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Structure of the mapping class groups of surfaces: a survey and a prospect

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Abstract In this paper, we survey recent works on the structure of the mapping class groups of surfaces mainly from the point of view of topology. We then discuss several possible directions for future research. These include the relation between the structure of the mapping class group and invariants of 3–manifolds, the unstable cohomology of the moduli space of curves and Faber's conjecture, cokernel of the Johnson homomorphisms and the Galois as well as other new obstructions, cohomology of certain infinite dimensional Lie algebra and characteristic classes of outer automorphism groups of free groups and the secondary characteristic classes of surface bundles. We give some experimental results concerning each of them and, partly based on them, we formulate several conjectures and problems.

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This paper is dedicated to Robion C Kirby on the occasion of his 60^{th} *birthday.*

1 Introduction

Let Σ_g be a closed oriented surface of genus $g \geq 2$ and let $\mathcal{M}_g$ be its mapping class group. This is the group consisting of path components of $\mathrm{Diff}_+ \Sigma_g$, which is the group of orientation preserving diffeomorphisms of Σ_g. $\mathcal{M}_g$ acts on the Teichmüller space $\mathcal{T}_g$ of Σ_g properly discontinuously and the quotient space $\mathbf{M}_g = \mathcal{T}_g/\mathcal{M}_g$ is the (coarse) moduli space of curves of genus g. $\mathcal{T}_g$ is known to be homeomorphic to $\mathbb{R}^{6g-6}$. Hence we have a natural isomorphism

$$H^*(\mathcal{M}_g;\mathbb{Q}) \cong H^*(\mathbf{M}_g;\mathbb{Q}).$$

On the other hand, by a theorem of Earle–Eells [20], the identity component of $\mathrm{Diff}_+ \Sigma_g$ is contractible for $g \geq 2$ so that the classifying space $\mathrm{BDiff}_+ \Sigma_g$

is an Eilenberg–MacLane space $K(\mathcal{M}_g,1)$. Therefore we have also a natural isomorphism

$$H^*(\mathrm{BDiff}_+\,\Sigma_g) \cong H^*(\mathcal{M}_g).$$

Thus the mapping class group serves as the orbifold fundamental group of the moduli space $\mathbf{M}_g$ and at the same time it plays the role of the *universal monodromy group* for oriented Σ_g–bundles. Any cohomology class of the mapping class group can be considered as a characteristic class of oriented surface bundles and, over the rationals, it can also be identified as a cohomology class of the moduli space.

The Teichmüller space $\mathcal{T}_g$ and the moduli space $\mathbf{M}_g$ are important objects primarily in complex analysis and algebraic geometry. Many important results concerning these two spaces have been obtained following the fundamental works of Ahlfors, Bers and Mumford. Because of the limitation of our knowledge, we only mention here a survey paper of Hain and Looijenga [43] for recent works on $\mathbf{M}_g$, mainly from the viewpoint of algebraic geometry, and a book by Harris and Morrison [53] for basic facts as well as more advanced results.

From a topological point of view, fundamental works of Harer [46, 47] on the homology of the mapping class group and also of Johnson (see [63]) on the structure of the Torelli group, both in early 80's, paved the way towards modern topological studies of $\mathcal{M}_g$ and $\mathbf{M}_g$. Here the Torelli group, denoted by $\mathcal{I}_g$, is the subgroup of $\mathcal{M}_g$ consisting of those elements which act on the homology of Σ_g trivially.

Slightly later, the author began a study of the classifying space $\mathrm{BDiff}_+\,\Sigma_g$ of surface bundles which also belongs to topology. The intimate relationship between three universal spaces, $\mathcal{T}_g, \mathbf{M}_g$ and $\mathrm{BDiff}_+\,\Sigma_g$ described above, imply that there should exist various interactions among the studies of these spaces which are peculiar to various branches of mathematics including the ones mentioned above. Although it is not always easy to understand mutual viewpoints, we believe that doing so will enhance individual understanding of these spaces.

