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Cosmetic Surgery on Knots

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Abstract This paper concerns the Dehn surgery construction, especially those Dehn surgeries leaving the manifold unchanged. In particular, we describe an oriented 1–cusped hyperbolic 3–manifold X with a pair of slopes r_1, r_2 such that the Dehn filled manifolds $X(r_1)$, $X(r_2)$ are oppositely oriented copies of the lens space $L(49,18)$, and there is no homeomorphism h of X such that $h(r_1)=h(r_2)$.

AMS Classification 57N10; 57M25, 57M50

Keywords Dehn surgery, Dehn filling, hyperbolic knots

1 Introduction

This paper concerns the Dehn surgery construction [5], in particular, those Dehn surgeries leaving the manifold unchanged, a phenomenon we call *cosmetic* surgery. In what follows it will be useful to consider the Dehn surgery construction in terms of the slightly more general notion of Dehn filling, recalled below.

Let X be an oriented 3–manifold with torus boundary $\partial X = T^2$, and let r be a slope on ∂X, that is, an isotopy class of unoriented, simple, closed curves in T^2. Define $X(r)$, the *r–Dehn filling* on X, as $X \cup (B^2 \times S^1)$ where the boundaries are identified by a homeomorphism taking ∂B^2 to the slope r. The connection with Dehn surgery is regained by considering $X(r_2)$ as a surgery on $M = X(r_1)$. Two Dehn fillings are considered *cosmetic* if there is a homeomorphism h between $X(r_1)$ and $X(r_2)$. The Dehn fillings are *truly cosmetic* if the homeomorphism h is orientation preserving, and *reflectively cosmetic* if h is orientation reversing. A pair of fillings $X(r_1)$ and $X(r_2)$ may be both truly cosmetic and reflectively cosmetic.

One's intuition might say that cosmetic surgeries must be rare, but examples are easy to find. As described in Rolfsen's beautiful book [18, Chapter 9] reflecting in the plane of the paper shows that p/q and $-p/q$ surgeries on an amphicheiral knot in the 3–sphere yield oppositely oriented copies of the same manifold, and performing $1/n$ surgery on the unknot always produces a consistently oriented 3–sphere. See Figure 1.

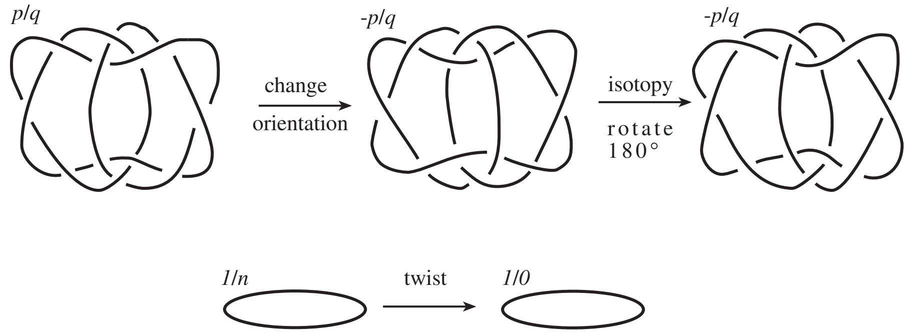

Figure 1

In both these cases, however, there exists a homeomorphism of the torally bounded knot exterior X which takes one surgery slope to the other. We call two slopes *equivalent* if such a homeomorphism exists. Truly cosmetic (resp. reflectively cosmetic) Dehn fillings $X(r_1)$ and $X(r_2)$ are considered *mundane* when there is an orientation preserving (resp. orientation reversing) homeomorphism of X taking r_1 to r_2. Cosmetic Dehn fillings which are not mundane are *exotic*.

There is one easy way to find exotic cosmetic fillings: swapping the sides of a Heegaard splitting of certain lens spaces can produce true and reflective exotic cosmetic surgeries on the unknot. In particular, p/q and p/q' surgeries yielding the lens spaces $L(p,q)$ and $L(p,q')$ where $qq' \equiv 1 \bmod p$ and $q' \neq q \bmod p$ perform this happy trick, for example $17/2$ and $17/9$ surgeries. So we require the space X not to be homeomorphic to $B^2 \times S^1$ in our definition of *exotic* cosmetic fillings.

