Geometry & Topology Monographs
Volume 2: Proceedings of the Kirbyfest
Pages 1–21

Combinatorial Dehn surgery on cubed and Haken 3–manifolds

I R Aitchison
J H Rubinstein

Abstract A combinatorial condition is obtained for when immersed or embedded incompressible surfaces in compact 3–manifolds with tori boundary components remain incompressible after Dehn surgery. A combinatorial characterisation of hierarchies is described. A new proof is given of the topological rigidity theorem of Hass and Scott for 3–manifolds containing immersed incompressible surfaces, as found in cubings of non-positive curvature.

AMS Classification 57M50; 57N10

Keywords 3–manifold, Dehn surgery, cubed manifold, Haken manifold

1 Introduction

An important property of the class of 3–dimensional manifolds with cubings of non-positive curvature is that they contain 'canonical' immersed incompressible surfaces (cf [3]). In particular these surfaces satisfy the 4–plane and 1–line properties of Hass and Scott (cf [11]) and so any P^2–irreducible 3–manifold which is homotopy equivalent to such a 3–manifold is homeomorphic to it (topological rigidity). In this paper we study the behaviour of these canonical surfaces under Dehn surgery on knots and links. A major objective here is to show that in the case of a cubed manifold with tori as boundary components, there is a simple criterion to tell if a canonical surface remains incompressible after a particular Dehn surgery. This result is very much in the spirit of the negatively curved Dehn surgery of Gromov and Thurston (cf [8]) and was announced in [4]. Many examples are given in [5].

The key lemma required is a combinatorial version of Dehn's lemma and the loop theorem for immersed surfaces of the type considered by Hass and Scott with an extra condition — the triple point property. We are able to give a simplified proof of the rigidity theorem of Hass and Scott for 3–manifolds containing immersed incompressible surfaces with this additional condition.

By analogy with the combinatorial Dehn's lemma and the loop theorem, we are also able to find a combinatorial characterisation of the crucial idea of hierarchies in 3–manifolds, as used by Waldhausen in his solution to the word problem [20]. In particular, if a list of embedded surfaces is given in a 3–manifold, with boundaries on the previous surfaces in the list, then a simple condition determines whether all the surfaces are incompressible and mutually boundary incompressible. This idea can be used to determine if the hierarchy persists after Dehn surgery — for example, if there are several cusps in a 3–manifold then we can tell if Dehn surgery on all but one cusp makes the final cusp remain incompressible (cf [4]).

One should also note the result of Mosher [15] that the fundamental group of an atoroidal cubed 3–manifold is word hyperbolic. Also in [2] it is shown that any cubed 3–manifold which has all edges of even degree is geometric in the sense of Thurston.

This research was supported by the Australian Research Council.

2 Preliminaries

In this section we introduce the basic concepts and definitions needed for the paper. All 3–manifolds will be compact and connected. All surfaces will be compact but not necessarily connected nor orientable. All maps will be assumed smooth.

Definition 2.1 A properly immersed compact surface not equal to a 2–sphere, projective plane or disk is *incompressible* if the induced map of fundamental groups of the surface to the 3–manifold is injective. We say that the surface is *boundary incompressible* if no essential arc on the surface with ends in its boundary is homotopic keeping its ends fixed into the boundary of the 3–manifold.

By the work of Schoen and Yau [18], any incompressible surface can be homotoped to be least area in its homotopy class, assuming that some Riemannian metric is chosen on the 3–manifold so that the boundary is *convex* (inward pointing mean curvature). Then by Freedman, Hass, Scott [9], after possibly a small perturbation, the least area map becomes a self-transverse immersion which lifts to a collection of embedded planes in the universal cover of the 3–manifold. Moreover any two of these planes which cross, meet in proper lines only, so that there are no simple closed curves of intersection. We will always assume that an incompressible surface is homotoped to satisfy these conditions.

Alternatively, the method of PL minimal surfaces can be used instead of smooth minimal surfaces (cf [14]). However it is interesting to arrive at this conclusion from other considerations, which we will do in Section 6, for the special surfaces associated with cubings of non-positive curvature. We will denote by $\mathcal{P}$ the collection of planes covering a given incompressible surface F in M.

Definition 2.2 F satisfies the *k–plane property* for some positive integer k if any subfamily of k planes in $\mathcal{P}$ has a disjoint pair. F satisfies the *1–line property* if any two intersecting planes in $\mathcal{P}$ meet in a single line. Finally, the *triple point property* is defined for surfaces F which already obey the 1–line property. This condition states that for any three planes P, P' and P'' of $\mathcal{P}$ which mutually intersect in lines, the three lines cross in an odd (and hence finite) number of triple points.

Remark The triple point property rules out the possibility that the common stabiliser of the three intersecting planes is non-trivial. Indeed, the cases of either three disjoint lines of intersection or three lines meeting in infinitely many triple points are not allowed. A non-trivial common stabiliser would require one of these two possibilities.

Definition 2.3 A 3–manifold is said to be *irreducible* if every embedded 2–sphere bounds a 3–cell. It is *P^2–irreducible* if it is irreducible and there are no embedded 2–sided projective planes.

From now on we will be dealing with 3–manifolds which are P^2–irreducible.

Definition 2.4 A 3–manifold will be called *Haken* if it is P^2–irreducible and either has non-empty incompressible boundary or is closed and admits a closed embedded 2–sided incompressible surface.

Definition 2.5 A compact 3–manifold admits a *cubing of non-positive curvature* (or just a *cubing*) if it can be formed by gluing together a finite collection of standard Euclidean cubes with the following conditions:

- Each edge of the resulting cell structure has degree at least four.

