ISSN 1464-8997

Geometry & Topology Monographs
Volume 1: The Epstein Birthday Schrift
Pages 451–478

Coordinates for Quasi-Fuchsian Punctured Torus Space

John R Parker
Jouni Parkkonen

Abstract We consider complex Fenchel–Nielsen coordinates on the quasi-Fuchsian space of punctured tori. These coordinates arise from a generalisation of Kra's plumbing construction and are related to earthquakes on Teichmüller space. They also allow us to interpolate between two coordinate systems on Teichmüller space, namely the classical Fuchsian space with Fenchel–Nielsen coordinates and the Maskit embedding. We also show how they relate to the pleating coordinates of Keen and Series.

AMS Classification 20H10; 32G15

Keywords Quasi-Fuchsian space, complex Fenchel-Nielsen coordinates, pleating coordinates

0 Introduction

In this note we study the holomorphic extension of the classical Fenchel–Nielsen coordinates of the Teichmüller space of once-punctured tori to the quasi-conformal deformation space of a Fuchsian group representing two punctured tori, quasi-Fuchsian punctured torus space. A *punctured torus group* $G = \langle S, T\rangle$ is a discrete, marked, free subgroup of $\mathrm{PSL}(2,\mathbb{C})$ with two generators whose commutator $K = T^{-1}S^{-1}TS$ is parabolic. This group acts naturally on the Riemann sphere by conformal transformations. The limit set $\Lambda(G)$ consists of all accumulation points of this action and is the smallest nonempty closed G–invariant subset of the Riemann sphere. Its complement is called the ordinary set $\Omega(G)$. The group G is called *quasi-Fuchsian* if its ordinary set $\Omega(G)$ consists of two simply connected components or equivalently if its limit set $\Lambda(G)$ is a topological circle. The space of all quasi-Fuchsian punctured torus groups up to conjugation within $\mathrm{PSL}(2,\mathbb{C})$ is called *quasi-Fuchsian punctured torus space* and will be denoted by $\mathcal{Q}$. The subset of $\mathcal{Q}$ consisting of groups whose limit set is a round circle is the space of all Fuchsian punctured torus groups. We call this *Fuchsian punctured torus space* and we will denoted it by $\mathcal{F}$. It is a copy of the Teichmüller space of the punctured torus.

Our approach to quasi-Fuchsian punctured torus groups is a combination of the classical Fenchel–Nielsen construction of Fuchsian groups and the gluing construction used by Kra in [12] for terminal b-groups. This is rather natural as Fuchsian groups form a real subspace inside the space of quasi-Fuchsian groups, and terminal b-groups form part of the boundary of the same space. We start with a Fuchsian group F of the second kind such that X_0, the quotient of the hyperbolic plane by F, is a sphere with a puncture and two infinite area ends with boundary geodesics of equal lengths. We then extend the group by adding a Möbius transformation that glues together the infinite area ends of the quotient to make a punctured torus. If the resulting group G is Fuchsian, this is the Fenchel–Nielsen construction. The construction is carried out in Section 1 and the Fenchel–Nielsen parameter is connected with the gluing parameter in Proposition 3.2. We can also regard F and G as acting on the Riemann sphere and we allow the Fenchel–Nielsen parameters to be complex. For other allowed values of the gluing parameter the resulting group G is a quasi-Fuchsian group bent along the geodesic in $\mathbb{H}^3$ corresponding to the boundary geodesics of X_0. The analysis of this bending, the associated shear, and their use for parametrising the deformation space of quasi-Fuchsian groups from different points of view is the main goal of the second half of the paper. We show that the resulting complexified Fenchel–Nielsen twist parameter can be interpreted as a complex shear as introduced by Parker and Series in [18] and that it has another natural interpretation as a $zw = t$ plumbing parameter as in Kra [12] The relationship between the various points of view is often easy at a conceptual level but can be hard to make explicit. In this paper we aim to make these connections as explicit as possible. Part of this involves writing down generators for punctured torus groups as matrices depending on parameters. This is useful for making explicit computations which we illustrate by drawing pictures of various slices through $\mathcal{Q}$.

One of the main themes of this paper will be a partial description of Keen–Series pleating invariants in terms of complex Fenchel–Nielsen parameters. For completeness we now give a brief account of pleating invariants [5, 8]. Unlike complex Fenchel–Nielsen coordinates these are not holomorphic coordinates but they do reflect the geometrical structure of the associated 3–manifold as well as the limit set of G. In particular, they may be used to determine the shape of the embedding of $\mathcal{Q}$ into $\mathbb{C}^2$ given by complex Fenchel–Nielsen coordinates. We will illustrate this with pictures of various slices through this embedding. Let G be a punctured torus group that is quasi-Fuchsian but not Fuchsian. We call such a group *strictly quasi-Fuchsian*. Consider $C(G)$, the the hyperbolic convex hull in $\mathbb{H}^3$ of the limit set of G (sometimes called the Nielsen region for G). This is a G–invariant, simply connected, convex subset of $\mathbb{H}^3$. Thus, its quotient $C(G)/G$ is a convex 3–manifold with boundary, whose fundamental group is G. In other words $C(G)/G$ is topologically, the product of a closed interval with a punctured torus. Each boundary component is topologically a punctured torus and naturally inherits a hyperbolic structure from the three manifold (this structure is different from the obvious hyperbolic structure on the corresponding component of $\Omega(G)/G$). This hyperbolic structure makes the boundary component into a pleated surface in the sense of Thurston. That is, it consists of totally geodesic flat pieces joined along a geodesic lamination, called the *pleating locus*, and which carries a natural transverse measure, the *bending measure*. The length l_μ of a measured lamination μ on a surface with a given hyperbolic structure, is the total mass on this surface of the measure given by the product of hyperbolic length along the leaves of μ with the transverse measure μ. For the punctured torus it is well known that measured geodesic laminations are projectively parametrised by the extended real line. If the support of the lamination is drawn on the square flat torus then this parameter is just the gradient. From this we see that the possible types of support that this lamination that can have fall into two categories. First, simple closed curves, sometimes called rational laminations because of their parametrisation by rational slopes on a square torus. The transverse measure is just the δ–measure on these curves. Secondly, laminations whose leaves are unbounded geodesic arcs and which correspond to "infinite words" in G. We refer to these as *infinite laminations*. They correspond to curves of irrational slope on a square torus and so are sometimes referred to as irrational laminations. The measure they carry is called bending measure. We remark that the pleating locus cannot be the same on both components of the convex hull boundary. This is an important observation. Most of the time in this paper, we will be concerned with the case where the pleating locus on one component of $\partial C(G)/G$ is a simple closed geodesic. In this case, there will be a constant angle across this geodesic between the two adjacent flat pieces. In this case, the lamination length is just the length of the geodesic in the hyperbolic structure on the convex hull boundary. Keen and Series show in [8] that a marked punctured torus group is determined by its pleating invariants, namely the projective classes (μ, l_μ), (ν, l_ν) where the supports of μ and ν are the pleating loci on the two components of $\partial C(G)$ and l_μ, l_ν are their lamination lengths.

Suppose that the pleating loci on both components of $\partial C(G)$ are simple closed curves γ, δ. The corresponding group elements necessarily have real trace (though this is not a sufficient condition). The collection of all groups in $\mathcal{Q}$ for which γ, δ are the pleating loci is called the *(rational) pleating plane* $\mathcal{P}_{\gamma,\delta}$. This is a two dimensional non-singular subset of $\mathcal{Q}$ and is parametrised by the lengths of the geodesics γ and δ (which in this case are the lamination lengths), see Theorem 2 of [8]. Keen and Series also define pleating planes for the cases where one or both of the pleating loci are infinite laminations. We will only make passing reference to such pleating planes.

We have been greatly helped by conversations with Linda Keen and Caroline Series. We would like to thank them for their help. We would also like to thank the referee for her/his comments which have improved the paper. The second author was supported by the Academy of Finland and by the foundation Magnus Ernroothin Säätiö of the Finnish Society of Sciences and Letters. Figures 4.1, 5.1 and 6.1 were drawn using a computer program developed by David Wright. The second author would like to thank him for his help in installing and using the program. Both authors would like to thank the Centre Emile Borel at the Institut Henri Poincaré for their hospitality.

1 Real Fenchel–Nielsen coordinates

In this section we show how to write down generators for Fuchsian punctured torus groups in terms of Fenchel–Nielsen coordinates. This section gives a foundation for the subsequent sections: In order to obtain complex Fenchel–Nielsen coordinates we simply keep the same normal form for the generators but make the parameters complex. The material in this section is quite standard, for a more complete discussion of Fenchel–Nielsen coordinates see Buser [2].

Let X be a punctured torus and $\gamma\subset X$ a simple closed geodesic. Then $X_0=X\setminus\gamma$ is a hyperbolic surface of genus 0 with one puncture and two geodesic boundary components of equal length, say l. X_0 can be realised as a quotient $X_0=N(G_0)/G_0$, where G_0 is a Fuchsian group of the second kind generated by two hyperbolic transformations with multiplier $\lambda=l/2\in\mathbb{R}_+$:

$$S=\begin{pmatrix}\cosh(\lambda) & \cosh(\lambda)+1\\ \cosh(\lambda)-1 & \cosh(\lambda)\end{pmatrix}\quad\text{and}\quad S'=\begin{pmatrix}\cosh(\lambda) & \cosh(\lambda)-1\\ \cosh(\lambda)+1 & \cosh(\lambda)\end{pmatrix}, \tag{1.1}$$

and $N(G_0)$ is the *Nielsen region* of G_0, that is, the hyperbolic convex hull in $\mathbb{H}$ of the limit set of G_0. For later reference we record that the fixed points of these transformations are $\operatorname{fix}S=\pm\coth(\lambda/2)$ and $\operatorname{fix}S'=\pm\tanh(\lambda/2)$. The transformations S and S' correspond to the boundary geodesics of X_0 and their product $K={S'}^{-1}S$ corresponds to the puncture. In other words

$$K={S'}^{-1}S=\begin{pmatrix}-1+2\cosh(\lambda) & 2\cosh(\lambda)\\ -2\cosh(\lambda) & -1-2\cosh(\lambda)\end{pmatrix} \tag{1.2}$$

is a parabolic transformation fixing -1.

