ISSN 1464-8997 **99**

Geometry & Topology Monographs
Volume 1: The Epstein Birthday Schrift
Pages 99–116

Controlled embeddings into groups that have no non-trivial finite quotients

Martin R Bridson

Abstract If a class of finitely generated groups $\mathcal{G}$ is closed under isometric amalgamations along free subgroups, then every $G \in \mathcal{G}$ can be quasi-isometrically embedded in a group $\widehat{G} \in \mathcal{G}$ that has no proper subgroups of finite index.

Every compact, connected, non-positively curved space X admits an isometric embedding into a compact, connected, non-positively curved space $\overline{X}$ such that $\overline{X}$ has no non-trivial finite-sheeted coverings.

AMS Classification 20E26, 20E06, 53C70; 20F32, 20F06

Keywords Finite quotients, embeddings, non-positive curvature

David Epstein's lucid writings, particularly those on automatic groups, had a strong influence on me when I was a graduate student. Since then, during many hours of enjoyable conversation, I have continued to benefit from his great insight into mathematics. It was therefore a great pleasure to speak at his birthday celebration and it is an equal pleasure to write an article for this volume.

0 Introduction

In this article I shall address the following general question: given a finitely generated group G that satisfies certain desirable properties, when can one embed G into a group which retains these desirable properties but does not have any non-trivial finite quotients? My interest in this question arises from a geometric problem that is the subject of Theorem C.

Our discussion begins with a general embedding theorem which is similar to results that were proved in the wake of the landmark paper by Higman, Neumann and Neumann [11]. The novel element in the result presented here is that we control the geometry of the embedding.

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Theorem A *Let $\mathcal{G}$ be a class of finitely generated groups. If $\mathcal{G}$ is closed under the operation of isometric amalgamation along finitely generated free groups, then every $G \in \mathcal{G}$ can be quasi-isometrically embedded in a group $\widehat{G} \in \mathcal{G}$ that has no proper subgroups of finite index.*

The definition of isometric amalgamation is given in Section 1. There are various interesting classes of groups that are closed under amalgamations along arbitrary finitely generated free groups, for example the class of all finitely presented groups, groups of type F_n, and groups of a given (cohomological or geometric) dimension $n \geq 2$. The benefit of restricting the geometry of the amalgamation becomes apparent when the defining properties of $\mathcal{G}$ are more geometric in nature. For example, the class of groups which satisfy a polynomial isoperimetric inequality is not closed under the operation of amalgamation along arbitrary finitely generated free groups (or indeed along quasi-isometrically embedded free groups), but it is closed under amalgamation along isometrically embedded subgroups (Corollary 4.2).

A refinement of the proof of Theorem A yields:

Theorem B *Every finitely presented group G can be embedded in a finitely presented group $\widehat{G}$ that has no non-trivial finite quotients and whose Dehn function $f_{\widehat{G}}$ satisfies:*

$$f_{\widehat{G}}(n) \leq n f_G(n).$$

One can (simultaneously) arrange for the isodiametric function of $\widehat{G}$ to be no greater than that of G.

Theorem A does not apply directly to the class of groups that arise as fundamental groups of compact non-positively curved spaces.[1] Nevertheless, using a more subtle argument based on the same blueprint of proof, in Section 3 we shall prove the following theorem. (We say that a covering $\widehat{Z} \to Z$ is 'non-trivial' if $\widehat{Z}$ is connected and $\widehat{Z} \to Z$ is not a homeomorphism.)

Theorem C *Every compact, connected, non-positively curved space X admits an isometric embedding into a compact, connected, non-positively curved space $\overline{X}$ such that $\overline{X}$ has no non-trivial finite-sheeted coverings. If X is a polyhedral complex of dimension $n \geq 2$, then one can arrange for $\overline{X}$ to be a complex of the same dimension.*

[1]Throughout this article we use the term 'non-positive curvature' in the sense of A.D. Alexandrov [3].

Any local isometry between compact non-positively curved spaces induces an injection on fundamental groups [3, II.4], so in the notation of Theorem C we have $\pi_1 X \hookrightarrow \pi_1 \overline{X}$. Since $\overline{X}$ has no non-trivial finite-sheeted coverings, $\pi_1 \overline{X}$ has no proper subgroups of finite index. Thus Theorem C gives a solution to our general embedding problem for the class of groups that arise as fundamental groups of compact non-positively curved spaces. An extension of Theorem C yields the corresponding result for groups that act properly and cocompactly on CAT(0) spaces (3.6).

The fundamental groups of the most classical examples of non-positively curved spaces, quotients of symmetric spaces of non-compact type, are residually finite. In 1995 Dani Wise produced the first examples of compact non-positively curved spaces whose fundamental groups have no non-trivial finite quotients [21]. He also constructed semihyperbolic groups that are not virtually torison free, cf (3.7). Subsequently, Burger and Mozes [5] constructed compact non-positively curved 2-complexes whose fundamental groups are simple. Fundamental groups of compact negatively curved spaces, on the other hand, are never simple [8], [16].

One might hope to prove an analogue of Theorem A in which the enveloping group $\widehat{G}$ is simple. However the techniques described in this article are clearly inadequate in this regard. Indeed, finitely presented simple groups have solvable word problems and hence so do their finitely presented subgroups. Thus if one wishes to embed a given finitely presented group G into a finitely presented simple group, then one must make essential use of the fact that G has a solvable word problem. Higman conjectures that the solvability of the word problem is the only obstruction to the existence of such an embedding [10] (cf [4], [17]).