Now things are much harder. Indeed, recent results in surgery theory suggest that examples of exotic cosmetic surgeries are few and far between. For example, the solutions to the knot complement problems in S^3 [11] and in $S^2 \times S^1$ [6], phrased in the language here, state that the 3–sphere S^3 and the manifold $S^2 \times S^1$ never arise via cosmetic surgery. So one thinks of cosmetic surgeries as occurring on knots in more general manifolds.

Similar results hold when the first Betti number of the manifold is positive. For example, Boileau, Domergue, and Mathieu [3] showed in 1995 that if X is irreducible and the core of the surgered solid torus is homotopically trivial in such an $X(r_1)$, then for r_2 distinct from r_1, the manifold $X(r_2)$ is never even simple homotopy equivalent to $X(r_1)$. An extension and sharpening of this result was given by M Lackenby [13] shortly thereafter. Assume that X is irreducible and atoroidal and, as before, that the core of the surgered torus is homotopically trivial in $X(r_1)$ with first Betti number positive. Then Lackenby's theorem shows that if at least one of the slopes r_2 and r_3 has a sufficiently high geometric intersection number with r_1, $X(r_2)$ and $X(r_3)$ are orientation preserving homeomorphic if and only if $r_2 = r_3$, and $X(r_2)$ and $X(r_3)$ are orientation reversing homeomorphic if and only if the surgery core is amphicheiral and $r_2 = -r_3$.

The assumption of homotopic triviality is important here, as illustrated by certain Seifert fibre spaces. As first shown by Mathieu [14], $9/1$ and $9/2$ surgeries on the right hand trefoil yield oppositely oriented copies of the same Seifert fibre space. The slopes are inequivalent since they have different distances from the meridian, and any homeomorphism of the knot exterior must take the meridian to itself. The existence of Seifert fibred cosmetic surgeries has a particularly nice picture in the Kirby calculus [18, Chapter 9] and depends on the existence of an exceptional fibre of index 2. The key property of such an exceptional fibre is that after orientation reversal, the type of an exceptional fibre of index 2 can be restored via a twist. See Figure 2 which begins with a surgery description of the trefoil knot K in S^3. (This comes from the Seifert fibration of S^3 over S^2 with a trefoil knot as regular fibre and two exceptional fibres of order 2 and 3 — see Seifert [20, sections 3,11], Montesinos [16, chapter 4].)

The same construction gives examples of exotic cosmetic surgeries on Seifert fibre spaces with positive first Betti number, starting with surgery descriptions of Seifert fibred spaces over higher genus surfaces as in [16, figure 12, p. 146].

This observation was also used by Rong [19] to classify the cosmetic surgeries where X carries a Seifert fibration and the surgery yields a Seifert fibre space. One consequence of the classification is that all the exotic cosmetic surgeries on Seifert fibre spaces are reflective. Another consequence of this technique is the fact that if a Seifert fibred X admits a pair of cosmetic surgeries yielding a Seifert fibre space, it admits an infinity of such pairs.

2 Cosmetic surgery on hyperbolic manifolds

From Seifert fibred cosmetic surgeries it is fairly easy to construct examples of cosmetic surgeries on graph manifolds, so the question arises as to whether

