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All Fuchsian Schottky groups are classical Schottky groups

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Abstract Not all Schottky groups of Möbius transformations are classical Schottky groups. In this paper we show that all Fuchsian Schottky groups are classical Schottky groups, but not necessarily on the same set of generators.

AMS Classification 20H10; 30F35, 30F40

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1 Introduction

A Schottky group of genus g is a group of Möbius transformations acting on the Riemann sphere $\overline{\mathbb{C}}$ generated by g elements $A_i, 1 \leq i \leq g$, each of which possesses a pair of Jordan curves $C_i, C_i' \subseteq \overline{\mathbb{C}}$, with the property that the $2g$ curves are mutually disjoint and that A_i maps C_i onto C_i' where the outside of C_i is sent onto the inside of C_i'. Direct use of combination theorems tells us that the resulting group is free on g generators, is discrete with a fundamental domain the region exterior to the $2g$ curves, and consists entirely of loxodromic and hyperbolic elements.

If in addition we can take all the Jordan curves to be geometric circles then the resulting group is called a classical Schottky group (or sometimes in order to be more specific we say it is classical on the generators $A_1, \ldots, A_g$). Marden [2] showed that not all Schottky groups are classical Schottky groups. Put very briefly, he argued that the algebraic limit of classical Schottky groups must be geometrically finite and so his isomorphism theorem implies that the ordinary set Ω of this limit cannot be empty. But most groups on the boundary of Schottky space have an empty ordinary set, so Schottky space strictly contains classical Schottky space. However, this argument is certainly non-constructive, raising the question of finding an explicit nonclassical Schottky group. Zarrow [7] claimed to have found such an example, but the paper of Sato [5] shows that it is in fact a classical Schottky group. A little later Yamamoto [6] did construct a nonclassical Schottky group.

The purpose of this paper is to show that if we examine the most straightforward cases where we might expect to find a counterexample, namely Fuchsian Schottky groups, then this approach is doomed to failure as all such groups are classical Schottky groups. Specifically we show that:

(**1**) Given a Fuchsian Schottky group G of any genus g then there exists a generating set for G of g hyperbolic Möbius transformations on which G is classical.

(**2**) The Fuchsian Schottky group G is classical on all possible generating sets if and only if $g = 2$ and G is generated by a pair of hyperbolic elements with intersecting axes.

(**3**) There exists a Fuchsian group which is Schottky on a particular generating set, but which cannot be classical on those generators.

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2 Proof of Main Theorem

Given any finitely generated Fuchsian group G (namely a discrete subgroup of $PSL(2,\mathbb{R})$) containing no elliptic elements, we form the quotient surface $S = U/G$ where U is the upper half plane. The complete hyperbolic surface S has ideal boundary $\partial S = (\overline{\mathbb{R}} \cap \Omega_G)/G$, where $\overline{\mathbb{R}}$ is the boundary of U in the Riemann sphere $\overline{\mathbb{C}}$ and Ω_G is the ordinary set of G. Note that G is Schottky if and only if S is a closed surface minus at least one hole (although S cannot be a one-holed sphere). This is because a Fuchsian group G with a quotient surface S as above must be free and purely hyperbolic, and this implies (see, say [3]) that G is indeed Schottky.

If S is a surface of genus n with h holes then G will be a free group of some rank r. The process of doubling S along its boundary corresponds to considering the quotient of the whole ordinary set Ω_G by G. As G is a Schottky group, Ω_G/G is topologically a closed surface of genus r. Therefore we conclude that $r = 2n + h - 1$ (with $n \geq 0, h \geq 1$ and $r \geq 1$).

The idea of the proof of theorem 1 is that given any such surface $S = U/G$, we find a particular reference surface, homeomorphic to S, which has a system of simple closed geodesics $\gamma_1, \dots, \gamma_r$ corresponding to a generating set for G. We also find disjoint complete simple geodesics $l_1, \dots, l_r$ on this reference surface which are properly embedded (they can be thought of as having their endpoints up the "spouts"), where l_i intersects γ_i once and is disjoint from γ_j $(j \neq i)$. We will find that if we cut along these geodesics $l_1, \dots, l_r$, a disc is obtained. We are then able to transfer these curves across to S. By viewing the process upstairs in the upper half plane U we get a fundamental domain for G, and then we can see directly that G is classical Schottky on our generating set.