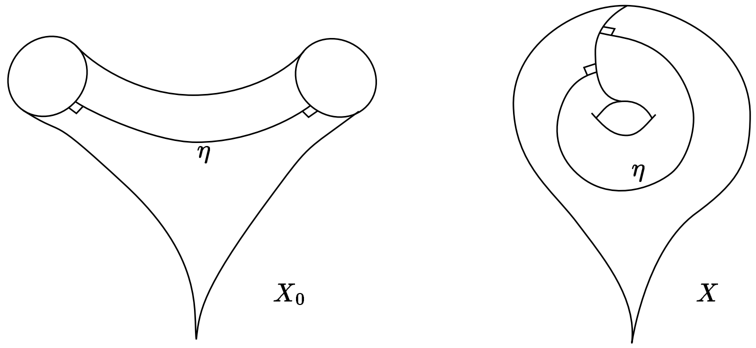

Figure 1.1 The Fenchel–Nielsen construction

The original surface X can be reconstructed by gluing together the geodesic boundary components of X_0. The gluing can be realised by adding to the group a hyperbolic Möbius transformation T that preserves $\mathbb{H}^2$. We form a new Fuchsian group, an HNN extension of G_0:

$$G = \langle G_0, T\rangle = (G_0) *_{\langle T\rangle} .$$

The transformation T is required to conjugate the cyclic subgroups $\langle S\rangle$ and $\langle S'\rangle$ in a manner compatible with the gluing operation:

$$T^{-1}ST = S'.$$

This condition fixes T up to one free parameter $\tau \in \mathbb{R}$, and T can be written in the form

$$T = \begin{pmatrix} \cosh(\tau/2)\coth(\lambda/2) & -\sinh(\tau/2) \\ -\sinh(\tau/2) & \cosh(\tau/2)\tanh(\lambda/2) \end{pmatrix}. \tag{1.3}$$

We recover the original (marked) surface with the correct geometry for exactly one parameter $\tau_0 \in \mathbb{R}$. However, the group G is a Fuchsian group for any real τ, and the parameter has a geometric interpretation: There is a unique simple geodesic arc η on X_0 perpendicular to both geodesic boundary curves. A distinguished lift of this arc to the universal covering $\mathbb{H}^2$ is the segment of the positive imaginary axis connecting $i\tanh(\lambda/2) \in \operatorname{axis}(S')$ and $i\coth(\lambda/2) \in \operatorname{axis}(S)$. Now T maps $i\tanh(\lambda/2)$ to a point on the axis of S, namely

$$T(i\tanh(\lambda/2)) = i\coth(\lambda/2)\big(\operatorname{sech}(\tau) + i\tanh(\tau)\big).$$

The (signed) hyperbolic distance of this point from $i\coth(\lambda/2)$ is exactly τ, the sign of τ is chosen to be positive if moving from $i\coth(\lambda/2)$ to $T\big(i\tanh(\lambda/2)\big)$ takes one in a positive (anti-clockwise) direction around the circle of radius $\coth(\lambda/2)$. The map $G \longmapsto (\lambda, \tau)$ is the *Fenchel–Nielsen coordinate* of the

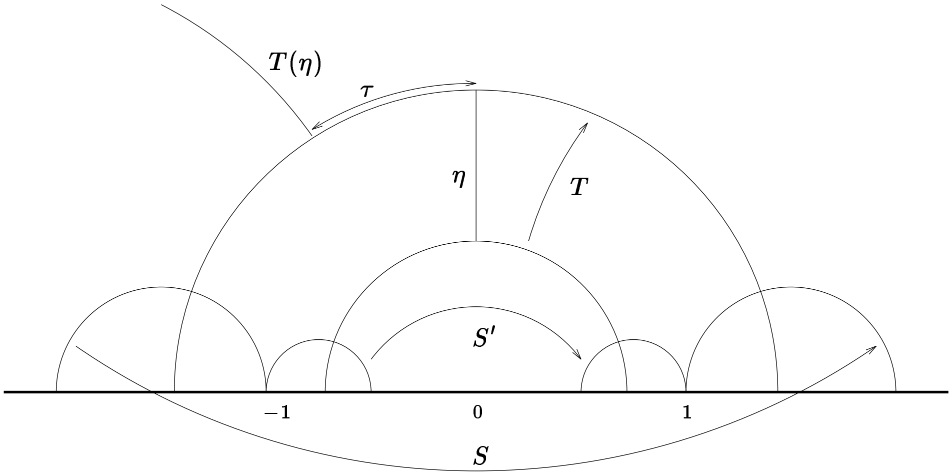

Figure 1.2 The fundamental domain

Teichmüller space of punctured tori. It defines a global real analytic parametrisation and identifies $\mathcal{F}$ with $\mathbb{R}_+ \times \mathbb{R}$ (see Buser [2]). Fenchel–Nielsen coordinates depend on the choice of an ordered pair of (homotopy classes of) simple closed curves on the punctured torus intersecting exactly once, that is a marking. We obtain different coordinates for different choices of marking. These choices are related by elements of the modular group. We investigate this in more detail in the next section. In [21] Waterman and Wolpert give computer pictures for the action of the modular group on Fenchel–Nielsen coordinates. They also give pictures of this action in another set of coordinates which can be easily derived from traces of generating triples.

Varying τ and keeping λ fixed is the Fenchel–Nielsen deformation considered by Wolpert in [22] and [23].

2 Complex Fenchel–Nielsen coordinates

The Teichmüller space of punctured tori seen as the space of Fuchsian groups representing a punctured torus, $\mathcal{F}$, is a natural subspace of the corresponding quasi-Fuchsian space, $\mathcal{Q}$. Kourouniotis [11] and Tan [20] showed that, for compact surfaces, the Fenchel–Nielsen coordinates can be complexified to give a global parametrisation of quasi-Fuchsian space. With this in mind we now suppose that λ and τ are complex. That is $(\lambda, \tau) \in \mathbb{C}_+ \times \mathbb{C}$ where $\mathbb{C}_+$ denotes those complex numbers with positive real part. With such λ and τ we consider groups generated by S and T with the normal forms (1.1) and (1.3). This means that S and T are now in $\mathrm{PSL}(2,\mathbb{C})$ rather than in $\mathrm{PSL}(2,\mathbb{R})$. The group $\langle S,T\rangle$ is not quasi-Fuchsian for all $(\lambda,\tau)\in\mathbb{C}_+\times\mathbb{C}$ but the *complex Fenchel–Nielsen coordinates* (λ,τ) do give global coordinates on $\mathcal{Q}$. We present a short proof of this fact using the stratification method developed by Kra and Maskit in [13].

Proposition 2.1 *The map* $h\colon \mathcal{Q}\to\mathbb{C}^2$ *given by* $h(G)=(\cosh^2(\lambda),e^\tau)$ *is a global complex analytic coordinate map on* $\mathcal{Q}$.

Proof Let $G=\langle A,B\rangle$ be a quasi-Fuchsian group of type $(1,1)$ generated by two loxodromic transformations A and B. Assume that the group is normalised so that 0 is the repelling fixed point, and ∞ is the attracting fixed point of A, and that $B(0)=1$. Let $x_1=B(\infty)$, and $x_2=B(1)$. Note that $x_1,x_2\in\Lambda(G)$.

We claim that G is determined by giving x_1 and x_2: Clearly B is determined, as we know how it maps three points. Also, from the normalisation we know that

$$A=\begin{pmatrix} a & 0\\ 0 & 1/a\end{pmatrix},\qquad B=\begin{pmatrix} x_1(x_2-1) & x_1-x_2\\ x_2-1 & x_1-x_2\end{pmatrix},$$

where $a\in\mathbb{C},\ |a|>1$. Now

$$\mathrm{tr}[A,B]=\frac{2a^2x_1-1-a^4}{a^2(x_1-1)}.$$

As $[A,B]$ is assumed to be a parabolic, solving for a^2 in the equation $\mathrm{tr}[A,B]=-2$ gives $a^2=2x_1-1\pm2\sqrt{x_1(x_1-1)}$. Only one of these solutions satisfies $|a|>1$. This fixes A. (The choice of the branch of the square root $a=\sqrt{a^2}$ does not affect A.)

Let us normalise the group $G=\langle S',T\rangle$ of Section 1 as above: We conjugate G with a transformation (here written as an element of $\mathrm{PGL}(2,\mathbb{C})$)

$$R=\begin{pmatrix} \cosh(\lambda)/(1-\cosh(\lambda)) & -\coth(\lambda)\\ 1/(1-\cosh(\lambda)) & \mathrm{csch}(\lambda)\end{pmatrix}.$$

This gives

$$S_0=RS'R^{-1}=\begin{pmatrix} e^\lambda & 0\\ 0 & e^{-\lambda}\end{pmatrix},$$

where we can assume $|e^\lambda|>1$, and

$$T_0=RTR^{-1}=\begin{pmatrix} \coth(\lambda)e^{-\tau/2} & \coth(\lambda)e^{\tau/2}\\ \mathrm{csch}(\lambda)\,\mathrm{sech}(\lambda)e^{-\tau/2} & \coth(\lambda)e^{\tau/2}\end{pmatrix}.$$

Now

$$x_1=\cosh^2(\lambda),\qquad x_2=\frac{1+e^\tau}{\mathrm{sech}^2(\lambda)+e^\tau}.$$

□

Remark 2.2 The choice $|e^\lambda| > 1$ implies $\lambda \in \mathbb{C}_+$. Unlike real Fenchel–Nielsen coordinates, there is no simple description of which pairs $(\lambda, \tau) \in \mathbb{C}_+ \times \mathbb{C}$ are in $h(\mathcal{Q})$, the image of quasi-Fuchsian space under the coordinate map. Using the pleating invariants of Keen and Series [8] one can determine how $h(\mathcal{Q})$ lies inside $\mathbb{C}^2$. In this paper we carry out part of this construction and illustrate our results by drawing slices through $\mathcal{Q}$ in Figure 5.1.

We now use the fact that $(\cosh^2(\lambda), e^\tau)$ give global coordinates to show that (λ, τ) give global coordinates on quasi-Fuchsian space. Let

$$\widetilde{\mathcal{FN}} = \left\{(\lambda, \tau) \in \mathbb{C}^2 \colon (\cosh^2(\lambda), e^\tau) \in h(\mathcal{Q})\right\},$$

where h is the map of Proposition 2.1. We denote by $\mathcal{FN}$ the component of $\widetilde{\mathcal{FN}}$ containing $\mathbb{R}_+ \times \mathbb{R}$. Our proof that (λ, τ) give global coordinates involves showing that there are no paths in $\widetilde{\mathcal{FN}}$ between two places where the parameters are different but the groups are the same.

Proposition 2.3 *Let $\gamma\colon [0,1] \longrightarrow \mathbb{C}_+ \times \mathbb{C}$ denote any path from $\gamma(0) = (\lambda_0, \tau_0)$ to $\gamma(1) = (\lambda_0 + m\pi i, \tau_0 + 2n\pi i)$ for any $(\lambda_0, \tau_0) \in \widetilde{\mathcal{FN}}$ and integers m and n not both zero. Then $\gamma([0,1])$ is not contained in $\widetilde{\mathcal{FN}}$.*

Proof We begin with the case $m = 1$ and $n = 0$.

