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Small surfaces and Dehn filling

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Abstract We give a summary of known results on the maximal distances between Dehn fillings on a hyperbolic 3–manifold that yield 3–manifolds containing a surface of non-negative Euler characteristic that is either essential or Heegaard.

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Keywords Dehn filling, hyperbolic 3–manifold, small surface

Dedicated to Rob Kirby on the occasion of his 60th birthday

0 Introduction

By a *small* surface we mean one with non-negative Euler characteristic, ie a sphere, disk, annulus or torus. In this paper we give a survey of the results that are known on the distances between Dehn fillings on a hyperbolic 3–manifold that yield 3–manifolds containing small surfaces that are either essential or Heegaard. We also give some new examples in this context.

In Section 1 we describe the role of small surfaces in the theory of 3–manifolds, and in Section 2 we summarize known results on the distances Δ between Dehn fillings on a hyperbolic 3–manifold M that create such surfaces. Section 3 discusses the question of how many manifolds M realize the various maximal values of Δ, while Section 4 considers the situation where the manifold M is *large* in the sense of Wu [53]. Finally, in Section 5 we consider the values of Δ for fillings on a hyperbolic manifold M with k torus boundary components, as k increases.

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1 Small surfaces and 3–manifolds

The importance of small surfaces in the theory of 3–manifolds is well known. For example, every 3–manifold (for convenience we shall assume that all 3–manifolds are compact and orientable) can be decomposed into canonical pieces by cutting it up along such surfaces.

For spheres, this is due to Kneser [38] (see also Milnor [41]), and goes as follows. If a 3–manifold M contains a sphere S which does not bound a ball in M, then S is *essential* and M is *reducible*. Otherwise, M is *irreducible*. Then any oriented 3–manifold M can be expressed as a connected sum $M_1 \# \ldots \# M_n$, where each M_i is either irreducible or homeomorphic to $S^2 \times S^1$. Furthermore, if we insist that no M_i is the 3–sphere, then the summands M_i are unique up to orientation-preserving homeomorphism.

Turning to disks, a properly embedded disk D in a 3–manifold M is said to be *essential* if ∂D does not bound a disk in ∂M. If M contains such a disk, ie, if ∂M is compressible, then M is *boundary reducible*; otherwise M is *boundary irreducible*. Then we have the following statement about essential disks in a 3–manifold, proved by Bonahon in [6]: In any irreducible 3–manifold M, if W is a maximal (up to isotopy) disjoint union of compression bodies on the components of ∂M, then W is unique up to isotopy, any essential disk in M can be isotoped (rel ∂) into W, and $\overline{M - W}$ is irreducible and boundary irreducible. Note that $\overline{M - W}$ is obtained from M by cutting M along a collection of essential disks that is maximal in the appropriate sense.

Now, let us say that a connected, orientable, properly embedded surface F, not a sphere or disk, in a 3–manifold M is *essential* if it is incompressible and not parallel to a subsurface of ∂M. With this definition, an essential surface may be boundary compressible. However, if F is an essential annulus and M is irreducible and boundary irreducible, then F is boundary incompressible.

Then, in an irreducible, boundary irreducible 3–manifold M, there is a canonical (up to isotopy) collection $\mathcal{F}$ of disjoint essential annuli and tori, such that each component of M cut along $\mathcal{F}$ is either a Seifert fiber space, an I–bundle over a surface, or a 3–manifold that contains no essential annulus or torus. This is the JSJ–decomposition of M, due to Jaco and Shalen [36] and Johannson [37].

Following Wu [53], let us call a 3–manifold *simple* if it contains no essential sphere, disk, annulus or torus. Then Thurston has shown [49], [50] that a 3–manifold M with non-empty boundary (other than B^3) is simple if and only if it is *hyperbolic*, in the sense that M with its boundary tori removed has a complete hyperbolic structure with totally geodesic boundary.

For closed 3–manifolds M, if $\pi_1(M)$ is finite then Thurston's geometrization conjecture [49], [50] asserts that M has a spherical structure. Equivalently, M is either S^3, a lens space, or a Seifert fiber space of type $S^2(p_1,p_2,p_3)$ with $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}>1$. (We shall say that a Seifert fiber space is *of type* $F(p_1,p_2,\dots,p_n)$ if it has base surface F and n singular fibers with multiplicities $p_1,p_2,\dots,p_n$.) Note that S^3 contains a Heegaard sphere, while a lens space contains a Heegaard torus. For closed 3–manifolds M with infinite fundamental group, there are two cases. If $\pi_1(M)$ has no $\mathbb{Z}\times\mathbb{Z}$ subgroup, then the geometrization conjecture says that M is hyperbolic. If $\pi_1(M)$ does have a $\mathbb{Z}\times\mathbb{Z}$ subgroup, then by work of Mess [40], Scott [47], [48], and, ultimately, Casson and Jungreis [12] and Gabai [20], M either contains an essential torus or is a Seifert fiber space of type $S^2(p_1,p_2,p_3)$.

Summarizing, we may say that if a 3–manifold is not hyperbolic then it either

(1) contains an essential sphere, disk, annulus or torus; or

(2) contains a Heegaard sphere or torus; or

(3) is a Seifert fiber space of type $S^2(p_1,p_2,p_3)$; or

(4) is a counterexample to the geometrization conjecture.

2 Distances between small surface Dehn fillings

Recall that if M is a 3–manifold with a torus boundary component T_0, and α is a *slope* (the isotopy class of an essential unoriented simple closed curve) on T_0, then the manifold obtained by α*–Dehn filling* on M is $M(\alpha)=M\cup V$, where V is a solid torus, glued to M along T_0 in such a way that α bounds a disk in V. If M is hyperbolic, then the set of *exceptional* slopes $E(M)=\{\alpha : M(\alpha)$ is not hyperbolic$\}$ is finite [49], [50], and we are interested in obtaining universal upper bounds on the size of $E(M)$. Note that if $\alpha\in E(M)$ then $M(\alpha)$ satisfies (1), (2), (3) or (4) above. Here we shall focus on (1) and (2), in other words, where $M(\alpha)$ contains a small surface that is either essential or Heegaard. (For results on case (3), see Boyer's survey article [7] and references therein, and also [10].)