This article is organized as follows. In Section 1 we describe some examples of groups that are not residually finite and define isometric amalgamation. In Section 2 we prove Theorem A. In Section 3 we discuss spaces of non-positive curvature and prove Theorem C. In Section 4 we examine the effect of isometric amalgamations on isoperimetric and isodiametric inequalities and prove Theorem B.

This article grew out of a lecture which I gave at the conference on Geometric Group Theory at Canberra in July 1996. I would like to thank the organizers of that conference. I would particularly like to thank Chuck Miller for arranging my visit and for welcoming me so warmly.

1 Residual finiteness and isometric amalgamation

A group G is said to be *residually finite* if for every non-trivial element $g \in G$ there is a finite group Q and an epimorphism $\phi\colon G \twoheadrightarrow Q$ such that $\phi(g) \neq 1$. As a first step towards producing groups with no finite quotients, we must gather a supply of groups that are not residually finite. The Hopf property provides a useful tool in this regard. A group H is said to be *Hopfian* if every epimorphism $H \twoheadrightarrow H$ is an isomorphism — in other words, if $N \subset H$ is normal and $H/N \cong H$ then $N = \{1\}$.

The following result was first proved by Malcev [14].

1.1 Proposition *If a finitely generated group is residually finite then it is Hopfian.*

Proof Let G be a finitely generated group and suppose that there is an epimorphism $\phi\colon G \to G$ with non-trivial kernel. We fix $g_0 \in \ker\phi \smallsetminus \{1\}$ and for every $n > 0$ we choose $g_n \in G$ such that $\phi^n(g_n) = g_0$.

If there were a finite group Q and a homomorphism $p\colon G \to Q$ such that $p(g_0) \neq 1$, then all of the maps $\phi_n := p\phi^n$ would be distinct, because $\phi_n(g_n) \neq 1$ whereas $\phi_m(g_n) = 1$ if $m > n$. But there are only finitely many homomorphisms from any finitely generated group to any finite group (because the images of the generators determine the map). □

1.2 Examples The following group was discovered by Baumslag and Solitar [6]:

$$\mathrm{BS}(2,3) = \langle a, t \mid t^{-1}a^2t = a^3\rangle.$$

The map $a \mapsto a^2, t \mapsto t$ is onto: a is in the image because $a = a^3a^{-2} = (t^{-1}a^2t)a^{-2}$. However this map is not an isomorphism: $[a, t^{-1}at]$ is a non-trivial element of the kernel. Meier [15] noticed that the salient features of this example are present in many other HNN extensions of abelian groups. Some of these groups were later studied by Wise [19], among them

$$T(n) = \langle a, b, t_a, t_b \mid [a,b] = 1,\ t_a^{-1}at_a = (ab)^n,\ t_b^{-1}bt_b = (ab)^n\rangle,$$

which is the fundamental group of a compact non-positively curved 2-complex (see (3.1)). If $n \geq 2$ then certain non-trivial commutators, for example $g_0 = [t_a(ab)t_a^{-1}, b]$, lie in the kernel of the epimorphism $T(n) \twoheadrightarrow T(n)$ given by $a \mapsto a^n, b \mapsto b^n, t_a \mapsto t_a, t_b \mapsto t_b$. The proof of (1.1) shows that g_0 has trivial image in every finite quotient of $T(n)$.

1.3 Definition of Isometric Amalgamation Let $H \subset G$ be a pair of groups with fixed finite generating sets. If, in the corresponding word metrics, $d_G(h,h') = d_H(h,h')$ for all $h,h' \in H$, then we say that H is *isometrically embedded* in G.

Consider a finite graph of groups (in the sense of Serre [18]). If one can choose finite generating sets for the vertex groups G_i and the edge groups $H_{i,j}$ such that the inclusions of the edge groups are all isometric embeddings, then we say that the fundamental group Γ of the graph of groups is obtained by an *isometric amalgamation of the* G_i *along the* $H_{i,j}$ or, more briefly, Γ *is an isometric amalgam of the* G_i.

Note that, with respect to the natural choice of generators, all of the vertex and edge groups are isometrically embedded in the amalgam. Note also that, even in the basic cases of HNN extensions and amalgamated free products, the above definition is more stringent than simply requiring that for each i,j there exist choices of generators (depending on i,j) with respect to which $H_{i,j} \hookrightarrow G_i$ is an isometric embedding.

Free products of finitely generated groups are (trivial) examples of isometric amalgams. One can also obtain both $G \times \mathbb{Z}$ and $G * \mathbb{Z}$ from G by isometric amalgamations: each is the fundamental group of a graph of groups with one vertex group G and one edge group; to obtain $G \times \mathbb{Z}$ one takes G as edge group and uses the identity map as the inclusions; to obtain $G * \mathbb{Z}$ one takes the edge group to be trivial.

1.4 Lemma *Let $\mathcal{G}$ be as in Theorem A and let $T(n)$ be as in (1.2). If $G \in \mathcal{G}$ then $G * T(n) \in \mathcal{G}$.*

Proof Fix a finite generating set $\mathcal{S}$ for G. As above $G * \mathbb{Z} \in \mathcal{G}$; let a be a generator of the $\mathbb{Z}$ free factor. The cyclic subgroup generated by a is isometrically embedded with respect to the generating system $\mathcal{S} \cup \{a\}$. We add a further stable letter b that commutes with a, thus obtaining $G * \mathbb{Z}^2 \in \mathcal{G}$.