Theorem 1 *Given a Fuchsian Schottky group G of any genus g then there exists a generating set for G of g hyperbolic Möbius transformations on which G is classical.*

Proof We prove the result by taking a standard Fuchsian classical Schottky group $G_{n,h}$ for each possible topological surface of genus n and h holes, and transfer the two sets of geodesics to curves on any other surface homeomorphic to $U/G_{n,h}$. These can be replaced by geodesics with all necessary properties preserved.

First consider $h = 1$. We choose $2n$ hyperbolic elements $A_1, \dots, A_{2n}$ so that their axes all intersect at the same point, and ensure that $G_{n,1} = \langle A_1, \dots, A_{2n} \rangle$ is classical Schottky by choosing the multipliers of the A_i in order to obtain for each group $\langle A_i \rangle$ a fundamental domain Δ_i consisting of the intersection of the exteriors of two geodesics L_i and $L_i' = A_i(L_i)$ so that all conditions of the free product combination theorem are satisfied; namely that

$$\Delta_i \cup \Delta_j = U \text{ for } i \neq j \text{ and } \bigcap_i \Delta_i \neq \emptyset.$$

Then we have a fundamental domain $\Delta_{n,1}$ (homeomorphic to a disc) for the discrete group $G_{n,1}$. There is one cycle of boundary intervals and so by the discussion above, the surface $S_{n,1} = U/G_{n,1}$ is indeed of genus n with boundary a circle.

We can project the axes of A_i down onto the surface to obtain our simple closed geodesics γ_i, and do the same with each L_i, which gives us the complete simple geodesic l_i right up to its two endpoints on the boundary. These have the appropriate properties mentioned earlier, and we see that the surface becomes a disc after cutting along all the geodesics $l_1, \dots l_{2n}$.

The group $G_{2,1}$ and the projection of these geodesics are illustrated in figures 1 and 2.

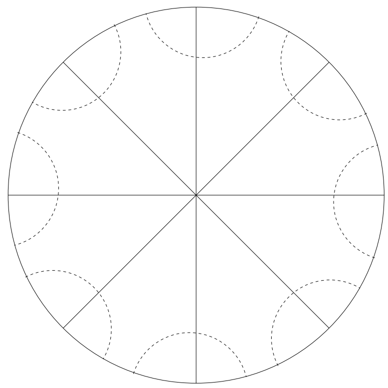

Figure 1

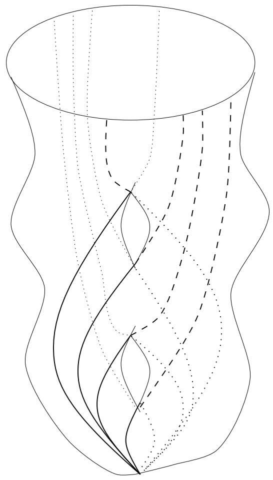

Figure 2

In order to construct $G_{n,h}$ when $h \geq 2$, take $G_{n,1}$ and choose an open interval I between one endpoint of some L_i and the nearest endpoint of a neighbouring geodesic L_j. This interval lies inside the ordinary set of $G_{n,1}$. Then inductively nest $h-1$ geodesics inside the previous one, so that each geodesic has endpoints in I. We then find hyperbolic transformations $A_{2n+1}, \ldots, A_{2n+h-1}$ with axes these geodesics and with each transformation having two geodesics L_i and $L'_i = A_i(L_i)$, where $2n+1 \leq i \leq 2n+h-1$, which it pairs. If these fundamental domains are correctly placed then $G_{n,h} = \langle A_1, \ldots A_{2n+h-1}\rangle$ is a discrete group having the correct quotient surface $S_{n,h} = U/G_{n,h}$ with a disc for a fundamental domain $\Delta_{n,h}$, where $\partial\Delta_{n,h}$ consists of $4n+2h-2$ geodesics L_i and L'_i, along with the same number of intervals of $\overline{\mathbb{R}}$. The geodesics and intervals alternate as we go round the boundary of the disc. Also the projections of these axes and of these paired geodesics which define γ_i and l_i have all the same properties as mentioned before. The case $n=1$, $h=5$ is pictured in figures 3 and 4.