Using the normalisation of Proposition 2.1 we have $T_0(\lambda_0, \tau_0) = T_0(\lambda_0 + \pi i, \tau_0)$. Also notice that $S_0(\lambda_0, \tau_0)$ and $S_0(\lambda_0 + \pi i, \tau_0)$ are the same in $\mathrm{PSL}(2,\mathbb{C})$ but differ by $-I$ in $\mathrm{SL}(2,\mathbb{C})$. They correspond to the two choices of square root for a^2 in Proposition 2.1. Thus moving along γ from (λ_0, τ_0) to $(\lambda_0 + \pi i, \tau_0)$ adds $i\pi$ to the multiplier of S_0. For more details of the relationship between multipliers and the different lifts of Möbius transformations in $\mathrm{PSL}(2,\mathbb{C})$ to matrices in $\mathrm{SL}(2,\mathbb{C})$ see the discussion in Section 1 of [18]. Let Π_1 be any hyperplane in $\mathbb{H}^3$ orthogonal to the axis of S_0 and let $\Pi_2 = S_0(\Pi_1)$ be its image under S_0. Because going along γ from (λ_0, τ_0) to $(\lambda_0 + \pi i, \tau_0)$ changes the multiplier of S_0 by πi then also Π_2 is rotated by 2π with respect to Π_1. We can think of going along γ as being the same as doing a Dehn twist of the annulus between $\partial\Pi_1$ and $\partial\Pi_2$ in $\widehat{\mathbb{C}}$.

Specifically we may decompose S_0 into a product of half turns (that is elliptic involutions in $\mathrm{PSL}(2,\mathbb{C})$ of order 2) as follows:

$$S_0 = \iota_1\iota_2 = \begin{pmatrix} 0 & e^\lambda \\ -e^{-\lambda} & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The geodesic fixed by $\iota_1(\lambda, \tau)$ has end points $\pm ie^\lambda$. Replacing (λ_0, τ_0) by $(\lambda_0 + \pi i, \tau_0)$ interchanges these end points. Equivalently this reverses the orientation of the geodesic. Therefore if Π_1 is hyperplane orthogonal to the axis of S_0 and containing the geodesic with end points $\pm i$ (that is the axis of ι_2) it is clear that its image under the ι_1 is rotated by 2π when we replace λ_0 by $\lambda_0 + \pi i$.

Let ξ_1 be any point of $\partial\Pi_1 \cap \Omega$ and $\xi_2 = S_0(\xi_1)$ be its image under S_0. Let α be any path in Ω joining ξ_1 and ξ_2. Now consider the homotopy H given by following α while (λ, τ) varies along γ. Denote the image of α at time t by α_t.

If the whole of γ were in $\mathcal{Q}$ then the homotopy H would induce an isotopy from $\Omega\big(G(\lambda_0, \tau_0)\big)$ to $\Omega\big(G(\lambda_0, \tau_0 + 2\pi i)\big)$. At each stage S_0 is loxodromic so Π_1 and Π_2 are disjoint and α_t consists of more than one point. Now α_0 and α_1 are both paths in $\Omega\big(G(\lambda_0, \tau_0)\big) = \Omega\big(G(\lambda_0 + \pi i, \tau_0)\big)$ joining ξ_1 and ξ_2. It is clear from the earlier discussion that the path $\alpha_1 {\alpha_0}^{-1}$ formed by going along α_1 and then backwards along α_0 winds once around the (closed) annulus between $\partial\Pi_1$ and $\partial\Pi_2$. This it separates the fixed points of S_0. This contradicts the fact that the limit set Λ is connected.

We can adapt this proof to cover the case where λ_0 is sent to $\lambda_0 + m\pi i$ for some non-zero integer m. This is done by observing that the path $\alpha_1 {\alpha_0}^{-1}$ now winds m times around the annulus between $\partial\Pi_1$ and $\partial\Pi_2$. Moreover this argument does not use the value of τ at each end of the path. It merely uses the fact that $T_0(\lambda_0, \tau_0) = T_0(\lambda_1, \tau_1)$ and so we may take $\tau_1 = \tau_0 + 2n\pi i$ without changing anything.

Thus we have proved the result when m and n are any integers with m not zero. It remains to prove the result when $m = 0$ and n is an integer other than zero. We do this as follows. Observe that, with the normalisation of (1.1) and (1.3), $S(\lambda_0, \tau_0) = S(\lambda_0, \tau_0 + 2\pi i)$ but $T(\lambda_0, \tau_0 + 2\pi i)$ and $T(\lambda_0, \tau_0)$ give distinct lifts in $\mathrm{SL}(2, \mathbb{C})$. As before we decompose T into a product of half turns as follows:

$$T = \iota_1 \iota_2 = \begin{pmatrix} \sinh(\tau/2) & \cosh(\tau/2)\coth(\lambda/2) \\ -\cosh(\tau/2)\tanh(\lambda/2) & -\sinh(\tau/2) \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The geodesic fixed by $\iota_1(\lambda, \tau)$ has end points

$$\frac{-\sinh(\tau/2) \pm i}{\cosh(\tau/2)\tanh(\lambda/2)}.$$

Replacing (λ_0, τ_0) by $(\lambda_0, \tau_0 + 2\pi i)$ interchanges these end points. The rest of the argument follows as before. □

The next two results are direct consequences of Propositions 2.1 and 2.3.

Corollary *The functions* $\cosh^2(\lambda)$ *and* e^τ *have well defined inverses in* $h(\mathcal{Q})$ *and so we can regard* (λ, τ) *is a global coordinate system for quasi-Fuchsian space.*

Corollary *The pair* $\big(\cosh(\lambda), \sinh(\tau/2)\big)$ *give global coordinates for quasi-Fuchsian space. In particular, the points where* $\sinh(\lambda) = 0$ *or* $\cosh(\tau/2) = 0$ *are not in* $\mathcal{FN}$.

Proof The first part follows from the previous corollary. We give a simple justification for the last statement. If $\sinh(\lambda) = 0$ then $\cosh(\lambda) = \pm 1$ and S is parabolic. Similarly if $\cosh(\tau/2) = 0$ then T is elliptic or else $\coth(\lambda)$ is infinite and S is parabolic as before. □

Complex Fenchel–Nielsen coordinates depend on the choice of a marking for the punctured torus, that is an ordered pair of generators for S. It is intuitively clear that changing this marking gives a biholomorphic change of the coordinates $\big(\cosh(\lambda), \sinh(\tau/2)\big)$. We now make this explicit.

Proposition 2.4 *Let* (S_0, T_0) *and* (S_1, T_1) *be any two generating pairs for a punctured torus group* G. *Let* (λ_0, τ_0) *and* (λ_1, τ_1) *be the corresponding complex Fenchel–Nielsen coordinates on* $\mathcal{Q}$. *Then the map*

$$\big(\cosh(\lambda_0), \sinh(\tau_0/2)\big) \longmapsto \big(\cosh(\lambda_1), \sinh(\tau_1/2)\big)$$

is a biholomorphic homeomorphism of $\mathcal{Q}$ *to itself.*

Proof A classical result of Nielsen [17] states that we can obtain the pair (S_1, T_1) from (S_0, T_0) by a sequence of elementary Nielsen-moves on the generators. As one of our aims is to make things explicit, we list these Nielsen moves and write down the effect that they have on the coordinates $\big(\cosh(\lambda), \sinh(\tau/2)\big)$. From this, it is clear that these changes of coordinate are holomorphic.

First, suppose that $(S', T') = (S, S^{\pm 1}T)$. Then

$$\cosh(\lambda') = \cosh(\lambda), \quad \sinh(\tau'/2) = \sinh(\tau/2)\cosh(\lambda) \mp \cosh(\tau/2)\sinh(\lambda).$$

Secondly, suppose that $(S', T') = (S, T^{-1})$. Then

$$\cosh(\lambda') = \cosh(\lambda), \quad \sinh(\tau'/2) = -\sinh(\tau).$$

Finally, suppose that $(S', T') = (T, S)$

$$\cosh(\lambda') = \frac{\cosh(\lambda)\cosh(\tau/2)}{\sinh(\lambda)}, \quad \sinh(\tau'/2) = \frac{-\sinh(\tau/2)\sinh(\lambda)}{\cosh(\tau/2)}.$$

□

3 Plumbing and earthquakes

In this section we show how the Fenchel–Nielsen construction is related to two standard constructions in Teichmüller theory, namely the $zw = t$ plumbing construction and to quake-bends. In particular, the Fenchel–Nielsen twist parameter is a special case of the quake-bend parameter and we show how to express the plumbing parameter in terms of Fenchel–Nielsen parameters.

Consider Teichmüller space of the punctured torus $\mathcal{F}$ with Fenchel–Nielsen coordinates as in Section 1. The motion through Teichmüller space obtained by fixing the length parameter λ but varying the shear τ is the Fenchel–Nielsen deformation (see [22]) which is the simplest example of an earthquake (see Waterman and Wolpert [21] and McMullen [16] for some other earthquakes). One may think of this as cutting along $\mathrm{Ax}(S)$ twisting and then regluing.

If we reglue so that along $\mathrm{Ax}(S)$ the two sides make a constant angle then we have an example of a *quake-bend* (see Epstein and Marden [4]). We can say that the group $G(\lambda,\tau)$ is obtained from $G(\lambda,0)$ by doing a quake-bend along S with parameter τ. That is, for $\lambda \in \mathbb{R}_+$, we take the Fuchsian group $G(\lambda,0)$ with generators

$$S = \begin{pmatrix} \cosh(\lambda) & \cosh(\lambda)+1 \\ \cosh(\lambda)-1 & \cosh(\lambda) \end{pmatrix}, \quad T = \begin{pmatrix} \coth(\lambda/2) & 0 \\ 0 & \tanh(\lambda/2) \end{pmatrix}.$$

This group has a fundamental domain rather like the one shown in Figure 1.2 except with $\tau = 0$ (the copy of the hyperbolic plane in question is the hyperplane in $\mathbb{H}^3$ whose boundary is the extended real axis). Let $Q(\tau)$ be a loxodromic map with the same fixed points as S and trace $2\cosh(\tau/2)$. Apply $Q(\tau)$ to that part of $\mathbb{H}^2$ lying above $\mathrm{Ax}(S)$, ie those points with $|z| > \coth(\lambda/2)$. What we have done is essentially cut along $\mathrm{Ax}(S)$ and reglued after performing a shear and a bend. Now repeat this construction along the axis of every conjugate of S. This is a quake-bend. For more details and a precise definition of what is involved, see [4]. A discussion of quake-bends and complex Fenchel–Nielsen coordinates in given in Section 5.3 of [7].