- Each link of a vertex of the cell structure is a triangulated 2–sphere so that any loop of edges of length three bounds a triangle, representing the intersection with a single cube, and there are no cycles of length less than three, other than a single edge traversed twice with opposite orientations.

Remark Note the conditions on the cubing are just a special case of Gromov's CAT(0) conditions.

Definition 2.6 The *canonical surface* S in a cubed 3–manifold is formed by gluing together the finite collection of squares, each obtained as the intersection of a plane with a cube, where the plane is midway between a pair of parallel faces of the cube.

Remarks (1) We consider cubed 3–manifolds which are closed or with boundaries consisting of incompressible tori and Klein bottles. In the latter case, the boundary surfaces are covered by square faces of the cubes. Such cubings are very useful for studying knot and link complements.

(2) In [3] it is sketched why the canonical surface is incompressible and satisfies the 4–plane, 1–line and triple point properties. We prove this in detail in the next section for completeness.

(3) Regions complementary to S are cones on unions of squares obtained by canonically decomposing each triangle in the link of a vertex into 3 squares.

(4) The complementary regions of S in (3) are polyhedral cells. Each such polyhedron Π has boundary determined by a graph Γ_Π whose edges correspond to arcs of double points of S and whose vertices are triple points of S. By construction, there is a unique vertex v in the original cubing which lies in the centre of Π: the graph Γ_Π is merely the graph on S^2 dual to the triangulation determined by the link of v in the cubing. The conditions on the cubing translate directly into the following statements concerning the graph Γ_Π.

   (a) Every face has degree at least 4.
   (b) Every embedded loop on S^2 meeting Γ_Π transversely in exactly two points bounds a disk cutting off a single arc of Γ_Π.
   (c) Every embedded loop on S^2 meeting Γ_Π transversely in exactly three points bounds a disk which contains a single vertex of degree 3 of Γ_Π. So the part of Γ_Π inside this disk can be described as a 'Y'.

Notation We will use S to denote a subset of M, ie, the image of the canonical surface and $\bar{S}$ to denote the domain of a map $f\colon \bar{S} \to M$ which has image $f(\bar{S}) = S$.

A similar notational convention applies for other surfaces in M.

3 The canonical surface

In this section we verify the properties listed in the final remark in the previous section.

Theorem 3.1 *The canonical surface S in a cubed 3–manifold M is incompressible and satisfies the 4–plane, 1–line and triple point properties. Moreover S is covered by a collection of embedded planes in the universal covering of M and two such planes meet at most in a single line. Also two such lines meet in at most a single point.*

Proof We show first that S is incompressible. Of course this follows by standard techniques, by thinking of M as having a polyhedral metric of non-positive curvature and using the Cartan–Hadamard Theorem to identify the universal covering with $\mathbf{R^3}$ (cf [7]). Since S is totally geodesic and geodesics diverge in the universal covering space, we see that S is covered by a collection of embedded planes $\mathcal{P}$.

However we want to use a direct combinatorial argument which generalises to situations in the next section where no such metric is obvious on M. Suppose that there is an immersed disk D with boundary C on S. Assume that D is in general position relative to S, so that the inverse image of S is a collection G of arcs and loops in the domain of the map $\bar{D} \to M$ with image D. G can be viewed as a graph with vertices of degree four in the interior of $\bar{D}$. Let v be the number of vertices, e the number of edges and f the number of faces of the graph G, where the faces are the complementary regions of G in $\bar{D}$. We assume initially that these regions are all disks.

An Euler characteristic argument gives that $v - e + f = 1$ and so since $2v \leq e$, there must be some faces with degree less than four. We define some basic homotopies on the disk D which change G to eventually decrease the number of vertices or edges. First of all assume there is a region in the complement of G adjacent to C with two or three vertices. In the former case we have a 2–gon D' of $\bar{D}$ with one boundary arc on C and the other on G. So D' has interior disjoint from S and its boundary lies on S. But by definition of S, any such a 2–gon can be homotoped into a double arc of S. For the 2–gon is contained in a cell in the closure of the complement of S. The cell has a polyhedral structure which can be described as the cone on the dual cell decomposition of a link of a vertex of the cubing. The two arcs of the 2–gon can be deformed into the 1–skeleton of the link and then define a cycle of length two. By definition such a cycle is an edge taken twice in opposite directions. We now homotop D until D' is pushed into the double arc of S and then push D slightly off this double arc. The effect is to eliminate the 2–gon D', ie one arc or edge of G is removed.

Next assume there is a region D'' of the complement of G bounded by three arcs, two of which are edges of G and one is in C. The argument is very similar to that in the previous paragraph. Note that when the boundary of D is pushed into the 1–skeleton of the link of some vertex of the cubing then it gives a 3–cycle which is the boundary of a triangle representing the intersection of the link with a single cube. Therefore we can slide D so that D'' is pushed into the triple point of S lying at the centre of this cube. Again by perturbing D slightly off S, D'' is removed and G has two fewer edges and one fewer vertex.

Finally to complete the argument we need to discuss how to deal with internal regions which are 1–gons, 2–gons or 3–gons. Now 1–gons cannot occur, as there are no 1–cycles in the link of a vertex. 2–gons can be eliminated as above. The same move as described above on 3–gons has the effect of inverting them, ie, moving one of the edges of the 3–gon past the opposite vertex. This is enough to finish the argument by the following observations.