Following Wu [53], let us say that a 3–manifold is of *type* S, D, A or T if it contains an essential sphere, disk, annulus or torus. Let us also say that it is of *type* S^H or T^H if it contains a Heegaard sphere or torus. Recall that the *distance* $\Delta(\alpha_1,\alpha_2)$ between two slopes on a torus is their minimal geometric intersection number. Then, for $X_i\in\{S,D,A,T,S^H,T^H\}$ we define

$$\Delta(X_1,X_2) = \max\{\Delta(\alpha_1,\alpha_2) : \text{there is a hyperbolic 3–manifold } M \text{ and slopes } \alpha_1,\alpha_2 \text{ on a torus component of } \partial M \text{ such that } M(\alpha_i) \text{ is of type } X_i,\ i=1,2\}.$$

The numbers $\Delta(X_1,X_2)$ are now known in almost all cases, and are summarized in Table 2.1.

	S	D	A	T	S^H	T^H
S	1	0	2	3	?	1
D		1	2	2	–	–
A			5	5	–	–
T				8	2	?
S^H					0	1
T^H						1

Table 2.1 $\Delta(X_1,X_2)$

(The entries $\Delta(X_1,X_2)$ for $X_1 = D$ or A and $X_2 = S^H$ or T^H are blank because the first case applies only to manifolds with boundary, while the second case applies only to closed manifolds.)

The upper bounds in the various cases indicated in Table 2.1 are due to the following. (S,S): Gordon and Luecke [28]; (S,D): Scharlemann [46]; (S,A): Wu [53]; (S,T): Oh [44], Qiu [45], and Wu [53]; (S,T^H): Boyer and Zhang [9]; (D,D): Wu [51]; (D,A): Gordon and Wu [33]; (D,T): Gordon and Luecke [31]; (A,A), (A,T), and (T,T): Gordon [22]; (T,S^H): Gordon and Luecke [29]; (S^H,S^H): Gordon and Luecke [27]; (S^H,T^H) and (T^H,T^H): Culler, Gordon, Luecke and Shalen [13].

References for the existence of examples realizing these upper bounds are as follows:

$\underline{(S,S)}$ An example of a hyperbolic 3–manifold, with two torus boundary components, having a pair of reducible Dehn fillings at distance 1, is given by Gordon and Litherland in [25]. By doing suitable Dehn filling along the other boundary component one obtains infinitely many hyperbolic 3–manifolds with a single torus boundary component, having reducible fillings at distance 1. Infinitely many such examples with two torus boundary components are given by Eudave-Muñoz and Wu in [15].

(S,A), (D,A) and (D,T) An example of a hyperbolic 3–manifold M, with two torus boundary components, with Dehn fillings $M(\alpha_1)$, $M(\alpha_2)$ such that $M(\alpha_1)$ is reducible and boundary reducible, $M(\alpha_2)$ is annular and toroidal, and $\Delta(\alpha_1,\alpha_2) = 2$, is given by Hayashi and Motegi in [35; section 12]. Infinitely many such examples are constructed by Eudave-Muñoz and Wu in [15].

(S,T) and (S,T^H) Boyer and Zhang point out in [8] and [9; Example 7.8], that the hyperbolic 3–manifold $M = W(6)$, obtained by 6–Dehn filling (using the usual meridian–latitude slope co-ordinates) on the exterior W of the Whitehead link, has the property that $M(1)$ is reducible, $M(4)$ is toroidal, and $M(\infty)$ is the lens space $L(6,1)$. Infinitely many such hyperbolic 3–manifolds M are given by Eudave-Muñoz and Wu in [15; Lemma 4.1 and Theorem 4.2]; ie, each M has Dehn fillings $M(\alpha_1)$, $M(\alpha_2)$, $M(\alpha_3)$ such that $M(\alpha_1)$ is reducible, $M(\alpha_2)$ is toroidal, $M(\alpha_3)$ is a lens space, $\Delta(\alpha_1,\alpha_2) = 3$, and $\Delta(\alpha_1,\alpha_3)$ $(= \Delta(\alpha_2,\alpha_3)) = 1$.

(D,D) Infinitely many examples of hyperbolic knots in a solid torus, with a non-trivial Dehn surgery yielding a solid torus, have been given by Berge [1] and [18].

(A,A) and (A,T) Miyazaki and Motegi [42] and, independently, Gordon and Wu [32], have shown that the exterior M of the Whitehead sister link has a pair of Dehn fillings $M(\alpha_1)$, $M(\alpha_2)$, each of which is annular and toroidal, with $\Delta(\alpha_1,\alpha_2) = 5$.

(T,T) Thurston has shown [49] that if M is the exterior of the figure eight knot then $M(4)$ and $M(-4)$ are toroidal.

(T,S^H) Infinitely many examples of hyperbolic knots in S^3 with half-integral toroidal Dehn surgeries are given by Eudave-Muñoz in [14].

(S^H,T^H) and (T^H,T^H) Infinitely many hyperbolic knots in S^3 with lens space surgeries are described by Fintushel and Stern in [16]. A general construction of such knots is given by Berge in [2], who has subsequently shown [3] that the knots listed in [2] are the only ones obtainable in this way. He has also suggested [2] that any knot in S^3 with a lens space surgery might be of this form.

There is a (unique) hyperbolic knot K in $S^1 \times D^2$ with two non-trivial surgeries which yield $S^1 \times D^2$; see [1]. Under an unknotted embedding of $S^1 \times D^2$ in S^3 with n meridional twists, the image of K is a hyperbolic knot K_n in S^3 with two lens space surgeries; see [1]. (The simplest example of this kind is the $(-2,3,7)$ pretzel knot, which is one of the knots constructed in [16].) Hence there are infinitely many hyperbolic 3–manifolds M with Dehn fillings $M(\alpha_1)$, $M(\alpha_2)$, $M(\alpha_3)$ such that $M(\alpha_1) \cong S^3$, $M(\alpha_2)$ and $M(\alpha_3)$ are lens spaces, and $\Delta(\alpha_1, \alpha_2) = \Delta(\alpha_1, \alpha_3) = \Delta(\alpha_2, \alpha_3) = 1$.

We see that only two values of $\Delta(X_1, X_2)$ are unknown, namely: $\Delta(S, S^H)$ and $\Delta(T, T^H)$. The conjectured values are $-\infty$ and 3, and the best bounds to date are 1 [26] and 5 [24], respectively.

The assertion that $\Delta(S, S^H) = -\infty$ says that no Dehn surgery on a hyperbolic knot in S^3 gives a reducible manifold. This would follow from the

Cabling Conjecture (González-Acuña and Short [21]) *If Dehn surgery on a non-trivial knot K in S^3 gives a reducible manifold then K is a cable knot.*

(Here, it is convenient to regard a torus knot as a cable of the unknot.)