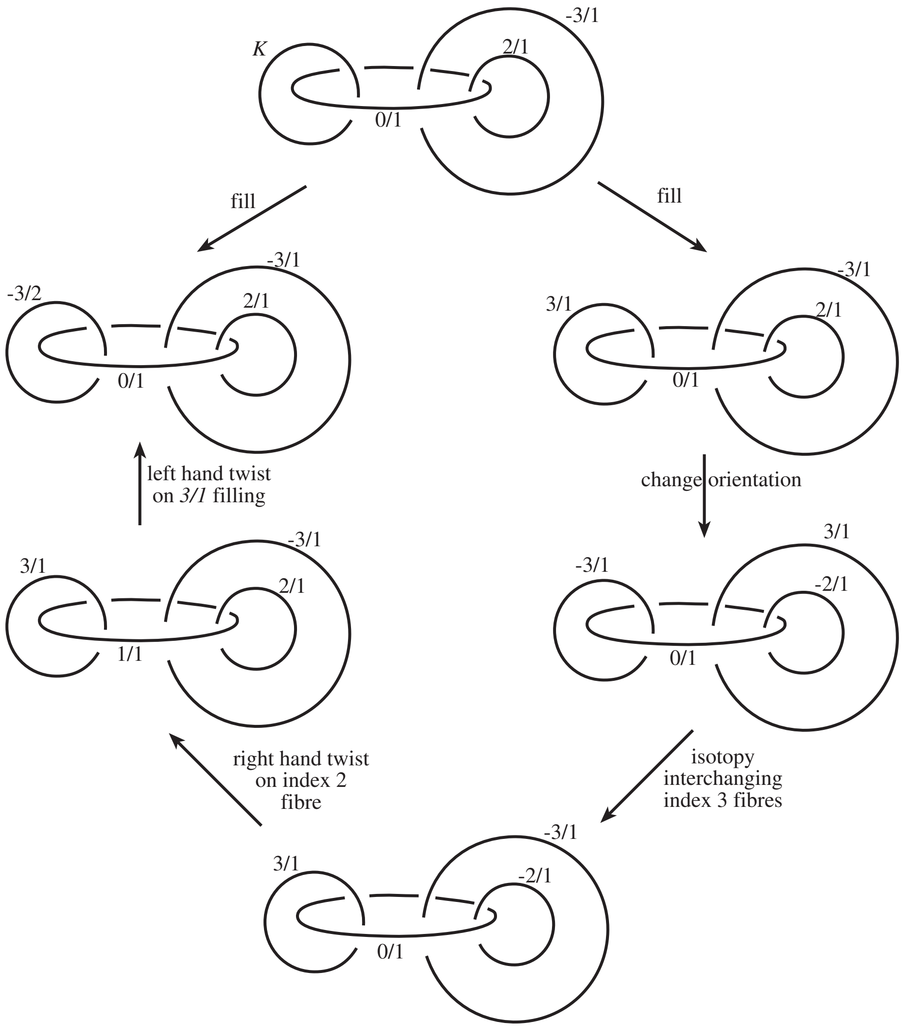

Figure 2

there are any hyperbolic examples. Such examples are unexpected because of the general theory of 1–cusped hyperbolic manifolds. For example, a theorem similar to M Lackenby's but for homotopically non-trivial knots and showing there can be only finitely many cosmetic surgeries on a hyperbolic knot is given below. Throughout this paper hyperbolic 3–manifolds are assumed to be *complete* and of *finite volume*.

Theorem 1 *Let X be a 1–cusped, orientable hyperbolic 3–manifold. Then there exists a finite set of slopes E on ∂X, such that if r_1 and r_2 are distinct slopes outside E, $X(r_1)$ and $X(r_2)$ homeomorphic implies that there exists an orientation reversing isometry h of X such that $h(r_1) = r_2$. In particular, if X is a classical knot complement, then the knot is amphicheiral and $r_1 = -r_2$.*

Proof By Thurston's theory of hyperbolic Dehn surgery [T1], we can choose E so that each filling outside E gives a hyperbolic manifold in which the surgery core circle is isotopic to the *unique* shortest closed geodesic. Assume r_1, r_2 are outside E and $X(r_1)$ is homeomorphic to $X(r_2)$. Then by Mostow rigidity there is an isometry $X(r_1) \to X(r_2)$ taking the core geodesic C_1 to core geodesic C_2. This restricts to a homeomorphism of X taking r_1 to r_2. The following lemma then completes the proof. □

Lemma 2 *Let X be a 1–cusped, orientable hyperbolic 3–manifold. If $h\colon X \to X$ is a homeomorphism which changes the slope of some peripheral curve, then h is orientation reversing. If X is a classical knot complement, then h takes each slope r to $-r$.*

Proof By Poincaré duality, the map induced by inclusion $H_1(\partial X; \mathbb{R}) \to H_1(X; \mathbb{R})$ has a 1–dimensional kernel K. The hyperbolic structure on the interior of X gives a natural Euclidean metric on $H_1(\partial X; \mathbb{R}) \cong \mathbb{R}^2$, defined up to similarity. By Mostow Rigidity, h is homotopic to an isometry, hence $h_*\colon H_1(\partial X; \mathbb{R}) \to H_1(\partial X; \mathbb{R})$ is an isometry. Further, K and its orthogonal complement are preserved by h_* so each is a $+1$ or -1 eigenspace for h_*. It follows that if h is orientation preserving then $h_* = \pm$ identity, so h does not change the slope of any peripheral curve.