First of all consider an arc of G which is a path with both ends on C and passes through vertices by using opposite edges at each vertex (degree 4), ie, corresponds to a double curve of $D \cap S$. If such an arc has a self intersection, it is easy to see there are embedded 2–gons or 1–gons consisting of subarcs. Choosing an innermost such a 2–gon or 1–gon, then there must be 'boundary' 3–gons (relative to the subdisk defined by the 2–gon or 1–gon) if there are intersections with arcs. Now push all 3–gons off such a 1–gon or 2–gon, starting with the boundary ones. Then we arrive at an innermost 1–gon or 2–gon with no arcs crossing it and can use the previous method to obtain a contradiction or to decrease the complexity of G. Similarly if two such arcs meet at more than one point, we can remove 3–gons from the interior of an innermost resulting 2–gon and again simplify G. Finally if such arcs meet at one point, we get 3–gons which can be made innermost. So in all cases G can be reduced down to the empty graph, with C then lying in a face of S. So C is contractible on S.

It remains to discuss the situation when some regions in the complement of G are not disks. In this case, there are components of G in the interior of $\bar{D}$. We simplify such an innermost component G' first by the same method as above, working with the subdisk $\bar{D}^*$ consisting of all the disks in the complement of G' and separated from C by G'. So we can get rid of G' and continue on until finally all of G is removed by a homotopy of D. Note that once D has no interior intersections with S then D can be homotoped into S as it lies in a single cell, which has the polyhedral structure of the cone on the dual of a link of a vertex of the cubing. This completes the argument showing that S is incompressible.

Next we wish to show why S has the 4–plane, 1–line and triple point properties. Before discussing this, it is helpful to discuss why the lifts of S to the universal covering are planes, without using the polyhedral metric. Suppose some lift of S to the universal covering $\mathcal{M}$ was not embedded. We know such a lift P is an immersed plane by the previous argument that S is incompressible. It is easy to see we can find an immersed disk D with boundary C on P which represents a 1–gon. There is one vertex where C crosses a double curve of P. But the same argument as in the previous paragraph applies to simplify the intersections of D with all the lifts P of S. We get a contradiction, as there cannot be a 1–gon with interior mapped into the complement of S. This establishes that all the lifts P are embedded as claimed.

It is straightforward now to show that any pair of such planes P, P' which intersect, meet in a single line. For if there is a simple closed curve of intersection, again the disk D bounded by this curve on say P can be homotoped relative to the other planes to get a contradiction. Similarly if there are at least two lines of intersection of P and P' then there is a 2–gon D with boundary arcs on P and P'. Again we can deform D to push its interior off $\mathcal{P}$ giving a contradiction. This establishes the 1–line property.

The 4–plane and triple point properties follow once we can show that any three planes of $\mathcal{P}$ which mutually intersect, meet in a single triple point. For then if four planes all met in pairs, then on one of the planes we would see three lines all meeting in pairs. But this implies there is a 3–gon between the lines and the same disk simplification argument as above, shows that this is impossible. There are two situations which could occur for three mutually crossing planes P, P' and P''. First of all, there could be no triple points at all between the three planes. In this case the 3–gon D with three boundary arcs joining the three lines of intersection on each of the planes can be used to give a contradiction. This follows by the same simplification argument, since the 3–gon can be homotoped to have interior disjoint from $\mathcal{P}$. Secondly there could be more than one triple point between the planes. But in this case, in say P, we would see two lines meeting more than once. Hence there would be 2–gons D in P between these lines. The interiors of such 2–gons can be homotoped off $\mathcal{P}$ and the resulting contradiction completes the argument for all the properties claimed in the theorem. ◻

Remarks (1) In the next section we will define a class of 3–manifolds which are almost cubed. These do not have nice polyhedral metrics arising from a simple construction like cubings but the same methods will work as in the above theorem. This is the basis of what we call a combinatorial version of Dehn's lemma and the loop theorem.

(2) In [6], other generalisations of cubings are given, where the manifold behaves as if 'on average' it has non-positive curvature. Again the technique of the above theorem applies in this situation, to deduce incompressibility of particular surfaces.

(3) A key factor in making the above method work is that there must always be faces of the graph G which are 2–gons or 3–gons. In particular the Euler characteristic argument to show existence of such regions breaks down once D is a 4–gon!

Definition 3.2 An immersed surface S is called *filling* if the closures of the complementary regions of S in M are all cells, for all least area maps in the homotopy class of S, for any metric on M.

It is trivial to see that for the canonical surface S in a cubed 3–manifold M, all the closures of the complementary regions are cells. A little more work checks that S is actually filling. In fact, one way to do this is to observe that any essential loop in M can be homotoped to a geodesic C in the polyhedral metric defined by the cubing. Then this geodesic lifts to a line in the universal covering $\mathcal{M}$. A geodesic line will meet the planes over S in single points or lie in such a plane. Note that the lines of intersection between the planes are also geodesics. So if the given geodesic line lies in some plane, then by the filling property, it meets some other line of intersection in a single point. Hence it meets the corresponding plane in one point.

A homotopy of S to a least area surface S' relative to some metric, will lift to a proper homotopy between collections of embedded planes $\mathcal{P}$ and $\mathcal{P}'$ in the universal covering. This latter homotopy cannot remove such essential intersections between the given geodesic line and some plane (all points only move a bounded distance, whereas the ends of the line are an unbounded distance from the plane and on either side of it). So we conclude that any essential loop must intersect S.

Therefore a complementary domain to S' must have fundamental group with trivial image in $\pi_1(M)$. The argument of [10] shows that for a least area map of an incompressible surface, all the complementary regions are π_1–injective. So such regions must be cells, as a cubed manifold is irreducible.

Remark Note that P^2–irreducibility for a cubed manifold can also be shown directly, since we can apply the same argument as above to simplify the intersections of an immersed sphere or projective plane with the canonical surface and eventually by a homotopy, achieve that the sphere or projective plane lies in a complementary cell.