With respect to $\mathcal{S} \cup \{a, b, (ab)^n\}$, the cyclic subgroups generated by a, b and $(ab)^n$ are all isometrically embedded. Thus $G * T(n)$ can be obtained from $G * \mathbb{Z}^2$ by an isometric amalgamation: the underlying graph of groups has one vertex group, $G * \mathbb{Z}^2$, there are two edges in the graph and both edge groups are cyclic; the homomorphism at one end of each edge sends the generator to $(ab)^n$, and the maps at the other ends are onto $\langle a \rangle$ and $\langle b \rangle$ respectively. $\square$

2 The proof of Theorem A

In order to clarify the exposition, we shall first prove a simplified version of Theorem A in which we do not examine the geometry of the amalgamations involved.

2.1 Lemma *Let $\mathcal{G}$ be a class of groups that is closed under the operation of amalgamation along finitely generated free groups. If $G \in \mathcal{G}$ is finitely generated, then it can be embedded in a finitely generated group $\widehat{G} \in \mathcal{G}$ that has no proper subgroups of finite index.*

Proof The following proof is chosen with Theorem A in mind (shorter proofs exist). A similar construction was used in [21].

Step 0 Replacing G by $G_0 = G * T(n)$ if necessary, we may assume that G contains an element of infinite order $g_0 \in G$ whose image in every finite quotient of G_0 is trivial (see (1.2)). Let $\{b_1, \ldots, b_n\}$ be a generating set for G_0. We replace G_0 by $G_1 = G_0 * \mathbb{Z}$, and take as generators $\mathcal{A}' := \{t, b_1 t, \ldots, b_n t\}$, where t generates the free factor $\mathbb{Z}$. We relabel the generators $\mathcal{A}' = \{a_0, \ldots, a_n\}$.

Step 1 We take an HNN extension of G_1 with n stable letters:

$$E_1 = \langle G_1, s_0, \ldots, s_n \mid s_i^{-1} a_i s_i = g_0^{p_i},\ i = 0, \ldots, n \rangle.$$

where the p_i are any non-zero integers. Now, since each a_i is conjugate to a power of g_0 in E_1, the only generators of E_1 that can survive in any finite quotient are the s_i. However, since there is an obvious retraction of E_1 onto the free subgroup generated by the s_i, the group E_1 still has plenty of finite quotients.

Step 2 We repeat the extension process, this time introducing stable letters τ_i to make the generators s_i conjugate to g_0:

$$E_2 = \langle E_1, \tau_0, \ldots, \tau_n \mid \tau_i^{-1} s_i \tau_i = g_0,\ i = 0, \ldots, n \rangle.$$

Step 3 Add a single stable letter σ that conjugates the free subgroup of E_2 generated by the s_i to the free subgroup of E_2 generated by the τ_i:

$$E_3 = \langle E_2, \sigma \mid \sigma^{-1} s_i \sigma = \tau_i,\ i = 0, \ldots, n \rangle.$$

At this stage we have a group in which all of the generators except σ are conjugate to g_0. In particular, every finite quotient of E_3 is cyclic.

Step 4 Because no power of a_0 lies in either of the subgroups of E_2 generated by the s_i or the τ_i, the normal form theorem for HNN extensions implies that $\{a_0, \sigma\}$ freely generates a free subgroup of E_3.

We define $\widehat{G}$ to be an amalgamated free product of two copies of E_3,

$$\widehat{G} = E_3 *_F \overline{E_3},$$

where $F = F(x,y)$ is a free group of rank two; the inclusion into E_3 is $x \mapsto a_0$ and $y \mapsto \sigma$, and the inclusion into $\overline{E}_3$ is $x \mapsto \overline{\sigma}$ and $y \mapsto \overline{a}_0$. All of the generators of $\widehat{G}$ are conjugate to a power of either g_0 or $\overline{g_0}$, and therefore cannot survive in any finite quotient. In other words, $\widehat{G}$ has no finite quotients. $\square$

The following lemma enables us to gauge the geometry of the embeddings in the preceding construction.

2.2 Lemma *Let G be a group with finite generating set $\mathcal{A}$, where no $a \in \mathcal{A}$ represents $1 \in G$.*

(1) *In any HNN extension of G with finitely many stable letters $s_0, \ldots, s_n$, the free subgroup generated by $S = \{s_0, \ldots, s_n\}$ is isometrically embedded with respect to $\mathcal{A} \cup S$. If $\langle a \rangle \subset G$ is isometrically embedded and has trivial intersection with the amalgamated subgroups of s_i then $\mathrm{gp}\{a, s_i\}$ is isometrically embedded in the HNN extension.*

(2) *If $H \subset G$ is isometrically embedded with respect to $\mathcal{A}$, then H is also isometrically embedded in any isometric amalgamation involving G as a vertex group (provided the amalgamation is isometric with respect to the same generating set $\mathcal{A}$).*

(3) *Let $g \in G \smallsetminus \{1\}$. The cyclic subgroups of $G * \langle t \rangle$ generated by t, by $[g,t]$, and by each (at) with $a \in \mathcal{A}$, are all isometrically embedded with respect to the choice of generators $\mathcal{A}^* = \{at, [g,t], t \mid a \in \mathcal{A}\}$.*

Proof (1) and (2) follow from the normal form theorem for graphs of groups [18].