In fact, the cabling conjecture and the assertion $\Delta(S, S^H) = -\infty$ are equivalent, since Scharlemann has shown [46] that the former is true for satellite knots.

Regarding $\Delta(T, T^H)$, the figure eight sister manifold M has slopes α_1, α_2 on ∂M such that $M(\alpha_1)$ is toroidal, $M(\alpha_2)$ is the lens space $L(5,1)$, and $\Delta(\alpha_1, \alpha_2) = 3$ [5]. In fact, there are infinitely many such hyperbolic manifolds M, and also infinitely many such M where $M(\alpha_2)$ is the lens space $L(7,2)$; see Section 3. On the other hand, it is shown in [24] that $\Delta(T, T^H) \leq 5$. Presumably $\Delta(T, T^H) = 3$: there is nothing in the argument of [24] to suggest that the bound of 5 obtained there is best possible, while 4 is not a Fibonacci number.

Question 2.1 *Is there a hyperbolic manifold with a toroidal filling and a lens space filling at distance 4 or 5?*

3 The manifolds realizing $\Delta(X_1, X_2)$

Having determined $\Delta(X_1, X_2)$, one can ask about the manifolds M that have fillings realizing $\Delta(X_1, X_2)$. Regarding the number of such manifolds, we have

Theorem 3.1 *In the cases where $\Delta(X_1, X_2)$ is known, there are infinitely many hyperbolic manifolds M realizing $\Delta(X_1, X_2)$, except when $(X_1, X_2) = (A, A), (A, T)$ or (T, T).*

This is well known when $\Delta(X_1, X_2) = 0$. References in the other cases are given in Section 2 above.

Turning to the exceptional cases (A, A), (A, T) and (T, T), the first two are simultaneously described in the following theorem. (Here, and in Theorem 3.3, Δ denotes $\Delta(\alpha_1, \alpha_2)$.)

Theorem 3.2 (Gordon–Wu [32], [34]) *Let M be a hyperbolic 3–manifold such that $M(\alpha_1)$ is annular and $M(\alpha_2)$ is annular (toroidal). Then there are:*

(1) *exactly one such manifold with $\Delta = 5$;*

(2) *exactly two such manifolds with $\Delta = 4$; and*

(3) *infinitely many such manifolds with $\Delta = 3$.*

The manifolds in (1) and (2) are the same in both the annular and the toroidal case. They are: in (1), the exterior of the Whitehead sister (or $(-2, 3, 8)$ pretzel) link, and in (2), the exteriors of the Whitehead link and the 2–bridge link associated with the rational number $3/10$.

Although the statements in Theorem 3.2 are identical in both cases (A, A) and (A, T), the proofs are necessarily quite different.

The next theorem describes the case (T, T).

Theorem 3.3 (Gordon [22]) *Let M be a hyperbolic 3–manifold such that $M(\alpha_1)$ and $M(\alpha_2)$ are toroidal. Then there are:*

(1) *exactly two such manifolds with $\Delta = 8$;*

(2) *exactly one such manifold with $\Delta = 7$;*

(3) *exactly one such manifold with $\Delta = 6$; and*

(4) *infinitely many such manifolds with $\Delta = 5$.*

Here the manifolds in (1), (2) and (3) are all Dehn fillings on the exterior W of the Whitehead link. Specifically, (using the usual meridian–latitude slope co-ordinates) they are: in (1), $W(1)$ and $W(-5)$ (these are the figure eight knot exterior and the figure eight sister manifold), in (2), $W(-5/2)$, and in (3), $W(2)$.

Of the two cases where $\Delta(X_1, X_2)$ is not known, namely $(X_1, X_2) = (S, S^H)$ and (T, T^H), recall that it is expected that there are no examples at all realizing (S, S^H). For the other case, (T, T^H), there are no examples known with $\Delta > 3$. However, the following theorem says that there are infinitely many examples with $\Delta = 3$.

Theorem 3.4 *For any integer $m > 0$ there are infinitely many hyperbolic 3–manifolds M with Dehn fillings $M(\alpha_1)$, $M(\alpha_2)$ such that $M(\alpha_1)$ is toroidal, $M(\alpha_2)$ is the lens space $L(6m \pm 1, 3m \mp 1)$, and $\Delta(\alpha_1, \alpha_2) = 3$.*

Proof We will construct these manifolds by suitably modifying the examples of hyperbolic manifolds with toroidal and reducible fillings at distance 3 given by Eudave-Muñoz and Wu in [15; section 4].

For $p, q \in \mathbb{Z}$ let $T_{p,q}$ be the tangle in the 3–ball $S^3 - \operatorname{Int} B$ shown in Figure 3.1, where $\boxed{n}$ denotes n positive half-twists, if $n \geq 0$, and $|n|$ negative half-twists, if $n < 0$; this is obtained from the tangle T_p shown in [15; Figure 4.1(a)] by adding q horizontal half-twists beneath the p vertical half-twists. Let $T_{p,q}(r)$ be the knot or link obtained by inserting into the 3–ball B the rational tangle parametrized (in the usual way) by $r \in \mathbb{Q} \cup \{\infty\}$. Let $M_{p,q}$ be the 2–fold branched covering of $T_{p,q}$. Thus $\partial M_{p,q}$ is a torus, and $M_{p,q}(r)$ is the 2–fold branched covering of $T_{p,q}(r)$.

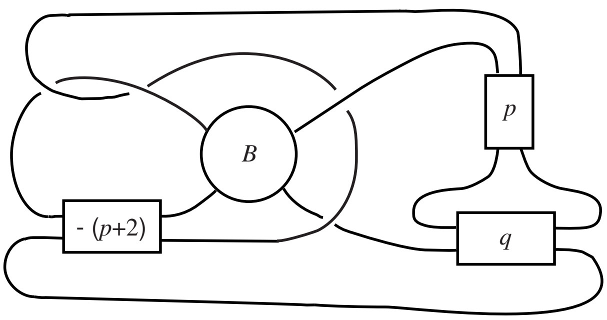

Figure 3.1

Assume that $p \geq 3$ and $q \neq 0$. Then, as in [15; Proof of Lemma 4.1], $M_{p,q}(\infty)$ is a non Seifert fibered, irreducible, toroidal manifold, $M_{p,q}(0)$ is the 2–fold branched cover of the 2–bridge knot corresponding to the rational number $1/(-(p+3) + 1/(-(p+1) + 1/q))$, ie, the lens space $L((p+3)(q(p-1)-1)+q, q(p-1)+1)$, and $M_{p,q}(1)$ and $M_{p,q}(1/2)$ are Seifert fiber spaces of type $S^2(p_1, p_2, p_3)$.