In this paper, we would like to survey some aspects of recent topological study of the mapping class group as well as the moduli space. More precisely, we focus on a study of the mapping class group which is related to the structure of the Torelli group $\mathcal{I}_g$ together with a natural action of the Siegel modular group $Sp(2g,\mathbb{Z})$ on some graded modules associated with the lower (as well as other) central series of $\mathcal{I}_g$. Here it turns out that explicit descriptions of Sp–invariant tensors of various Sp–modules using classical symplectic representation theory, along the lines of Kontsevich's fundamental works in [85, 86], and also Hain's recent work [41] on $\mathcal{I}_g$ using mixed Hodge structures can play very important roles. These two points will be reviewed in section 4 and section 5, respectively. In the final section (section 6), we describe several experimental results, with sketches of proofs, by which we would like to propose some possible directions for future research.

This article can be considered as a continuation of our earlier papers [113, 114, 117].

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2 $\mathcal{M}_g$ as an extension of the Siegel modular group by the Torelli group

Let us simply write H for $H_1(\Sigma_g, \mathbb{Z})$. We have the intersection pairing

$$\mu\colon\ H \otimes H \longrightarrow \mathbb{Z}$$

which is a non-degenerate skew symmetric bilinear form on H. The natural action of $\mathcal{M}_g$ on H, which preserves this pairing, induces the classical representation

$$\rho_0\colon\ \mathcal{M}_g \longrightarrow \operatorname{Aut} H.$$

If we fix a symplectic basis of H, then $\operatorname{Aut} H$ can be identified with the Siegel modular group $Sp(2g, \mathbb{Z})$ so that we can write

$$\rho_0\colon\ \mathcal{M}_g \longrightarrow Sp(2g, \mathbb{Z}).$$

The *Torelli group*, denoted by $\mathcal{I}_g$, is defined to be the kernel of ρ_0. Thus we have the following basic extension of three important groups

$$1 \longrightarrow \mathcal{I}_g \longrightarrow \mathcal{M}_g \longrightarrow Sp(2g, \mathbb{Z}) \longrightarrow 1. \tag{1}$$

Associated to each of these groups, we have various moduli spaces. Namely the (coarse) moduli space $\mathbf{M}_g$ of genus g curves for $\mathcal{M}_g$, the moduli space $\mathbf{A}_g$ of principally polarized abelian varieties for $Sp(2g, \mathbb{Z})$ and the *Torelli space* $\mathbf{T}_g$ for $\mathcal{I}_g$. Here the Torelli space is defined to be the quotient of the Teichmüller space $\mathcal{T}_g$ by the natural action of $\mathcal{I}_g$ on it. Since $\mathcal{I}_g$ is known to be torsion free, $\mathbf{T}_g$ is a complex manifold. We have holomorphic mappings between these moduli spaces

$$\mathbf{T}_g \longrightarrow \mathbf{M}_g \longrightarrow \mathbf{A}_g$$

where the first map is an infinite ramified covering and the second map is injective by the theorem of Torelli.

By virtue of the above facts, we can investigate the structure of $\mathcal{M}_g$ (or that of $\mathbf{M}_g$) by combining individual study of $\mathcal{I}_g$ and $Sp(2g,\mathbb{Z})$ (or $\mathbf{T}_g$ and $\mathbf{A}_g$) together with some additional investigation of the action of $Sp(2g,\mathbb{Z})$ on the structure of the Torelli group or Torelli space. Here it turns out that the symplectic representation theory can play a crucial role. However, before reviewing them, let us first recall the fundamental works of D Johnson on the structure of $\mathcal{I}_g$ very briefly (see [63] for details) because it is the starting point of the above method.