In [3] it is observed that a 3–manifold has a cubing of non-positive curvature if and only if it has a filling incompressible surface satisfying the 4–plane, 1–line and triple point properties. This follows immediately from the work of Hass and Scott [11]

4 Combinatorial Dehn's lemma and the loop theorem

Our aim here is to define a class of 3–manifolds which are almost cubed and for which one can still verify that the canonical surface is imcompressible and satisfies similar properties to that for cubings. In fact the canonical surface here satisfies the 4–plane, 1–line and triple point properties, but not necessarily the filling property. So as in [10], the closures of the complementary regions of S are π_1–injective handlebodies. These surfaces naturally arise in [5], where we investigate surgeries on certain classes of simple alternating links containing such closed surfaces in their complements.

Definition 4.1 Suppose that S is an immersed closed surface in a compact 3–manifold M. We say that the closure of a connected component of the complement in S of the double curves of S, is a *face* of S.

Definition 4.2 Suppose that D is an immersed disk in M with boundary C on an immersed closed surface S and interior of D disjoint from S. Assume also that there are no such disks D in M for which C crosses the double curves of S exactly once. We say that D *is homotopically trivial relative to* S, if one of the following three situations hold:

(1) If C has no intersections with the double curves of S, then D can be homotoped into a face of S, keeping its boundary fixed.

(2) Any 2–gon D (ie, C meets the singular set of S in two points) is homotopic into a double curve of S, without changing the number of intersections of C and the double curves of S and keeping C on S.

(3) Any 3–gon D is homotopic into a triple point of S, without changing the number of intersections of C and the double curves of S and keeping C on S.

Remark Note that these conditions occur in Johannson's definition of boundary patterns in [12], [13].

Theorem 4.3 *Assume that S is an immersed closed surface in a compact 3–manifold M. Suppose that all the faces of S are polygons with at least four sides. Also assume that any embedded disk D with boundary C on S and interior disjoint from S, with C meeting the double curves two or three times, is homotopically trivial relative to S and there is no such disk with C crossing the double curves once.*

Then S is incompressible with the 4–plane, 1–line and triple point properties. Moreover S lifts to a collection of embedded planes in the universal cover of M and each pair of these planes meets in at most a single line. If three planes mutually intersect in pairs, then they share a single triple point. Also the closures of components of the complement of S are π_1–injective. Finally if S is incompressible with the 4–plane, 1–line and triple point properties then S can be homotoped to satisfy the above set of conditions.

Proof The proof is extremely similar to that for Theorem 3.1 so we only remark on the ideas. First of all the conditions in the statement of Theorem 4.3 play the same role as the link conditions in the definition of a cubing of non-positive curvature. So we can homotop disks which have boundary on S to reduce the graph of intersection of the interior of the disk with S. In this way, 2–gons and 3–gons can be eliminated, as well as compressing disks for S. This is the key idea and the rest of the argument is entirely parallel to Theorem 3.1. Note that the closures of the complementary regions of S are π_1–injective, by essentially the same proof as in [10].

The only thing that needs to be carefully checked, is why it suffices to assume that only embedded disks in the complementary regions need to be examined, to see that any possibly singular n–gons, for $2 \leq n \leq 3$, are homotopically trivial and there are no singular 1–gons.

Suppose that we have a properly immersed disk D in a complementary region, with boundary meeting the set of double curves of S at most three times and D is not homotopic into a double arc, a triple point or a face of S. If this disk is not homotopic into the boundary of the complementary region, we can apply Dehn's lemma and the loop theorem to replace the singular disk by an embedded one. Moreover since the boundary of the new disk is obtained by cutting-and-pasting of the old boundary curve, we see the new curve also meets the set of double curves of S at most three times. So this case is easy to handle: it does not happen.

Next assume that the singular disk is homotopic into the boundary surface T of the complementary region. (Note we include the possibility here that the complementary region is a ball and T is a 2–sphere). Let C be the boundary curve of the singular disk and let N be a small regular neighbourhood of C in T. Thus C is null homotopic in T. Notice that there are at most three double arcs of S crossing N. Now fill in the disks D' bounded by any contractible boundary component C' of N in T, to enlarge N to N'. Since C shrinks in T, it is easy to see by Van Kampen's theorem, that C contracts also in N'. Also if C' meets the double arcs of S, we see the picture in D' must be either a single arc or three arcs meeting at a single triple point, or else we have found an embedded disk contradicting our assumption. For we only need to check that C' cannot meet the double arcs in at least four points. If C' did have four or more intersection points with the singular set of S, then one of the double arcs crossing N has both ends on C'. But this is impossible, as there would be a cycle in the graph of the double arcs on T, which met the contractible curve C once.

Finally we notice that there must be some disks D' which meet the double arcs; in fact at least one point on the end of each double arc in the boundary of N must be in such a disk. For otherwise it is impossible for C to shrink in N', as there is an essential intersection at one point with such an arc. (This immediately shows the possibility that C crossed the double curves once cannot happen). So there are either one or two disks D' with a single arc and at most one such disk with three arcs meeting in a triple point. But the latter case means that C can be shrunk into the triple point and the former means C can be homotoped into the double arc of S in N' by an easy examination of the possibilities.

Hence this shows that it suffices to consider only embedded disks when requiring the properties in Theorem 4.3. This is very useful in applications in [5].

To show the converse, assume we have an incompressible surface which has the 4–plane, 1–line and triple point properties. Notice that in the paper of Hass and Scott [11], the triple point property is enough to show that once the number of triple points has been minimised for a least area representative of S, then the combinatorics of the surface are rigid. So we get that S has exactly the properties as in Theorem 4.3. □

Remark Theorem 4.3 can be viewed as a singular version of Dehn's lemma and the loop theorem. For we have started with an assumption that there are no embedded disks of a special type with boundary on the singular surface S and have concluded that S is incompressible, ie π_1–injective. In [6] other variants on this theme are given.