The normal form theorem for free products tells us that if we write $[g,t]^n$ as a word in the generators $\mathcal{A} \cup \{t\}$, then that word must contain at least $2n$ occurences of $t^{\pm 1}$. Each of the elements of $\mathcal{A}^*$ contains at most two occurences of $t^{\pm 1}$, therefore $d_{\mathcal{A}^*}(1, [g,t]^n) = n$.

If a word over $\mathcal{A} \cup \{t\}$ equals $(at)^n$ in $G * \langle t \rangle$, then its exponent sum in t must be n. Therefore, since each of the generators in $\mathcal{A}^*$ has t-exponent sum 1 or 0, we have $d_{\mathcal{A}^*}(1, (at)^n) = n$. $\square$

2.3 The Proof of Theorem A We follow the proof of (2.1). What we must ensure is that at each stage the embedding which we described can be performed by means of an *isometric* amalgamation.

First we choose a finite generating set $\mathcal{A}$ for $G_0 = G * T(n)$ so that $G \hookrightarrow G_0$ is an isometric embedding, and we fix an element $g \in G_0$ whose image is trivial in every finite quotient of G_0. Then as generators for $G_1 = G_0 * \langle t \rangle$ we take $\mathcal{A}^* := \{at, [g,t], t \mid a \in \mathcal{A}\}$. Note the difference with (2.1) — we have included $[g,t]$. Define $g_0 = [g,t]$.

Lemma 2.2(3) assures us that the amalgamations carried out in Step 1 of the proof of (2.1) are along isometrically embedded subgroups provided that we take all $p_i = 1$. And parts (1) and (2) of Lemma 2.2 imply that the amalgamations carried out in Steps 2, 3 and 4 of (2.1) are also along isometrically embedded subgroups. Thus we obtain the desired group $\widehat{G} \in \mathcal{G}$ that has no finite quotients.

We have the inclusions $G \subset G_0 \subset G_1 \subset \widehat{G}$. The third inclusion was constructed to be an isometric embedding. The first and second inclusions are obviously isometric embeddings with respect to natural choices of generators. But it does not follow that $G \hookrightarrow \widehat{G}$ is an isometric embedding, because at the end of Step 0 of the proof we switched from the obvious set of generators for G_1 to a less natural set that was suited to our purpose. On the other hand, for any finitely generated group H, the identity map between the metric spaces obtained by endowing H with different word metrics is bi-Lipschitz. Thus, $G \subset \widehat{G}_0$ is a quasi-isometric embedding (with respect to any choice of word metrics). $\square$

For future reference we note:

2.4 Lemma *The cyclic subgroups generated by all of the stable letters introduced in the above construction are isometrically embedded in* $\widehat{G}$.

3 The non-positively curved case

The proof that we shall give of Theorem C is entirely self-contained except that we do not prove the basic facts about non-positively curved spaces that are listed (3.2). One could shorten the proof of Theorem C considerably by using the complexes constructed in [21] or [5] in place of Lemmas 3.3 and 3.5. However those constructions are rather complicated, so we feel that there is benefit in presenting a more direct account.

The example given in (4.3(2)) shows that the class of groups which act properly and cocompactly on spaces of non-positive curvature does not satisfy the conditions of Theorem A. Nevertheless, with appropriate attention to detail, one can use the blueprint of our proof of Theorem A to prove Theorem C, and this is what we shall do. First we need to know that there exists a compact non-positively curved 2-complex whose fundamental group is not residually finite.

3.1 Wise's Examples [19] Let

$$T(n) = \langle a, b, t_a, t_b \mid [a, b] = 1,\ t_a^{-1} a t_a = (ab)^n,\ t_b^{-1} b t_b = (ab)^n \rangle.$$

In Section 1 we saw that if $n \geq 2$ then this group is not Hopfian and therefore not residually finite. $T(n)$ is the fundamental group of the non-positively curved 2-complex $X(n)$ that one constructs as follows: take the (skew) torus formed by identifying opposite sides of a rhombus with sides of length n and small diagonal of length 1; the loops formed by the images of the sides of the rhombus are labelled a and b respectively; to this torus attach two tubes $S \times [0,1]$, where S is a circle of length n; one end of the first tube is attached to the loop labelled a and one end of the second tube is attached to the loop labelled b; in each case the other end of the tube wraps n times around the image of the small diagonal of the rhombus.

Any complex obtained by attaching tubes along local geodesics in the above manner is non-positively curved in the natural length metric (see [3, II.11]). We shall need the following additional facts concerning metric spaces of non-positive curvature; see [3] for details.

3.2 Proposition *Let X be a compact, connected, geodesic space of non-positive curvature. Fix $x \in X$.*

(1) *Each homotopy class in $\pi_1(X, x)$ contains a unique shortest loop based at x. This based loop is the unique local geodesic in the given homotopy class.*

(2) *Each conjugacy class in $\pi_1(X, x)$ is represented by a closed geodesic in X (ie a locally isometric embedding of a circle). In other words, every loop in X is freely homotopic to a closed geodesic (which need not pass through x). If two closed geodesics are freely homotopic then they have the same length.*

(3) *$\pi_1(X, x)$ is torsion-free.*

(4) *Metric graphs are non-positively curved.*

(5) *The induced path metric on the 1-point union of two non-positively curved spaces is again non-positively curved.*

(6) *If X is a compact non-positively curved space, Z is a compact length space and $i_1, i_2\colon Z \to X$ are locally isometric embeddings, then, when*

endowed with the induced path metric, the quotient of $X \cup (Z \times [0,L])$ *by the equivalence relation generated by* $i_1(z) \sim (z,0)$ *and* $i_2(z) \sim (z,L)$ *is non-positively curved. Moreover, if* L *is greater than the diameter of* X*, then* X *is isometrically embedded in the quotient.*

A particular case of (6) that we shall need is where X is the disjoint union of spaces X_1 and X_2, and Z is a circle. In this case the quotient is obtained by joining X_1 to X_2 with a cylinder whose ends are attached along closed geodesics.

