of marked hyperbolic structures on S such that the complex conjugate of the corresponding complex structure is equal to the structure on the second component of Ω_Γ/Γ. We identify $\mathcal{T}(\overline{S})$ with $\mathcal{T}(S)$ by the above correspondence from now on. By this correspondence, the Fuchsian representations of $\pi_1(S)$ are mapped onto the diagonal of $\mathcal{T}(S) \times \mathcal{T}(S)$.

Thurston introduced a natural compactification of a Teichmüller space in [27], which is called the Thurston compactification nowadays. Let S be a hyperbolic surface of finite area. Let $\mathcal{S}$ denote the set of free homotopy classes of simple closed curves on S. Let $P\mathbf{R}_+^{\mathcal{S}}$ denote the projective space obtained from the space $\mathbf{R}_+^{\mathcal{S}}$ of non-negative functions on $\mathcal{S}$. We endow $P\mathbf{R}_+^{\mathcal{S}}$ with the quotient topology of the weak topology on $\mathbf{R}_+^{\mathcal{S}} \setminus \{0\}$. The Teichmüller space $\mathcal{T}(S)$ is embedded in $P\mathbf{R}_+^{\mathcal{S}}$ by taking $g \in \mathcal{T}(S)$ to the class represented by a function whose value at $s \in \mathcal{S}$ is the length of the closed geodesic in the homotopy class. The closure of the image of $\mathcal{T}(S)$ in $P\mathbf{R}_+^{\mathcal{S}}$ is homeomorphic to the ball and defined to be the Thurston compactification of $\mathcal{T}(S)$. The boundary of $\mathcal{T}(S)$ corresponds to "the space of projective laminations" in the following way.

A compact subset of S consisting of disjoint simple geodesics is called a geodesic lamination. A geodesic lamination endowed with a transverse measure which is invariant under a homotopy along leaves is called a measured lamination. The subset of a measured lamination λ consisting of the points $x \in \lambda$ such that any arc containing x at the interior has a positive measured with respect to the transverse measure is called the support of λ. We can easily see that the support of a measured lamination λ is a geodesic lamination. The set of measured laminations with the weak topology with respect to measures on finite unions of arcs is called the measured lamination space and denoted by $\mathcal{ML}(S)$. The set of simple closed geodesics with positive weight is dense in $\mathcal{ML}(S)$. For a measured lamination (λ, μ), where μ denotes the transverse measure, and a homotopy class of simple closed curves σ, we define their intersection number $i(\lambda, \sigma)$ to be

$\inf_{s\in\sigma}\mu(s)$. (We also use the notation $i(\lambda, s)$ to denote $i(\lambda,[s])$.) By defining the value at $\sigma \in \mathcal{S}$ to be $i(\lambda,\sigma)$, we can define a map $\iota\colon \mathcal{ML}(S) \to \mathbf{R}_+^{\mathcal{S}}$. By projectivising the both spaces, we have a map $\bar\iota\colon \mathcal{PL}(S) \to P\mathbf{R}_+^{\mathcal{S}}$, where $\mathcal{PL}(S)$ denotes the projectivization of $\mathcal{ML}(S)$, ie $(\mathcal{ML}(S)\setminus\{\emptyset\})/(0,\infty)$. It can be proved that in fact $\bar\iota$ is an embedding and coincides with the boundary of the image of $\mathcal{T}(S)$, that is, the boundary of the Thurston compactification of $\mathcal{T}(S)$. Refer to Fathi et al [11] for further details of these facts.

Let e be a geometrically infinite tame end of the non-cuspidal part of a hyperbolic 3–manifold M, which faces a frontier component Σ of a core. From now on, we always assume that every frontier component of a core is incompressible in M. By the definition of geometrically infinite tame end, there exists a sequence of simple closed curves $\{\gamma_i\}$ on Σ such that the closed geodesic homotopic to γ_i tends to e as $i\to\infty$. Consider the sequence $\{[\gamma_i]\}$ (the projective classes represented by $\{\gamma_i\}$) in $\mathcal{PL}(\Sigma)$. (We identify γ_i with the closed geodesic homotopic to γ_i with respect to some fixed hyperbolic structure on Σ.) Since $\mathcal{PL}(\Sigma)$ is compact, the sequence $\{[\gamma_i]\}$ converges to a projective lamination $[\lambda]\in\mathcal{PL}(\Sigma)$ after taking a subsequence. Such a measured lamination λ is called an ending lamination of e. (The original definition is due to Thurston [26].) An ending lamination is maximal (ie it is not a proper sublamination of another measured lamination), and connected. (Thurston [26], see also Ohshika [17].) If both λ and λ' are ending laminations of an end e, their intersection number $i(\lambda,\lambda')$ is equal to 0 (essentially due to Thurston [26] and Bonahon [7]). We shall give a proof of this fact, based on Bonahon's result in section 3. By the maximality, this implies that $|\lambda| = |\lambda'|$ where $|\lambda|$ denotes the support of λ.

In this paper, we shall deal with a hyperbolic 3–manifold $M = \mathbf{H}^3/\Gamma$ with a homotopy equivalence $\phi\colon S \to M$ preserving cusps. In this case, M has a core which is homeomorphic to $S\times I$. For a homotopy equivalence $\phi\colon S\to M$ and a lamination λ, its image $\phi(\lambda)$ is homotopic to a unique lamination on $S\times\{t\}$ for both $t = 0,1$. When the measured lamination homotopic to $\phi(\lambda)$ is an ending lamination, we say that $\phi(\lambda)$ represents an ending lamination. For an end e of M, the *end invariant* of e is defined to be a projective lamination $[\lambda]$ on S such that $\phi(\lambda)$ represent an ending lamination of e when e is geometrically infinite, and the point in the Teichmüller space corresponding to the conformal structure of the component of Ω_Γ/Γ when e is geometrically finite.

Now, let e_1, e_2 be the two ends of M_0 which are contained in the "upper complement" and the "lower complement" of a core respectively with respect to the orientation give on M and S. We define the *end invariant* of $M = \mathbf{H}^3/\Gamma$ to be a pair (χ, υ), where χ is the end invariant of e_1 and υ that of e_2. This means in particular that when Γ is a quasi–Fuchsian group, the end invariant is equal to $Q(\Gamma,\phi) \in \mathcal{T}(S) \times \mathcal{T}(S)$, where $Q\colon QF(S) \to \mathcal{T}(S) \times \mathcal{T}(S)$ is the Ahlfors–Bers map with the second factor $\mathcal{T}(\overline{S})$ identified with $\mathcal{T}(S)$.

Let S be a hyperbolic surface of finite area and M a complete hyperbolic 3–manifold. A pleated surface $f\colon S \to M$ is a continuous map which is totally geodesic in $S - \ell$ for some geodesic lamination ℓ on S such that the path metric induced by f coincides with the hyperbolic metric on S. We say that a sequence of pleated surfaces with base point $\{f_i\colon (S_i, x_i) \to (M_i, y_i)\}$ converges geometrically to a pleated surface with base point $f\colon (S,x) \to (M,y)$ when there are (K_i, r_i)–approximate isometries ρ_i between (M_i, v_i) and (M, v), and $\overline{\rho}_i$ between (S_i, w_i) and (S, w) such that $K_i \to 1$ and $r_i \to \infty$ as $i \to \infty$ and $\{\rho_i \circ f_i \circ \overline{\rho}_i^{-1}\}$ converges to f uniformly on every compact subset of S, where v_i, v, w_i, w are base-frames on x_i, x, y_i, y respectively. The space of pleated surfaces has the following compactness property due to Thurston whose proof can be found in Canary–Epstein–Green [9].

Proposition 2.3 *For any sequence of pleated surfaces with base point $\{f_i\colon (S_i, x_i) \to (M_i, y_i)\}$ such that the injectivity radius at y_i is bounded away from 0 as $i \to \infty$, there exists a subsequence which converges geometrically.*

We say that a (measured or unmeasured) geodesic lamination λ on S is realized by a pleated surface f when λ is mapped totally geodesically by f. A measured lamination λ lying on a component of the frontier of a core of M represents an ending lamination of an end of M_0 if and only if there is no pleated surface (homotopic to the inclusion) realizing λ. (This follows from Proposition 5.1 in Bonahon [7] which we shall cite below as Proposition 2.5.)

We shall use the following two results of Bonahon [7] several times in this paper. The first is Proposition 3.4 in his paper.

Lemma 2.4 (Bonahon) *Let M be a complete hyperbolic 3–manifold. Let S be a properly embedded incompressible surface in the non-cuspidal part M_0. Then there exists a constant C with the following property. Let α^*, β^* be closed geodesics in M which are homotopic to closed curves α, β on S by homotopies coming to the same side of S, and are located at distance at least D from S. Suppose that neither α^* nor β^* intersects a Margulis tube whose axis is not itself, α^* or β^*. Then we have*

$$i(\alpha,\beta) \le Ce^{-D}\text{length}(\alpha)\text{length}(\beta) + 2.$$

The second is Proposition 5.1 in Bonahon's paper. Before stating the proposition, we need to define some terms used there. A train track on a surface S is a graph with C^1–structure such that all edges coming to a vertex are tangent mutually there. Furthermore we impose the condition that there is no component of the complement which is the interior of a monogon or a bigon or an annulus without angle. We call edges of a train track branches and vertices switches. A regular neighbourhood of a train track τ can be foliated by arcs transverse to τ. Such a neighbourhood is called a tied neighbourhood of τ, and the arcs are called ties. We say that a geodesic lamination λ is carried by a train track τ when a tied neighbourhood of τ can be isotoped to contain λ so that each leaf of λ should be transverse to the ties.