Definition 4.4 We say that M is *almost cubed* if it is P^2–irreducible and contains a surface S as in Theorem 4.3.

It is interesting to speculate as to how large is the class of almost cubed 3–manifolds. We do not know of any specific examples of compact P^2–irreducible 3–manifolds with infinite π_1 which are not almost cubed.

Corollary 4.5 *Suppose that M is a compact P^2–irreducible 3–manifold with boundary, which is almost cubed, ie, there is a canonical surface S in the interior of the manifold. Assume also that the complementary regions of S include collars of all components of the boundary. Suppose that a handlebody is glued onto each boundary component of M to give a new manifold M'. If the boundary of every meridian disk, when projected onto S, meets the double curves at least four times, then M' is almost cubed.*

Proof This follows immediately from Theorem 4.3, by observing that since the boundary of every meridian disk meets the double curves at least four times, there are no non-trivial n–gons in the complement of S in M' for $n = 2, 3$ and no 1–gons. Hence M' is almost cubed, as S in M' has similar properties to S in M. □

Remark Examples satisfying the conditions of the corollary are given in [5]. In particular such examples occur for many classes of simple alternating link complements. In [1], the class of well-balanced alternating links are shown to be almost cubed and so the corollary applies.

5 Hierarchies

Our aim in this section is to give a similar treatment of hierarchies, to that of cubings and almost cubings. The definition below is motivated by the special hierarchies used by Waldhausen in his solution of the word problem in the class of Haken 3–manifolds [20]. Such hierarchies were extensively studied by Johannson in his work on the characteristic variety theorem [12] and also in [13].

Definition 5.1 A *hierarchy* is a collection $\mathcal{S}$ of embedded compact 2–sided surfaces $S_1, S_2, \dots, S_k$, which are not 2–spheres, in a compact P^2–irreducible 3–manifold, with the following properties:

(1) Each S_i has boundary on the union of the previous S_j for $j \leq i-1$.

(2) If an embedded polygonal disk D intersects $\mathcal{S}$ in its boundary loop C only and C meets the boundary curves of $\mathcal{S}$ in at most 3 points, then D is homotopic into either an arc of a boundary curve, a vertex or a surface of $\mathcal{S}$, where ∂D is mapped into $\mathcal{S}$ throughout the homotopy.

(3) Assume an embedded polygonal disk D intersects $\mathcal{S}$ in its boundary loop C only and C has only one boundary arc λ on the surface S_j, where j is the largest value for surfaces of $\mathcal{S}$ met by C. Then λ is homotopic into the boundary of S_j keeping the boundary points of λ fixed.

Remarks (1) Note that Waldhausen shows that for a Haken 3–manifold, one can always change a given hierarchy into one satisfying these conditions by the following simple procedures:

- Assuming that all S_j have been picked for $j < i$, then first arrange that after M is cut open along all the S_j to give M_{i-1}, the boundary of S_i is chosen so that there are no triangular regions cut off between the 'boundary pattern' of M_{i-1} (ie, all the boundary curves of surfaces S_j with $j < i$) and the boundary curves of S_i. This is done by minimising the intersection between the boundary pattern of M_{i-1} and ∂S_i.
- It is simple to arrange that S_i is boundary incompressible in M_{i-1}, by performing boundary compressions if necessary. So there are no 2–gons between S_i and S_j for any $j < i$.
- Finally one can see that there are no essential triangular disks in M_{i-1}, with one boundary arc on S_i and the other two arcs on surfaces S_j for values $j \leq i-1$, by the boundary incompressibility of S_i as in the previous step.

(2) Notice we are not assuming the hierarchy is complete, in the sense that the complementary regions are cells (in the case that M is closed) or cells and collars of the boundary (if M is compact with incompressible boundary).

(3) The next result is a converse statement, showing that the conditions above imply that the surfaces do form a hierarchy.

Theorem 5.2 *Assume that $S_1, S_2, \ldots, S_k$ is a sequence $\mathcal{S}$ of embedded compact 2–sided surfaces, none of which are 2–spheres, in a compact P^2–irreducible 3–manifold M with the following properties:*

(1) *Each S_i has boundary on the union of the previous S_j for $j \leq i-1$.*

(2) *If an embedded polygonal disk D intersects $\mathcal{S}$ in its boundary loop C only and C meets the boundary curves of $\mathcal{S}$ in at most 3 points, then D is homotopic into either an arc of a boundary curve, a vertex or a surface of $\mathcal{S}$.*

(3) *Assume an embedded polygonal disk D intersects $\mathcal{S}$ in its boundary loop C only and C has only one boundary arc λ on the surface S_j, where j is the largest value for surfaces of $\mathcal{S}$ met by C. Then λ is homotopic into the boundary of S_j keeping the boundary points of λ fixed.*

Then each of the surfaces S_i is incompressible and boundary incompressible in the cut open manifold M_{i-1} and $\mathcal{S}$ forms a hierarchy for M as above.

Proof The argument is very similar to those for Theorems 3.1 and 4.3 above, so we outline the modifications needed.

Suppose there is a compressing or boundary compressing disk D for one of the surfaces S_i. We may assume that all the previous S_j are incompressible and boundary incompressible by induction. Consider G, the graph of intersection of D with the previous S_j, pulled back to the domain of D. Then G is a degree three graph; however at each vertex there is a 'T' pattern as one of the incident edges lies in some S_j and the other two in the same surface S_k for some $k < j$. We argue that the graph G can be simplified by moves similar to the ones in Theorems 3.1 and 4.3.