3.3 Lemma *There exists a compact, connected, non-positively curved 2-complex* K *with basepoint* $x_0 \in K$ *such that:*

(1) *there is an element* $g_0 \in \pi_1(K,x_0)$ *whose image in every finite quotient of* $\pi_1(K,x_0)$ *is trivial;*

(2) $\pi_1(K,x_0)$ *is generated by a finite set of elements each of which is represented by a closed geodesic that passes through* x_0 *and has integer length;*

(3) g_0 *is represented by a closed geodesic of length 1 that passes through* x_0*.*

Proof Let X be a compact, connected, 2-complex of non-positive curvature and let $g_0 \in \pi_1 X$ be a non-trivial element whose image in every finite quotient of $\pi_1 X$ is trivial (the spaces $X(n)$ of (3.1) give such examples). We choose a point x_0 on a closed geodesic that represents the conjugacy class of g_0. Suppose that $\pi_1(X,x_0)$ is generated by $\{b_1,\dots,b_n\}$, let β_i be the shortest loop based at x_0 in the homotopy class b_i, and let l_i be the length of β_i. Let l_0 be the length of the closed geodesic representing g_0. Replacing g_0 by a proper power if necessary, we may assume that $l_0 > l_i$ for $i = 1,\dots,n$.

Consider the following metric graph Λ: there are $(n+1)$ vertices $\{v_0,\dots,v_n\}$ and $2n$ edges $\{e_1,\varepsilon_1,\dots,e_n,\varepsilon_n\}$; the edge e_i connects v_0 to v_i and has length $(l_0-l_i)/2$; the edge ε_i is a loop of length l_0 based at v_i. We obtain the desired complex K by gluing Λ to X, identifying v_0 with x_0, and then scaling the metric by a factor of l_0 so that the closed geodesic representing $g_0 \in \pi_1(K,x_0)$ has length 1.

Let $\gamma_i \in \pi_1(K,x_0)$ be the element given by the geodesic c_i that traverses e_i, crosses ε_i, and then returns along e_i, that is $c_i = e_i\varepsilon_i\overline{e}_i$, where the overline denotes reversed orientation. Note that $\pi_1(K,x_0)$ is the free product of $\pi_1(X,x_0)$ and the free group generated by $\{\gamma_1,\dots,\gamma_n\}$. As generating set for $\pi_1(K,x_0)$ we choose $\{b_i\gamma_i, b_i\gamma_i^2 \mid i=1,\dots,n\}$.

According to parts (4) and (5) of the preceding proposition, K has non-positive curvature. Moreover, the concatenation of any non-trivial locally geodesic loop in X, based at x_0, and any non-trivial locally geodesic loop in Λ based at v_0 is a closed geodesic in K. Thus $\beta_i c_i$ and $\beta_i e_i \varepsilon_i^2 \overline{e}_i$ are closed geodesics in K; the former has length 2 and the latter has length 3; the former represents $b_i \gamma_i$ and the latter represents $b_i \gamma_i^2$. □

3.4 The proof of Theorem C Given a compact, connected, non-positively curved space X we must isometrically embed it in a compact, connected, non-positively curved space $\overline{X}$ whose fundamental group has no non-trivial finite quotients. Moreover the embedding must be such that if X is a complex of dimension at most $n \geq 2$ then so is $\overline{X}$. We give two constructions, the first in outline and the second in detail.

First Proof We form the 1-point union of X with one of the complexes $X(n)$ described in (3.1) thus ensuring that some element g_0 of the fundamental group has trivial image in every finite quotient. We then apply the construction of (3.3), gluing a metric graph to our space to obtain a space X' whose fundamental group is generated by elements represented by closed geodesics that pass through a basepoint on a closed geodesic representing g_0. To complete the proof one follows the argument of Lemma 3.5 with X' in place of K (taking the cylinders attached to be sufficiently long so that X is isometrically embedded in the resulting space, 3.2(6)).

Second Proof Choose a finite set of generators for $\pi_1 X$, and let $c_1, \ldots, c_N$ be closed geodesics in X representing the conjugacy classes of these elements. Lemma 3.5 gives a compact non-positively curved 2-complex K_4 whose fundamental group has no finite quotients; fix a closed geodesic c_0 in K_4. Take N copies of K_4 and scale the metric on the i-th copy so that the length of c_0 in the scaled metric is equal to the length $l(c_i)$ of c_i. Then glue the N copies of K_4 to X using cylinders $S_i \times [0, L]$ where S_i is a circle of length $l(c_i)$; the ends of $S_i \times [0, L]$ are attached by arc length parametrizations of c_0 and c_i respectively. Call the resulting space $\overline{X}$.

Part (6) of (3.2) assures us that $\overline{X}$ is non-positively curved, and if the length L of the gluing tubes is sufficiently large then the natural embedding $X \hookrightarrow \overline{X}$ will be an isometry.

It remains to construct K_4.

3.5 Lemma *There exists a compact non-positively curved 2-complex K_4 whose fundamental group has no finite quotients.*

Proof Let K be as in (3.3). We mimic the argument of (2.1), with $\pi_1(K, x_0)$ in the rôle of G_1. At each stage we shall state what the fundamental group of the complex being constructed is; in each case this is a simple application of the Seifert-van Kampen theorem.