When λ is a measured lamination and carried by a train track τ, the transverse measure induces a weight system on the branches of τ, by defining the weight of a branch to be the measure of ties intersecting the branch. We can easily prove that such a weight system is uniquely determined by λ and τ. Conversely a weight system w on a train track τ satisfying the switch condition that the sum of weights on incoming branches and the sum of those on outgoing branches coincides at each switch, determines a unique measured lamination such that the weight system which it induces on τ is equal to w. Refer to Penner–Harer [23] for more precise definitions and explanations for these facts.

A continuous map f from a surface S to a hyperbolic manifold M is said to be adapted to a tied neighbourhood N_τ of a train track τ on S when each branch of τ is mapped to a geodesic arc in M and each tie of N_τ is mapped to a point. Consider a map f adapted to a tied neighbourhood of a train track τ. For a weight system w on τ, we define the length of $f(\tau, w)$ to be $\sum w_b \mathrm{length}(f(b))$, where the sum is taken over all the branches of τ, and w_b denotes the weight on b assigned by w. For a measured lamination λ carried by τ, if it induces a weight system w on τ, we define the length of $f(\lambda)$ to be the length of $f(\tau, w)$.

For two branches b, b' meeting at a switch σ from opposite directions, the exterior angle $\theta(f(b, b'))$ between b, b' with respect to f is the exterior angle formed by $f(b)$ and $f(b')$ at $f(\sigma)$. The weight system w determines the weight flowing from b to b'. Let $b_1, \ldots, b_p$ and $b'_1, \ldots, b'_q$ be the branches meeting at a switch σ with $b_1, \ldots, b_p$ coming from one direction and $b'_1, \ldots, b'_q$ from the other. The exterior angle at $f(\sigma)$ is the sum of $w_{k,l}\theta(f(b_k, b'_l))$ for all $k = 1, \ldots, p, l = 1, \ldots, q$, where $w_{k,l}$ denotes the weight flowing from b_k to b'_l. The quadratic variation of angle at $f(\sigma)$ is the sum of $w_{k,l}\theta^2 f(b_k, b'_l)$ in the same situation as above. The total curvature of $f(\tau, w)$ is defined to be the sum of the exterior angles at all the images of switches on τ. Similarly, the quadratic variation of angle for $f(\tau, w)$ is defined to be the sum of the quadratic variations of angle at all switches.

Proposition 2.5 (Bonahon) *Let M be a complete hyperbolic 3–manifold and S a hyperbolic surface of finite type. Let $\phi\colon S \to M$ be a continuous incompressible map taking cusps to cusps, and λ a measured lamination on S. Then the one of the following two cases occurs and they are mutually exclusive.*

(1) *For any $\epsilon > 0$, there is a map ϕ_ϵ homotopic to ϕ, which is adapted to a train track carrying λ such that* $\mathrm{length}(\phi_\epsilon(\lambda)) < \epsilon$.

(2) *For any ϵ, there is a map ϕ_ϵ homotopic to ϕ, which is adapted to a train track τ carrying λ by a weight system ω, with the following property: The total curvature and the quadratic variation of angle for $\phi_\epsilon(\tau, w)$ are less than ϵ. Furthermore such a map ϕ_ϵ satisfies the following: There are $\delta > 0, t < 1$ such that $\delta \to 0, t \to 1$ as $\epsilon \to 0$, and for any simple closed curve γ such that $[\gamma]$ is sufficiently close to $[\lambda]$ in $\mathcal{PL}(S)$, the closed geodesic γ^* homotopic to $\phi(\gamma)$ in M has a part of length at least $t\mathrm{length}\phi_\epsilon(\gamma)$ which lies within distance δ from $\phi_\epsilon(\gamma)$.*

We can easily see that the first alternative exactly corresponds to the case when λ represents an ending lamination, and that the second alternative holds if and only if there is a pleated surface realizing λ. Taking this into account, the proposition implies in particular the following. First, in the situation as in the proposition, $\phi(\lambda)$ represents an ending lamination of an end of M_0 if and only if it is not realized by a pleated surface homotopic to ϕ since the two alternatives are exclusive.

Secondly, if λ is an ending lamination, then any measured lamination λ' with the same support as λ is also an ending lamination. This is because a train track carrying λ also carries λ' and if the condition (1) holds for λ, it equally holds for the weight system corresponding to λ'.

There is another proposition which we shall make use of essentially in our proof. The proposition is an application of Thurston's covering theorem which originally appeared in [26] (see also [19] for its proof, and Canary [8] for its generalization).

Proposition 2.6 (Thurston) *Let S be a hyperbolic surface of finite area. Let $\{(\Gamma_i, \phi_i)\}$ be a sequence of Kleinian groups in $AH_p(S)$ converging to (G, ψ). Let Γ_∞ be a geometric limit of $\{\Gamma_i\}$ after taking a subsequence, and let $q\colon \mathbf{H}^3/G \to \mathbf{H}^3/\Gamma_\infty$ be the covering map associated with the inclusion $G \subset \Gamma_\infty$. Suppose that $(\mathbf{H}^3/G)_0$ has a geometrically infinite end e. Then there exists a neighbourhood E of e such that $q|E$ is a proper embedding.*

3 The main theorem

Our main theorem on a sufficient condition for Kleinian groups isomorphic to surface groups to diverge in the deformation spaces is the following.

Theorem 3.1 *Let S be a hyperbolic surface of finite area. Let $\{(\Gamma_i, \phi_i)\}$ be a sequence of Kleinian groups in $AH_p(S)$ with isomorphisms $\phi_i\colon \pi_1(S) \to \Gamma_i$ inducing homotopy equivalences $\phi_i\colon S \to \mathbf{H}^3/\Gamma_i$. Let (χ_i, υ_i) be an end invariant of (Γ_i, ϕ_i). Suppose that $\{\chi_i\}$ and $\{\upsilon_i\}$ converge in either the Thurston compactification of the Teichmüller space $\mathcal{T}(S)$ or the projective lamination space $\mathcal{PL}(S)$ to maximal connected projective laminations $[\mu]$ and $[\nu]$ with the same support. Then $\{(\Gamma_i, \phi_i)\}$ does not converge in $AH_p(S)$.*

Let us briefly sketch the outline of the proof of our main theorem. Note that we can assume by taking a subsequence that all the Γ_i are the same type of the three; quasi-Fuchsian groups or totally degenerate b–groups or totally doubly degenerate groups. We consider here only the case when all the Γ_i are quasi-Fuchsian. The proof is by reductio ad absurdum. Suppose that our sequence $\{(\Gamma_i, \phi_i)\}$ converges in $AH_p(S)$. Then we have the algebraic limit (G, ψ) which is a subgroup of a geometric limit Γ_∞. By applying the continuity of the length function on $AH_p(S) \times \mathcal{ML}(S)$, which will be stated and proved in Lemma 4.2, we shall show that $\psi(\mu)$ represents an ending lamination of an end e_μ in $(\mathbf{H}^3/G)_0$. We shall take a neighbourhood E_μ of e_μ which can be projected homeomorphically by the covering map $q\colon \mathbf{H}^3/G \to \mathbf{H}^3/\Gamma_\infty$ to a neighbourhood of an end of $(\mathbf{H}^3/\Gamma_\infty)_0$ using Proposition 2.6. Let S_0 denote the non-cuspidal part of S. We shall then show that deep inside E_μ there is an embedded surface $f'(S_0)$ homotopic to $\psi|S_0$ such that every pleated surface homotopic to $q \circ \psi$ touching $q \circ f'(S_0)$ is contained in $q(E_\mu)$.

By projecting f' to $\mathbf{H}^3/\Gamma_\infty$ and pulling back by an approximate isometry, we get an embedded surface $f_i\colon S_0 \to \mathbf{H}^3/\Gamma_i$ which is homotopic to ϕ_i converging to an embedded surface $f_\infty\colon S_0 \to \mathbf{H}^3/\Gamma_\infty$ geometrically which is the projection of f'. By using a technique of interpolating pleated surfaces due to Thurston, we shall show that there is a pleated surface $k_i\colon S \to \mathbf{H}^3/\Gamma_i$ homotopic to ϕ_i which intersects $f_i(S_0)$ at an essential simple closed curve. These pleated surfaces converge geometrically to a pleated surface $k_\infty\colon S' \to \mathbf{H}^3/\Gamma_\infty$, where S' is an open incompressible surface on S. The condition that the limit surface k_∞ touches $f_\infty(S)$ forces k_∞ to be a pleated surface from S, and to be lifted to a pleated surface to $\mathbf{H}^3/G$ which realizes a measured lamination with the same support as μ. This will contradict the fact that $\psi(\mu)$ represents an ending lamination.

4 Ending laminations and pleated surfaces

In this section, we shall prove lemmata basically due to Thurston which will be used in the proof of our main theorem.

Throughout this section, $\{(\Gamma_i, \phi_i)\}$ denotes a sequence as in Theorem 3.1. Suppose that $\{(\Gamma_i, \phi_i)\}$ converges to (G, ψ) in $AH_p(S)$ where $\psi\colon \pi_1(S) \to G$ is an isomorphism. (Our proof of Theorem 3.1 is by reductio ad absurdum. Therefore we assumed above the contrary of the conclusion of Theorem 3.1.) We also use this symbol ψ to denote the homotopy equivalence from S to $\mathbf{H}^3/G$ corresponding to the isomorphism. We can assume that ϕ_i converges to ψ as representations by taking conjugates if necessary.

Now let $\tilde{z} \in \mathbf{H}^3$ be a point and $\tilde{v}$ be a frame based on $\tilde{z}$. Then $\tilde{z}, \tilde{v}$ are projected by the universal covering maps to z_i, v_i of $\mathbf{H}^3/\Gamma_i$ and z, v of $\mathbf{H}^3/G$. Since we assumed that $\{\Gamma_i\}$ converges algebraically to G, we can assume by passing through a subsequence that $\{\Gamma_i\}$ converges geometrically to a Kleinian group Γ_∞ which contains G as a subgroup. Let v_∞, z_∞ be the images in $\mathbf{H}^3/\Gamma_\infty$ of $\tilde{v}, \tilde{z}$ by the universal covering map.