One can perform this construction for irrational measured laminations. In this case the new measure is obtained by multiplying the initial bending measure by the quake-bend parameter. This gives a way of generalising the Fenchel–Nielsen twist parameter τ analogous to the way lamination length generalises the hyperbolic length of a simple closed curve.

We now relate these ideas by extending the $zw = t$*–plumbing construction* to this situation. Essentially the same construction was used by Earle and Marden [3] and Kra [12] in the case of punctured surfaces and it was extended by Arés [1] and Parkkonen [19] for surfaces with elliptic cone points.

Let X_0 be a punctured cylinder (as in Section 1). Assume that the boundary geodesics γ_1 and γ_2 corresponding to boundary components b_1 and b_2 have equal length $l = 2\lambda > 0$. Let U_1 and U_2 be neighbourhoods of, respectively, the ends of X_0 corresponding to γ_1 and γ_2. Let γ_{12} be the shortest geodesic arc connecting the two boundary components, and let

$$\mathcal{A}_\lambda = \{\zeta \in \mathbb{C} \mid e^{-\pi^2/\lambda} < |\zeta| < 1\}$$

with its hyperbolic metric of constant curvature -1. The curve $\{|z| = e^{-\pi^2/2\lambda}\}$ is the unique geodesic in $\mathcal{A}_\lambda$ with this metric.

We define local coordinates at the ends of X_0 by

$$z\colon U_1 \to \mathcal{A}_\lambda \quad \text{and} \quad w\colon U_2 \to \mathcal{A}_\lambda$$

by requiring that the maps are isometries and that the segments $\gamma_{12} \cap U_1$ and $\gamma_{12} \cap U_2$ are mapped into $\mathcal{A}_\lambda \cap \mathbb{R}_+$. These conditions define the maps z and w uniquely.

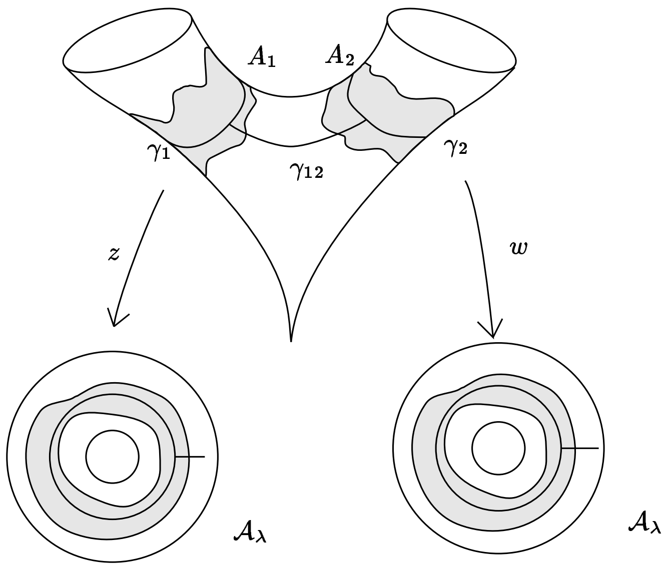

Figure 3.1 The $zw = t$ plumbing construction

If $A \subset X_0$ is an annulus homotopic to a boundary component b of X_0, we call the component of ∂A separating the other component of ∂A from b, the outer boundary of A. The remaining component of ∂A is the inner boundary of A. Assume there are annuli $A_i \subset U_i$ and a holomorphic homeomorphism $f\colon A_1 \to A_2$ so that

$$z(x)w(f(x)) = t$$

for some constant $t \in \mathbb{C}$ and f maps the outer boundary of A_1 to the inner boundary of A_2. The outer boundaries bound annuli on X_0. Remove these annuli to form a new Riemann surface X_{trunc}. Define

$$X_t := X_{\mathrm{trunc}}/\sim,$$

where the equivalence is defined by setting

$$x \sim y \Longleftrightarrow z(x)w(y) = t.$$

We say that X_t was obtained from X_0 by the *$zw = t$ plumbing construction with plumbing or gluing parameter t*. If the annuli A_i can be chosen to be collar neighbourhoods of the boundary geodesics γ_i, we say that the plumbing is *tame*.

Next we show that the Fenchel–Nielsen twist parameter is naturally associated with a plumbing parameter:

Lemma 3.1 *If G is in $\mathcal{Q}$ with $\lambda \in \mathbb{R}_+$ then $t = e^{-\pi^2/\lambda}\, e^{-\pi i\tau/\lambda} = e^{i\pi\mu}$ where $\mu = (i\pi - \tau)/\lambda$.*

Proof Let $\Pi\colon \mathbb{H}^2 \to \mathbb{H}^2/G_0$ be the canonical projection. Let $\widetilde{\gamma_1}$ be the geodesic in $\mathbb{H}^2$ connecting the fixed points of S and $\widetilde{\gamma_2}$ the geodesic connecting the fixed points of S'. Now the boundary geodesics for which the gluing will be done are $\gamma_i = \Pi(\widetilde{\gamma_i})$. The local coordinates are given by

$$z(P) = \exp\left(\frac{\pi i}{\lambda}\log\left(\frac{\Pi^{-1}(P)\sinh(\lambda/2) + \cosh(\lambda/2)}{-\Pi^{-1}(P)\sinh(\lambda/2) + \cosh(\lambda/2)}\right)\right),$$

and

$$w(Q) = \exp\left(\frac{\pi i}{\lambda}\log\left(\frac{\Pi^{-1}(Q)\cosh(\lambda/2) - \sinh(\lambda/2)}{\Pi^{-1}(Q)\cosh(\lambda/2) + \sinh(\lambda/2)}\right)\right).$$

Substituting for T we see, after simplifying, that

$$z\big(T(Q)\big) = \exp\left(\frac{\pi i}{\lambda}\log\left(\frac{e^{-\tau}\,\Pi^{-1}(Q)\cosh(\lambda/2) + \sinh(\lambda/2)}{-\Pi^{-1}(Q)\cosh(\lambda/2) + \sinh(\lambda/2)}\right)\right).$$

Thus $z(T(Q))\; w(Q) = \exp(-\pi^2/\lambda - \pi i\tau/\lambda)$ as claimed. □

The same proof also yields the following:

Proposition 3.2 *The classical Fenchel–Nielsen construction is a $zw = t$ plumbing construction for a parameter t of modulus $e^{-\pi^2/\lambda}$.*

4 λ–slices

In this section we keep λ real but allow τ to be complex. When $\theta = \mathrm{Im}(\tau)$ is in the interval $(0, \pi]$ we will show that the axis of S is the pleating locus on one component of the convex hull boundary and when $\theta \in [-\pi, 0)$ then it is the pleating locus on the other component. We will show that τ has an interpretation as a *complex shear* along the pleating locus, $\mathrm{Ax}(S)$, see Parker and Series [18]. The complex shear σ is defined as follows. The imaginary part of σ, which we require to be in the interval $(-\pi, \pi)$, is the bending angle on the convex hull boundary across $\mathrm{Ax}(S)$. The real part of σ defined as follows. Let η be the unique simple geodesic arc in the convex hull boundary from $\mathrm{Ax}(S)$ to itself and orthogonal to $\mathrm{Ax}(S)$ at both ends. Then we form a curve in the convex hull boundary in the homotopy class specified by T by going along η and then along $\mathrm{Ax}(S)$. The real part of the complex shear is the signed distance we go along $\mathrm{Ax}(S)$. This definition is made precise on page 172 of [18]. The theorems of this section should be compared with the constructions found in [12] and section 2.2 of [5]. We also note that one may use the formulae of [18] to show that, when λ is real, the imaginary part of σ cannot be $\pm\pi$, Proposition 7.1 of [8].

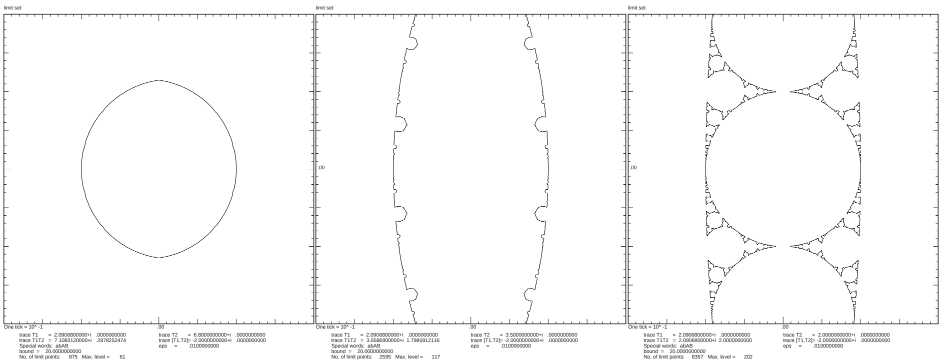

Figure 4.1 Limit sets of groups in a λ–slice

Let us fix $\lambda > 0$. Consider the set

$$\{\tau \in \mathbb{C} \mid (\lambda, \tau) \in \mathcal{FN}\}$$

The λ*–slice* $\mathcal{Q}_\lambda$ is defined to be the component of this set containing the points where $\tau \in \mathbb{R}$ (compare with the quake-bend planes of [8]). We wish to obtain an estimate for the allowed values of τ for each λ. In order to do this we will construct pleating coordinates on each λ–slice. A first approximation can be achieved by estimating the values of $\theta = \mathrm{Im}(\tau)$ that correspond to tame plumbing constructions. The following theorem is an explicit version of Theorem 6.1 of [7]. Specifically, we show that the constant ϵ of that theorem can be taken as $\theta_0 = 2\arccos\big(\tanh(\lambda)\big)$ (compare Section 6 of [9]). Because the point $(\lambda, i\theta_0)$ is on the boundary of quasi-Fuchsian space, there can be no larger uniform bound on $\mathrm{Im}(\tau)$ that ensures discreteness. The fact that θ is the imaginary part of the quake-bend will follow from Theorem 4.2.