Johnson proved in [62] that $\mathcal{I}_g$ is finitely generated for all $g \geq 3$ by constructing explicit generators for it. Before this work, a homomorphism

$$\tau \colon\ \mathcal{I}_g \longrightarrow \Lambda^3 H/H$$

was introduced in [61] which generalized an earlier work of Sullivan [136] extensively and is now called the Johnson homomorphism. Here $\Lambda^3 H$ denotes the third exterior power of H and H is considered as a natural submodule of $\Lambda^3 H$ by the injection

$$H \ni u \longmapsto u \wedge \omega_0 \in \Lambda^3 H$$

where $\omega_0 \in \Lambda^2 H$ is the symplectic class (in homology) defined as $\omega_0 = \sum_i x_i \wedge y_i$ for any symplectic basis x_i, y_i $(i = 1, \cdots, g)$ of H.

Let $\mathcal{K}_g \subset \mathcal{M}_g$ be the subgroup of $\mathcal{M}_g$ generated by all Dehn twists along *separating* simple closed curves on Σ_g. It is a normal subgroup of $\mathcal{M}_g$ and is contained in the Torelli group $\mathcal{I}_g$. In [64], Johnson proved that $\mathcal{K}_g$ is exactly equal to $\operatorname{Ker}\tau$ so that we have an exact sequence

$$1 \longrightarrow \mathcal{K}_g \longrightarrow \mathcal{I}_g \stackrel{\tau}{\longrightarrow} \Lambda^3 H/H \longrightarrow 1. \tag{2}$$

Finally in [65], he determined the abelianization of $\mathcal{I}_g$ for $g \geq 3$ in terms of certain combination of τ and the totality of the Birman–Craggs homomorphisms defined in [12]. The target of the latter homomorphisms are $\mathbb{Z}/2$ so that the first rational homology group of $\mathcal{I}_g$ (or more precisely, the abelianization of $\mathcal{I}_g$ modulo 2 torsions) is given simply by τ. Namely we have an isomorphism

$$\tau \colon\ H_1(\mathcal{I}_g;\mathbb{Q}) \cong \Lambda^3 H_{\mathbb{Q}}/H_{\mathbb{Q}}$$

where $H_{\mathbb{Q}} = H \otimes \mathbb{Q}$.

Problem 2.1 Determine whether the Torelli group $\mathcal{I}_g$ $(g \geq 3)$ is finitely presentable or not. If the answer is yes, give an explicit finite presentation of it.

It should be mentioned here that Hain [41] proved that the Torelli Lie algebra $\mathfrak{t}_g$, which is the Malcev Lie algebra of $\mathcal{I}_g$, is finitely presentable for all $g \geq 3$. Moreover he gave an explicit finite presentation of $\mathfrak{t}_g$ for any $g \geq 6$ which turns out to be very simple, namely there arise only quadratic relations. Here a result of Kabanov [68] played an important role in bounding the degrees of relations. More detailed description of this work as well as related materials will be given in section 5.

On the other hand, in the case of $g = 2$, Mess [99] proved that $\mathcal{I}_2 = \mathcal{K}_2$ is an infinitely generated free group. Thus we can ask

Problem 2.2 (i) Determine whether the group $\mathcal{K}_g$ is finitely generated or not for $g \geq 3$.

(ii) Determine the abelianization $H_1(\mathcal{K}_g)$ of $\mathcal{K}_g$.

We mention that $\mathcal{K}_g$ is far from being a free group for $g \geq 3$. This is almost clear because it is easy to construct subgroups of $\mathcal{K}_g$ which are free abelian groups of high ranks by making use of Dehn twists along mutually disjoint separating simple closed curves on Σ_g. More strongly, we can show, roughly as follows, that the cohomological dimension of $\mathcal{K}_g$ will become arbitrarily large if we take the genus g sufficiently large. Let

$$\tau_g(2)\colon\ \mathcal{K}_g \longrightarrow \mathfrak{h}_g(2)$$

be the second Johnson homomorphism given in [115, 118] (see section 5 below for notation). Then it can be shown that the associated homomorphism

$$\tau_g(2)^*\colon\ H^*(\mathfrak{h}_g(2)) \longrightarrow H^*(\mathcal{K}_g)$$

is non-trivial by evaluating cohomology classes coming from $H^*(\mathfrak{h}_g(2))$, under the homomorphism $\tau_g(2)^*$, on abelian cycles of $\mathcal{K}_g$ which are supported in the above free abelian subgroups.