Let c_0 be the closed geodesic of length 1 representing g_0. Let $\{a_0, \ldots, a_n\}$ be the generators given by 3.3(2), let α_i be the closed geodesic through x_0 that represents a_i, and suppose that α_i has length p_i. For each i, we glue to K a cylinder $S_{p_i} \times [0,1]$, where S_{p_i} is a circle of length p_i, with basepoint v_i; one end of the cylinder is attached to α_i while the other end wraps p_i-times around c_0, and $v_i \times \{0,1\}$ is attached to x_0. Let K_1 be the resulting complex. By the Seifert-van Kampen theorem, $\pi_1(K_1, x_0) = E_1$, in the notation of (2.1). Part (6) of (3.2) implies that K_1 is non-positively curved.

The images in K_1 of the paths $v_i \times [0,1]$ give an isometric embedding into K_1 of the metric graph Y that has one vertex and n edges of length 1; call the corresponding free subgroup $F_1 \subset E_1$ (it is the subgroup generated by the s_i in (2.1)).

Step 2 of (2.1) is achieved by attaching n cylinders of unit circumference $S_1 \times [0,1]$ to K_1, the ends of the i-th cylinder being attached to c_0 and to the image of $v_i \times [0,1]$. The resulting complex K_2 has $\pi_1(K_2, x_0) = E_2$. As in the previous step, the free subgroup $F_2 \subset E_2$ generated by the basic loops that run along the new cylinders is the π_1-image of an isometric embedding $Y \to K_2$. (This F_2 is the subgroup generated by the τ_i in (2.1).)

To achieve Step 3 of (2.1), we now glue $Y \times [0,L]$ to K_2 by attaching the ends according to the isometric embeddings that realize the embeddings $F_1, F_2 \subset \pi_1(K_2, x_0)$. This gives us a compact non-positively curved complex K_3 with fundamental group E_3 (in the notation of (2.1)). Let v be the vertex of Y, observe that $v \times \{0, L\}$ is attached to $x_0 \in K_3$, and let $\sigma \in \pi_1(K_3, x_0)$ be the homotopy class of the loop $[0,L] \to K_3$ given by $t \mapsto (v,t)$.

We left open the choice of L, the length of the mapping cylinder in Step 3, we now specify that it should be p_0, the length of the geodesic representing the generator a_0. An important point to observe is that the angle at x_0 between the image of $v \times [0,L]$ and any path in $K_1 \subset K_3$ is π. Thus the free subgroup $\mathrm{gp}\{a_0, \sigma\}$ is the π_1-image in $\pi_1(K_3, x_0)$ of an isometry from the metric graph Z with one vertex (sent to x_0) and two edges of length $L = p_0$. In fact, we have two such isometries $Z \to K_3$, corresponding to the free choice we have of which edge of Z to send to the image of $v \times [0,L]$. We use these two maps to realize Step 4 of the construction on (2.1): we apply part (6) of (3.2) with X equal to the disjoint union of two copies of K_3 and with the two maps $Z \to K_3$

employed as the local isometries i_1, i_2, the image of one of the maps being in each component of X. The resulting space is the desired complex K_4. □

By gluing non-positively curved orbi-spaces (in the sense of Haefliger [9]), or by performing equivariant gluing, one can extend Theorem C to include groups with torsion. We refer the reader to [3, II.11] for the technical tools that make this adaptation straightforward.

3.6 Theorem *If a group G acts properly and cocompactly by isometries on a $CAT(0)$ space Y then one can embed G in a group $\widehat{G}$ that acts properly and cocompactly by isometries on a $CAT(0)$ space $\overline{Y}$ and has no proper subgroups of finite index. If Y is a polyhedral complex of dimension $n \geq 2$ then so is $\overline{Y}$.*

Since the group G need not be torsion-free, (3.6) shows in particular that there exist compact non-positively curved orbihedra, with finite local groups, that are not finitely covered by any polyhedron (where 'covered' refers to covering in the sense of orbispaces and 'polyhedron' means an orbihedron whose local groups are trivial). We close our discussion of non-positively curved spaces with an explicit example to illustrate this point. The first examples of this type were discovered by my student Wise [20], and the following example is essentially contained in his work.

3.7 A semihyperbolic group that is not virtually torison-free

In the hyperbolic plane $\mathbb{H}^2$ we consider a regular quadrilateral Q with vertex angles $\pi/4$. Let α and β be hyperbolic translations that identify the opposite sides of Q. Then Q is a fundamental domain for the action of $G = \mathrm{gp}\{\alpha, \beta\}$; the commutator $[\alpha, \beta]$ acts as a rotation through π at one vertex of Q, and away from the orbit of this vertex the action of G is free. Thus the quotient orbifold $V = \mathbb{H}^2/G$ is a torus with one singular point, and at that singular point the local group is $\mathbb{Z}_2$.

Let $X(n)$ and $T(n)$ be as in (3.1) and fix a closed geodesic c in the homotopy class of a non-trivial element g_0 in the kernel of a self-surjection $T(n) \twoheadrightarrow T(n)$. We scale the metric on $X(n)$ so that this geodesic has length $l = |\alpha| = |\beta|$. Then we take a copy of $X(n)$ and consider the orbispace $\overline{V}$ obtained by gluing it to V using a tube $S_l \times [0,1]$ one end of which is glued to c and the other end of which is glued to the image in V of the axis of α.