If X is a classical knot complement, then h preserves both the longitude and meridian up to sign (using the solution to the knot complement problem [11]). Hence, if h is orientation reversing, each slope is taken to its negative. □

The theorem strongly suggests that finding cosmetic surgeries on hyperbolic manifolds that yield hyperbolic manifolds is hard, perhaps impossible. Instead one looks to find cosmetic surgeries that yield non-hyperbolic manifolds, such as the lens spaces. Another reason to look here is that there are cosmetic surgeries in the solid torus. In fact, Berge [1] and Gabai ([6], [7]) classify those knots in $B^2 \times S^1$ which have inequivalent slopes which fill to $B^2 \times S^1$.

From their classification, these knots are certain 1–bridge braids and, with a unique exception where there are three such slopes, these knots have exactly two slopes which yield a solid torus when filled. The meridian of this new solid torus, however, is quite different from the meridian of the original solid torus. Indeed, as originally shown by C Gordon [9], if one performs p/q–surgery on a knot in a solid torus and again obtains a solid torus, the meridian of this new solid torus is given by the slope $p/(k^2q)$, where k is the winding number of the original knot in the solid torus. Now, from such a knot in a solid torus it is easy to construct a torally bounded 3–manifold having a pair of slopes yielding lens spaces by attaching a solid torus to the outside of the solid torus in which the knot lies. The lens spaces produced are determined by the relation of this last attaching curve to the meridians described above. This has led to a conjectural classification of the fillings on hyperbolic knot complements in the 3–sphere which yield lens spaces, see [10].

Of course, the purpose here is to construct examples where filling on the appropriate slopes yields homeomorphic lens spaces. This requirement produces a severe number theoretic obstruction, namely, that the outside attaching slope r be simultaneously a (p,q) and (p,q') curve with $qq' \equiv \pm 1 \bmod p$ or $q \equiv \pm q' \bmod p$ with respect to our two meridians. (See [17] or [4].) The number p is, of course, the geometric intersection number of r and the meridians, and is commonly referred to as the *distance* between the two slopes. This already puts powerful restrictions on the allowable slopes as given a pair of slopes $r_1 = a_1/b_1$ and $r_2 = a_2/b_2$, there are exactly two slopes equidistant from r_1 and r_2. For this one notes that the geometric intersection number is just $|a_1b_2 - b_1a_2|$, so if $r = c/d$ is equidistant from r_1 and r_2 then either $a_1d - b_1c = a_2d - b_2c$ or $a_1d - b_1c = b_2c - a_2d$. In the former case it follows that $c/d = (a_1 - a_2)/(b_1 - b_2)$ and in the latter that $c/d = (a_1 + a_2)/(b_1 + b_2)$.

The precise lens spaces produced are then determined by the following.

Lemma 3 *Let M be the manifold obtained by a/b and c/d Dehn fillings on the two boundary components of $T^2 \times [0,1]$, where $\gcd(a,b) = \gcd(c,d) = 1$. Let a^*, b^* be integers such that $a^*b - b^*a = 1$. Then M is the lens space $L(p,q)$ where $\dfrac{p}{q} = \dfrac{bc - ad}{a^*d - b^*c}$.*

Proof The linear automorphism of T^2 given by the matrix

$$\begin{bmatrix} a^* & a \\ b^* & b \end{bmatrix}^{-1} = \begin{bmatrix} b & -a \\ -b^* & a^* \end{bmatrix}$$

takes the slopes $\dfrac{a}{b}$ and $\dfrac{c}{d}$ to $\dfrac{0}{1}$ and $\dfrac{bc - ad}{a^*d - b^*c}$. □

A further restriction on r arises from hyperbolicity. Considering the complements of these braids in the solid torus as link exteriors, one notes that of the six types listed in the Berge–Gabai classification only four are atoroidal and acylindrical, and hence by Thurston ([22], [23]) correspond to 2–cusped hyperbolic manifolds. If one wishes to construct hyperbolic examples by the above construction, the slope r must lie in the hyperbolic region of the Dehn surgery space of these manifolds. After filling on r, there is also the question as to whether the slopes r_1 and r_2 are indeed inequivalent.