Now given any Fuchsian Schottky group G with quotient surface S and boundary ∂S, there exists a homeomorphism

$$h\colon S_{n,h} \cup \partial S_{n,h} \mapsto S \cup \partial S$$

for some n and h. We also have natural continuous projections

$$\begin{array}{llll} p\colon & U \cup (\Omega_{G_{n,h}} \cap \overline{\mathbb{R}}) & \mapsto & S_{n,h} \cup \partial S_{n,h} \\ q\colon & U \cup (\Omega_G \cap \overline{\mathbb{R}}) & \mapsto & S \cup \partial S \end{array}$$

where p and q are both covering maps, and both domains are simply connected

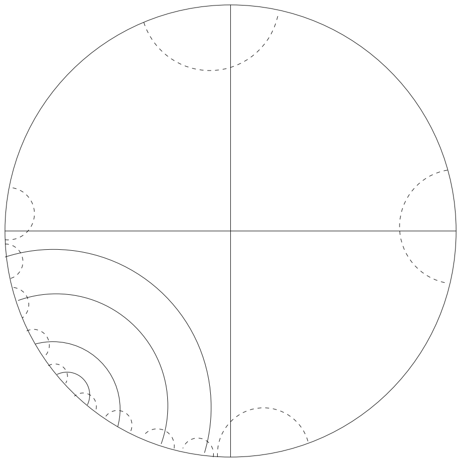

Figure 3

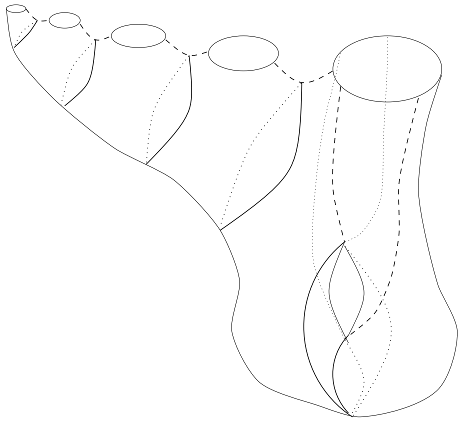

Figure 4

covering spaces of their images (where the elementary neighbourhoods of points downstairs are open discs, or half discs for points on the boundary).

By the lifting theorem, we have a continuous map

$$H\colon U \cup (\Omega_{G_{n,h}} \cap \overline{\mathbb{R}}) \mapsto U \cup (\Omega_G \cap \overline{\mathbb{R}})$$

which is a lift of hp, so that $hp = qH$. By reversing p and q, we see that H is a homeomorphism.

Take any element $g \in G_{n,h}$. This is a deck transformation of p and so $pg = p$. Conjugating g by H, we have $q(HgH^{-1}) = q$, thus HgH^{-1} is a deck transformation of q and therefore H defines an isomorphism of $G_{n,h}$ onto G by conjugation.

Note that H maps U to U and $\Omega_{G_{n,h}} \cap \overline{\mathbb{R}}$ to $\Omega_G \cap \overline{\mathbb{R}}$, because it is a lift of h which sends boundary points to and from boundary points. Therefore the image under H of the fundamental domain $\Delta_{n,h}$ is a disc in U. But $H(\partial\Delta_{n,h})$ will consist of $4n+2h-2$ disjoint closed intervals of $\overline{\mathbb{R}}$, along with curves $H(L_i)$ and $H(L'_i)$ lying entirely in U apart from their endpoints which are also endpoints of these intervals of $\overline{\mathbb{R}}$. We find that the order in which the images under H of the L_i, L'_i and the intervals appear around $\partial H(\Delta_{n,h}) = H(\partial\Delta_{n,h}) \subseteq U \cup (\Omega_G \cap \overline{\mathbb{R}})$ is the same as the original order around $\partial\Delta_{n,h}$ (or the opposite order if H is orientation reversing).