The hyperbolic manifolds with base frame $\{(\mathbf{H}^3/\Gamma_i, v_i)\}$ converge in the sense of Gromov to $(\mathbf{H}^3/\Gamma_\infty, v_\infty)$. Let $q\colon \mathbf{H}^3/G \to \mathbf{H}^3/\Gamma_\infty$ be the covering associated with the inclusion $G \subset \Gamma_\infty$. Then $q(z) = z_\infty$ and $dq(v) = v_\infty$.

Consider the case when at least one end e of $(\mathbf{H}^3/\Gamma_i)_0$ is geometrically finite. Let Σ_i be the boundary components of the convex core of $\mathbf{H}^3/\Gamma_i$ facing e which corresponds to a component of the quotient of the region of discontinuity $\Omega^0_{\Gamma_i}/\Gamma_i$. Let $h_i\colon S \to \Sigma_i$ be a homeomorphism homotopic to ϕ_i. Now by the assumption of Theorem 3.1, the marked conformal structures of $\Omega^0_{\Gamma_i}/\Gamma_i$ converge to either $[\mu]$ or $[\nu]$, say $[\mu]$. Then we have the following.

Lemma 4.1 *There exist an essential simple closed curve γ_i on Σ_i, and a sequences of positive real numbers $\{r_i\}$ going to 0 such that $r_i \mathrm{length}_{\Sigma_i}(\gamma_i) \to 0$ and $\{r_i(h_i^{-1}(\gamma_i)) \in \mathcal{ML}(S)\}$ converges to a measured lamination with the same support as the measured lamination μ, where we regard $h_i^{-1}(\gamma_i)$ as an element in $\mathcal{ML}(S)$.*

Proof Let m_i be the point in $\mathcal{T}(S)$ determined by the marked conformal structure on $\Omega^0_{\Gamma_i}/\Gamma_i$. By Sullivan's theorem proved in Epstein–Marden [10], the assumption in Theorem 3.1 that $m_i \to [\mu]$ implies that the marked hyperbolic structures g_i on S induced by h_i from those on Σ_i as subsurfaces in $\mathbf{H}^3/\Gamma_i$ also converge to $[\mu]$ as $i \to \infty$ in the Thurston compactification of $\mathcal{T}(S)$.

Let γ_i be the shortest essential closed curves on Σ_i with respect to the hyperbolic metrics induced from $\mathbf{H}^3/\Gamma_i$. Consider the limit $[\mu_0]$ of $\{[h_i^{-1}(\gamma_i)]\}$ in $\mathcal{PL}(S)$ passing through a subsequence if necessary. Then there are bounded sequences of positive real numbers r_i such that $r_i h_i^{-1}(\gamma_i) \to \mu_0$ in $\mathcal{ML}(S)$. Suppose that $i(\mu, \mu_0) \neq 0$. Then by the "fundamental lemma" 8-II-1 in Fathi–Laudenbach–Poenaru, we should have $\text{length}(r_i h_i^{-1}(\gamma_i)) \to \infty$. On the other hand, since γ_i is the shortest essential closed curve with respect to g_i, we see that $\text{length}_{g_i}(h^{-1}(\gamma_i)) = \text{length}_{\Sigma_i}(\gamma_i)$ is bounded. This implies that $r_i \text{length}(h_i^{-1}(\gamma_i))$ is also bounded as $i \to \infty$, which is a contradiction. Thus we have proved that $i(\mu, \mu_0) = 0$.

As μ is assumed to be maximal and connected, this means that $|\mu| = |\mu_0|$. In particular μ_0 is not a simple closed curve, and we can see the sequences $\{r_i\}$ must go to 0 as $i \to \infty$. □

The next lemma, which asserts the continuity of the lengths of realized measured laminations, appeared in Thurston [28]. The following proof is based on Proposition 2.5 due to Bonahon. Soma previously suggested a possibility of such a proof.

Lemma 4.2 *Let $L\colon AH_p(S) \times \mathcal{ML}(S) \to \mathbf{R}$ be the function such that $L((\Gamma, \phi), \lambda)$ is the length of the realization of λ on a pleated surface homotopic to ϕ when such a pleated surface exists, otherwise set $L((\Gamma, \phi), \lambda) = 0$. Then L is continuous.*

Proof Let $\{(G_i, \psi_i)\} \in AH_p(S)$ be a sequence which converges to $(G', \psi') \in AH_p(S)$, and let $\{\lambda_j\}$ be measured laminations on S converging to λ'. We shall prove that L is continuous at $((G', \psi'), \lambda')$. We can take representatives for elements of the sequence so that the representations $\{\psi_i\}$ converge to ψ'. Fix a base frame $\tilde{v}$ on $\mathbf{H}^3$ and let w_i be the base frame of $\mathbf{H}^3/G_i$ which is the projection of $\tilde{v}$ by the universal covering map. Since G_i converges algebraically, the injectivity radius at the basepoint under w_i is bounded away from 0 as $i \to \infty$. By compactness of geometric topology (see Corollary 3.1.7 in Canary–Epstein–Green [9]) and the diagonal argument, we can see that for any large $r > 0$ and small $\epsilon > 0$, there exists i_0 such that for any $i > i_0$, there exists a Kleinian group H' containing G' and a $((1+\epsilon), r)$–approximate isometry $\rho_i\colon B_r(\mathbf{H}^3/G_i, w_i) \to B_r(\mathbf{H}^3/H', w')$, where B_r denotes an r–ball. (Note that the group H' may depend on i since a geometric limit exists only after taking a subsequence.)

First suppose that λ' can be realized by a pleated surface homotopic to ψ'. Then by Proposition 2.5, for any small $\delta > 0$, there exists a train track τ with a weight system ω carrying λ' and a continuous map $f\colon S \to \mathbf{H}^3/G'$ homotopic to ψ' which is adapted to a tied neighbourhood N_τ of τ such that the total curvature and the quadratic variation of angle for $f(\tau, \omega)$ are less than δ.

For a Kleinian group H' containing G', by composing the covering $q\colon \mathbf{H}^3/G' \to \mathbf{H}^3/H'$ to f, we get a map with the same property homotopic to $q \circ \psi'$. We take r and ϵ so that for any geometric limit H', the r-ball centred at the base point under w' contains the image of $q \circ f$ and so that if we pull back $q \circ f$ by a $((1+\epsilon), r)$-approximate isometry and straighten the images of branches to geodesic arcs, the image of (τ, ω) has the total curvature and the quadratic variation of angle less than 2δ. Then for $i > i_0$, there exists a map $f_i\colon S \to \mathbf{H}^3/G_i$ homotopic to ϕ_i which is adapted to τ such that $f_i(\tau, \omega)$ has total curvature and quadratic variation of angle less than 2δ. Again by Proposition 2.5, this implies that there is a neighbourhood U of λ' in $\mathcal{ML}(S)$ such that for any weighted simple closed curve γ in U, there exist $\nu_U > 0$ depending on U, $\eta_\delta > 0$, and $t_\delta < 1$ depending on δ such that $\nu_U \to 0$ as U gets smaller and $\eta_\delta \to 0, t_\delta \to 1$ as $\delta \to 0$, and the following holds. We can homotope γ so that $N_\tau \cap \gamma$ corresponds the weight system ω' (which may not satisfy the switch condition since γ may not be homotoped into N_τ) whose value at each branch differs from that of ω at most ν_U, and the closed geodesic γ_i^* homotopic to $\psi_i(\gamma)$ has a part with length $t_\delta \mathrm{length}(\gamma_i^*)$ which lies within distance η_δ from $f^i(\tau \cap \gamma)$. The same holds for f and the closed geodesic γ^* homotopic to $\psi(\gamma)$.

It follows that there is a positive real number ζ depending on ϵ, δ, U which goes to 0 as $\epsilon \to 0, \delta \to 0$ and U gets smaller remaining to be a neighbourhood of λ', such that if γ, γ' are weighted simple closed curves in U, then $|\mathrm{length}(\gamma_i^*) - \mathrm{length}(\gamma'^*)| < \zeta$, where γ'^* is the closed geodesic in $\mathbf{H}^3/G'$ homotopic to $\psi(\gamma')$. Since the set of weighted simple closed curves is dense in $\mathcal{ML}(S)$ and any realization of measured lamination can be approximated by realizations of simple closed curves, this implies our lemma in the case when λ' is realizable by a pleated surface homotopic to ψ'.

Next suppose that λ' is not realizable by a pleated surface homotopic to ψ'. This means that λ' is an ending lamination of an end of $(\mathbf{H}^3/G')_0$. By a result of Thurston in [26] (see also Lemma 4.4 in [17]), it follows that λ' is maximal and connected. In this case the alternative (i) of Proposition 2.5 holds. Hence for any small $\epsilon > 0$, there exists a train track τ carrying λ' with weight ω and a continuous map f homotopic to ψ' which is adapted to a tied neighbourhood N_τ of τ, such that λ' can be homotoped so that the length of $f(\tau,\omega)$ is less than δ. Then by the same argument as the last paragraph, there exists i_0 such that if $i > i_0$ there exists a map f_i adapted to N_τ such that $f_i(\tau,\omega)$ has length less than 2δ.