Both these points are exemplified by a family of braids for which the construction gives distinct fillings yielding homeomorphic lens spaces. These examples arise from the Berge braids of "Type IV" and the Pythagorean triples of the form $(s,t,u) = (2k+1, 2k(k+1), 2k^2+2k+1)$. In particular, denoting by W_n the product of the first $n-1$ standard braid generators, the braids $W_s^1 W_t^{-s}$ give rise to a family of torally bounded 3–manifolds each with one pair of slopes yielding oppositely oriented copies of $L(u+s, s+2) = L(2(k+1)^2, 2k+3)$ and another pair of slopes yielding oppositely oriented copies of $L(u-s, s-2) = L(2k^2, 2k-1)$. However, it turns out that each of these torally bounded 3–manifolds has the structure of an amphicheiral graph manifold. This was initially suggested by computer calculations via SnapPea [We], then confirmed, after using lots of string, by a direct analysis via the Montesinos trick ([15], [2]). So these examples are neither hyperbolic nor surgeries on inequivalent slopes.

3 A surprising example

The preceding discussion makes the following example even more striking. Our construction will produce an oriented 1–cusped hyperbolic manifold X with an exotic pair of reflective cosmetic Dehn fillings. The filled manifolds $X(r_1)$ and $X(r_2)$ will be oppositely oriented copies of the lens space $L(49,18)$, and the cusped manifold X will be presented naturally as a knot exterior in $S^2 \times S^1$.

Begin with the now infamous 1–bridge braid $W_3^{-1} W_7^3$ in a solid torus. This braid and its mirror image are the unique 1–bridge braids with three distinct slopes which yield a solid torus when filled (see Figure 3). When the solid torus T in which the braid lies is considered to be a standard torus in the 3–sphere S^3, these special slopes are given by $1/0$, $18/1$, and $19/1$. Filling on these slopes produces solid tori whose meridians are represented by the $1/0$, $18/49$, and $19/49$ slopes on the boundary of T.

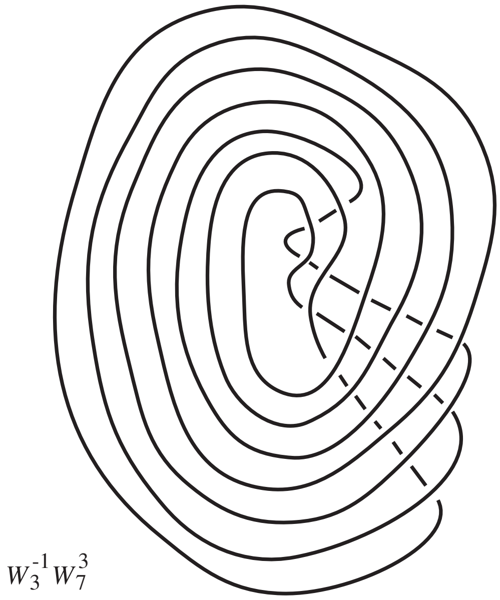

Figure 3

For the pair $r_1 = 18/49$ and $r_2 = 19/49$ it follows from our discussion above that the slopes $1/0$ and $37/98$ are equidistant to r_1 and r_2. Of course, attaching a solid torus to the p/q curve on a standard solid torus in S^3 is the same as performing q/p surgery on the core of the complementary solid torus. Choosing $1/0$ then, our braid becomes a knot in $S^2 \times S^1$ representing 7 in $\pi_1 \cong \mathbb{Z}$. An analysis by SnapPea [24] shows that the exterior of this knot is hyperbolic and that there is no homeomorphism taking the slope $18/1$ to $19/1$. (Alternatively, the Montesinos trick can be used to show this knot exterior to be atoroidal and acylindrical.)

So if filling on these slopes does indeed produce homeomorphic manifolds, we have the example we seek. But now an easy exercise, using Lemma 3 or the Kirby calculus (see Figure 4), shows that filling on these slopes produces the manifolds $L(49, -18)$ and $L(49, -19)$.

Since $(-18)\cdot(-19) = 342$ and $7\cdot 49 = 343$, the classification of lens spaces shows that one has obtained oppositely oriented copies of the lens space $L(49, 18)$.

Remark In SnapPea's notation, these cosmetic surgeries are given by the $(0, 1)$ and $(1, 1)$ fillings on the cusped hyperbolic manifold $m172$.