Theorem 4.1 *Let $\theta_0 \in (0, \pi)$ be defined by the equation $\cos(\theta_0/2) = \tanh(\lambda)$. Then for $\mathrm{Im}(\tau) = \theta \in (-\theta_0, \theta_0)$ the group G is a quasi-Fuchsian punctured torus group.*

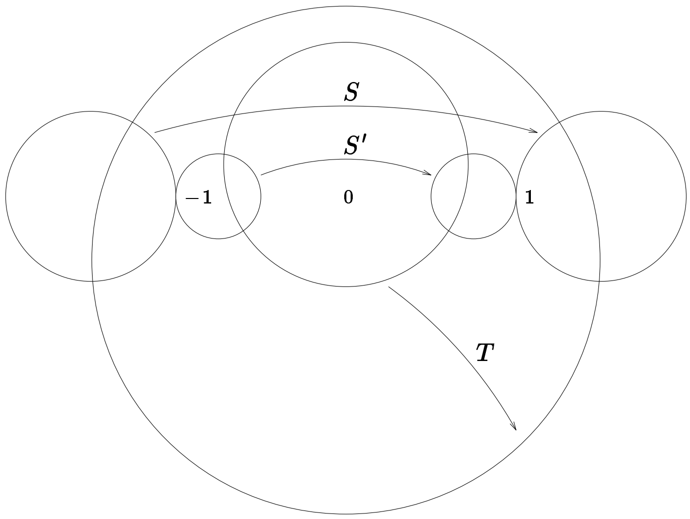

Figure 4.2 The construction for the combination theorem

Proof It is easy to check that the circle with centre at $i\tanh(\lambda/2)\tan(\theta/2)$ and radius $\tanh(\lambda/2)\sec(\theta/2)$ is mapped by T to the circle with centre at $-i\coth(\lambda/2)\tan(\theta/2)$ and radius $\coth(\lambda/2)\sec(\theta/2)$. Moreover these circles are mapped to themselves under $\langle S'\rangle$ and $\langle S\rangle$ respectively (the circles pass through the fixed points of S' and S). Providing the two circles are disjoint then the annulus between them is a fundamental domain for $\langle T\rangle$. It is easy to check that the circles are disjoint if and only if $\cos(\theta/2) > \tanh(\lambda)$, that is $\theta \in (-\theta_0, \theta_0)$. When this happens we can use Maskit's second combination theorem [14, 15] to show that G is discrete, has a fundamental domain with two components each of which glues up to give a punctured torus and G is quasi-Fuchsian. $\square$

For a positive real number λ, suppose that G is a quasi-Fuchsian punctured torus group. The ordinary set of G has two components. There is an obvious way to label these as the "top" and "bottom" components so that, for the case when G is Fuchsian, the upper half plane is the "top"component. In what follows, we give a result that enables us to make this definition precise. Namely in Lemmas 4.3 and 4.4, we show that either the "top" component contains the upper half plane or the "bottom" component contains the lower half plane (or both, in which case the group would be Fuchsian). When G is strictly quasi-Fuchsian there are two components to the convex hull boundary facing these two components of the ordinary set. We label them "top" and "bottom" as well (this notation is also used by Keen and Series on page 370 of [7]). Both of these components is a pleated surface and so we may speak of the pleating locus on the "top" and "bottom". The following theorem may be thought of as a generalisation of Proposition 6.2 of [18].

Theorem 4.2 *For any parameter in a λ slice ($\lambda \in \mathbb{R}$) with $\theta \in (0,\pi)$ (respectively $\theta \in (-\pi, 0)$) the pleating locus on the "bottom" (respectively "top") surface is S and τ (respectively $-\tau$) is the complex shear along S with respect to the curve T as defined in [18].*

Intuitively this should be clear as we are keeping λ real and bending away from $\mathrm{Ax}(S)$. As we are only bending along one curve the result is convex. In the general case we could not expect a Fenchel–Nielsen complex twist to always be the complex shear on the convex hull boundary as we may bend along different curves in different directions. In what follows we only consider the case $\theta > 0$. By symmetry this is sufficient. The proof will be by way of several lemmas.

Lemma 4.3 *If $\theta \in (0, \theta_0)$ then the lower half plane $\mathbb{L}$ is contained in $\Omega(G)$.*

Proof We will consider the lower half plane $\mathbb{L}$ with its Poincaré metric. We then use plane hyperbolic geometry to prove the result.

Let D^* be the fundamental region for the action of $F = \langle S, S' \rangle$ on $\mathbb{L}$ formed by the intersection of $\mathbb{L}$ with the exterior of the isometric circles for S and S'. That is

$$D^* = \{z \in \mathbb{L} \colon |(\cosh(\lambda) + \varepsilon_1)z + \varepsilon_2 \cosh(\lambda)| \geq 1 \text{ for all choices of } \varepsilon_1, \varepsilon_2 = \pm 1\}.$$

We are now going to consider various hypercycles (that is arcs of circles) with endpoints at the fixed points of S and S'. To begin with, let c_0 and c_0' be the semicircles centred at 0 of radius $\coth(\lambda/2)$ and $\tanh(\lambda/2)$. Clearly these are the Poincaré geodesics joining the fixed points of S and S' respectively. Let D_0 be the subset of $\mathbb{L}$ between these two semi-circles:

$$D_0 = \{z \in \mathbb{L}: \tanh(\lambda/2) \le |z| \le \coth(\lambda/2)\}\,.$$

The Nielsen region $N(F)$ of $F = \langle S, S'\rangle$, that is the hyperbolic convex hull of $\Lambda(F)$ in $\mathbb{L}$, is

$$N(F) = \bigcup_{g\in F} g(D^* \cap D_0).$$

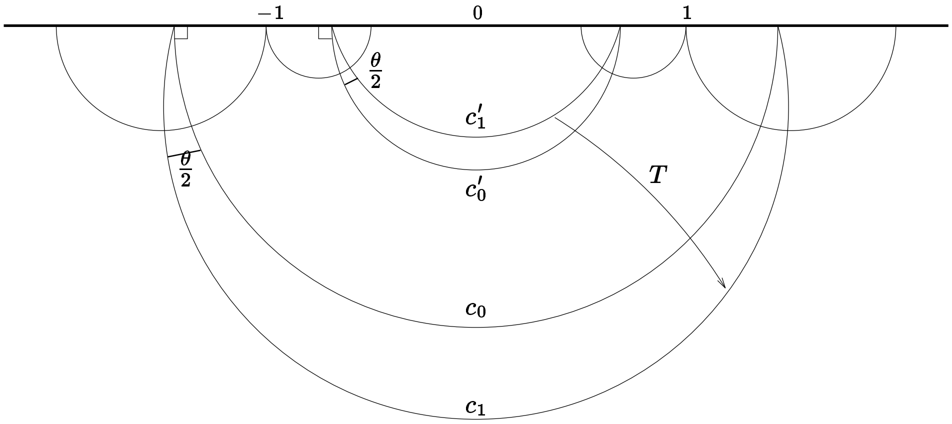

Figure 4.3 The construction in the lower half plane

Now consider the circular arcs c_1 and c'_1 in $\mathbb{L}-D_0$ with endpoints at $\pm\coth(\lambda/2)$ and $\pm\tanh(\lambda/2)$ which make an angle $\theta/2$ with c_0 and c'_0 respectively. In other words c_1 is the arc of the circle centred at $-i\coth(\lambda/2)\tan(\theta/2)$ with radius $\coth(\lambda/2)\sec(\theta/2)$ lying in the lower half plane. Similarly c'_1 is the intersection of $\mathbb{L}$ with the circle centred at $i\tanh(\lambda/2)\tan(\theta/2)$ with radius $\tanh(\lambda/2)\sec(\theta/2)$. Figure 4.3 shows c_1 and c'_1. Observe that c_1 and c'_1 are a constant distance $d(\theta)$ from c_0 and c'_0 where

$$d(\theta) = \log\bigl(\sec(\theta/2) + \tan(\theta/2)\bigr).$$

Denote the lune between c_0 and c_1 by $B(\theta)$ and the lune between c'_0 and c'_1 by $B'(\theta)$. Let D_1 be the subset of the lower half plane lying between c_1 and c'_1. Now D_1 is just the intersection of $\mathbb{L}$ with the fundamental region for T considered in Theorem 4.1. One of the consequences of Maskit's combination theorem is that $D^* \cap D_1$ is contained in $\Omega(G)$. (It is at this point that we have used $\theta < \theta_0$.) Let $N(\theta)$ be the union of all F translates of $D^* \cap D_1$:

$$N(\theta) = \bigcup_{g\in F} g(D^* \cap D_1).$$

It is clear that $N(\theta)$ is just the $d(\theta)$ neighbourhood of $N(F)$. Since $D^* \cap D_1$ is contained in $\Omega(G)$ then so is $N(\theta)$.

We are going to mimic this construction with more arcs. For each n with $n\theta < \pi$, let c_n and c'_n be the circular arcs in $\mathbb{L} - D_0$ with endpoints at $\pm\coth(\lambda/2)$ and $\pm\tanh(\lambda/2)$ making an angle of $n\theta/2$ with c_0 and c'_0 respectively. That is c_n is the arc of a circle with centre at $-i\coth(\lambda/2)\tan(n\theta/2)$ and radius $\coth(\lambda/2)\sec(n\theta/2)$ and c'_n is the arc of a circle with centre at $i\tanh(\lambda/2)\tan(n\theta/2)$ and radius $\tanh(\lambda/2)\sec(n\theta/2)$. As before, c_n is a constant distance $d(n\theta)$ from c_0 and c'_n is the same distance from c'_0. We define D_n, the subset of $\mathbb{L}$ between c_n and c'_n, and the lunes $B(n\theta)$ and $B'(n\theta)$ as before. Let

$$N(n\theta) = \bigcup_{g \in F} g(D^* \cap D_n).$$

Again $N(n\theta)$ is the $d(n\theta)$ neighbourhood of $N(F)$.

Furthermore, let n_0 be the integer with $(n_0 - 1)\theta < \pi \leq n_0\theta$. We define arcs c_{n_0} and c'_{n_0} which are now in the closed upper half plane. We also define $B(n_0\theta)$, $B'(n_0\theta)$ and $N(n_0\theta)$ geometrically but remark that these no longer have any metrical properties. An important observation is that $\mathbb{L}$ is contained in $N(n_0\theta)$.

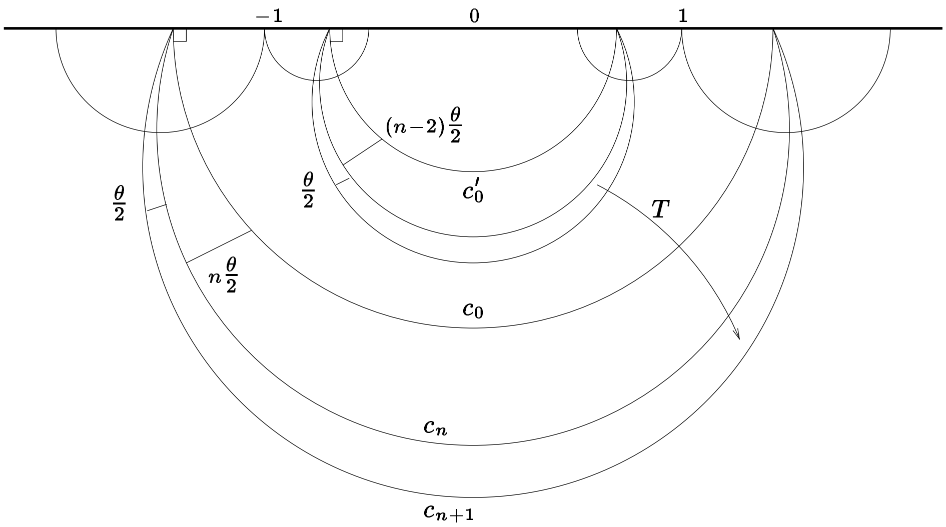

Figure 4.4 The inductive step

The rest of the proof follows by an induction from $n = 1$ up to $n = n_0$. We claim that, for $1 \leq n < n_0$ that if $B(n\theta)$ and $B'(n\theta)$ are in $\Omega(G)$ then so are

$B\big((n+1)\theta\big)$ and $B'\big((n+1)\theta\big)$. This in turn means that $N\big((n+1)\theta\big)$ is in $\Omega(G)$. In particular $N(n_0\theta)$, which contains $\mathbb{L}$, is in $\Omega(G)$.