Since $\{\lambda_j\}$ converges to λ' and λ' is maximal, λ_j is carried by τ for sufficiently large j with weight ω_j whose values at branches are close to those of ω. Hence there exists j_0 such that $f_i(\tau,\omega_j)$ is less than 3δ if $j > j_0$. As the length of realization of λ_j by a pleated surface homotopic to ψ_i is less than that of $f_i(\tau,\omega_j)$, this implies our lemma in the case when λ' cannot be realized by a pleated surface homotopic to ψ'. □

The following is a well-known result of Thurston appeared in [26] and also a corollary of Lemma 2.4 due to Bonahon. Nevertheless, as its proof is not so straightforward when sequences of closed geodesics intersect Margulis tubes non-trivially, we shall prove here that Lemma 2.4 implies this lemma.

Lemma 4.3 *Let M be a hyperbolic 3–manifold. Let e be a geometrically infinite tame end of the non-cuspidal part M_0. Let λ, λ' be measured laminations on a frontier component T of a core, which faces e. Suppose that both λ and λ' are ending laminations of the end e. Then the supports of λ and λ' coincide.*

Proof Let s_j and s'_j be simple closed curves on T such that for some positive real numbers x_j and y_j, we have $x_j s_j \to \lambda$, $y_j s'_j \to \mu$ and such that the closed geodesics s_j^* homotopic to s_j and ${s'_j}^*$ homotopic to s'_j tend to the end e as $j \to \infty$. If there exists a constant $\epsilon_0 > 0$ such that neither s_j^* nor ${s'_j}^*$ intersects an ϵ_0–Margulis tube whose axis is not s_j^* or ${s'_j}^*$ itself, then we can apply Lemma 2.4 and the proof is completed.

Next suppose that for at least one of s_j^* and ${s'_j}^*$ (say s_j^*), a constant as ϵ_0 above does not exist. We shall prove that we can replace s_j with another simple closed curve to which we can apply Lemma 2.4. By assumption, there exist closed geodesics ξ_j whose lengths go to 0 and such that s_j^* intersect the ϵ_j–Margulis tube whose axis is ξ_j, where $\epsilon_j \to 0$. Let $h_j\colon (T,\sigma_j) \to M$ be a pleated surface homotopic to the inclusion whose image contains s_j^* as the image of its pleating locus, where σ_j is the hyperbolic structure on T induced by h_j. Put a base point y_i on T which is mapped into s_j^* but outside the ϵ_0–Margulis tubes by h_j. Let $h_\infty\colon ((T',\sigma_\infty), y_\infty) \to (M', y_\infty)$ be the geometric limit of $\{h_j : (T', y_j) \to (M, h_j(y_j))\}$ after taking a subsequence, where T' is an incompressible subsurface in T. We shall first show that T' cannot be the entire of T.

Suppose that $T' = T$ on the contrary. Let l be the geodesic lamination on (T, σ_∞) which is the geometric limit of the closed geodesic on (T, σ_j) corresponding to s_j^* as $j \to \infty$. Since l cannot approach to a cusp (as $T = T'$), it is compact. Therefore we can take a point in the intersection of s_j^* and the ϵ_j–Margulis tube which converges to a point x on $h_\infty(l)$ associated with the geometric convergence of $\{h_j\}$ to h_∞ as $j \to \infty$. Then for any small ϵ, there is an essential closed curve passing x with length less than ϵ which can be obtained by pushing forward by an approximate isometry an essential loop intersecting s_j^* of length less than ϵ_j for sufficiently large j. This is a contradiction.

Thus there is an extra cusp for h_∞. Let c be a simple closed curve on T' representing an extra cusp. Let $\overline{\rho}_j : B_{r_j}((T, \sigma_j), y_j) \to B_{r_j}((T', \sigma_\infty), y_\infty)$ be an approximate isometry associated with the geometric convergence of $\{h_j\}$ to h_∞. Let c_j be a simple closed curve on T which is homotopic to $\overline{\rho}_j^{-1}(c)$. Let l' be a measured lamination to which $\{r_j c_j\}$ converges for some positive real numbers r_j. Let c_j^+ be the closed geodesic on (T, σ_j) homotopic to c_j. Let α be a measured lamination to whose projective class the hyperbolic structures σ_j converge, after passing through a subsequence if necessary. Then as $\mathrm{length}_{\sigma_j}(c_j^+)$ goes to 0 as $j \to \infty$, we have $i(\alpha, l') = 0$ by Lemma 3.4 in [17]. By the same reason, considering $\{s_j\}$, we have $i(\lambda, \alpha) = 0$. Since λ is maximal and connected, these imply that the supports of λ and l' coincide. In particular, l' is an ending lamination of the end for which λ is an ending lamination.

Because the length of the closed geodesic c_j^+ goes to 0 as $j \to \infty$, the closed geodesic homotopic to $h_j(c_j)$, whose length is at most the length of c_j^+, must be the axis of an ϵ_0–Margulis tube for sufficiently large j. Thus we can replace s_j with c_j, and and by the same fashion, we can replace s'_j with another simple closed curve if necessary. We can apply Lemma 2.4 for such simple closed curves. □

5 Proof of the main theorem

We shall complete the proof of Theorem 3.1 in this section. Recall that under the assumption for the reductio ad absurdum, we have (G, ψ) which is the algebraic limit of $\{(\Gamma_i, \phi_i)\}$.

Lemma 5.1 *In the situation of Theorem 3.1, the non-cuspidal part* $(\mathbf{H}^3/G)_0$ *of the hyperbolic 3–manifold* $\mathbf{H}^3/G$ *has a geometrically infinite tame end for which* $\psi(\mu)$ *represents an ending lamination.*

Proof Suppose first that the end e^i of $(\mathbf{H}^3/\Gamma_i)_0$ corresponding to the first factor of the end invariant is geometrically finite. Then by Lemma 4.1, there exists a sequence of weighted simple closed curves $r_i\gamma_i$ on S converging to μ such that for the closed geodesic γ_i^* in $\mathbf{H}^3/\Gamma_i$ homotopic to $\phi_i(\gamma_i)$, we have $r_i\mathrm{length}(\gamma_i^*) \to 0$. By the continuity of length function L on $AH_p(S)\times\mathcal{ML}(S)$ (Lemma 4.2), we have $L((G,\psi),\mu)$ is 0, which means that μ cannot be realized by a pleated surface homotopic to ψ. As we assumed that μ is maximal and connected, there must be a geometrically infinite tame end of $(\mathbf{H}^3/G)_0$ with ending lamination represented by $\psi(\mu)$. This last fact, originally due to Thurston, can be proved using Bonahon's result: by Proposition 2.5, if $L((G,\psi),\mu)=0$ and μ is maximal and connected, then for any sequence of simple closed curves δ_j on S whose projective classes converge to that of μ, the closed geodesics δ_j^* homotopic to $\psi(\delta_j)$ tend to an end of $(\mathbf{H}^3/G)_0$. This means that $\psi(\mu)$ is an ending lamination for a geometrically tame end of $(\mathbf{H}^3/G)_0$.

Next suppose that the end e^i is geometrically infinite. Then χ_i is represented by a measured lamination μ_i which represents an ending lamination of e^i, hence $L((\Gamma_i,\phi_i),\mu_i)=0$. We can assume that μ_i lies on the unit ball of $\mathcal{ML}(S)$ with respect to the metric induced from some fixed hyperbolic structure on S. Then μ_i converges to a scalar multiple of μ since we assumed that $\chi_i=[\mu_i]$ converges to $[\mu]$. By the continuity of L, this implies that $L((G,\psi),\mu)=0$ and that $\psi(\mu)$ represents an ending lamination for $(\mathbf{H}^3/G)_0$. □

We shall denote the end in Lemma 5.1, for which $\psi(\mu)$ represents an ending lamination, by e_μ.

Recall that $q\colon \mathbf{H}^3/G \to \mathbf{H}^3/\Gamma_\infty$ is a covering associated with the inclusion. Now by Proposition 2.6, the end e_μ has a neighbourhood E_μ such that $q|E_\mu$ is a proper embedding. Since μ is maximal and connected, the end e_μ has a neighbourhood homeomorphic to $S_0\times\mathbf{R}$, where S_0 is the non-cuspidal part of S Hence by refining E_μ, we can assume that E_μ is also homeomorphic to $S_0\times\mathbf{R}$.

Lemma 5.2 *We can take an embedding* $f'\colon S_0 \to E_\mu$ *homotopic to* $\psi|S_0$ *whose image is contained in the convex core such that for any pleated surface* $g\colon S\to\mathbf{H}^3/\Gamma_\infty$ *homotopic to* $q\circ\psi$ *with non-empty intersection with* $qf'(S_0)$, *we have* $g(S)\cap(\mathbf{H}^3/\Gamma_\infty)_0\subset q(E_\mu)$.

Proof Fix a constant $\epsilon_0 > 0$ less than the Margulis constant. There exists a constant K such that for any hyperbolic metric on S, the diameter of S modulo the ϵ_0–thin part is bounded above by K. (This can be easily seen by considering the moduli of S.)