Also, $T_{p,q}(1/3)$ is the knot K_q shown in Figure 3.2; compare [15; Figure 4.1(f)]. Thus K_q is the 2–bridge knot corresponding to the rational number $1/(-2 + 1/(-q+1/3)) = (1-3q)/(6q+1)$. Hence $M_{p,q}(1/3)$ is (up to orientation) the lens space $L(6q+1, 3q-1)$. Setting $m = |q|$ gives the lens spaces described in the theorem. Note that $\Delta(\infty, 1/3) = 3$.

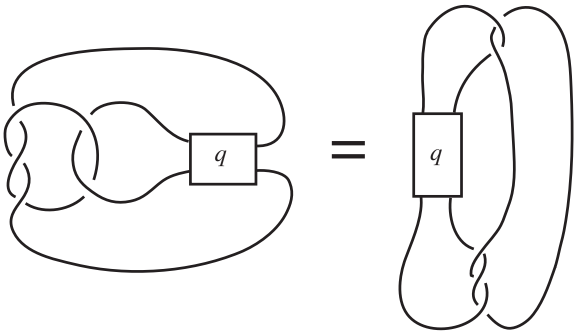

Figure 3.2

It remains to show that for any q ($\neq 0$) there are infinitely many distinct hyperbolic manifolds of the form $M_{p,q}$. But the proof given by Eudave-Muñoz and Wu of the corresponding assertion for their manifolds M_p [15; Proof of Theorem 4.2], applies virtually unchanged in our present situation, the only modifications necessary being to replace the reference to [26] by one to [13], and to delete the references to [28] and [9]. □

Question 3.1 *For which lens spaces L are there infinitely many hyperbolic 3–manifolds M with Dehn fillings $M(\alpha_1)$, $M(\alpha_2)$ such that $M(\alpha_1)$ is toroidal, $M(\alpha_2)$ is homeomorphic to L, and $\Delta(\alpha_1, \alpha_2) = 3$?*

4 Large Manifolds

Wu has shown [53] that for manifolds M which are *large* in the sense that $H_2(M, \partial M - T_0) \neq 0$, the bounds in Table 2.1 can often be improved. (Note that M is *not* large if and only if it is a $\mathbb{Q}$–homology $S^1 \times D^2$ or a $\mathbb{Q}$–homology $T^2 \times I$.) Thus we define (for $X_i \in \{S, D, A, T\}$)

$$\begin{aligned}\Delta^*(X_1, X_2) = \max\{\Delta(\alpha_1, \alpha_2) :\ & \text{there is a large hyperbolic 3–manifold } M \\ & \text{and slopes } \alpha_1, \alpha_2 \text{ on a torus component of } \partial M \\ & \text{such that } M(\alpha_i) \text{ is of type } X_i,\ i = 1, 2\}.\end{aligned}$$

(It is clear that if M is large then $M(\alpha)$ can never contain a Heegaard sphere or torus.) Then the values of $\Delta^*(X_1, X_2)$ are as shown in Table 4.1.

	S	D	A	T
S	0	0	1	1
D		1	$1-2$	1
A			4	4
T				$4-5$

Table 4.1 $\Delta^*(X_1, X_2)$

The following are references for the fact that the relevant entries in Table 4.1 are upper bounds for $\Delta^*(X_1, X_2)$.

(S, S) For manifolds with boundary a union of tori this is due to Gabai [17; Corollary 2.4]. The general case follows from this by a trick due to John Luecke; see [53; Remark 4.2].

(S, D), (D, D) and (D, A) Here the upper bounds are the same as those for $\Delta(X_1, X_2)$ in Table 2.1.

(S, A), (S, T) and (D, T) These are due to Wu [53; Theorems 4.1 and 4.6].

(A, A) and (A, T) By [34] and [32] (see Theorem 3.2), the only hyperbolic manifold with annular/annular or annular/toroidal fillings at distance 5 is the Whitehead sister link exterior, which is a $\mathbb{Q}$–homology $T^2 \times I$.

(T, T) By [22] (see Theorem 3.3), the only hyperbolic manifolds with a pair of toroidal fillings at distance greater than 5 are the fillings $W(1)$, $W(-5)$, $W(-5/2)$ and $W(2)$ on the Whitehead link exterior W. These are all $\mathbb{Q}$–homology $S^1 \times D^2$'s.

References for the fact that the relevant entries in Table 4.1 are lower bounds for $\Delta^*(X_1, X_2)$ are as follows.

(S, T) and (D, T) In [53; Example 4.7] Wu gives the example of the Borromean rings exterior M, which has $M(\infty)$ reducible and boundary reducible and $M(0)$ toroidal.

(S, A) and (D, A) In [53; Example 4.8] Wu constructs a hyperbolic manifold M whose boundary consists of four tori, with slopes α_1 and α_2 such that $M(\alpha_1)$ is reducible and boundary reducible, $M(\alpha_2)$ is annular, and $\Delta(\alpha_1, \alpha_2) = 1$.

<u>(D,D)</u> Berge [4] and Gabai [19] have given examples of simple manifolds M with distinct slopes α_1 and α_2 such that $M(\alpha_i)$ is a handlebody of genus $g \geq 2$, $i = 1, 2$.

<u>(A,A), (A,T) and (T,T)</u> It is shown in [32; Lemma 7.1] that the Whitehead link exterior M has fillings $M(\alpha_1)$, $M(\alpha_2)$, each of which is annular and toroidal, with $\Delta(\alpha_1, \alpha_2) = 4$. Since the Whitehead link has linking number zero, M is large.

The two unknown values of $\Delta^*(X_1, X_2)$ in Table 4.1 give rise to the following questions.