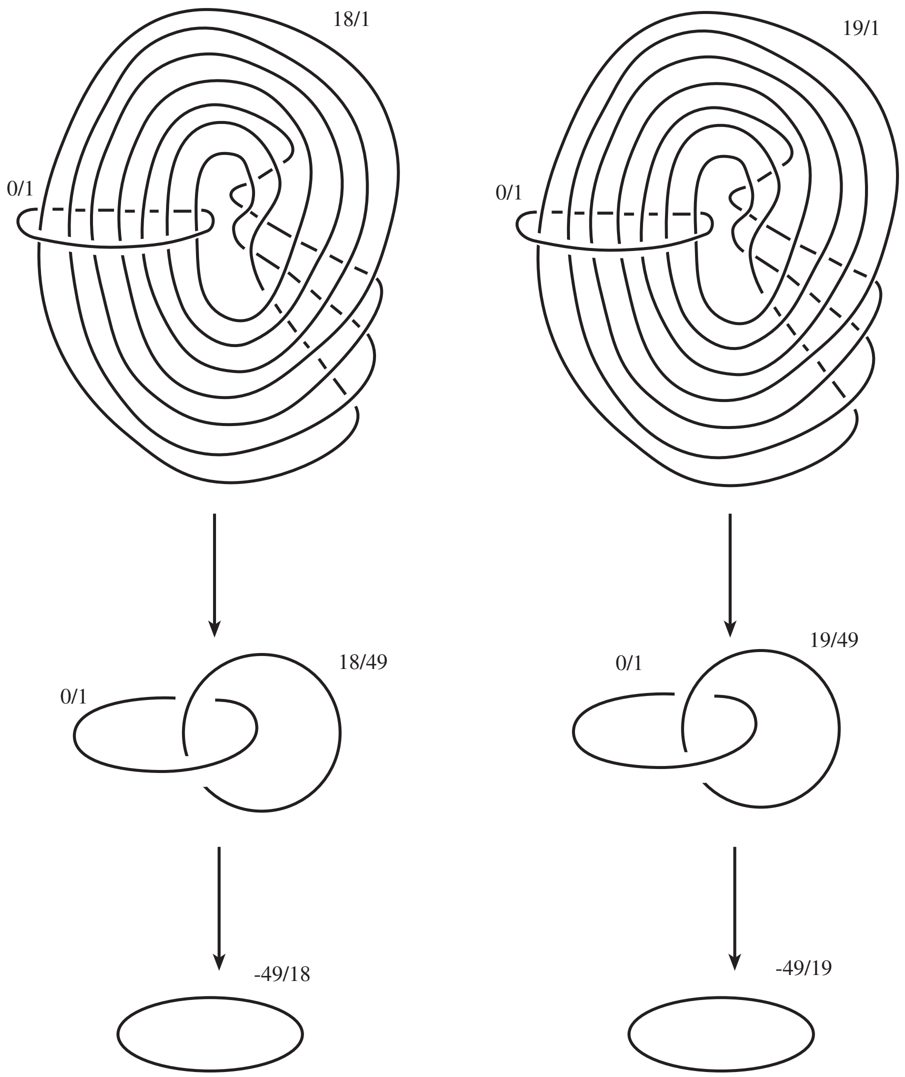

Figure 4

4 Concluding Remarks

To date this is the only known example of exotic cosmetic surgery on a hyperbolic knot exterior. As it and all the Seifert fibred examples are reflective, it seems appropriate to make the following conjecture, given in problem 1.81 of Kirby's problem list [12].

Conjecture 1 (Cosmetic surgery conjecture) Exotic cosmetic surgeries are never truly cosmetic.

An equivalent form of this conjecture is the

Conjecture 2 (Oriented knot complement conjecture) If K_1 and K_2 are knots in a closed, oriented 3–manifold M whose complements are homeomorphic via an orientation-preserving homeomorphism, then there exists an orientation-preserving homeomorphism of M taking K_1 to K_2.

We close with two further conjectures and a comment. Our earlier theorem suggests the following:

Conjecture 3 Cusped hyperbolic manifolds admit no cosmetic fillings, true or reflective, yielding hyperbolic manifolds.

Conjecture 4 Closed geodesics in a hyperbolic 3–manifold are determined by their complements (even allowing orientation reversing homeomorphisms).