Note that since the end e_μ is geometrically infinite, it has a neighbourhood contained in the convex core. Take $t \in \mathbf{R}$ large enough so that $S_0 \times \{t\} \subset E_\mu$ is contained in the convex core and the distance from $S_0 \times \{t\}$ to the frontier of E_μ in $(\mathbf{H}^3/G)_0$ modulo the ϵ_0–thin part is greater than $2K$. Choose f' homotopic $\psi|S_0$ so that its image is $S_0 \times \{t\}$. Then the distance between $qf'(S_0)$ and the frontier of $q(E_\mu)$ modulo the ϵ_0–thin part is also greater than $2K$. Suppose that a pleated surface $g\colon S \to \mathbf{H}^3/\Gamma_\infty$ touches $qf'(S_0)$. Then $g(S)$ cannot meet the frontier of $q(E_\mu)$ since the ϵ_0–thin part of S' with respect to the hyperbolic structure induced by g is mapped into the ϵ_0–thin part of $\mathbf{H}^3/\Gamma_\infty$, and any path on $g(S)$ has length less than K modulo the ϵ_0–thin part of $\mathbf{H}^3/\Gamma_\infty$. Also it is impossible for $g(S)$ to go into the cuspidal part of $\mathbf{H}^3/\Gamma_\infty$ and come back to the non-cuspidal part since the intersection of $g(S)$ with the cuspidal part of $\mathbf{H}^3/\Gamma_\infty$ is contained in a neighbourhood of cusps of $g(S)$. This means that without meeting the frontier of $q(E_\mu)$, the pleated surface $g(S)$ cannot go outside $q(E_\mu)$ in $(\mathbf{H}^3/\Gamma_\infty)_0$. Thus the intersection of such a pleated surface with $(\mathbf{H}^3/\Gamma_\infty)_0$ must be contained in $q(E_\mu)$. □

We denote $q \circ f'$ by $f_\infty\colon S_0 \to (\mathbf{H}^3/\Gamma_\infty)_0$. Pulling back this embedding f_∞ by an approximate isometry ρ_i for sufficiently large i, we get an embedding $f_i\colon S_0 \to (\mathbf{H}^3/\Gamma_i)_0$. Since f_∞ comes from the surface homotopic to ψ in the algebraic limit, for sufficiently large i, the surface f_i is homotopic to ϕ_i.

Consider the case when $(\mathbf{H}^3/\Gamma_i)_0$ has a geometrically finite end; that is Γ_i is either quasi-Fuchsian or a totally degenerate b–group. As in the previous section, let Σ_i be a boundary components of the convex core of $\mathbf{H}^3/\Gamma_i$, and let $h_i\colon S \to \Sigma_i$ be a homeomorphism homotopic to ϕ_i. The homeomorphisms h_i can also be regarded as pleated surfaces in $\mathbf{H}^3/\Gamma_i$. Let μ_i be the bending locus of h_i, to which we give transverse measures with full support so that μ_i should converge to measured laminations as $i \to \infty$ after taking subsequences. (Since the unit sphere of the measured lamination space is compact, this is always possible. Also if h_i happens to be totally geodesic, we can set μ_i to be any measured lamination on S.)

Lemma 5.3 *Suppose that Γ_i is either quasi-Fuchsian or a totally degenerate b–group as above. The sequence of the measured laminations $\{\mu_i\}$ converges to a measured lamination with the same support as μ after taking subsequences.*

Proof Let μ' be a limit of $\{\mu_i\}$ after taking a subsequence. If $i(\mu, \mu') = 0$, we have nothing to prove any more because μ is maximal and connected. Now assume that $i(\mu', \mu) \neq 0$. Then, by the fact that the marked hyperbolic structure on Σ_i converges to $[\mu]$ as $i \to \infty$ and Lemma 3.4 in [17], we have $\mathrm{length}_{\Sigma_i}(\mu_i) \to \infty$. On the other hand, by the continuity of the length function L on $AH_p(S) \times \mathcal{ML}(S)$ (Lemma 4.2), we have

$$\mathrm{length}_{\mathbf{H}^3/\Gamma_i}(\phi_i(\mu_i)) = L((\Gamma_i, \phi_i), \mu_i) \to L((G, \psi), \mu') = \mathrm{length}_{\mathbf{H}^3/G}(\psi(\mu')) < \infty$$

where $\mathrm{length}_{\mathbf{H}^3/\Gamma_i}(\phi_i(\mu_i))$ denotes the length of the image of μ_i realized by pleated surface homotopic to ϕ_i etc. Since μ_i is mapped by h_i into the bending locus of Σ_i, it is realized by h_i, hence $\mathrm{length}_{\Sigma_i}(\mu_i) = \mathrm{length}_{\mathbf{H}^3/\Gamma_i}(\phi_i(\mu_i))$. This is a contradiction. □

Now we assume that Γ_i is quasi-Fuchsian. Then there are two boundary components Σ_i, Σ_i' of the convex core of $\mathbf{H}^3/\Gamma_i$, and homeomorphisms $h_i\colon S \to \Sigma_i \subset \mathbf{H}^3/\Gamma_i$ and $h_i'\colon S \to \Sigma_i' \subset \mathbf{H}^3/\Gamma_i$ homotopic to ϕ_i which are regarded as pleated surfaces. We have two measured laminations of unit length μ_i and μ_i' whose supports are the bending loci of h_i and h_i'. By Lemma 5.3, the sequence of the measured laminations $\{\mu_i\}$ converges to a measured lamination μ' and $\{\mu_i'\}$ converges to a measured lamination μ'' such that $|\mu'| = |\mu''| = |\mu|$. As the space of transverse measures on a geodesic lamination is connected (or more strongly, convex with respect to the natural PL structure), we can join μ' and μ'' by an arc $\alpha\colon I = [0,1] \to \mathcal{ML}(S)$ such that $|\alpha(t)| = |\mu|$. Join μ_i and μ_i' by an arc $\alpha_i\colon I \to \mathcal{ML}(S)$ which converges to the arc α joining μ' and μ''.

Next suppose that Γ_i is a totally degenerate b–group. We can assume without loss of generality that the first factor χ_i of the end invariant represents an ending lamination and the second v_i a conformal structure. Then we have a pleated surface $h_i\colon S \to \Sigma_i$ homotopic to ϕ_i whose image is the boundary of the convex core. Let μ_i be a measured lamination of the unit length whose support is equal to that of the bending locus as before. Again by Lemma 5.3, we see that $\{\mu_i\}$ converges to a measured lamination μ' with the same support as μ. Let μ_i' be a measured lamination of the unit length representing the class χ_i. By the assumption of Theorem 3.1, the sequence $\{\mu_i'\}$ converges to a measured lamination μ'' with the same support as μ. As in the case of quasi-Fuchsian group, we join μ' and μ'' by an arc α, and then join μ_i and μ_i' by an arc α_i which does not pass an ending lamination for $\mathbf{H}^3/\Gamma_i$ at the interior so that it will converge to α uniformly.

In the case when Γ_i is a totally doubly degenerate group, both χ_i and v_i are represented by ending laminations. Let μ_i representing χ_i and μ_i' representing

v_i be measured laminations of the unit length. Then by assumption, $\{\mu_i\}$ and $\{\mu_i'\}$ converge to measured laminations μ' and μ'' with the same support as μ. As before we join μ' and μ'' by an arc α, and μ_i, μ_i' by α_i which does not pass an ending lamination of $\mathbf{H}^3/\Gamma_i$ at the interior so that $\{\alpha_i\}$ converges to α.

Next we shall consider constructing for each i a homotopy consisting of pleated surfaces and negatively curved surfaces in $\mathbf{H}^3/\Gamma_i$ as in Thurston [26]. What we shall have is a homotopy between h_i and h_i' in the case when Γ_i is quasi-Fuchsian; a half-open homotopy $\hat{H}_i\colon S \times [0,1) \to \mathbf{H}^3/\Gamma_i$ such that $\hat{H}_i(S \times \{t\})$ tends to the unique geometrically infinite end as $r \to 1$ in the case when Γ_i is a totally degenerate b–group; and an open homotopy $\hat{H}_i\colon S \times (0,1) \to \mathbf{H}^3/\Gamma_i$ such that $\hat{H}_i(S \times \{t\})$ tends to one end as $t \to 0$ and to the other as $t \to 1$ in the case when Γ_i is a totally doubly degenerate group. To construct such a homotopy, we need the notion of rational depth for measured laminations due to Thurston. An alternative approach to construct such a homotopy using singular hyperbolic triangulated surfaces can be found in Canary [8].

A train track τ is called birecurrent when the following two conditions are satisfied. (This definition is due to Penner–Harer [23].) (1) The τ supports a weight system which is positive on each branch b of τ. (2) For each branch b of τ, there exists a multiple curve σ (ie a disjoint union of non-homotopic essential simple closed curves) transverse to τ which intersects b such that $S - \tau - \sigma$ has no bigon component.

A birecurrent train track which is not a proper sub-train track of another birecurrent train track is said to be complete.

Any measured lamination is carried by some complete train track. (Refer to Corollary 1.7.6 in [23].) The weight systems on a complete train track gives rise to a coordinate system in the measured lamination space. (See Lemma 3.1.2 in [23].) The rational depth of a measured lamination is defined to be the dimension of the rational vector space of linear functions with rational coefficients vanishing on the measured lamination with respect to a coordinate system associated with a complete train track carrying the measured lamination. This definition is independent of the choice of a coordinate system since functions corresponding to coordinate changes are linear functions with rational coefficients. The set of measured laminations with rational depth n has codimension n locally at regular points. In particular a generic arc in the measured lamination space does not pass a measured lamination with rational depth more than 1.

Now perturb α and α_i to a piece-wise linear path with respect to the PL structure of $\mathcal{ML}(S)$ determined by complete train tracks fixing the endpoints so that for each $t \in I$, the measured lamination $\alpha_i(t)$ is not an ending lamination and has rational depth 0 or 1, and that for each i there exist only countably many values t for which $\alpha_i(t)$ has rational depth 1.

The following lemma was first proved in section 9 in Thurston [26]. A fairly detailed proof can be found there.

Lemma 5.4 *If a measured lamination has rational depth* 0*, then each component of its complement is either an ideal triangle or a once-punctured monogon except when* S *is a once-punctured torus. In the case when* S *is a once-punctured torus, each component of the complement is an ideal once-punctured bigon. A pleated surface* $f\colon S \to M$ *realizing a measured lamination* ζ *of rational depth* 0 *is unique among the maps in the homotopy class, and every sequence of homotopic pleated surfaces realizing measured laminations converging to* ζ *converges to the pleated surface realizing* ζ*.*

Proof First we shall show that each complementary region of a measured lamination of rational depth 0 is either an ideal triangle or an ideal once-punctured monogon unless S is a once-punctured torus.