Thus all we have do is prove the claim, which we now do. Since $B(n\theta)$ and $B'(n\theta)$ are contained in $\Omega(G)$ then so is $N(n\theta)$. Consider $T^{-1}B\big((n+1)\theta\big)$. Since c_{n+1} makes an angle of $n\theta/2$ with c_1 and T acts conformally on $\widehat{\mathbb{C}}$ we see that $T^{-1}(c_{n+1})$ makes an angle of $n\theta/2$ with $T^{-1}(c_1)=c_1'$, see Figure 4.4. In other words $T^{-1}(c_{n+1})$ is a hypercycle a constant distance $d\big((n-1)\theta\big)$ from c_0' (also it is not c_{n-1}'). This means that $T^{-1}(c_{n+1})$, and hence also $T^{-1}B\big((n+1)\theta\big)$, is contained within the $d(n\theta)$ neighbourhood of $N(F)$, that is $N(n\theta)$. Since $N(n\theta)$ was assumed to be in $\Omega(G)$, we see that $T^{-1}B\big((n+1)\theta\big)$ and hence also $B\big((n+1)\theta\big)$ is contained in $\Omega(G)$, as claimed. We remark that if $n>n_0$ then $T^{-1}(c_{n+1})$ lies in the closed upper half plane and the argument breaks down. A similar argument shows that $B'\big((n+1)\theta\big)$ is also contained in $\Omega(G)$. This completes the proof. □

Lemma 4.4 *If $\tau\in\mathcal{Q}_\lambda$ and $\theta\in(0,\pi)$ then the pleating locus on the "bottom" surface is S.*

Proof Suppose first that $\theta\in(0,\theta_0)$. From Lemma 4.3 we see that $\mathbb{L}$ is contained in $\Omega(G)$. Thus the geodesic plane in $\mathbb{H}^3$ with boundary the real axis is a support plane for $\partial C(G)$. Moreover the image of this plane under T must also be a support plane for $\partial C(G)$. As the intersection of these two planes is the axis of S we have the result.

Now consider $\tau=t+i\theta\in\mathcal{Q}_\lambda$ and $\theta\in[\theta_0,\pi)$. We proceed as in Proposition 5.4 of [5]. Suppose that S is not the pleating locus for the bottom surface. Consider a path α in $\mathcal{Q}_\lambda$ joining τ with $\tau'=t'+i\theta'$ where $\theta'\in(0,\theta_0)$. Without loss of generality, suppose that if $\tau\in\alpha$ then $\mathrm{Im}(\tau)\geq\theta'>0$. We know that at τ' the pleating locus on the bottom surface is S. Using the standard identification of projective measured laminations on the punctured torus with the extended real line (with the topology given by stereographic projection of the usual topology on the circle) then Keen and Series show that the pleating locus is continuous with respect to paths in $\mathcal{Q}$ [6]. Therefore there are points on the path α for which the pleating locus is a projective measured lamination arbitrarily close to γ_∞. In particular there are points where the pleating locus is γ_m for $m\in\mathbb{Z}$ which corresponds to $W_m=S^{-m}T\in G$ (in the next section we will give more details of how to associate words with simple closed curves). In particular, this group element must have real trace. In other words there is a point of α where $\mathrm{tr}(S^{-m}T)=2\cosh(\tau/2+m\lambda)\coth(\lambda)$ is real, and so

$$0=\sinh(t/2+m\lambda)\sin(\theta/2).$$

As $\theta \in [\theta', \pi)$ we see that $\sin(\theta/2) \neq 0$. Thus $t/2 + m\lambda = 0$ and $\mathrm{tr}(S^{-m}T) = 2\cos(\theta/2)$. This means $S^{-m}T$ is elliptic and so τ is not in $\mathcal{Q}_\lambda$ after all. □

Lemma 4.5 *With S and T as in the theorem and $\theta \in (0, \pi)$ (respectively $\theta \in (-\pi, 0)$) the complex shear σ along S with respect to T is $\sigma = \tau$ (respectively $\sigma = -\tau$).*

Proof The trace of T is

$$\cosh(\tau/2)\big(\coth(\lambda/2) + \tanh(\lambda/2)\big) = 2\cosh(\tau/2)\coth(\lambda).$$

Writing $\mathrm{tr}(T) = 2\cosh\big(\lambda(T)\big)$ and $\mathrm{tr}(S) = 2\cosh\big(\lambda(S)\big)$ the formula (I) of [18] gives the complex shear along S with respect to T as σ where

$$\begin{aligned}\cosh(\sigma/2) &= \cosh\big(\lambda(T)\big)\tanh\big(\lambda(S)\big)\\ &= \cosh(\tau/2)\coth(\lambda)\tanh(\lambda)\\ &= \cosh(\tau/2).\end{aligned}$$

Thus σ and τ agree up to sign and addition of multiples of $2\pi i$. Since $\mathrm{Im}(\sigma)$ is in $(0, \pi)$ we find that $\sigma = \tau$ when $\theta = \mathrm{Im}(\tau) > 0$ and $\sigma = -\tau$ when $\theta < 0$. □

5 Pleating rays on λ–slices

We have shown that on a λ–slice the pleating locus on one component of the convex hull boundary is γ_∞ which corresponds to S. We now investigate the intersection of each λ–slice with the rational pleating plane associated to the simple closed curves γ_∞ and $\gamma_{p/q}$. We call this intersection a *pleating ray*. Part of the this section will be a justification of this name.

In order to obtain pleating rays on each λ–slice, we follow the arguments in [5], many of which are inherently two-dimensional in nature. These arguments have been superseded by more general arguments in [8]. We give these arguments to help the reader interpret Figure 5.1 and Figure 6.1 without having to refer to [5] or [8]. But, since these arguments are not new, we shall not give all the details. Furthermore, we indicate how one may use pleating rays on λ–slices to obtain the rational pleating planes. This is the simplest part of the construction of pleating coordinates. The more complicated parts are treated at length in [8].

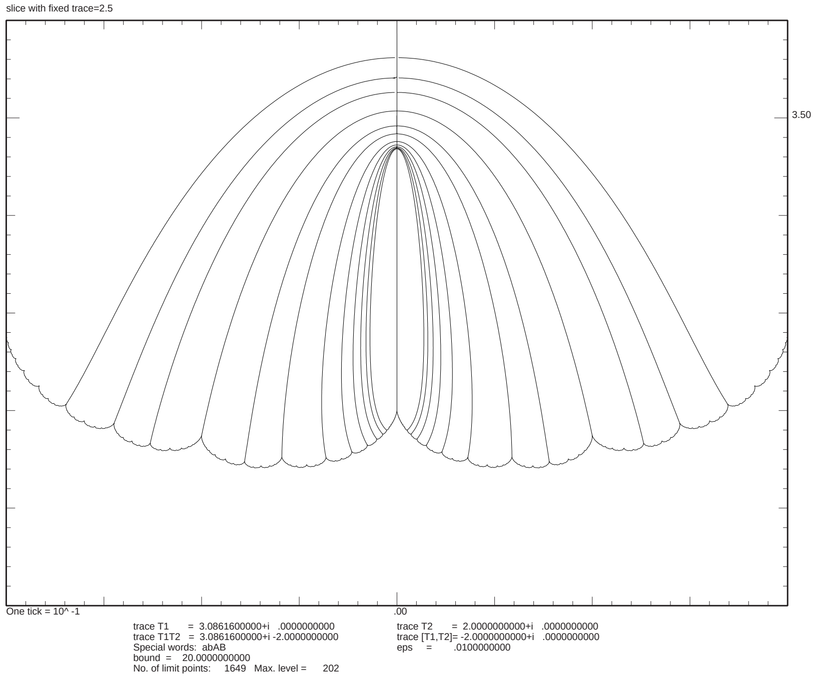

Figure 5.1 Part of a slice through $\mathcal{Q}$ with λ held to be real and fixed. In this case $\cosh(\lambda) = 5/4$. This figure shows the image of the slice under the 2 to 1 map $\tau \longmapsto i\,\mathrm{tr}\,T = 2i\cosh(\tau/2)\coth(\lambda) = \frac{10}{3}i\cosh(\tau/2)$. The figure shows pleating rays for this slice, see [8] or Section 6. The vertical line from $10i/3$ upwards represents Fuchsian space (which has been folded onto itself at the point corresponding to a rectangular torus). Observe that the pleating rays meet Fuchsian space orthogonally.

In what follows, we assume that the pleating locus on one component of the convex hull boundary is γ_∞, represented by S, and the pleating locus the other is also a simple closed curve, $\gamma_{p/q}$ for some $p/q \in \mathbb{Q}$. There is a special word $W_{p/q} \in G = \langle S, T\rangle$ corresponding to the homotopy class of simple closed curves $[\gamma_{p/q}]$. These words are defined recursively in [24] (see also Section 3.1 of [5]) but of course, we now need to use the generators S and T defined (1.1) and (1.3). First, $W_\infty = S^{-1}$, $W_m = S^{-m}T$ for $m \in \mathbb{Z}$. If $qr - ps = 1$ then we inductively define $W_{(p+r)/(q+s)} = W_{r/s}W_{p/q}$.

For each $\gamma_{p/q}$ the *p/q–pleating ray* $\mathcal{P}^\lambda_{p/q,\infty}$ on $\mathcal{Q}_\lambda$ is defined to be the those points of $\mathcal{Q}_\lambda$ for which the pleating locus is $\gamma_{p/q}$ on the "top" and γ_∞ on the "bottom". Thus these points have $\mathrm{Im}(\tau) \in (0,\pi)$, Theorem 4.2. Likewise $\mathcal{P}^\lambda_{\infty,p/q}$ consists of those points in $\mathcal{Q}_\lambda$ where the pleating locus on the "top" surface is γ_∞ and that on the "bottom" is $\gamma_{p/q}$. Such points have $\mathrm{Im}(\tau) \in$

$(-\pi, 0)$. This discussion may be summarised in the following result which should be compared to Theorem 5.1 of [5].