First of all, note that by an innermost region argument, there must be either an innermost 0–gon, 2–gon or a triangular component of the closure of the complement of G. For we can cut up D first using the arcs of intersection with S_1, then S_2 etc. Using the first collection of arcs, there is clearly an outermost 2–gon region in D. Next the second collection of arcs is either disjoint from this

2–gon or there is an outermost 3–gon. At each stage, there must always be an outermost 2–gon or 3–gon. (Of course any simple closed curves of intersection just start smaller disks which can be followed by the same method. If such a loop is isolated, one gets an innermost 0–gon which is readily eliminated, by assumption).

By supposition, such a 2–gon or 3–gon can be homotoped into either a boundary curve or into a boundary vertex of some S_j. We follow this by slightly deforming the map to regain general position. The complexity of the graph is defined by listing lexicographically the numbers of vertices with a particular label. The label of each vertex is given by the subscript of the first surface of $\mathcal{S}$ containing the vertex. (cf [13] for a good discussion of this lexicographic complexity). The homotopy above can be readily seen to reduce the complexity of the graph. Note that the hypotheses only refer to embedded n–gons but as in the proof of Theorem 4.3 it is easy to show that if there are only trivial embedded n–gons for $n < 4$, then the same is true for immersed n–gons, using Dehn's lemma and the loop theorem. Similarly, the hypothesis (3) of the theorem can be converted to a statement about embedded disks, using Dehn's lemma and the loop theorem, since cutting open the manifold using the previous surfaces, converts the polygon into a 2–gon. This completes the proof of Theorem 5.2. □

Example 5.3 Consider the Borromean rings complement M in the 3–sphere. We can find such a hierarchy easily as follows:

Start with a peripheral torus as S_1, ie, one of the boundary tori of the complement M. Next choose S_2 as an essential embedded disk with boundary on S_1, with a tube attached to avoid the intersections with one of the other components C_1 of B, the Borromean rings. Now cut M open along S_1 and S_2 to form a collar of S_1 and another component M_2. It is easy to see that M_2 is a genus 2 handlebody and B has two components C_1 and C_2 inside this, forming a sublink B'. Moreover these two loops are generators of the fundamental group of M_2, since they are dual (intersect in single points) to two disjoint meridian disks for M_2. Finally the loops are readily seen to be linked once in M_2.

Now we can cut M_2 open using a separating meridian disk with a handle attached, so as to avoid B'. This surface S_3 can also be viewed as another spanning disk with boundary on S_1 having a tube attached to miss the other component C_2 of B'. So there is a nice symmetry between S_2 and S_3. Finally we observe that when M is cut open along S_1, S_2 and S_3 to form M_3, this is a pair of genus two handlebodies, each of which contains one of C_1 and C_2 as a core curve for one of the handles, plus the collar of S_1. So we can choose

S_4 and S_5 to be non-separating meridian disks for these handlebodies, disjoint from B, so that M_5 consists of collars of the three boundary tori.

Notice that the 'boundary patterns' on each of these tori, ie, the boundary curves of the surfaces in the hierarchies, consist of two contractible simple closed curves and four arcs, two of which have boundary on each of these loops. The pairs of arcs are 'parallel', in that the whole boundary pattern divides each torus into six regions, two 0–gon disks, two 4–gon disks and two annuli (cf Figure 4 in [4]).

As a corollary then to Theorem 5.2, we observe that any non-trivial surgery on each of the two components C_2 and C_3 of B gives meridian disks S_6 and S_7 which meet the boundary pattern at least four times. Hence these surfaces form a hierarchy for the surgered manifold and so all such surgeries on C_2 and C_3 have the result that the peripheral torus S_1 remains incompressible.

Remarks (1) This can be proved by other methods but the above argument is particularly revealing, not using any hyperbolic geometry.

(2) A similar example of the Whitehead link is discussed in [4] and it is interesting to note the boundary pattern found there (Figure 4) is exactly the same as here.

(3) A very significant problem is to try to use the above characterisation of hierarchies to find some type of polyhedral metric of non-positive curvature, similar to cubings. This would give a polyhedral approach to Thurston's uniformisation Theorem for Haken manifolds [19].

6 Topological rigidity of cubed manifolds

In this section we give a different approach to the result of Hass and Scott [11] that if a closed P^2–irreducible 3–manifold is homotopy equivalent to a cubed 3–manifold then the manifolds are homeomorphic. This can be viewed as a polyhedral version of Mostow rigidity, which says that complete hyperbolic manifolds of finite volume which are homotopy equivalent are isometric, in dimensions greater than 2. We refer to this as topological rigidity of cubed 3–manifolds. Our aim here is to show that this rigidity theorem can be shown without resorting to the least area methods of Freedman Hass Scott [9], but can be obtained by a direct argument more like Waldhausen's original proof of rigidity for Haken 3–manifolds [21]. Note that various generalisations of Hass and Scott's theorem have been obtained recently by Paterson [16], [17] using different methods.

Theorem 6.1 *Suppose that M is a compact 3–manifold with a cubing of non-positive curvature and M' is a closed P^2–irreducible 3–manifold which is homotopy equivalent to M. Then M and M' are homeomorphic.*

Proof Note that the cases where either M has incompressible boundary or is non-orientable, are not so interesting, as then M and M' are Haken and the result follows by Waldhausen's theorem [21]. So we restrict attention to the case where M and M' are closed and orientable.

Our method is a mixture of those of [11] and [21] and we indicate the steps, which are all quite standard techniques.