Suppose that S is not a once-punctured torus and that a measured lamination ζ has a complementary region which is neither an ideal triangle nor an ideal once-punctured monogon. Then, we can construct a birecurrent train track τ carrying ζ whose complement has a component which is neither a triangle nor a once-punctured monogon. (Refer to section 1.7 in [23].)

A birecurrent train track is maximal if and only if every component of its complement is either a triangle or a once-punctured monogon, and non-maximal birecurrent train track is a sub-train track of a complete train track. (Theorem 1.3.6 in [23].) Hence there exists a complete train track τ' containing τ as a proper sub-train track. Since there is a branch of τ' through which ζ does not pass after homotoping ζ so that it is carried by τ', it follows that with respect to the coordinate system corresponding to τ', the measured lamination ζ has rational depth at least 1.

In the case when S is a once-punctured torus, again Theorem 1.3.6 in [23] says that a birecurrent train track is maximal if and only if its (unique) complementary region is a once-punctured bigon. Thus the same argument as above also implies our claim in the case of once-punctured torus.

Next we shall show the uniqueness of realization of a measured lamination of rational depth 0. Let f, g be two pleated surfaces realizing a measured lamination ζ of depth 0. The pleated surfaces f, g induce hyperbolic metrics m_1, m_2 respectively on S. (These may differ as we do not know if f and g coincide.) The measured lamination ζ is homotopic to measured geodesic laminations ζ_1 with respect to m_1 and ζ_2 with respect to m_2. Consider the universal covers $p_1\colon \mathbf{H}^2 \to (S, m_1)$ and $p_2\colon \mathbf{H}^2 \to (S, m_2)$. Let $\tilde{\zeta}_1$ be $p_1^{-1}(\zeta_1)$ and let $\tilde{\zeta}_2$ be $p_2^{-1}(\zeta_2)$.

The pleated surfaces f, g are lifted to maps $\tilde{f}, \tilde{g}\colon \mathbf{H}^2 \to \mathbf{H}^3$. Since ζ has compact support, there is a homeomorphism from S to S homotopic to the identity which takes ζ_1 to ζ_2 and is equal to the identity near cusps. Also for a homotopy between f and g, the distance moved by the homotopy on the compact set ζ has an upper bound. These imply that for each leaf l of $\tilde{\zeta}$ the images of the corresponding leaves l_1 of $\tilde{\zeta}_1$ by $\tilde{f}$ and l_2 of $\tilde{\zeta}_2$ by $\tilde{g}$ are within a bounded distance. Since both $\tilde{f}(l_1)$ and $\tilde{g}(l_2)$ are geodesics in $\mathbf{H}^3$ and two geodesics lying in bounded distance coincides in $\mathbf{H}^3$, these two images must coincide. Hence we have a map $q\colon \mathbf{H}^2 \to \mathbf{H}^2$ equivariant with respect to the action of $\pi_1(S)$ with the property $\tilde{f}|\tilde{\zeta}_1 = \tilde{g} \circ q|\tilde{\zeta}_1$ which maps $\tilde{\zeta}_1$ to $\tilde{\zeta}_2$ isometrically.

It remains to prove that q extends to an equivariant isometry $\overline{q}$ of $\mathbf{H}^2$ with the property $\tilde{f} = \tilde{g} \circ \overline{q}$. Since ζ has rational depth 0, each of its complementary regions is either an ideal triangle or an ideal once-punctured monogon unless S is a once-punctured torus. An ideal triangle on S is lifted to that on $\mathbf{H}^2$. Since the three sides of the triangle are mapped to geodesics by $\tilde{f}$ or $\tilde{g}$, the triangle must be mapped totally geodesically. Considering that there is only one isometry type of ideal triangles, we can see that this implies q can be extended to ideal triangle complementary components without problem.

For complementary regions which are ideal once-punctured monogon, or ideal once-punctured bigon in the case when S is a once-punctured torus, we have to use the fact that pleated surfaces are totally geodesics near cusps. (This is proved in Proposition 9.5.5 in Thurston [26].) Once we know this, we can subdivide such regions into ideal triangles by adding geodesics tending to cusps on S, which are mapped to geodesics by f or g. Since each cusp of S is mapped to the same cusp of M by f and g, we can arrange them so that the lifts of these added geodesics should be compatible with q. Hence by extending the map finally to ideal triangles, we get a map $\overline{q}$ as we wanted.

Finally let us prove the last sentence of our lemma. Let ξ_j be measured laminations converging to ζ, and f_j a pleated surface realizing ξ_j. Since ζ can be realized by a pleated surface, the alternative (2) of Proposition 2.5 should be valid for ζ. We shall show that if there is no compact set in M which intersects all the images of f_j, then we can see that the alternative (2) of Proposition 2.5 fails to hold for ζ.

Suppose that the alternative (2) of Proposition 2.5 holds for ζ. Then for any $\delta > 0$ and $t < 1$, there exist a map $f_\delta\colon S \to M$ homotopic to f such that for any simple closed curve γ whose projective class is close to that of ζ, the closed geodesic γ^* homotopic to $f_\delta(\gamma)$ has a part of length at least $t\mathrm{length}(\gamma^*)$ which is contained in the δ–neighbourhood of $f_\delta(\zeta)$. Note that as $\delta \to 0$, this map f_δ converges to a pleated surface realizing ζ, which must be equal to f. (Refer to [20] for a further explanation.) On the other hand, since ξ_j is also realized by a pleated surface homotopic to f, the alternative (2) holds also for ξ_j. Then we have a surface f_j^δ with the same property as f_δ above replacing ζ with ζ_j. Since we assumed that f_j tends to an end of M, we can have surfaces $f_j^{\delta_j}$ going to an end and a simple closed curve γ_j whose projective class is close to that of ζ_j such that a large part of the closed geodesic γ_j^* is contained in the δ_j–neighbourhood of $f_j^{\delta_j}(S)$. This is a contradiction because γ_j^* must also have a large part contained in the δ–neighbourhood of $f_\delta(S)$ which remains in a neighbourhood of $f(S)$.

Thus the surfaces $f_j(S)$ remain to intersect a compact set, hence converge to a pleated surface g homotopic to f uniformly on any compact set of S. (Theorem 5.2.18 in Canary–Epstein–Green [9].) The pleated surface g realizes a geodesic lamination ζ_∞ which is a geometric limit of $\{\zeta_j\}$ regarded as geodesic laminations forgetting the transverse measures. It is known that ζ_∞ contains the support of ζ. (See for example Lemma 5.3.2. in [9].) Thus g also realizes ζ, and by the uniqueness of such pleated surfaces proved above, we see that $f = g$, which means that $\{f_j\}$ converges to f uniformly on any compact set of S. $\square$

Now let $H_i\colon S \times I, S \times [0,1), S \times (0,1) \to \mathbf{H}^3/\Gamma_i$ (depending on the type of Γ_i; a quasi-Fuchsian group or a totally degenerate b–group or a totally doubly degenerate) be a map such that for each $t \in I, [0,1), (0,1)$, the map $H_i(\ ,t)\colon S \to \mathbf{H}^3/\Gamma_i$ is a pleated surface realizing $\alpha_i(t)$. Then H_i is continuous with respect to t by Lemma 5.4 except at values t where $\alpha(t)$ has rational depth 1, which are countably many. Since we made α_j piece-wise linear we can see that the right and left limits exists even at t where $\alpha_i(t)$ has depth 1. (This can be seen by considering a complete train track giving a coordinate near $\alpha_i(t)$.) As was shown in Thurston [26], (see section 4 in [20] for an explanation), at such a point of discontinuity t, the left limit and the right limit differ only in a complementary region R of $\alpha_i(t)$ which is either an ideal quadrilateral or an ideal once-punctured bigon except when S is a once-punctured torus. Then we can modify H_i to a continuous homotopy $\hat{H}_i$ by interpolating negatively curved surfaces realizing $\alpha_i(t_0)$ between $\lim_{t\to t_0-0} H_i(\ ,t)$ and $\lim_{t\to t_0+0} H_i(\ ,t)$ at each t_0 where $\alpha_i(t_0)$ has rational depth 1 as in Thurston [26]. These negatively curved surface coincide with the left and the right limit outside R where the right and left limit differ.

We need to prove that a family of surfaces thus obtained is continuous with respect to the parameter. The only case that we have to take care of is when the values t_k for which $\alpha_i(t_k)$ has depth 1 accumulates to a point $t_0 \in I$. The negatively curved surfaces interpolated at t_k have the same image as a pleated surface realizing $\alpha(t_k)$ outside a complementary region R_k. The image of R_k by the left limit pleated surface and the right limit surface bound an ideal tetrahedron if R_k is an ideal quadrilateral or a solid torus with cusps if R_k is an ideal once-punctured bigon in the case when S is not a once-punctured torus. The form of R_k gets thinner and thinner as $k \to \infty$ since t_k accumulates. (This can again be seen by considering a coordinate chart given by a complete train track.) This implies that the trajectories of the homotopy between the left limit and the right limit, which are contained in the ideal tetrahedron or the solid torus have length going to 0 as $k \to \infty$. Even in the case when S is a once-punctured torus, a similar argument can work although we need to take more cases into account. Thus we can see that $\hat{H}(\ ,t)$ is continuous with respect t even at the point t_0 to which depth-1 points t_k accumulate.

Lemma 5.5 *For each i, there is a pleated surface $k_i\colon S \to \mathbf{H}^3/G$ homotopic to ϕ_i touching $f_i(S_0)$ which realizes a measured lamination $\overline{\mu}_i$ such that $\{\overline{\mu}_i\}$ converges to a measured lamination $\overline{\mu}$ with the same support as μ after taking a subsequence. Moreover we can choose k_i so that $k_i^{-1}(f_i(S_0))$ contains an essential component relative to cusps.*

Remark 1 Although the last sentence of this lemma is not necessary for our purpose now, it will be used for our forthcoming work in [21]. Also Canary's result on filling a convex core by pleated surfaces in [8] will suffice to prove only the former part of our lemma.