Question 4.1 *Is there a large hyperbolic manifold with a boundary reducible filling and an annular filling at distance 2?*

Question 4.2 *Is there a large hyperbolic manifold with two toroidal fillings at distance 5?*

5 Manifolds with boundary a union of tori

Restricting attention to hyperbolic 3–manifolds whose boundary components are tori, we can consider what happens to the maximal distances between exceptional fillings as the number of boundary components increases. More precisely, we can define, for $X_i \in \{S, D, A, T\}$,

$$\Delta^k(X_1, X_2) = \max\{\Delta(\alpha_1, \alpha_2) : \text{there is a hyperbolic 3–manifold } M \text{ such that } \partial M \text{ is a disjoint union of } k \text{ tori, and slopes } \alpha_1, \alpha_2 \text{ on some component of } \partial M \text{, such that } M(\alpha_i) \text{ is of type } X_i,\ i = 1, 2\}.$$

This is defined for $k \geq 1$ if $X_1, X_2 \in \{S, T\}$, and for $k \geq 2$ otherwise.

Since a 3–manifold with more than two torus boundary components is large, we have

$$\Delta^*(X_1, X_2) \geq \Delta^k(X_1, X_2) \quad \text{if} \quad k \geq 3.$$

If a 3–manifold whose boundary consists of ℓ tori contains an essential disk, then it also contains an essential sphere, provided $\ell \geq 2$, and if it contains an essential annulus, and is irreducible, then it also contains an essential torus, provided $\ell \geq 4$. Hence

$$\Delta^k(S, X) \geq \Delta^k(D, X), \text{ if } k \geq 3 \;;$$
$$\Delta^k(T, X) \geq \Delta^k(A, X), \text{ if } k \geq 5 \text{ and } \Delta^k(A, X) > \Delta^k(S, X).$$

Now suppose M is a hyperbolic 3–manifold with slopes α_1, α_2 on some torus component of ∂M such that $M(\alpha_i)$ is of type X_i, where $X_i = S, D, A$ or T, $i = 1,2$. Let F_i be the corresponding essential surface in $M(\alpha_i)$, $i = 1,2$. If there is a torus component T of ∂M which does not meet F_1 or F_2, then known results imply that there are infinitely many slopes β on T such that F_i remains essential in $M(\alpha_i)(\beta)$, $i = 1,2$. Since $M(\beta)$ is hyperbolic for all but finitely many β, there are infinitely many slopes β such that $M(\beta)$ is hyperbolic and $M(\beta)(\alpha_i)$ is of type X_i, $i = 1,2$. Thus

$$\Delta^{k-1}(X_1, X_2) \geq \Delta^k(X_1, X_2),$$

provided k is large enough that there is guaranteed to be a boundary component which misses F_1 and F_2; this depends on the pair X_1, X_2.

The values of $\Delta^2(X_1, X_2)$ and $\Delta^3(X_1, X_2)$ are shown in Tables 5.1 and 5.2.

	S	D	A	T
S	1	0	2	$2-3$
D		1	2	2
A			5	5
T				5

Table 5.1 $\Delta^2(X_1, X_2)$

	S	D	A	T
S	0	0	1	1
D		0	1	1
A			3	3
T				$3-5$

Table 5.2 $\Delta^3(X_1, X_2)$

The upper bounds for $\Delta^2(X_1, X_2)$ in Table 5.1 are the same as the upper bounds for $\Delta(X_1, X_2)$ in Table 2.1, except for (T,T). This case follows from [22] (see Theorem 3.3), since the manifolds listed there with a pair of toroidal fillings at distance greater than 5 all have a single boundary component.

References for examples realizing the (lower) bounds in Table 5.1 are among those listed for Table 2.1 in Section 2, ie, (S,S): [25], [15]; (S,T), (D,A) and (D,T): [35], [15]; (D,D): [1], [18]; (A,A), (A,T) and (T,T): [42], [32].

Turning to $\Delta^3(X_1, X_2)$, the upper bounds are the same as those for $\Delta^*(X_1, X_2)$ (see Table 4.1), except in the cases (D,A), (A,A) and (A,T). For (D,A), we have $\Delta^3(D,A) \leq \Delta^3(S,A) \leq 1$, while the facts that $\Delta^3(A,A) \leq 3$ and $\Delta^3(A,T) \leq 3$ follow from [34] and [32] respectively; see Theorem 3.2.

References for examples realizing the lower bounds in Table 5.2 are as follows.

<u>(S,T) and (D,T)</u> [53; Example 4.7]; see Section 4 above.

<u>(S,A)</u> Let M be the hyperbolic manifold constructed by Wu in [53; Example 4.8], with four torus boundary components, and slopes α_1, α_2 (on T_0, say) such that $M(\alpha_1)$ is reducible, $M(\alpha_2)$ is annular, and $\Delta(\alpha_1,\alpha_2)=1$. By doing a suitable Dehn filling on the boundary component T_1 which is neither T_0 nor either of the components containing the boundary components of the annulus in $M(\alpha_2)$, we get a hyperbolic 3–manifold M' with three torus boundary components, such that $M'(\alpha_1)$ is reducible and $M'(\alpha_2)$ is annular. Another example is given in Theorem 5.1 below.

<u>(D,A)</u> See Theorem 5.1. (Note that although in Wu's example [53; Example 4.7] $M(\alpha_1)$ is also boundary reducible, it is T_1 that is compressible in $M(\alpha_1)$, so we cannot use the argument given above in the case (S,A) to conclude that $\Delta^3(D,A)=1$.)

<u>(A,A), (A,T) and (T,T)</u> In [32; Section 7] is described a hyperbolic 3–manifold M, called the *magic manifold*, which is the exterior of a certain 3–component link in S^3 and has Dehn fillings $M(\alpha_1)$, $M(\alpha_2)$, each of which is annular and toroidal, with $\Delta(\alpha_1,\alpha_2)=3$.

The following theorem shows that $\Delta^3(D,A)=1$.

Theorem 5.1 *There exists a hyperbolic 3–component link in S^3 whose exterior M has Dehn fillings $M(\alpha_1), M(\alpha_2)$ such that $M(\alpha_1)$ is boundary reducible, $M(\alpha_2)$ is annular, and $\Delta(\alpha_1,\alpha_2)=1$.*

Proof Let $L=K_1\cup K_2\cup K_3$ be the 3–component link illustrated in Figure 5.1. Let M be the exterior of L.