$\overline{V}$ inherits the structure as a (non-positively curved) orbihedron in which the only singular point is the original one; at this singular point the local structure is as it was in V. The fundamental group $\widehat{G}$ of $\overline{V}$ is $G *_{\mathbb{Z}} T(n)$, where the amalgamation identifies $g_0 \in T(n)$ with $\alpha \in G$. Now, g_0 has trivial image in every finite quotient of $T(n)$, therefore $[\alpha,\beta] = [g_0,\beta]$ has trivial image in every finite quotient of $\widehat{G}$. It follows that $[\alpha,\beta]$, which has order two, lies in every subgroup of $\widehat{G}$ that has finite index.

In the case $n = 2$, the group $\widehat{G}$ has the following presentation:

$$\langle a,b,s,t,\alpha,\beta \mid \alpha = [s^{-1}(ab)s,b],\ [a,b] = [\alpha,\beta]^2 = 1,\ t^{-1}bt = s^{-1}as = (ab)^2\rangle.$$

4 Isoperimetric inequalities

Isoperimetric inequalities for finitely presented groups $G = \langle \mathcal{A} \mid \mathcal{R}\rangle$ measure the complexity of the word problem. If a word w in the free group $F(\mathcal{A})$ represents the identity in G, then there is an equality

$$w = \prod_{i=1}^{N} x_i^{-1} r_i x_i$$

in $F(\mathcal{A})$, where $r_i \in \mathcal{R}^{\pm 1}$. Isoperimetric inequalities give upper bounds on the integer N in a minimal such expression. The bounds are given as a function of the length of w, and the function $f_G\colon \mathbb{N} \to \mathbb{N}$ giving the optimal bound is called the *Dehn function* of the presentation. If there is a constant $K > 0$ such that the functions $g, h\colon \mathbb{N} \to \mathbb{N}$ satisfy $g(n) \leq K\, h(Kn) + Kn$, then one writes $g \preceq h$. It is not difficult to show (see [1] for example) that the Dehn functions of different finite presentations of a fixed group are $\simeq$ equivalent, where $f \simeq g$ means that $f \preceq g$ and $g \preceq f$.

As an alternative measure of complexity for the word problem, instead of trying to bound the integer N in the above equality one might seek to bound the length of the conjugating elements x_i. In this case the function giving the optimal bound is called the *isodiametric function* of the group, which we write $\Phi_G(n)$. Again, this function is $\simeq$ independent of the chosen presentation (see [7]).

We refer the reader to [7] for more information and references concerning Dehn functions and isodiametric functions and their (useful) interpretation in terms of the geometry of van Kampen diagrams.

4.1 Proposition *If G is an isometric amalgam of a finite collection $\{G_i \mid i \in I\}$ of finitely presented groups, then the Dehn function $f_G(n)$ of G is $\preceq n^2 + n \max_i f_{G_i}(n)$.*

Proof A diagrammatic version of the proof is given in (4.3(3)), here we present a more algebraic proof.

By definition, G is the fundamental group of a finite graph of groups. For the sake of notational convenience we shall assume that there are no loops in the graph of groups under consideration. The proof in the general case is entirely similar but notationally cumbersome.

Thus we have a finite tree with vertex set I and a set of edges $\mathcal{E} \subset I \times I$. At the vertex indexed i the vertex group is G_i. Let $H_{i,j}$ be the edge group associated to $(i,j) \in \mathcal{E}$. By definition, (1.3), there are finite generating sets $\mathcal{A}_i$ for the G_i and subsets $\mathcal{B}_{i,j} \subset \mathcal{A}_i$ with specified bijections $\phi_{i,j}\colon \mathcal{B}_{i,j} \to \mathcal{B}_{j,i}$ for each $(i,j) \in \mathcal{E}$; the set $\mathcal{B}_{i,j}$ generates $H_{i,j}$, each of the inclusions $H_{i,j} \hookrightarrow G_i$ is isometric with respect to these choices of generators, and $\phi_{i,j} = \phi_{j,i}^{-1}$.

We fix finite presentations $\langle \mathcal{A}_i \mid \mathcal{R}_i \rangle$ for the G_i. Then,

$$G \cong \langle \mathcal{A} \mid \mathcal{R},\ \phi_{i,j}(b) = b, \forall b \in \mathcal{B}_{i,j} \rangle,$$

where $\mathcal{A} = \coprod_i \mathcal{A}_i$, $\mathcal{R} = \coprod_i \mathcal{R}_i$, and (i,j) runs over $\mathcal{E}$

Let W be a word in the generators $\mathcal{A}$. Suppose that W is identically equal to a product $u_1 \dots u_m$, where each u_k is a word over one of the alphabets $\mathcal{A}_{i(k)}$ and each $\mathcal{A}_{i(k)} \neq \mathcal{A}_{i(k+1)}$. Under these circumstances W is said to have *alternating length* m. The normal form theorem for amalgamated free products [13] (or more generally graph products [18]) ensures that this notion of length is well-defined. It also tells us that if $W = 1$ in G then at least one of the subwords u_k is equal in $G_{i(k)}$ to a word ω in the generators $\mathcal{B}_{i(k),i(k\pm1)}$. Because $H_{i(k),i(k\pm1)}$ is isometrically embedded in $G_{i(k)}$, we can replace u_k by ω without increasing the length of W. This can be done at the cost of applying at most $f_{G_{i(k)}}(2|u_k|)$ relations. We apply $|\omega|$ relations to replace each letter b of ω with $\phi_{i(k),i(k\pm1)}(b)$. Then, without applying any more relations, we group ω together with the neighbouring word $u_{k\pm1}$. The net effect of this operation is to reduce the alternating length of W without increasing its actual length. By repeating this operation fewer than $|W|$ times we can replace W by a word W' with $|W'| \leq |W|$ that involves letters from only one of the alphabets $\mathcal{A}_i$. Since W' represents the identity in G_i, we can then reduce W' to the empty word by applying at most $f_{G_i}(|W'|)$ relators from $\mathcal{R}_i$.