Proposition 5.1 *On each λ–slice $\mathcal{Q}_\lambda$ and for $p/q \in \mathbb{Q}$ the pleating rays $\mathcal{P}^\lambda_{p/q,\infty}$ and $\mathcal{P}^\lambda_{\infty,p/q}$ each consist of a non-empty, connected, non-singular arc on which $\mathrm{tr}(W_{p/q})$ is real and which meet $\mathcal{F}$ orthogonally at the same point from the opposite side. Their other end-points lie on the boundary of $\mathcal{Q}_\lambda$ and at these points $|\,\mathrm{tr}(W_{p/q})| = 2$.*

Some rational pleating rays are shown in the pictures Figures 5.1 and 6.1. It can be observed that the pleating rays are non-singular connected arcs that meet Fuchsian space orthogonally.

Sketch proof This is an adaptation of ideas in [5] and [8]. First we fix a particular λ–slice $\mathcal{Q}_\lambda$. In Theorem 4.2 we showed that γ_∞, represented by S, is the pleating locus on one component of the convex hull boundary. For definiteness we take this to be the "bottom" component. By symmetry all our arguments go through when the pleating loci are the other way round.

It was shown in Corollary 6.4 of [18] that, when the complex shear is purely imaginary, the pleating locus on the "top" component is T (that is γ_0). Using a change of generators (marking) as in Proposition 2.4, it follows that, when the real part of the complex shear is $-2m\lambda$, for an integer m, then the pleating locus on the "top" component is $S^{-m}T$ (that is γ_m). Consider the line where $\mathrm{Im}(\tau) = \theta_0/2$. Such groups are all quasi-Fuchsian (Theorem 4.1) and at $\tau = -2m\lambda + i\theta_0/2$ the pleating locus is γ_m for $m \in \mathbb{Z}$. Thus, by the continuity of the pleating locus, see [6], as we move along this line we find points whose pleating locus is given by any real parameter. This shows that any real pleating ray on $\mathcal{Q}_\lambda$ is non-empty.

It is clear that $\mathcal{P}^\lambda_{p/q,\infty}$ is contained in the real locus of $\mathrm{tr}(W_{p/q})$. We now investigate how this real locus meets Fuchsian space. Any brach of the real locus of $\mathrm{tr}(W_{p/q})$ contained in $\mathcal{Q}_\lambda - \mathcal{F}$ meets $\mathcal{F}$ in a singularity of $\mathrm{tr}(W_{p/q})$. A result of Wolpert, page 226 of [23], says that the second derivative of $\left|\mathrm{tr}(W_{p/q})\right|$ with respect to τ along Fuchsian space is strictly positive. (We have used here that $\gamma_{p/q}$ and γ_∞ are both simple and they intersect.) Thus $\mathrm{tr}(W_{p/q})$ has a unique singularity in $\mathcal{F}$ and this singularity is quadratic. Hence the branches of its real locus must meet orthogonally. In particular there is one branch meeting $\mathcal{F}$ at this point on which $\mathrm{Im}(\tau) > 0$ and one brach where $\mathrm{Im}(\tau) < 0$.

For $0 < p/q < 1$ the pleating ray $\mathcal{P}^\lambda_{p/q,\infty}$ (which is non-empty) must be contained in the open set bounded by $\mathcal{F}$, that is $\mathrm{Im}(\tau) = 0$; the pleating rays

$\mathcal{P}^\lambda_{0,\infty}$, that is $\mathrm{Re}(\tau)=0$, and $\mathcal{P}^\lambda_{1,\infty}$, that is $\mathrm{Re}(\tau)=-2\lambda$; and the boundary of $\mathcal{Q}_\lambda$. The pleating ray must be a union of connected components of the intersection of this set with the real locus of $\mathrm{tr}(W_{p/q})$. The proof of this statement follows Proposition 5.4 of [5]. A similar argument has been used in Lemma 4.4 so we will not repeat it. It is clear that if the pleating locus on the "top" is $\gamma_{p/q}$ and if $|\,\mathrm{tr}(W_{p/q})|>2$ then the group is in the interior of $\mathcal{Q}$. Thus, moving along $\mathcal{P}^\lambda_{p/q,\infty}$ in the direction of increasing $|\,\mathrm{tr}(W_{p/q})|$ we cannot reach the boundary of $\mathcal{Q}$ and so we must reach $\mathcal{F}$. It follows that $\mathcal{P}^\lambda_{p/q,\infty}$ is connected and non-singular. If not, there would be at least two branches of $\mathcal{P}^\lambda_{p/q,\infty}$ on which $|\,\mathrm{tr}(W_{p/q})|$ is increasing. But there is only one branch that meets $\mathcal{F}$, a contradiction. A similar analysis takes care of other p/q.

Finally, when $|\,\mathrm{tr}(W_{p/q})|=2$ the pleating ray reaches the boundary of $\mathcal{Q}_\lambda$ and the curve $\gamma_{p/q}$ has become parabolic. This completes our sketch proof of Proposition 5.1. □

In order to obtain the pleating planes associated to the pairs γ_∞, $\gamma_{p/q}$ we must vary λ. As we do this, the pleating rays on each λ–slice now sweep out the whole pleating plane. Keen and Series prove that this gives a connected, non-singular two dimensional subset of $\mathcal{Q}$. In order to obtain pleating planes associated to other pairs of curves we use the change of coordinates given in Proposition 2.4. Specifically, if the pleating loci we are interested in are $\gamma_{a/b}$ and $\gamma_{c/d}$ which intersect $q=ad-bc\neq 0$ times then there is a sequence of Nielsen moves taking the pair $(\gamma_\infty,\gamma_{p/q})$ to the pair $(\gamma_{a/b},\gamma_{c/d})$. Associated to these Nielsen moves is a biholomorphic change of coordinates on $\mathcal{Q}$ and the pleating plane associated to $\gamma_{a/b}$ and $\gamma_{c/d}$ is the image under this change of coordinates of the pleating plane associated to γ_∞ and $\gamma_{p/q}$.

We conclude this section with a discussion of how one may take data associated to one component of the convex hull boundary and find information about the other component. At first sight it does not seem clear how this could be done. But, at least when the pleating locus on one component is a simple closed curve, this follows from the relationship between complex Fenchel–Nielsen coordinates and Keen–Series pleating invariants. Let G be a strictly quasi-Fuchsian punctured torus group. Suppose that the pleating locus on one component of the convex hull boundary is a simple closed curve γ of length λ. Then we can construct Fenchel–Nielsen coordinates relative to a generating pair S, T where γ is represented by S. The complex Fenchel–Nielsen coordinates are given purely in terms of data associated to the component of the convex hull boundary on which γ is the pleating locus. By considering the associated λ–slice $\mathcal{Q}_\lambda$, we can find the Keen–Series pleating invariants for G in terms of the complex Fenchel–Nielsen coordinates. We have not mentioned lamination length on $\mathcal{Q}_\lambda$

in the above discussion. It suffices to remark that when the lamination on the other component of the convex hull boundary is also a simple closed curve given by $W \in G$, then the lamination length can be easily found from $\mathrm{tr}(W)$. For irrational pleating rays, we just use a continuity argument. In particular, we can determine information about the pleating on the other component of the convex hull boundary (this generalises Corollary 6.4 of [18], where it is shown that if the pleating locus on one component of $\partial C(G)/G$ is S and the complex shear is purely imaginary then the pleating locus on the other component is T). Moreover, if the pleating locus on the other component of the convex hull boundary is also a simple closed curve, we can use a sequence of Nielsen moves (see Proposition 2.4) to determine the Fenchel–Nielsen coordinates with respect to δ. In fact this is very straightforward.

On the other hand, suppose the pleating locus is an infinite measured lamination μ with lamination length l_μ. The projective class (μ, l_μ) (see [8]) generalises the choice of simple closed curve with δ–measure and the hyperbolic length of that curve. It follows from the work of Epstein–Marden, [4], that the group is completely determined by (μ, l_μ) and the quake-bend parameter τ (see [7, 8] for a discussion of the quake-bend parameter for quasi-Fuchsian punctured torus groups). These generalise the Fenchel–Nielsen coordinates for an infinite lamination. However, it does not seem that there is a straightforward way to go explicitly from these parameters to the pleating invariants or to the corresponding parameters on the other component of the convex hull boundary.

6 Degeneration to the Maskit embedding

In the previous sections we have considered what happens when λ is a fixed real positive number. In this section, we consider what happens when $\lambda = 0$. We should expect the complex shear to tend to $i\pi$ as λ tends to 0 (compare Theorem 4.1(i) of [18], see Proposition 6.1 below). This means that complex Fenchel–Nielsen coordinates degenerate. In this section we show that by using the plumbing parameter instead, we obtain the *Maskit embedding* of Teichmüller space, denoted $\mathcal{M}$ (see [24, 5]). This is defined to be the space of free Kleinian groups G on two generators S, T up to conjugation, such that each group has the following properties. First, the generator S and the commutator $K = T^{-1}S^{-1}TS$ are both parabolic. Secondly, the components of the ordinary set are of two kinds. Namely, a simply connected, G–invariant component whose quotient is a punctured torus; and also infinitely many round discs whose stabilisers are thrice punctured sphere groups, all conjugate within G. In other words these groups are terminal b-groups. This space is a holomorphically parametrised copy of the Teichmüller space of a punctured torus.

There is a standard normal form for the generators in terms of a parameter μ, see [24, 5], which is

$$S_0 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad T_0 = \begin{pmatrix} -i\mu & -i \\ -i & 0 \end{pmatrix}. \tag{6.1}$$

The goal of this section is to show that as we let λ tend to zero, the normal form for S and T given in (1.1), (1.3) degenerate to generators of groups in the Maskit embedding (6.1). Moreover, the λ slices $\mathcal{Q}_\lambda$ with their pleating rays tend to the Maskit embedding with its pleating rays. We illustrate this with a series of pictures which should be compared to Figure 1 of [5]. There is a discussion of how the Maskit embedding lies on the boundary of quasi-Fuchsian space on page 190 of [18].

