Geometry & Topology Monographs
Volume 2: Proceedings of the Kirbyfest
Pages 335–342

Order 2 Algebraically Slice Knots

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Abstract The concordance group of algebraically slice knots is the subgroup of the classical knot concordance group formed by algebraically slice knots. Results of Casson and Gordon and of Jiang showed that this group contains in infinitely generated free (abelian) subgroup. Here it is shown that the concordance group of algebraically slice knots also contain elements of finite order; in fact it contains an infinite subgroup generated by elements of order 2.

AMS Classification 57M25; 57N70, 57Q20

Keywords Concordance, concordance group, slice, algebraically slice

1 Introduction

The classical knot concordance group, $\mathcal{C}$, was defined by Fox [5] in 1962. The work of Fox and Milnor [6], along with that of Murasugi [18] and Levine [14, 15], revealed fundamental aspects of the structure of $\mathcal{C}$. Since then there has been tremendous progress in 3– and 4–dimensional geometric topology, yet nothing more is now known about the underlying group structure of $\mathcal{C}$ than was known in 1969. In this paper we will describe new and unexpected classes of order 2 in $\mathcal{C}$.

What is known about $\mathcal{C}$ is quickly summarized. It is a countable abelian group. According to [15] there is a surjective homomorphism of ϕ: $\mathcal{C} \to \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$. The results of [6] quickly yield an infinite set of elements of order 2 in $\mathcal{C}$, all of which are mapped to elements of order 2 by ϕ.

The results just stated, and their algebraic consequences, present all that is known concerning the purely algebraic structure of $\mathcal{C}$ in either the smooth or topological locally flat category. For instance, one can conclude that elements of order 2 detected by homomorphisms to $\mathbb{Z}_2$, such as the Fox–Milnor examples, are not evenly divisible, but it remains possible that any given countable abelian group is a subgroup of $\mathcal{C}$, including such groups as the infinite direct sum of copies of $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$. Most succinctly, we know that $\mathcal{C}$ is isomorphic to a direct sum $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus G$, but all that is known about G is that it is countable and abelian. In particular, Fox and Milnor's original question on the existence of torsion of order other than 2 remains completely open. Other basic questions regarding the structure of $\mathcal{C}$ appear in [10, 12, 13].

More is known about the pair $(\mathcal{C}, \phi)$. Casson and Gordon [2, 3] showed that the kernel of Levine's homomorphism, $\mathcal{A}$ (the *concordance group of algebraically slice knots*), is nontrivial; Jiang showed that Casson and Gordon's examples provide an infinitely generated free subgroup of $\mathcal{A}$.

(We should observe here that Alexander polynomial 1 knots are known to represent classes in $\mathcal{A}$. Freedman's work [7] implies that all such knots are topologically locally flat slice. Donaldson's work [4] implies that some such knots are not smoothly slice. Needless to say, the accomplishments of both [7] and [4] have been revolutionary in the study of 4–manifolds. However, it is perhaps surprising that neither has revealed any further group theoretic structure of either $\mathcal{C}$ or the pair $(\mathcal{C}, \phi)$.)

It is becoming clear that any unexpected complexity in $\mathcal{C}$ will appear in $\mathcal{A}$, that is, among algebraically slice knots: if any odd torsion exist, it obviously must be in $\mathcal{A}$, and recent work [17] showing that infinite collections of knots that map to elements of order 4 under Levine's homomophism are of infinite order in $\mathcal{C}$ supports the conjecture that any 4–torsion must also be in $\mathcal{A}$.

In this paper we construct an infinite family of order 2 elements in $\mathcal{A}$. These are the first such examples, and the first examples of any type showing that $\mathcal{A}$ has any structure beyond that demonstrated by Jiang. Our methods apply in the smooth setting, as did the original work of Casson and Gordon, but work of Freedman [8] shows that they apply in the topological locally flat category also.

Thanks are due to Larry Taylor for pointing out the particular problem being addressed here, and to Zhenghan Wang for observing a simplification of our original construction.