Step 1 By Theorem 3.1, if S is the canonical surface for the cubed manifold M and M_S is the cover corresponding to the fundamental group of S, then S lifts to an embedding denoted by S again in M_S. For by Theorem 3.1, all the lifts of S to the universal covering $\mathcal{M}$ are embedded planes and $\pi_1(S)$ stabilises one of these planes with quotient the required lift of S. Let $f\colon M' \to M$ be the homotopy equivalence and assume that f has been perturbed to be transverse to S. Denote the immersed surface $f^{-1}(S)$ by S'. Notice that f lifts to a map $\tilde{f}\colon \mathcal{M}' \to \mathcal{M}$ between universal covers and so all the lifts of S' to $\mathcal{M}'$ are properly embedded non-compact surfaces. In fact, if M'_S is the induced cover of M' corresponding to M_S, ie, with fundamental group projecting to $f^{-1}(\pi_1(S))$, then there is an embedded lift, denoted S' again, of S' to M'_S, which is the inverse image of the embedded lift of S.

The first step is to surger S' in M'_S to get a copy of S as the result. We will be able to keep some of the nice properties of S by this procedure, especially the 4–plane property. This will enable us to carry out the remainder of the argument of Hass and Scott quite easily. For convenience, we will suppose that S is orientable. The non-orientable case is not difficult to derive from this; we leave the details to the reader. (All that is necessary is to pass to a 2–fold cover of M_S and M'_S, where S lifts to its orientable double covering surface.)

Since f is a homotopy equivalence, so is the lifted map $f_S\colon M'_S \to M_S$. Hence if S' is not homeomorphic to S, then the induced map on fundamental groups of the inclusion of S' into M'_S has kernel in $\pi_1(S')$. So we can compress S' by Dehn's lemma and the loop theorem. On the other hand, the ends of M_S pull back to ends of M'_S and a properly embedded line going between the ends of M_S meeting S once, pulls back to a similar line in M'_S for S'. Hence S' represents a non-trivial homology class in M'_S and so cannot be completely compressed. We conclude that S' compresses to an incompressible surface S'' separating the ends of M'_S.

Now we claim that any component S^* of S'' which is homologically non-trivial, must be homeomorphic to S. Also the inclusion of S^* induces an isomorphism on fundamental groups to M'_S. The argument is in [9], for example, but we repeat it for the benefit of the reader. The homotopy equivalence between M'_S and S induces a map $f\colon S^* \to S$ which is non-zero on second homology. So f is homotopic to a finite sheeted covering. Lifting S^* to the corresponding finite sheeted cover of M'_S, we get a number of copies of S^*, if the map $S^* \to M'_S$ is not a homotopy equivalence. Now the different lifts of S^* must all separate the two ends of the covering of M'_S and so are all homologous. (Note as the second homology is cyclic, there are exactly two ends). But then any compact region between these lifts projects onto M'_S and so M'_S is actually compact, unless S is non-orientable, which has been ruled out.

Finally to complete this step, we claim that if S^* is projected to M' and then lifted to $\mathcal{M}'$, then the result is a collection of embedded planes $\mathcal{P}^*$ satisfying the 4–plane property. Notice first of all that all the lifts $\mathcal{P}'$ of S' to $\mathcal{M}'$ satisfy the '4–surface' property. In other words, if any subcollection of four components of $\mathcal{P}'$ are chosen, then there must be a disjoint pair. This is evident as S' is the pull-back of S and so $\mathcal{P}'$ is the pull-back of $\mathcal{P}$. Then the 4–plane property clearly pulls-back to the '4–surface' property as required.

Now we claim that as S' is surgered and then a component S^* is chosen, this can be done so that the 4–surface property remains valid. For consider some disk D used to surger the embedded lift S' in M'_S. By projecting to M' and lifting to $\mathcal{M}'$, we have a family of embedded disks surgering the embedded surfaces $\mathcal{P}'$.

It is sufficient to show that one such D can be selected so as to miss all the surfaces P'' in $\mathcal{P}'$ which are disjoint from a given surface P' containing the boundary of D, as the picture in the universal covering is invariant under the action of the covering translation group. This is similar to the argument in [10]. First of all, if D meets any such a surface P'' in a loop which is non-contractible on P'', we can replace D by the subdisk bounded by this loop. This subdisk has fewer curves of intersection with $\mathcal{P}'$ than the original. Of course the subdisk may not be disjoint from its translates under the stabiliser of P''. However we can fix this up at the end of the argument. We relabel P'' by P' if this step is necessary.

Suppose now that D meets any surface P'' disjoint from P' in loops C which are contractible on P''. Choose an innermost such a loop. We would like to do a simple disk swap and reduce the number of such surfaces P'' disjoint from P' met by D. Note we do not care if the number of loops of intersection goes up during this procedure. However we must be careful that no new planes are intersected by D. So suppose that C bounds a disk D'' on P'' met by some plane P_1 which is disjoint from P', but D does not already meet P_1. Then clearly P_1 must meet D'' in a simple closed curve in the interior of D''. Now we can use the technique of Hass and Scott to eliminate all such intersections in P'. For by the 4–surface property, either such simple closed curves are isolated (ie not met by other surfaces) or there are disjoint embedded arcs where the curves of intersection of the surfaces cross an innermost such loop. But then we can start with an innermost such 2–gon between such arcs and by simple 2–gon moves, push all the arcs equivariantly outside the loop. At each stage we decrease the number of triple points and eventually can eliminate the contractible double curves. The conclusion is that eventually we can pull D off all the surfaces disjoint from P'.