Proof Recall that $f'(S_0)$ is contained in the convex core of $\mathbf{H}^3/G$. Then we can assume that $f_i(S_0)$ is also contained in the convex core of $\mathbf{H}^3/\Gamma_i$. It follows that $f_i(S_0)$ is contained in the image of $\hat{H}_i$.

By perturbing $f_i(S_0)$ if necessary, we can assume that $\hat{H}_i$ is transverse to $f_i(S_0)$ and that $\hat{H}_i^{-1}(f_i(S_0))$ is an embedded surface in $S \times (0,1)$. Let F

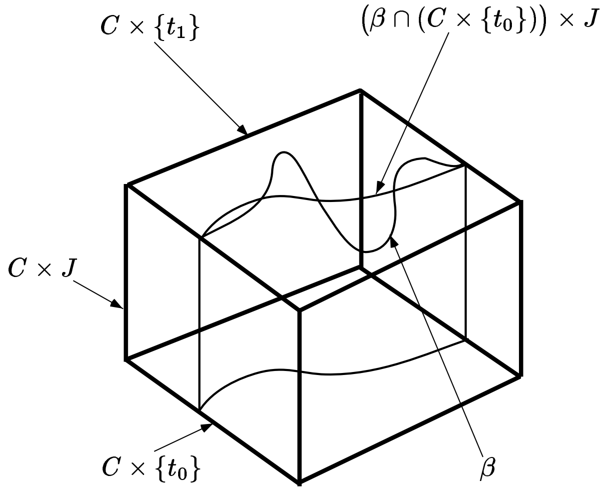
be a component of $\hat{H}_i^{-1}(f_i(S_0))$ which separates $S \times \{0\}$ from $S \times \{1\}$. It is easy to see such a component exists because in the case when Γ_i is quasi-Fuchsian, Σ_i and Σ_i' lie in different components of $\mathbf{H}^3/\Gamma_i - f_i(S_0)$, in the case when Γ_i is a b–group, $f_i(S_0)$ separates a geometrically infinite end from Σ_i, and in the case when Γ_i is doubly degenerate, $f_i(S_0)$ separates two ends. Then $\pi_1(F)$ is mapped onto $\pi_1(S)$ by the homomorphism induced by inclusion, hence $(\hat{H}_i|F)_\#\colon \pi_1(F) \to \pi_1(f_i(S_0))$ is surjective.

We can assume that for each $t \in I$, the intersection $(S \times \{t\}) \cap F$ is at most one dimensional by perturbing $f_i(S_0)$ again if necessary. Then there exists $t_0 \in I$ such that $(S \times \{t_0\}) \cap F$ contains a simple closed curve K which represents a non-trivial element of $\pi_1(S)$ relatively to the punctures of S. If $\hat{H}_i(\ ,t_0)$ is a pleated surface, we simply let k_i be $\hat{H}_i(\ ,t_0)$. In this case, $k_i^{-1}(f_i(S_0))$ contains K, which is essential relatively to the cusps. The pleated surface k_i realizes a measured lamination $\alpha_i^{t_0}$ in the image of α_i, which we let be $\overline{\mu}_i$. The measured lamination $\overline{\mu}_i = \alpha_i^{t_0}$ converges after taking a subsequence to a measured lamination in $\alpha(I)$ hence with the same support as μ.

Suppose that $\hat{H}_i(\ ,t_0)$ is an interpolated negatively curved surface. Let $\alpha_i^{t_0}$ be the measured lamination of rational depth 1 realized by $\hat{H}_i(\ ,t_0)$. We can assume that $\hat{H}_i(\ ,t_0)(\alpha_i^{t_0})$ is transverse to $f_i(S_0)$ again by a perturbation of $f_i(S_0)$ without changing the homotopy class of K. Let $J = [t_0,t_1] \subset I$ be an interval such that $\hat{H}_i(\ ,[t_0,t_1))$ are interpolated negatively curved surfaces and $\hat{H}_i(\ ,t_1)$ is a pleated surface realizing $\alpha_i^{t_0}$. Let C be a component of the complement of $\alpha_i^{t_0}$ which is not an ideal triangle. Since $\alpha_i^{t_0}$ has rational depth 1, such a component is unique and every simple closed curve in C is either represents a cusp or homotopic to $\mathrm{Fr}C$.

On the other hand, by the construction of interpolated surfaces, $\hat{H}_i|\mathrm{FrC} \times J$ is constant with respect to $t \in J$. If $C \times \{t_0\}$ does not intersect K, the pleated surface $\hat{H}_i(\ ,t_1) \cap F$ contains a simple closed curve homotopic to K, and we can let $\hat{H}_i(\ ,t_1)$ be k_i. Suppose that $C \times \{t_0\}$ intersects K.

First consider the case when S is not a once-punctured torus. Then C is either simply connected or an ideal once-punctured monogon. Consider a component β of $(C \times J) \cap F$ intersecting K. Since K does not represent a cusp, each component of $\beta \cap (C \times \{t_0\}) \cap K$ must be an open arc. Since $\hat{H}_i|\mathrm{Fr}C \times J$ is constant with respect to $t \in J$, the component β must be isotopic to $\{\beta \cap (C \times \{t_0\})\} \times J$ fixing $\overline{\beta} \cap (\mathrm{Fr}C \times J)$. This implies that there exists a component K' of $S \times \{t_1\} \cap F$ such that $\hat{H}_i(K',t_1)$ is homotopic in $\mathbf{H}^3/\Gamma_i$ to $\hat{H}_i(K,t_0)$ on $f_i(S_0)$. Hence by letting $\hat{H}_i(\ ,t_1)$ be k_i, we get a surface as we wanted.



Next suppose that S is a once-punctured torus. The only case to which the argument above cannot be applied is one when C is a once-punctured open annulus and K is contained in $C \times \{t_0\}$. By isotoping $f_i(S_0)$ if necessary we can assume that all the components of $(C \times J) \cap F$ are annuli. Still there is a possibility that the component of $(C \times J) \cap F$ containing K is an annulus which is parallel into $C \times \{t_0\}$, and our argument above would break down. If there is another essential (ie incompressible and not boundary-parallel, where we regard $C \times \partial J$ as the boundary,) component of $(C \times J) \cap F$, then we can retake K so that K lies on its boundary and our argument above can be applied. Suppose that all the components are inessential. Then consider another interval $J' = [t_2, t_0] \subset I$, such that $\hat{H}_i(\ ,t)$ is an interpolated surface if $t \in (t_2, t_0]$ and $\hat{H}_i(\ ,t_2)$ is a pleated surface realizing $\alpha_i^{t_0}$. Again we can assume that all the components of $(C \times J') \cap F$ are annuli. Then some component of $(C \times J') \cap F$ is essential because otherwise F cannot be a surface separating $S \times \{0\}$ from $S \times \{1\}$. Hence by the argument as before, retaking K, we can assume that the component of $(S \times J') \cap F$ containing K intersects $S \times \{t_2\}$ by a simple closed curves homotopic to K.

Thus in either case, we can get a pleated surface k_i realizing $\alpha_i^{t_0}$, which is either $\hat{H}_i(\ ,t_1)$ or $\hat{H}_i(\ ,t_2)$, and which intersects $f_i(S_0)$ so that the inverse image of $f_i(S_0)$ has a non-contractible component that is not homotopic to a cusp. $\square$

Proof of Theorem 3.1 Consider a geometric limit $k_\infty\colon S' \to \mathbf{H}^3/\Gamma_\infty$ of the sequence of pleated surfaces $k_i\colon S \to \mathbf{H}^3/\Gamma_i$ constructed above. (Here S' is an open incompressible surface on S.) By construction, $k_\infty(S)$ intersects $f_\infty(S_0)$.

Suppose that S' is not equal to S. Then there is a frontier component c of S' on S which does not represent a cusp of S. Note that we can apply the same argument as Lemma 5.2, and prove that $k_\infty(S)$ does not meet the frontier of $q(E_\mu)$. Now since $k_\infty(c)$ is homotopic to a cusp component of $\mathbf{H}^3/\Gamma_\infty$ which can be reached from $q(E_\mu)$, it is homotopic to the image of a cusp of S by f_∞. By pulling back a homotopy by an approximate isometry, this implies that $k_i(c)$ is homotopic to the image of a cusp by f_i. Since both k_i and f_i are homotopic to ϕ_i, this means that c is homotopic to a cusp of S. This is a contradiction.

Thus S' must be equal to S, and we have a limit pleated surface $k_\infty\colon S \to \mathbf{H}^3/\Gamma_\infty$ touching $f_\infty(S_0)$. By Lemma 5.2, we see that $k_\infty(S) \cap (\mathbf{H}^3/\Gamma_\infty)_0$ is contained in $q(E_\mu)$. Therefore k_∞ can be lifted to a pleated surface $k'\colon S \to \mathbf{H}^3/G$ whose intersection with $(\mathbf{H}^3/G)_0$ is contained in E_μ.