2 Basic building blocks

The basic idea of the construction of algebraically slice order 2 knots is to take the connected sums of pairs of algebraically concordant negative amphicheiral knots. If the knots aren't concordant then the connected sum will be of order 2 in $\mathcal{A}$. The trick is to find an infinite collection of such pairs so that it is possible to prove that they have the desired properties. Our examples, J_n, will be built as connected sums of pairs of knots K_T that we first examine. For an arbitrary knot, T, consider the knot K_T illustrated in Figure 1 along with a surgery diagram of M_T, the 2–fold branched cover of S^3 branched over K_T, drawn using the algorithm of Akbulut and Kirby [1]. The illustration indicates that K_T bounds a genus one Seifert surface so that one band in the surface has the knot T tied in it and the other band has the mirror image of T, $-T$, tied in it. (More precisely, in Figure 1 the tangle $-T$ is obtained from the tangle T by changing all the crossings.) The bands are twisted so that K_T has Seifert form

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

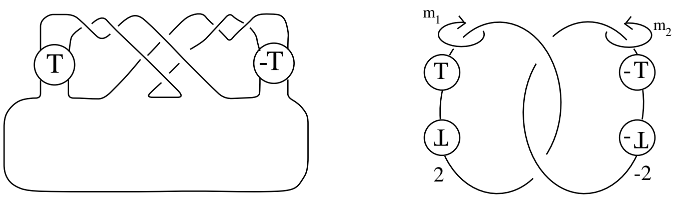

Figure 1

Lemma 2.1 *The knot K_T is of order 2 in the knot concordance group, $\mathcal{C}$.*

Proof Changing all the crossings in K_T is easily seen to have the effect of simply reversing its orientation; that is, K_T is negative amphicheiral and is hence of order 1 or 2 in $\mathcal{C}$. (Stated differently, $K_T = -K_T$, so K_T is of order at most 2.)

The Alexander polynomial of K_T is $t^2 - 3t + 1$, which is irreducible, since the discriminant, 5, is not a perfect square. According to [6] this obstructs a knot from being slice. It follows that the order of K_T is exactly 2. □

We will need to understand the 2–fold branched cover of S^3, M_T.

Lemma 2.2 *$H_1(M_T) = \mathbb{Z}_5$ and is generated by the meridian labeled m_1 in Figure 1. Any homomorphism $\phi\colon H_1(M_T) \to \mathbb{Z}_5$ taking value a on m_1 takes value $3a$ on m_2.*

Proof A relation matrix for the homology $H_1(M_T)$ with respect to m_1 and m_2, computed using its surgery presentation, is given by the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}.$$

This matrix presents the cyclic group $\mathbb{Z}_5$. The relations imply that $m_2 = -2m_1$, or $m_2 = 3m_1$. □

Our final calculation regarding K_T is of its Casson–Gordon invariants. Associated to any representation χ: $H_1(M_T) \to \mathbb{Z}_5$ there is a rational invariant, denoted $\sigma_1(\tau(K_T, \chi))$ in [2]. We do not need the exact value of this invariant, but only its dependence on T. Let K_0 be the Figure 8 knot, obtained when T is trivial. Work of Litherland [16] on computing τ for satellite knots yields the desired result, Lemma 2.3 below. Here, σ_p, $p \in \mathbb{Q}/\mathbb{Z}$, denotes the Tristram–Levine signature of a knot [19], given by the signature of the hermetian matrix

$$(1 - e^{p2\pi i})V + (1 - e^{-p2\pi i})V^t,$$

where V denotes the Seifert matrix of the knot.

Lemma 2.3 *If χ_a: $H_1(M_T) \to \mathbb{Z}_5$ takes value a on m_1 then*

$$\sigma_1(\tau(K_T, \chi_a)) = \sigma_1(\tau(K_0, \chi)) + 2\sigma_{a/5}(T) + 2\sigma_{3a/5}(-T).$$

Proof The knot K_T is a satellite knot with companion T, winding number 0, and with orbit K'_{-T}, the knot formed from K_T by removing the knot T in the left band. Applying σ_1 to the equation given as Corollary 2 in [16] gives:

$$\sigma_1(\tau(K_T, \chi_a)) = \sigma_1(\tau(K'_{-T}, \chi_a)) + 2\sigma_{a/5}(T).$$

Repeating this companionship argument to remove the $-T$ from the band in K'_{-T} yields the desired result. The second signature is evaluated at $3a/5$ because, according to Lemma 2.2, χ takes that value on m_2. (In Litherland's notation, $\chi(x_i) = 3a$, viewed as an element of $\mathbb{Z}_5$.) □

Remark We should remind the reader here that both the Casson–Gordon invariant and the Tristram–Levine signature functions are symmetric under a sign change; that is, $\sigma_1(\tau(K_T, \chi_a)) = \sigma_1(\tau(K_T, \chi_{-a}))$ and $\sigma_p(T) = \sigma_{-p}(T)$.

Our ultimate examples, J_i, will be of the form $K_0 \# K_{T_i}$ for particular T_i which yield nontrivial signature values in the formula of Lemma 2.3. Explicit examples will be obtained by letting T_i be connected sums of $(2,7)$–torus knots, so we conclude this section with the following computation.