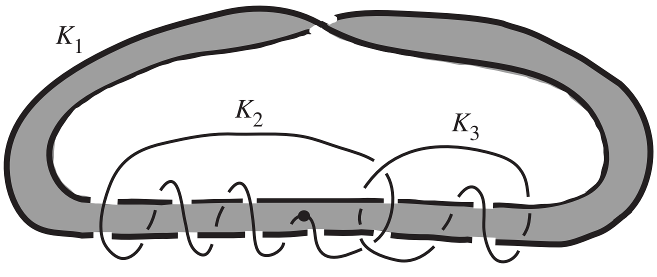

Figure 5.1

Claim *M is hyperbolic.*

Proof First, since (with appropriate orientations) we have linking numbers $\mathrm{lk}(K_1, K_2) = 5$, $\mathrm{lk}(K_1, K_3) = 2$, M is irreducible. Second, it follows easily from [11], again by considering linking numbers, that M is not a Seifert fiber space. Hence it suffices to show that M is atoroidal.

So let T be an essential torus in M. We see from Figure 5.1 that K_1 bounds a Möbius band B that is punctured once by K_2 and is disjoint from K_3. This gives rise to a once-punctured Möbius band F in M. By an isotopy of T, we may suppose that T intersects F transversely in a finite disjoint union of simple closed curves, each being orientation preserving and essential in F. Hence we can choose an orientation reversing curve C in F such that $C \cap T = \emptyset$. Up to isotopy in F, there are two possibilities for C (because of the puncture), but in each case we see from Figure 5.1 that the link $L' = C \cup K_2 \cup K_3$ has a connected, prime, alternating diagram, and is not a $(2, q)$ torus link, and hence by [39], is hyperbolic. It follows that T is either

(i) compressible in $S^3 - L'$; or

(ii) parallel in $S^3 - L'$ to $\partial N(C)$; or

(iii) parallel in $S^3 - L'$ to $\partial N(K_2)$; or

(iv) parallel in $S^3 - L'$ to $\partial N(K_3)$.

In case (i), let D be a compressing disk for T in $S^3 - L'$. Then T bounds a solid torus V in S^3 containing D. Since T is incompressible in $S^3 - L$, D must meet K_1. Hence $K_1 \subset V$. We now distinguish two subcases: (a) K_1 is not contained in a ball in V; and (b) K_1 is contained in a ball in V.

In subcase (a), since K_1 is unknotted in S^3, it follows that V is also, and hence, since T is incompressible in $S^3 - L$, we must have K_2 or $K_3 \subset S^3 - V$. If any component of L' were contained in V, then it would lie in a ball in V, and so L' would be a split link. Hence $L' \subset S^3 - V$. But $K_1 \cup C$ is a Hopf link, and so K_1 is a core of V, contradicting the essentiality of T in M.

In subcase (b), first note that since each of C, K_2 and K_3 has non-zero linking number with K_1, we must have $C \cup K_2 \cup K_3 \subset V$, and hence V is knotted in S^3. Now consider $T \cap B = T \cap F$; any component of $T \cap B$ either bounds a disk in B containing the point $K_2 \cap B$, or is parallel in B to K_1. If there are components of the first type, let γ be one that is innermost in B; thus γ bounds a disk E in B which meets K_2 in a single point and has interior disjoint from T. If γ were inessential on T, then we would get a 2–sphere in S^3 meeting K_2 transversely in a single point, which is impossible. Hence E is a meridian disk of V. But D is a meridian disk of V which misses K_2, so again we get a contradiction. It follows that each component of $T \cap B$ is parallel in

B to K_1. If $T \cap B \neq \emptyset$, then the annulus in B between K_1 and an outermost component γ of $T \cap B$ defines an isotopy of K_1, fixing K_3, which takes K_1 to γ. But since the meridian disk D of V misses K_3, K_3 lies in a ball in V, and hence $\mathrm{lk}(\gamma, K_3) = 0$. Since $\mathrm{lk}(K_1, K_3) = 2$, this is a contradiction.

We therefore have $T \cap B = \emptyset$. Thus $B \subset V$, and T is an essential torus in $S^3 - \mathrm{Int}\, N(B \cup K_2 \cup K_3)$. Now $B \cup K_2 \cup K_3$ collapses to the graph $\Gamma \subset S^3$ shown in Figure 5.2, and $S^3 - \mathrm{Int}\, N(\Gamma)$ is homeomorphic to the exterior of the tangle t in B^3 shown in Figure 5.3. Since t is not a split tangle, $\partial B^3 - t$ is incompressible in $B^3 - t$. (To see that t is not split, observe that if it were, it would be a trivial 2–string tangle together with a meridional linking circle of one of the components. Hence any 2–component link, with each component unknotted, obtained by capping off (B^3, t) with a trivial tangle, would be a Hopf link. But joining the N and E, and S and W, arc endpoints of t in the obvious way gives the 2–bridge link corresponding to the rational number $5/18$.) Also, two copies of (B^3, t) may be glued together along their boundaries so as to get a link in S^3 that has a connected, prime, alternating diagram. By [39], the exterior of this link is atoroidal, and hence the exterior of t in B^3 is also atoroidal. This contradiction completes the proof of subcase (b), and hence of case (i).

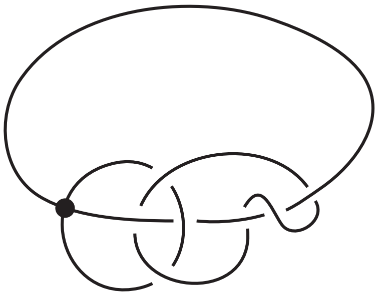

Figure 5.2

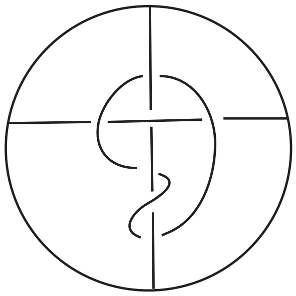

Figure 5.3

In case (ii), T bounds a solid torus V in S^3 with C as a core, and $K_1 \subset V$. Hence $\mathrm{lk}(K_1, K_2) = 5$ is a multiple of $\mathrm{lk}(C, K_2) = 2$ or 3, a contradiction. Similarly, in case (iii) we get that $\mathrm{lk}(K_1, C) = 1$ is a multiple of $\mathrm{lk}(K_2, C) = 2$ or 3, and in case (iv), that $\mathrm{lk}(K_1, K_2) = 5$ is a multiple of $\mathrm{lk}(K_3, K_2) = 0$.

This completes the proof of the claim.