Now since k_i is homotopic to the pull-back of $k_\infty = q \circ k'$ by an approximate isometry for sufficiently large i, and k_i is homotopic to ϕ_i, we see that k' must be homotopic to ψ. As k_i realizes $\overline{\mu}_i$ and $\{\overline{\mu}_i\}$ converges to $\overline{\mu}$, the pleated surfaces k_∞ and k' realize $\overline{\mu}$. As $|\overline{\mu}| = |\mu|$, by changing the transverse measure, μ can also be realized by k'. On the other hand, by Lemma 5.1, $\psi(\mu)$ is an ending lamination hence μ cannot be realized by a pleated surface homotopic to ψ. This is a contradiction. Thus we have completed the proof of Theorem 3.1. □

6 Strong convergence of surface groups

In Theorems 9.2, 9.6.1 in Thurston [26], it is stated and roughly proved that if a sequence of Kleinian groups, which are isomorphic to a freely indecomposable Kleinian group (ie satisfying the condition $(*)$ introduced by Bonahon) without accidental parabolics preserving the parabolicity, converges algebraically to a Kleinian group without accidental parabolic elements, then the convergence is strong (ie the geometric limit coincides with the algebraic limit.) (See also Canary [8].) We gave its detailed proof in Ohshika [19] except for the case when the Kleinian group is algebraically isomorphic to a surface group. The reason why we did not include the case of surface group there is that it would necessitate to prove that for a convergent sequence, the hyperbolic structures on the two boundary components cannot degenerate to the same point in the Thurston boundary. As this is proved in Theorem 3.1, we can give the proof for the case of surface group here. Let (Γ_i, ϕ_i) be a Kleinian group without accidental parabolic elements with isomorphism $\phi_i\colon \pi_1(S) \to \Gamma_i \subset PSL_2\mathbf{C}$ for a hyperbolic surface of finite area S. Thurston's original proof in [26] in this case consists of proving that the projectivized bending laminations of two boundary components of the convex cores of $\mathbf{H}^3/\Gamma_i$ cannot converge to projective lamination with the same support. This is exactly the argument on which our proofs of the main theorems are based.

Corollary 6.1 *Let S be a hyperbolic surface of finite area. Let (Γ_i, ϕ_i) be a Kleinian group without accidental parabolic elements with isomorphism $\phi_i\colon \pi_1(S) \to \Gamma_i \subset PSL_2\mathbf{C}$. Suppose that $\{(\Gamma_i, \phi_i)\}$ converges algebraically to a Kleinian group (G, ψ) without accidental parabolic elements. Then G is also the geometric limit of $\{\Gamma_i\}$. In other words, $\{\Gamma_i\}$ converges strongly to G.*

Proof We have only to prove that every subsequence of $\{(\Gamma_i, \phi_i)\}$ has a subsequence which converges strongly to (G, ψ). Since a subsequence of $\{(\Gamma_i, \phi_i)\}$ satisfies the condition of Corollary 6.1, we only need to show that $\{(\Gamma_i, \phi_i)\}$ in the statement of the corollary has a subsequence strongly converging to (G, ψ).

By taking a subsequence, we can assume that all of the $\{(\Gamma_i, \phi_i)\}$ are either quasi-Fuchsian or totally degenerate groups or totally doubly degenerate groups, and that $\{\Gamma_i\}$ converges geometrically to a Kleinian group Γ_∞. Suppose first that all of the $\{\Gamma_i\}$ are quasi-Fuchsian. Let $(m_i, n_i) \in \mathcal{T}(S) \times \mathcal{T}(S)$ be $Q((\Gamma_i, \phi_i))$. If both $\{m_i\}$ and $\{n_i\}$ converge inside the Teichmüller space (after taking a subsequence), $\{(\Gamma_i, \phi_i)\}$ converges to a quasi-Fuchsian group strongly as is well known. (See for example Jørgensen–Marden [13].) Assume that one of $\{m_i\}$ and $\{n_i\}$, say $\{m_i\}$, does not converge inside the Teichmüller space and converges to a projective lamination $[\lambda]$ in the Thurston compactification of the Teichmüller space, and that the other, say $\{n_i\}$, converges inside the Teichmüller space. Then G is a b–group. By the same argument as the proof of Lemma 5.1, the measured lamination λ cannot be realized by a pleated surface homotopic to ψ. If λ is not maximal and connected, as is shown in Thurston [26] or Lemma 4.4 in [17], G has an accidental parabolic element, which contradicts our assumption. Hence λ is maximal and connected, $\psi(\lambda)$ represents an ending lamination of the geometrically infinite tame end of $(\mathbf{H}^3/G)_0$, and G is a totally degenerate b–group. Let Σ_i be a boundary component of the convex core of $\mathbf{H}^3/\Gamma_i$ corresponding to the ideal boundary component with the structure n_i. Then as is shown in [19], the pleated surface Σ_i converges geometrically to a boundary component Σ_∞ of the convex core of $\mathbf{H}^3/\Gamma_\infty$ which can be lifted to a boundary component Σ of the convex core of $\mathbf{H}^3/G$, which must be the whole boundary of the convex core as G is a totally degenerate b–group. Hence a neighbourhood of the geometrically finite end of $(\mathbf{H}^3/G)_0$ is mapped homeomorphically to that of a geometrically finite end of $(\mathbf{H}^3/\Gamma_\infty)_0$ by the covering projection $q\colon \mathbf{H}^3/G \to \mathbf{H}^3/\Gamma_\infty$. On the other hand, by Proposition 2.6, there is also a neighbourhood of the geometrically tame end of $(\mathbf{H}^3/G)_0$ which is mapped homeomorphically to a neighbourhood of a geometrically infinite tame end of $(\mathbf{H}^3/\Gamma_\infty)_0$ by q. This implies that $G = \Gamma_\infty$.

Next assume that neither $\{m_i\}$ nor $\{n_i\}$ converges inside the Teichmüller space. After taking a subsequence, we can assume that $\{m_i\}$ converges to a projective lamination $[\lambda] \in \mathcal{PL}(S)$ and $\{n_i\}$ converges to a projective lamination $[\mu] \in \mathcal{PL}(S)$ in the Thurston compactification of the Teichmüller space. Since neither λ nor μ can be realized by a pleated surface homotopic to ψ by Lemma 5.1, they must be maximal and connected again by Thurston [26] or Lemma 4.4 in [17]. Then we can apply Theorem 3.1 to our situation and see that the support of λ is different from that of μ. This implies that the end of $(\mathbf{H}^3/G)_0$ with ending lamination represented by $\psi(\mu)$ is different from one with ending lamination represented by $\psi(\lambda)$ by Lemma 4.3, hence G is totally doubly degenerate. Let e_λ and e_μ denote the two distinct ends of $(\mathbf{H}^3/G)_0$ with ending laminations represented by $\psi(\lambda)$ and $\psi(\mu)$ respectively. By Proposition 2.6, there are neighbourhoods E_λ of e_λ and E_μ of e_μ such that $q|E_\lambda$ and $q|E_\mu$ are homeomorphisms to neighbourhoods of ends of $(\mathbf{H}^3/\Gamma_\infty)_0$. As $(\mathbf{H}^3/G)_0$ has only two ends, this can happen only when $\Gamma_\infty = G$ or G is a subgroup of Γ_∞ of index 2. We can see that the latter cannot happen by Lemma 2.3 in [19] (this fact is originally due to Thurston [26]). Thus we have proved our corollary when all of $\{\Gamma_i\}$ are quasi-Fuchsian.

Next assume that all the Γ_i are totally degenerate b–groups. Let m_i be the marked hyperbolic structure on S determined by the conformal structure of $\Omega_{\Gamma_i}/\Gamma_i$, and let λ_i be an ending lamination of unit length of the geometrically infinite tame end of $(\mathbf{H}^3/\Gamma_i)_0$. We can assume that $\{\lambda_i\}$ converges to a measured lamination λ after taking a subsequence. By the same argument as before, λ is maximal and connected, and $\psi(\lambda)$ represents an ending lamination of $(\mathbf{H}^3/G)_0$ by Lemma 5.1. First assume that $\{m_i\}$ converges inside the Teichmüller space. Then as before, the boundary Σ_i of the convex core of $\mathbf{H}^3/\Gamma_i$ converges geometrically to a boundary component Σ_∞ of the convex core of $\mathbf{H}^3/\Gamma_\infty$ which can be lifted to a boundary component Σ of the convex core of $\mathbf{H}^3/G$. Hence G is a totally degenerate b–group, and a neighbourhood of the geometrically finite end of $(\mathbf{H}^3/G)_0$ is mapped homeomorphically by q to a neighbourhood of an end of $\mathbf{H}^3/\Gamma_\infty$. Then as before, using Proposition 2.6, we can conclude that $G = \Gamma_\infty$.

Next assume that $\{m_i\}$ does not converge inside the Teichmüller space. Then after taking a subsequence, we can assume that $\{m_i\}$ converges to a projective lamination $[\mu]$. By the same argument as before, we can see that μ is maximal and connected, and $\psi(\mu)$ represents an ending lamination. Then by Theorem 3.1, we can see that the support of λ is different from that of μ. Hence G is totally doubly degenerate, and by Proposition 2.6, we conclude that $G = \Gamma_\infty$. Finally suppose that all the Γ_i are totally doubly degenerate. Let λ_i and μ_i be measured laminations of the unit length such that $\phi_i(\lambda_i)$ and $\phi_i(\mu_i)$ represent ending laminations of the two geometrically infinite tame ends of $(\mathbf{H}^3/\Gamma_i)_0$. By taking a subsequence, we can assume that $\{\lambda_i\}$ converges to a measured lamination λ and $\{\mu_i\}$ converges to a measured lamination μ in $\mathcal{ML}(S)$. Then as before, both λ and μ are maximal and connected, and $\psi(\lambda)$ and $\psi(\mu)$ represent ending laminations of $(\mathbf{H}^3/G)_0$. By Theorem 3.1, we can see that the support of λ is different from that of μ. Hence the end of $(\mathbf{H}^3/G)_0$ with ending lamination represented by $\psi(\lambda)$ is different from that with ending lamination represented by $\psi(\mu)$ by Lemma 4.3, which implies that G is totally doubly degenerate. Then by Proposition 2.6 again, we conclude that $G = \Gamma_\infty$, and the proof is completed. □

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