Consider the limit of S as λ tends to zero:

$$S_0 = \lim_{\lambda \to 0} \begin{pmatrix} \cosh(\lambda) & \cosh(\lambda)+1 \\ \cosh(\lambda)-1 & \cosh(\lambda) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Similarly the limit of $S' = T^{-1}ST$ as λ tends to zero is:

$$S'_0 = \lim_{\lambda \to 0} \begin{pmatrix} \cosh(\lambda) & \cosh(\lambda)-1 \\ \cosh(\lambda)+1 & \cosh(\lambda) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

The parabolic transformations S_0 and S'_0 generate the level 2 principal congruence subgroup of $\mathrm{PSL}(2,\mathbb{Z})$, a torsion-free triangle group. A comparison of the plumbing parameter calculated in Lemma 3.1 with the corresponding result for terminal b-groups (see Kra [12; Section 6.4]) suggests that, in order to study the degeneration of quasi-Fuchsian groups in $\bigcup_{\lambda>0} \mathcal{Q}_\lambda$ as $\lambda \longrightarrow 0$, it is useful to make a change of parameters

$$\mu = \frac{i\pi - \tau}{\lambda}.$$

We refer to μ as the *plumbing parameter*. In terms of this parameter the matrix T can be written as

$$T = \begin{pmatrix} -i\sinh(\lambda\mu/2)\coth(\lambda/2) & -i\cosh(\lambda\mu/2) \\ -i\cosh(\lambda\mu/2) & -i\sinh(\lambda\mu/2)\tanh(\lambda/2) \end{pmatrix}. \tag{6.2}$$

Using Lemma 3.1, we see that (1.1) and (6.2) give a parametrisation of the generators of G in terms of a length parameter and a plumbing parameter. The following result on the limit groups, which should be compared to Theorem 4.1(i) of [18], now follows rather easily:

Proposition 6.1 *Consider a sequence of groups where λ tends to zero but μ remains fixed. Then the complex shear along S tends to $i\pi$.*

Proof The conclusion is immediate from the definition of μ: $\tau = i\pi - \mu\lambda \to i\pi$ as $\lambda \to 0$. □

We now show that when λ tends to zero with μ being kept fixed we obtain the standard form for group generators in the Maskit embedding.

Proposition 6.2 *Assume that $\mu \in \mathcal{Q}_\lambda$ for small λ. As λ tends to zero the group with parameter (λ, μ) tends to the terminal b-group representing punctured torus on its invariant component with parameter μ.*

Proof We have already seen that S_0 and S_0' have the correct form.

Let μ be fixed. For small λ we have

$$\sinh(\lambda\mu/2)\coth(\lambda/2) = \big(\lambda\mu/2 + O(\lambda^2)\big)\big(2/\lambda + O(1)\big) = \mu + O(\lambda).$$

Therefore we have

$$\lim_{\lambda\to 0}\big(\sinh(\lambda\mu/2)\coth(\lambda/2)\big) = \mu.$$

This means that the limit as λ tends to zero of T is

$$\begin{aligned} T_0 &= \lim_{\lambda\to 0}\begin{pmatrix} -i\sinh(\lambda\mu/2)\coth(\lambda/2) & -i\cosh(\lambda\mu/2) \\ -i\cosh(\lambda\mu/2) & -i\sinh(\lambda\mu/2)\tanh(\lambda/2) \end{pmatrix} \\ &= \begin{pmatrix} -i\mu & -i \\ -i & 0 \end{pmatrix}. \end{aligned}$$

The limiting matrices S_0 and T_0 are just the usual group generators of terminal b-groups in the Maskit embedding $\mathcal{M}$ of Teichmüller space of the punctured torus. □

The convergence of λ–slices to $\mathcal{M}$ is illustrated in Figure 6.1.

Remarks 6.3 (a) The plumbing construction is tame when $\mathrm{Im}(\tau) = \theta \in (0, \theta_0)$ or equivalently $\mathrm{Im}(\mu) \in \big((\pi - \theta_0)/\lambda, \pi/\lambda\big)$. For small λ we have $\theta_0 = \pi - 2\lambda + O(\lambda)^2$. As λ tends to zero this interval tends to $(2, \infty)$, which is the condition for tame plumbing in the Maskit slice, Section 6.2 of [12] or Proposition 2.3 of [24].

(b) In the (λ, μ) parameters, Fuchsian space corresponds to the union of the lines $\mathrm{Im}(\mu) = \pi/\lambda$. When $\lambda \to 0$, $\mathrm{Im}\,\mu \to \infty$, that is, the closure of Fuchsian space touches $\mathcal{M}$ at the boundary point corresponding to the parameter $\mu = \infty$ (see page 191 of [18]).

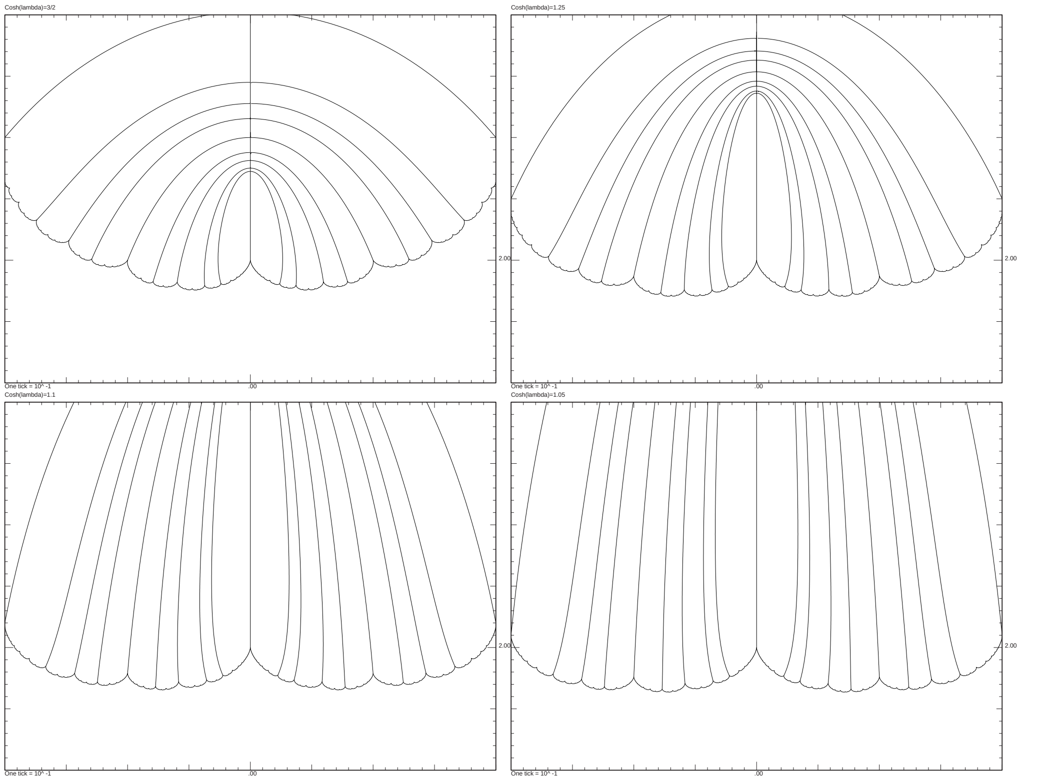

Figure 6.1 λ–slices for $\mathrm{tr}(T) = 3,\ 2.5,\ 2.2$ and 2.1 drawn with a collection of rational pleating rays

References

[1] **P Arés**, *Coordinates for Teichmüller spaces of b-groups with torsion*, Ann. Acad. Sci. Fenn. 20 (1995) 279–300

[2] **P Buser**, *Geometry and Spectra of Compact Riemann Surfaces*, Birkhauser (1992)

[3] **C J Earle**, **A Marden**, *Geometric complex coordinates for Teichmüller space*, in preparation

[4] **D B A Epstein**, **A Marden**, *Convex hulls in hyperbolic space, a theorem of Sullivan and measured pleated surfaces*, from: "Analytic and geometric aspects of hyperbolic space", D B A Epstein (editor), LMS Lecture Notes 111, Cambridge University Press, (1987) 112–253

[5] **L Keen**, **C Series**, *Pleating coordinates for the Maskit embedding of the Teichmüller space of punctured tori*, Topology 32 (1993) 719–749

[6] **L Keen**, **C Series**, *Continuity of convex hull boundaries*, Pacific J. Math. 168 (1995) 183–206

[7] **L Keen**, **C Series**, *How to bend pairs of punctured tori*, from: "Lipa's legacy", J Dodziuk and L Keen (editors), Contemporary Mathematics 211 (1997) 359–387

[8] **L Keen**, **C Series**, *Pleating invariants for punctured torus groups, Warwick preprint 10/1998*

[9] **C Kourouniotis**, *Deformations of hyperbolic structures* Math. Proc. Cambridge Phil. Soc. 98 (1985) 247–261

[10] **C Kourouniotis**, *The geometry of bending quasi-Fuchsian groups*, from: "Discrete groups and geometry", W J Harvey and C Maclachlan (editors), LMS Lecture Notes 173, Cambridge University Press, (1992) 148–164

[11] **C Kourouniotis**, *Complex length coordinates for quasi-Fuchsian groups*, Mathematika 41 (1994) 173–188

[12] **I Kra**, *Horocyclic coordinates for Riemann surfaces and moduli spaces I: Teichmüller and Riemann spaces of Kleinian groups*, Journal Amer. Math. Soc. 3 (1990) 500–578

[13] **I Kra**, **B Maskit**, *The deformation space of a Kleinian group*, American J. Math. 103 (1980) 1065–1102

[14] **B Maskit**, *On Klein's combination theorem II*, Trans. AMS 131 (1968) 32–39

[15] **B Maskit**, *Kleinian Groups*, Springer–Verlag, (1987)

[16] **C McMullen**, *Complex earthquakes and Teichmüller theory*, J. Amer. Math. Soc. 11 (1998) 283–320

[17] **J Nielsen**, *Die Isomorphismen der allgemeinen unendlichen Gruppe mit zwei Erzeugenden*, Math. Ann. 78 (1918) 385–397

[18] **J R Parker**, **C Series**, *Bending formulae for convex hull boundaries*, J. d'Analyse Math. 67 (1995) 165–198

[19] **J Parkkonen**, *Geometric complex analytic coordinates for deformation spaces of Koebe groups*, Ann. Acad. Sci. Fenn. Math. Diss. 102 (1995) 1–50

[20] **S P Tan**, *Complex Fenchel-Nielsen coordinates for quasi-Fuchsian structures*, International J. Math. 5 (1994) 239–251

[21] **P L Waterman**, **S A Wolpert**, *Earthquakes and tessellations of Teichmüller space*, Trans. AMS 278 (1983) 157–167

[22] **S A Wolpert**, *The Fenchel–Nielsen deformation*, Annals of Math. 115 (1982) 501–528

[23] **S A Wolpert**, *On the symplectic geometry of deformations of a hyperbolic surface*, Annals of Math. 117 (1983) 207–234

[24] **D J Wright**, *The shape of the boundary of Maskit's embedding of the Teichmüller space of once punctured tori*, preprint

Department of Mathematical Sciences, University of Durham
Durham DH1 3LE, UK

Department of Mathematics, University of Jyväskylä
P.O.Box 35, FIN-40351 Jyväskylä, Finland

Email: `J.R.Parker@durham.ac.uk, parkkone@math.jyu.fi`

Received: 12 November 1997 Revised: 12 August 1998