Let T_0 be the boundary component of M corresponding to the component K_1 of L. Then, since $L - K_1$ is the 2-component unlink, $M(\infty)$ is boundary reducible. Also, the Möbius band B bounded by K_1, which is punctured once by K_2, has boundary slope 2. Hence $M(2)$ contains a Möbius band whose boundary is a meridian of K_2. Hence (see [25; Proof of Proposition 1.3, Case (1)]) $M(2) \cong X \cup_T Q$, where Q is a $(1,2)$–cable space, glued to X along a torus T, with $Q \cap \partial M(2) = \partial N(K_2)$. Since $M(\infty)$ is boundary reducible, $M(0)$ is irreducible, by [46], and hence T is incompressible in X. Therefore $M(2)$ is annular. □

Regarding the one unknown value of $\Delta^2(X_1, X_2)$ in Table 5.1 we have the following question.

Question 5.1 *Is there a hyperbolic manifold with boundary a union of two tori, having a reducible filling and a toroidal filling at distance 3?*

Similarly, the one unknown value of $\Delta^3(X_1, X_2)$ in Table 5.2 leads to the following question.

Question 5.2 *Is there a hyperbolic manifold with boundary a union of three tori, having two toroidal fillings at distance* 4 *or* 5*?*

Seeing the values in the tables for $\Delta(X_1, X_2)$, $\Delta^2(X_1, X_2)$ and $\Delta^3(X_1, X_2)$ decreasing leads one to ask if $\Delta^k(X_1, X_2)$ is eventually zero; equivalently, if a hyperbolic 3–manifold with k torus boundary components has at most one exceptional Dehn filling (on any given boundary component) for k sufficiently large. However, the following two theorems show that this is not the case.

The first is essentially due to Wu [53].

Theorem 5.2 (Wu [53]) *For any* $k \geq 4$ *there are infinitely many hyperbolic 3–manifolds* M *such that* ∂M *consists of* k *tori, with Dehn fillings* $M(\alpha_1)$, $M(\alpha_2)$ *such that* $M(\alpha_1)$ *is reducible and boundary reducible,* $M(\alpha_2)$ *is annular and toroidal, and* $\Delta(\alpha_1, \alpha_2) = 1$.

Proof This is essentially Example 4.8 of [53]. We simply modify Wu's construction by taking X to be a simple manifold with ∂X a genus 2 surface together with $(k-4)$ tori. Then $M = M_1 \cup_P X$ is simple, with ∂M consisting of k tori. Let T_0 be the component of ∂M corresponding to K_1 in [53; Figure 4.2]. Then $M(\infty)$ is reducible and boundary reducible, and $M(0)$ is annular. It remains to show that $M(0)$ is toroidal. Now $M(0)$ is irreducible (since M is large and $M(\infty)$ is reducible), and hence $M(0)$ will be toroidal unless $k=4$ and $M(0) \cong$ (pair of pants) $\times S^1$. But since $M_1(\infty)$ is reducible, $M_1(0)$ is boundary irreducible by [46], and hence $M(0)$ contains an incompressible genus 2 surface, so we are done. □

Remark Examples as in Theorem 5.2 can also be obtained by generalizing the construction given in the proof of Theorem 5.1 to links with $k \geq 4$ components.

It follows from Theorem 5.2 that $\Delta^k(X_1, X_2) \geq 1$ for $k \geq 4$, where $X_1 \in \{S, D\}$ and $X_2 \in \{A, T\}$. The next theorem shows that for $k \geq 4$, $\Delta^k(A, A)$, $\Delta^k(A, T)$ and $\Delta^k(T, T)$ are ≥ 2.

Theorem 5.3 *For any $k \geq 4$ there exists a k–component hyperbolic link in $S^2 \times S^1$ whose exterior M has Dehn fillings $M(\alpha_1), M(\alpha_2)$, each of which is annular and toroidal, with $\Delta(\alpha_1, \alpha_2) = 2$.*

Proof Consider the tangle in $S^2 \times I$ illustrated in Figure 5.4, consisting of three arcs and a closed loop K_1 (The tangle is shown lying in the solid cylinder $D^2_+ \times I$, where we regard S^2 as the union of two hemispheres $D^2_+ \cup D^2_-$.) Gluing together the two ends $S^2 \times \{0\}$ and $S^2 \times \{1\}$, in such a way that the pairs of points $\{a, a'\}$, $\{b, b'\}$ and $\{c, c'\}$ are identified, we obtain a 2–component link $L = K_1 \cup K_2$ in $S^2 \times S^1$. For convenience we have chosen the knot K_2 to be the (reflection of the) one considered by Nanyes in [43], so that we can appeal to some of the properties of K_2 established there.

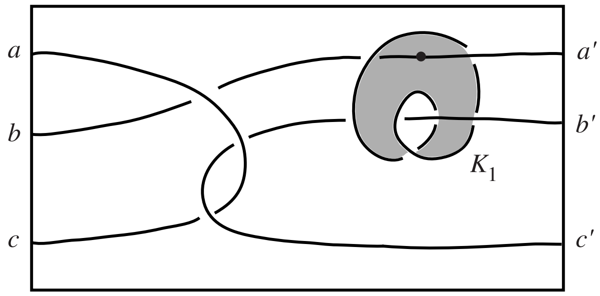

Figure 5.4

We see from Figure 5.4 that K_1 bounds a Möbius band, with boundary slope 2, which is punctured once by K_2. Hence, doing 2–Dehn filling on the exterior of L along the boundary component T_0 corresponding to K_1, we get a manifold containing a Möbius band, whose boundary is a meridian of K_2.

Redrawing K_1 as in Figure 5.5, we also see that K_1 bounds a disk, with boundary slope 0, which K_2 intersects in two points, with the same sign. Hence 0–Dehn filling the exterior of L along T_0 gives a manifold that contains an annulus, whose boundary consists of two coherently oriented meridians of K_2.

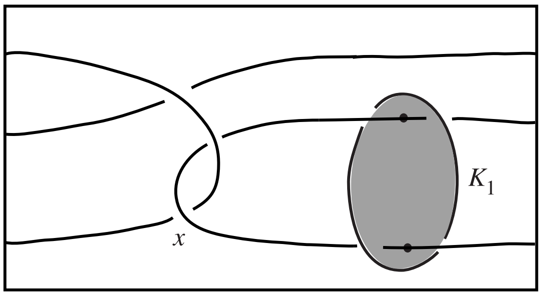

Figure 5.5

One can show that L is hyperbolic, and the idea is to enlarge L to a $k-$ component hyperbolic link L_k, $k \geq 4$, without disturbing the Möbius band and annulus described above. We do this by successively inserting $(k-2)$ additional components $K_3, \ldots, K_k$ in a small neighborhood of the crossing x indicated in Figure 5.5, as follows. First insert K_3 around x as shown in Figure 5.6; then, in the same manner, insert K_4 around one of the crossings of K_3 with (say) K_2; then insert K_5 around one of the crossings of K_4 with K_3 (say), etc.. Let M denote the exterior of L_k in $S^2 \times S^1$. Then we still have that $M(2)$ contains a Möbius band, and $M(0)$ contains an annulus, as described earlier.