The total number of relators applied in the reduction of W to W' is fewer than $m|W| + m \max_i f_{G_i}(|W|)$, where m is the alternating length of W. Therefore the total number of relators that we had to apply in reducing W to the empty word was less than $|W|^2 + |W| \max_i f_{G_i}(|W|)$. $\square$

4.2 Corollary *The class of groups that satisfy a polynomial isoperimetric inequality is closed under the formation of isometric amalgamations along finitely generated subgroups.*

4.3 Remarks

(1) If instead of considering isometric amalgamations we considered the fundamental groups of graphs of groups in which the edge groups were only quasi-isometrically embedded, then the above proof would break down at the point where we noted that $|W'| \leq |W|$. In fact Proposition 4.1 would be false under this weaker hypothesis: consider the Baumslag-Solitar groups for example.

(2) Let D be the direct product of the free group on $\{a, b\}$ and the free group on $\{c, d\}$. Let $L = \mathrm{gp}\{ac, bc\}$. For a suitable choice of generators, L is isometrically embedded in D. It is shown in [2] and [3] that $D *_L D$ has a cubic Dehn function, whereas D has a quadratic Dehn function. Thus, in general, isometric amalgamations may increase the polynomial degree of Dehn functions.

(3) The proof of (4.1) can be recast as an induction argument in which one proves that the area of a minimal van Kampen diagram for W is $m(\max_i f_{G_i}(|W|) + |W|)$, where m is the alternating length of W. This admits a simple geometric proof which we shall now sketch.

Draw a circle labelled by W, divide it into m subarcs according to the decomposition of W as an alternating word. Maintaining the notation established in the proof of (4.1), we draw a chord in the disc connecting the endpoints of the circular arc labelled by u_k. We label the chord by a geodesic word $\omega \in \mathcal{B}^*_{i(k),i(k\pm1)}$ that is equal to u_k in G. We fill the subdisc with boundary labelled $u_k\omega^{-1}$ using a minimal-area van Kampen diagram over the given presentation of $G_{i(k)}$. We then attach to the chord labelled ω faces corresponding to relators of the type $\phi_{i(k),i(k\pm1)}(b)$; the effect of this is to replace ω by the corresponding word in the generators $\mathcal{B}_{i(k\pm1),i(k)}$. By induction, we may fill the remaining subdisc with a van Kampen diagram of area no greater than $(m-1)(\max_i f_{G_i}(|W|) + |W|)$. We may choose u_k so that $2|u_k| \leq |W|$, and hence $|u_k| + |\omega| \leq |W|$. Therefore the area of the whole diagram is no greater than $m\,(\max_i f_{G_i}(|W|) + |W|)$, completing the induction.

A simple induction on alternating length, in the manner of (4.3(3)), allows one to show that (with respect to the finite presentations considered in (4.1)) every null-homotopic word W of alternating length m bounds a van Kampen diagram in which every vertex can be joined to the basepoint of the diagram by a path in the 1-skeleton that has length at most $|W| + \max_i \Phi_{G_i}(|W|)$. Thus:

4.4 Proposition *If G is an isometric amalgam of a finite collection $\{G_i \mid i \in I\}$ of finitely presented groups, then the isodiametric function $\Phi_G(n)$ of G is $\preceq \max_i \Phi_{G_i}(n)$.*

4.5 The Proof of Theorem B Given an infinite finitely presented group G, we replace it by $G * \mathbb{Z}$. This does not change the Dehn function or the isodiametric function of G but it allows us to assume that G is generated by a finite set of elements $\{a_i, \ldots, a_r\}$ such that each $\langle a_i \rangle$ is isometrically embedded in G (see 2.2(3)).

The fundamental group S of any of the spaces $\overline{X}$ yielded by Theorem C will satisfy a quadratic isoperimetric inequality and a linear isodiametric inequality [3, III]. At the level of π_1, the proof of Theorem C was exactly parallel to that of (2.1), so Lemma 2.4 implies that S contains an isometrically embedded infinite cyclic subgroup $\langle s \rangle$.

The group $\widehat{G}$ whose existence is asserted in Theorem B is obtained by taking an amalgamated free product of G and m copies of S: the cyclic subgroup $\langle s \rangle$ in the i-th copy of S is identified with $\langle a_i \rangle \subset G$. In other words, $\widehat{G}$ is the fundamental group of a tree of groups in which there is one vertex of valence m, with vertex group G, and m vertices of valence 1, each with vertex group S; each edge group is infinite cyclic and the generator of the i-th edge group is mapped to $s \in S$ and $a_i \in G$.

Proposition 4.1 tells us that the Dehn function of $\widehat{G}$ is $\preceq n f_G(n)$, and Proposition 4.4 tells us that the isodiametric function of $\widehat{G}$ is no worse than that of G. □

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Mathematical Institute, 24–29 St Giles', Oxford, OX1 3LB

Email: `bridson@maths.ox.ac.uk`

Received: 16 November 1997