Remarks (3) $\Rightarrow$ (4) but they are not equivalent as it may happen that the core of one of the surgeries is not isotopic to a closed geodesic. Some evidence for these conjectures has been provided by a computer search: No manifold in the Hodgson–Weeks census of 11,031 low-volume closed, orientable hyperbolic 3–manifolds is obtained by two inequivalent fillings on a manifold in the Hildebrand–Weeks census of 4,815 cusped, orientable hyperbolic 3-manifolds triangulated by at most 7 ideal simplices. (All of these hyperbolic manifolds are incorporated in SnapPea [24].)

Question Does there exist a pair of exotic cosmetic fillings which are simultaneously true and reflective?

One can also ask the above questions with homeomorphism replaced by homotopy equivalence or simple homotopy equivalence (see [3] for a version which is different because the knots are null-homotopic). We note that there is a hyperbolic knot exterior in a lens space with a pair of slopes which yield non-homeomorphic but orientation preserving homotopy equivalent lens spaces. It is obtained by replacing the $1/0$ slope in our construction above by $37/98$.

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References

[1] **J Berge**, *The knots in $D^2 \times S^1$ which have nontrivial Dehn surgeries that yield $D^2 \times S^1$*, Topology and its Applications, 38 (1991) 1–20, and private communication

[2] **S Bleiler**, *Prime tangles and composite knots*, from: "Knot theory and Manifolds", Springer–Verlag Lecture notes in Mathematics, 1144 (1985) 1–13

[3] **M Boileau**, **M Domergue**, **Y Mathieu**, *Chirurgies de Dehn le long de nœuds nul homotopes*, preprint

[4] **E J Brody**, *The topological classification of the lens spaces*, Ann. of Math. 71 (1960) 163–184

[5] **M Dehn**, *Über die Topologie des dreidimensionalen Raumes*, Math. Ann. 69 (1910) 137–168

[6] **D Gabai**, *Foliations and the topology of 3-manifolds: II*, J. Differential Geom. 26 (1987) 461–478

[7] **D Gabai**, *Surgery on knots in solid tori*, Topology, 28 (1989) 1–6

[8] **D Gabai**, *1-bridge braids in solid tori*, Topology Appl. 37 (1990) 221–235

[9] **C McA Gordon**, *Dehn surgery on satellite knots*, Transactions AMS, 275 (1983) 687–708

[10] **C McA Gordon**, *Dehn surgery on knots*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990) Math. Soc. Japan, Tokyo (1991) 631–642

[11] **C McA Gordon**, **J Luecke**, *Knots are determined by their complements*, J. Amer. Math. Soc. 2 (1989) 371–415

[12] **R Kirby**, *Problems in low-dimensional topology*, AMS/IP Studies in Advanced Mathematics, volume 2, part 2, Amer. Math. Soc. (1997)

[13] **M Lackenby**, *Dehn surgery on knots in 3-manifolds*, J. Amer. Math. Soc. 10 (1997) 835–864

[14] **Y Mathieu**, *Sur les nœuds qui ne sont pas déterminés par leur complément et problemes de chirurgie dans les variétés de dimension 3*, PhD thesis, Marseille (1990)

[15] **J M Montesinos**, *Surgery on links and double branched coverings of S^3*, Annals of Math. Studies, 84 (1975) 227-260

[16] **J M Montesinos**, *Classical Tessellations and Three-Manifolds*, Springer (1987)

[17] **K Reidemeister**, *Homotopieringe und Linsenräume*, Abh. Math. Sem. Univ. Hamburg, 11 (1935) 102–109

[18] **D Rolfsen**, *Knots and Links*, Publish or Perish Inc. (1976)

[19] **Y W Rong**, *Some knots not determined by their complements*, from: "Quantum topology", Ser. Knots Everything, 3, World Sci. Publishing, River Edge, NJ, (1993) 339–353

[20] **H Seifert**, *Topologie dreidimensionaler gefaserter Räume*, Acta Math. 60 (1933) 147–238, English translation in: H Seifert and W Threlfall, *A textbook of topology*, Academic Press, (1980)

[21] **W P Thurston**, *The geometry and topology of* 3*-manifolds*, Princeton Univ. Math. Dept. (1979)

[22] **W P Thurston**, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. 6 (1982) 357–381

[23] **W P Thurston**, *Hyperbolic structures on* 3*-manifolds, I*, Annals of Math. 124 (1986) 203–246

[24] **J Weeks**, *SnapPea: a computer program for creating and studying hyperbolic 3-manifolds*, available from `http://www.northnet.org/weeks`

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