By setting $B_i = HA_iH^{-1}$ we obtain a generating set for G, and because A_i sends the geodesic L_i to L'_i, we see that B_i sends the curve $H(L_i)$ to the curve

$H(L_i')$. Also it is easy to check that the disc $H(\Delta_{n,h})$ is a fundamental domain for the action of G on U. In particular, the intersection of the exteriors in U of $H(L_i)$ and $H(L_i')$ is a fundamental domain for $\langle B_i \rangle$. We replace these two curves by geodesics M_i and $M_i' = B_i(M_i)$ which have the same endpoints. Just as in [1], this gives us $2n+h-1$ pairs of geodesics freely homotopic to the curves they replaced, and paired by a generating set B_i with another fundamental domain D_i for each group $\langle B_i \rangle$ that lies between these two geodesics. The free product combination theorem can be applied to $\langle B_1 \rangle, \ldots \langle B_{2n+h-1} \rangle$, as $D_i \cup D_j = U$ for $i \neq j$ and $\bigcap_i D_i \neq \emptyset$. We can see this by looking at the endpoints of the geodesics which have not been changed when passing from curves. Therefore, by reflecting this picture in the real axis, the group G is generated by elements B_i, each of which possesses a pair of mutually disjoint geometric circles C_i and C_i', with the outside of C_i being sent by B_i onto the inside of C_i'. By definition, G is a classical Schottky group. □

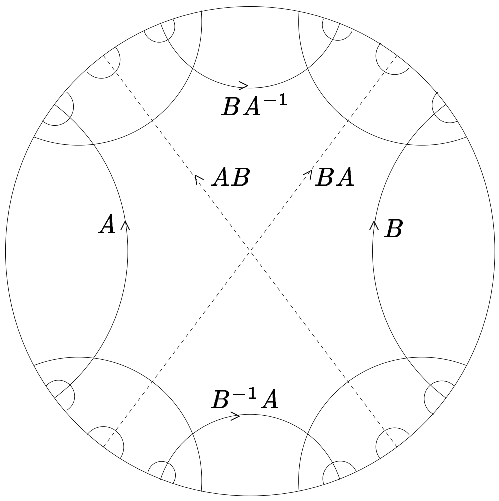

Figure 5

3 Proof of other Theorems

Suppose we are given any two hyperbolic elements A and B with different axes. We want to know when $G = \langle A, B \rangle$ is free, discrete and purely hyperbolic (hence Schottky). This problem falls naturally into two cases.

(**A**) The two hyperbolic elements have intersecting axes. Then it is well known that G is free, discrete and purely hyperbolic if and only if the commutator

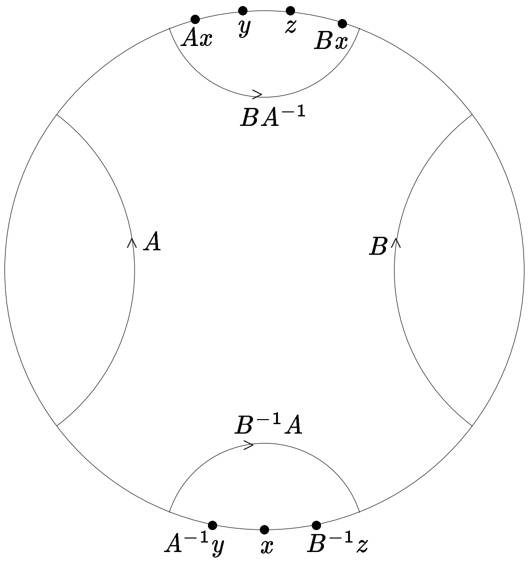

Figure 6

$ABA^{-1}B^{-1}$ is hyperbolic. See for instance [4] where this is shown by explicitly exhibiting two pairs of geometric circles, one paired by A and one by B. In this case the quotient surface is a one holed torus and, as any generating pair will have intersecting axes, we see that G is classical on every possible generating pair.

Alternatively we can see this directly from section 1 by using the fact that there will exist a homeomorphism from our standard surface to the quotient surface of G that takes the two simple closed geodesics γ_1, γ_2 onto two curves freely homotopic to the simple closed geodesics corresponding to any generating pair of G.

(**B**) The hyperbolic elements have non-intersecting axes. If so then all generating pairs of G must have non-intersecting axes, or else we are back in case (A).

First suppose G is a classical Schottky group on these two generators A and B. Without loss of generality we can replace any generator by its inverse so that we get a picture such as the one in figure 5, with the arrows on the two generators in the same direction. The quotient surface is a three holed sphere. Note that the axis of AB projects down onto a "figure of eight" geodesic, and so this group cannot be classical on the generating pair $\langle A, AB\rangle$.