Lemma 2.4 *For the T the $(2,7)$–torus knot and $a \neq 0$ mod 5, $\sigma_{a/5}(T) + \sigma_{3a/5}(-T) = 4$ or -4 depending on whether $a = \pm 2$ mod 5 or $a = \pm 1$ mod 5.*

Proof The signature function of T, σ_p, is given (by definition) as the signature of the form $(1 - e^{p2\pi i})V + (1 - e^{-p2\pi i})V^t$ where V is a Seifert matrix for T:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The signature of this form is easily computed to be:

$$\sigma_p(T) = \begin{cases} 0, & \text{if } 0 < p < 1/14) \\ 2, & \text{if } 1/14 < p < 3/14 \\ 4, & \text{if } 3/14 < p < 5/14 \\ 6, & \text{if } 5/14 < p < 7/14. \end{cases}$$

The result now follows, since one of the signatures of T that appears will be either 2 or 6 while the signature of $-T$ that appears will be either -6 or -2, respectively. □

3 Infinite 2–torsion among algebraically slice knots

Definition *Let T represent the $(2,7)$–torus knot, let $T_i = \#_i T$, let $K_i = K_{T_i}$ and let $J_i = K_0 \# K_i$.*

If $i = 0$ then $\#_i T$ denotes the unknot. It follows that J_0 is the connected sum of the Figure 8 knot with itself and is hence slice. Also note that the definition makes sense for $i < 0$, letting $\#_i T$ denote the connected sum of $-i$ copies of the $(2,-7)$–torus knots in that case.

Lemma 3.1 *Each knot, J_i, is algebraically slice and of order at most two in $\mathcal{C}$ (and hence also in $\mathcal{A}$).*

Proof Since each K_i is of order two, the connected sum of two of them is of order 1 or 2. Also, since all the K_i have the same Seifert form, the Seifert form of an order two knot, the connected sum of two of them is algebraically slice. □

Theorem 3.2 *For $i \neq j$, the knots J_i and J_j are not concordant.*

Proof For arbitrary i and j, if J_i and J_j are concordant then $J_i \# J_j$ is slice. Expanding, we have that $K_0 \# K_i \# K_0 \# K_j$ is slice. Since K_0 is of order 2, this implies that $K_i \# K_j$ is slice.

The 2–fold branched cover of S^3 branched over $K_i \# K_j$ has first homology splitting naturally as $\mathbb{Z}_5 \oplus \mathbb{Z}_5$. There is a $\mathbb{Q}/\mathbb{Z}$ valued nonsingular symmetric linking form on this homology group. Since the covering space is a connected sum, the linking form splits along the given direct sum decomposition. Furthermore, the linking form takes the same value on the merdians m_1 in each of the two individual branched covering spaces, since the surgery matrices determine the linking form and are the same for both covering spaces.

According to Casson and Gordon, if $K_i \# K_j$ is slice, there is some vector $v \in \mathbb{Z}_5 \oplus \mathbb{Z}_5$ with self linking 0 so that for any $\mathbb{Z}_5$ valued character that vanishes on v the associated Casson–Gordon invariant must vanish. With the given linking form, v must be a multiple of either $(2,1)$ or $(2,-1)$, and hence we can consider the χ that takes value 1 on the m_1 generator of $H_1(M_{T_i})$ and value ± 2 on the m_1 generator of $H_1(M_{T_j})$.

Applying Lemma 2.3 along with the additivity of Casson–Gordon invariants [9] shows that if $K_i \# K_j$ is slice then

$$\begin{aligned}\sigma_1(\tau(M_{T_0}, \chi_1)) + 2\sigma_{1/5}(T_i) + 2\sigma_{2/5}(-T_i) + \sigma_1(\tau(M_{T_0}, \chi_2))& \\ + 2\sigma_{2/5}(T_j) + 2\sigma_{1/5}(-T_j) &= 0.\end{aligned}$$

Consider first the case that $i = 0 = j$. In this case $K_i \# K_j$ is slice, so the above formula yields that

$$\sigma_1(\tau(M_{T_0}, \chi_1)) + \sigma_1(\tau(M_{T_0}, \chi_2)) = 0.$$

Hence we can simplify the equation to find that if $K_i \# K_j$ is slice, then

$$2\sigma_{1/5}(T_i) + 2\sigma_{2/5}(-T_i) + 2\sigma_{2/5}(T_j) + 2\sigma_{1/5}(-T_j) = 0.$$

From Lemma 2.4 this simplifies to be $8(j - i) = 0$. Clearly, if $i \neq j$ this yields a contradiction. □

Corollary 3.3 *$\mathcal{A}$ contains a subgroup isomorphic to $\mathbb{Z}_2^\infty$.* □

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Received: 13 August 1998 Revised: 26 February 1999