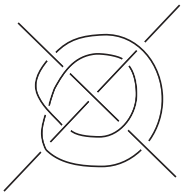

Figure 5.6

We shall show that M is hyperbolic, and that $M(2)$ and $M(0)$ are annular and toroidal.

First, let t be the tangle in $S^2 \times I$ that corresponds to the link L_k, ie, the tangle obtained from that illustrated in Figure 5.5 by inserting the components $K_3, \ldots, K_k$ as described above. Let N be the exterior of t in $S^2 \times I$.

Claim 1 *N is irreducible and atoroidal.*

Proof The arc of t with endpoints a' and b may be isotoped away from the rest of t, so N is homeomorphic to the exterior of the tangle t_0 in $D^2 \times I \cong B^3$, obtained from that shown in Figure 5.7 by inserting $K_3, \ldots, K_k$.

Gluing two copies of (B^3, t_0) along their boundaries, in such a way that the arc endpoints a and c in each copy are identified with b' and c' respectively in the other copy, we obtain a link in S^3 with a diagram that is connected, prime and alternating. Therefore, by [39], the exterior of this link is irreducible and atoroidal. Moreover, since t_0 is not a split tangle, $\partial B^3 - t_0$ is incompressible in $B^3 - t_0$. It follows that the exterior N of t_0 in B^3 is also irreducible and atoroidal.

This completes the proof of Claim 1.

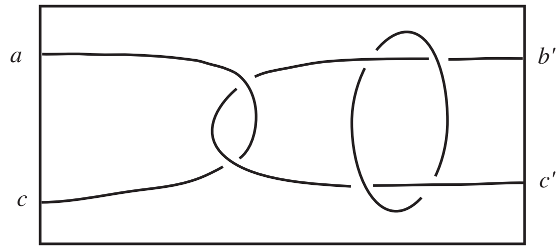

Figure 5.7

Claim 2 *M is hyperbolic.*

Proof The two thrice-punctured spheres $P_i = S^2 \times \{i\} - \text{Int } N(t)$, $i = 0, 1$, are incompressible in the exterior of t in $S^2 \times I$, as they are incompressible in the exterior in $S^2 \times I$ of the three arcs that make up K_2; see [43]. Let P be the thrice-punctured sphere $P_1 = P_2$ in M. Since N is irreducible by Claim 1, and P_1 and P_2 are incompressible in N, it follows that M is irreducible.

If M were a Seifert fiber space, then the incompressible surface P would be horizontal, which is impossible since M has at least four boundary components.

Hence it suffices to show that M is atoroidal. So let T be an essential torus in M, which we isotop to minimize the number of components of $T \cap P$. Then no component of $T \cap P$ is inessential in P, and hence either $T \cap P = \emptyset$, or some component γ of $T \cap P$ bounds a disk D in the 2–sphere $S = S^2 \times \{0\} = S^2 \times \{1\}$, such that D meets K_2 transversely in a single point and has interior disjoint from T. Now γ is essential on T, otherwise we get a 2–sphere in $S^2 \times S^1$ meeting K_2 in a single point, contradicting [43]. Hence compressing T along D gives a 2–sphere Σ meeting K_2 in two points. Since K_2 is locally unknotted (see [43]), Σ bounds a 3–ball B in $S^2 \times S^1$ such that $(B, B \cap K_2) \cong (B^3, B^1)$. Let T' be the boundary of the solid torus $V = B - \text{Int } N(K_2)$. Note that T is obtained from Σ by adding a tube. If this tube lies in B, then T is isotopic to

T'; if it lies outside B, then T is parallel, in the exterior of K_2, to $\partial N(K_2)$. Since T is essential in M, in both cases we must have $(L - K_2) \cap B \neq \emptyset$. If T' were compressible in $V - L$, then L would be a split link, contradicting the fact that M is irreducible. Hence T' is an essential torus in M. Note also that T' may be isotoped off P. This gives an essential torus in N, contradicting Claim 1.

This completes the proof of Claim 2.

Claim 3 *$M(2)$ and $M(0)$ are annular and toroidal.*

Proof As observed above, $M(2)$ contains a Möbius band. Hence, as in the proof of Theorem 5.1, $M(2) \cong X \cup_T Q$, where Q is a $(1,2)$–cable space, glued to X along a torus T. If T is incompressible in X, then $M(2)$ is annular and toroidal. On the other hand, if T compresses in X, then $M(2)$ is reducible.

Now consider $M(0)$. First note that, since $M(2)$ is either annular or reducible, and $\Delta^k(S,A) = 1$, $\Delta^k(S,S) = 0$, for $k \geq 4$, $M(0)$ is irreducible. Now, as we saw earlier, $M(0)$ contains an annulus A, whose boundary components are coherently oriented on $\partial N(K_2)$. It follows that A is not boundary parallel in $M(0)$. If A were compressible in $M(0)$, then $M(0)$ would be boundary reducible, and hence reducible, a contradiction. We conclude that $M(0)$ is annular. Now, since $\Delta^k(S,A) = 1$, $k \geq 4$, $M(2)$ cannot be reducible, and hence it is annular and toroidal.

Finally, tubing A along $\partial N(K_2)$ gives a Klein bottle F in $M(0)$. The boundary of a regular neighborhood of F is a torus T which is essential since $M(0)$ is irreducible. Hence $M(0)$ is toroidal.

This completes the proof of Claim 3 and hence of Theorem 5.3. □

Theorems 5.2 and 5.3 (together with Theorems 3.2 and 3.3) show that the values of $\Delta^k(X_1, X_2)$, $k \geq 4$, are as indicated in Table 5.3.

	S	D	A	T
S	0	0	1	1
D		0	1	1
A			$2-3$	$2-3$
T				$2-5$

Table 5.3 $\Delta^k(X_1, X_2)$, $k \geq 4$

Question 5.3 *What are the values of $\Delta^k(A,A)$, $\Delta^k(A,T)$ and $\Delta^k(T,T)$ for $k \geq 4$?*

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