Theorem 2 *A group G that has a quotient surface which is not a one holed* *torus cannot be classical on all generating sets.*

Proof We have already considered any G generated by two elements. Given any G generated by three or more elements, we can find a pair of generators with non-intersecting axes, and use the above argument on the subgroup generated by this pair. As the subgroup is not classical on all generating sets, nor is G. □

Finally we show the existence of a Fuchsian group generated by two elements which is Schottky, but not classical, on this generating pair.

Lemma 1 *A group $G = \langle A, B\rangle$ (where A and B are hyperbolic elements with non-intersecting axes, oriented as in figure 5) is classical on $\langle A, B\rangle$ if and only if both fixed points of $B^{-1}A$ lie in the interval between the repelling fixed points of A and B.*

Proof If we know G is classical on $\langle A, B\rangle$ then we can build up a pattern of nested circles as in figure 5, and see the location of the fixed points of the axes directly. Conversely if we only have information as in figure 6 then we consider the image of a suitable point x under the generators.

The axis of $B^{-1}A$ is sent to the axis of BA^{-1} by both generators, and also note that the arrows on BA^{-1} and $B^{-1}A$ are as in the picture (for instance consider the image of a fixed point of A). Then we choose any x inside the interval enclosed by the axis of $B^{-1}A$, and mark it and its images under A and B. We can take any two points y and z in the interval between Ax and Bx, and use these as endpoints for the geometric circles we require.

We can see that $A^{-1}y$ will be closer than x to the repelling fixed point of A, and similarly with $B^{-1}z$ and B. This gives us four endpoints $y, z, A^{-1}y$ and $B^{-1}z$, one for each circle. We have four more endpoints to mark but this choice is totally arbitrary: merely pick any point in the interval between A's fixed points, along with its image under A, and do the same for B too. This provides us with our two pairs of circles which show that G is discrete, and classical on $\langle A, B\rangle$. □

Theorem 3 *The Fuchsian group in figure 7, which is Schottky on the generators A and B, is not classical on them.*

Proof The exterior F of the two pairs of curves C_A, C'_A (paired by A) and C_B, C'_B (paired by B) is a fundamental domain, and is sent by the element BA^{-1} inside the circle $C(= B(C_A))$. The attracting fixed point of BA^{-1} must lie inside C and therefore it separates the fixed points of A. □

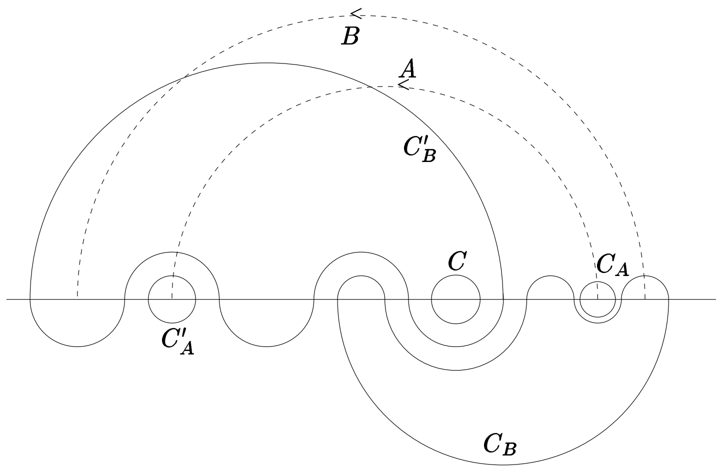

Figure 7

References

[1] **M Dehn**, **J Stillwell**, *Appendix: The Dehn–Nielsen Theorem*, from: "Papers on Group Theory and Topology", Springer–Verlag (1987) 363–395

[2] **A Marden**, *The geometry of finitely generated Kleinian groups*, Ann. of Math. 99 (1974) 383–462

[3] **B Maskit**, *A characterisation of Schottky groups*, J. Analyse Math. 19 (1967) 227–230

[4] **N Purzitsky**, *Two–Generator Discrete Free Products*, Math. Z. 126 (1972) 209–223

[5] **H Sato**, *On a paper of Zarrow*, Duke Math. J. 57 (1988) 205–209

[6] **H Yamamoto**, *An example of a nonclassical Schottky group*, Duke Math. J. 63 (1991) 193–197

[7] **R Zarrow**, *Classical and nonclassical Schottky groups*, Duke Math. J. 42 (1975) 717–724

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