Finally to fix up the disk D relative to the action of the stabiliser of P', project P' to the compact surface in M'_S. Now we see that D may project to an immersed disk, but all the lifts of this immersed disk with boundary on P' are embedded and disjoint from the surfaces in $\mathcal{P}'$ which miss P'. We can now apply Dehn's lemma and the loop theorem to replace the immersed disk by an embedded one in M'_S. This is obtained by cutting and pasting, so it follows immediately that any lift of the new disk with boundary on P' misses all the surfaces which are disjoint from P' as desired. This completes the first step of the argument.

Step 2 The remainder of the argument follows that of Hass and Scott closely. By Step 1 we have a component S^* of the surgered surface which gives a subgroup of $\pi_1(M')$ mapped by f isomorphically to the subgroup of $\pi_1(M)$ corresponding to $\pi_1(S)$. Also S^* is embedded in the cover M'_S and all the lifts have the 4–plane property in $\mathcal{M}'$. All that remains is to use Hass and Scott's triple point cancellation technique to get rid of redundant triple points and simple closed curves of intersection between the planes over S^*. Eventually we get a new surface, again denoted by S^*, which is changed by an isotopy in M'_S and has the 1–line and triple point properties. It is easy then to conclude that an equivariant homeomorphism between $\mathcal{M}'$ and $\mathcal{M}$ can be constructed.

Note that this case is the easy one in Hass and Scott's paper, as the triple point property means that triangular prism regions cannot occur and so no crushing of such regions is necessary. □

Corollary 6.2 *If $f\colon M' \to M$ is a homotopy equivalence and M has an immersed incompressible surface S satisfying the 4–plane property, then the method of Theorem 6.1 shows that M' also has an immersed incompressible* *surface S' satisfying the 4–plane property and f induces an isomorphism from $\pi_1(S')$ to $\pi_1(S)$.* □

Remark This can be shown to be true by least area methods also, but it is interesting to have alternative combinatorial arguments. Least area techniques give the result also for the k–plane property, for all k. However there is no direct information about how the surface is pulled back, as in the above argument.

References

[1] **I R Aitchison**, **E Lumsden**, **J H Rubinstein**, *Cusp structure of alternating links*, Invent. Math. 109 (1992) 473–494

[2] **I R Aitchison**, **S Matsumoto**, **J H Rubinstein**, *Immersed surfaces in cubed manifolds*, Asian J. of Math. 1 (1997) 85–95

[3] **I R Aitchison**, **J H Rubinstein**, *An introduction to polyhedral metrics of non-positive curvature*, from: "Geometry of Low Dimensional Manifolds: 2", London Math. Soc. Lecture Notes Series 151, Cambridge University Press (1990) 127–161

[4] **I R Aitchison**, **J H Rubinstein**, *Incompressible surfaces and the topology of 3–dimensional manifolds*, J. Aust. Math. Soc. (Series A) 55 (1993) 1–22

[5] **I R Aitchison**, **J H Rubinstein**, *Immersed incompressible surfaces in alternating link complements*, in preparation

[6] **I R Aitchison**, **J H Rubinstein**, **M Sageev**, *Separability properties of some immersed incompressible surfaces in 3–manifolds*, in preparation

[7] **W Ballmann**, **M Gromov**, **V Schroeder**, *Manifolds of non-positive curvature*, Progress in Math. 61, Birkhauser (1985)

[8] **S Bleiler**, **C Hodgson**, *Spherical space forms and Dehn surgery*, from: "Knots 90", de Gruyter (1992) 425–434

[9] **M Freedman**, **J Hass**, **P Scott**, *Least area incompressible surfaces in 3–manifolds*, Invent. Math. 71 (1983) 609–642

[10] **J Hass**, **J H Rubinstein**, **P Scott**, *Compactifying covers of closed 3–manifolds*, J. Diff. Geom. 30 (1989) 817–832

[11] **J Hass**, **P Scott**, *Homotopy equivalences and homeomorphisms of 3–manifolds*, Topology, 31 (1992) 493–517

[12] **K Johannson**, *Homotopy equivalences of 3–manifolds with boundaries*, Lecture Notes in Math. 761, Springer (1978)

[13] **K Johannson**, *On the loop and sphere theorem*, from: "Low Dimensional Topology", Proc. of the Conf. at Univ. of Tennessee, Conf. Proc. and Lecture Notes on Geom. and Top. 3, International Press (1994) 47–54

[14] **W Jaco**, **J H Rubinstein**, *PL minimal surfaces in 3–manifolds*, J. of Diff. Geom. 27 (1988) 493–524

[15] **L Mosher**, *Geometry of cubulated 3–manifolds*, Topology, 34 (1995) 789–813

[16] **J Paterson**, *The classification of 3–manifolds with 4–plane quasi-finite immersed surfaces*, preprint

[17] **J Paterson**, *Immersed one-to-one projective surfaces in 3–manifolds*, preprint

[18] **R Schoen**, **S T Yau**, *Existence of incompressible minimal surfaces and the topology of 3–dimensional manifolds with non-negative scalar curvature*, Annals of Math. (2) 110 (1979) 127–142

[19] **W P Thurston**, *The geometry and topology of 3–manifolds*, Princeton University (1978)

[20] **F Waldhausen**, *On irreducible 3–manifolds which are sufficiently large*, Annals of Math. 87 (1968) 56–88

[21] **F Waldhausen**, *The word problem in fundamental groups of sufficiently large 3–manifolds*, Annals of Math. 88 (1968) 272–280

Department of Mathematics and Statistics, University of Melbourne
Parkville, Vic 3052, Australia

Email: `iain@ms.unimelb.edu.au, rubin@ms.unimelb.edu.au`

Received: 24 September 1998 Revised: 14 November 1999