In section 6.6, we will consider the cohomological structure of the group $\mathcal{K}_g$ from a hopefully deeper point of view which is related to the secondary characteristic classes of surface bundles introduced in [119].

3 The stable cohomology of $\mathcal{M}_g$ and the stable homotopy type of $\mathbf{M}_g$

Let $\pi\colon \overline{\mathbf{C}}_g \to \overline{\mathbf{M}}_g$ be the universal family of stable curves over the Deligne–Mumford compactification of the moduli space $\mathbf{M}_g$. In [122], Mumford defined certain classes

$$\kappa_i \in A^i(\overline{\mathbf{M}}_g)$$

in the Chow algebra (with coefficients in $\mathbb{Q}$) of the moduli space $\overline{\mathbf{M}}_g$ by setting $\kappa_i = \pi_*(c_1(\omega)^{i+1})$ where ω denotes the relative dualizing sheaf of the morphism π. On the other hand, in [107] the author independently defined certain integral cohomology classes

$$e_i \in H^{2i}(\mathcal{M}_g;\mathbb{Z})$$

of the mapping class group $\mathcal{M}_g$ by setting $e_i = \pi_*(e^{i+1})$ where

$$\pi\colon \ \mathrm{EDiff}_+\,\Sigma_g \to \mathrm{BDiff}_+\,\Sigma_g$$

is the universal oriented Σ_g–bundle and $e \in H^2(\mathrm{EDiff}_+\,\Sigma_g;\mathbb{Z})$ is the Euler class of the relative tangent bundle of π. As was mentioned in section 1, there exists a natural isomorphism $H^*(\mathcal{M}_g;\mathbb{Q}) \cong H^*(\mathbf{M}_g;\mathbb{Q})$ and it follows immediately from the definitions that $e_i = (-1)^{i+1}\kappa_i$ as an element of these *rational* cohomology groups. The difference in signs comes from the fact that Mumford uses the first Chern class of the relative dualizing sheaf of π while our definition uses the Euler class of the relative tangent bundle. These classes κ_i, e_i are called tautological classes or Mumford–Morita–Miller classes.

In this paper, we use our notation e_i to emphasize that we consider it as an integral cohomology class of the mapping class group rather than an element of the Chow algebra of the moduli space. A recent work of Kawazumi and Uemura in [78] shows that the integral class e_i can play an interesting role in a study of certain cohomological properties of finite subgroups of $\mathcal{M}_g$.

Let

$$\Phi\colon \ \mathbb{Q}[e_1, e_2, \cdots] \longrightarrow \lim_{g\to\infty} H^*(\mathcal{M}_g;\mathbb{Q}) \tag{3}$$

be the natural homomorphism from the polynomial algebra generated by e_i into the stable cohomology group of the mapping class group which exists by virtue of a fundamental result of Harer [47]. It was proved by Miller [102] and the author [108], independently, that the homomorphism Φ is *injective* and we have the following well known conjecture (see Mumford [122]).

Conjecture 3.1 The homomorphism Φ is an isomorphism so that

$$\lim_{g\to\infty} H^*(\mathcal{M}_g;\mathbb{Q}) \cong \mathbb{Q}[e_1, e_2, \cdots].$$

We would like to mention here a few pieces of evidence which support the above conjecture. First of all, Harer's explicit computations in [46, 49, 51] verify the conjecture in low degrees. See also [4] for more recent development. Secondly, Kawazumi has shown in [72] (see also [71, 73]) that the Mumford–Morita–Miller classes occur naturally in his algebraic model of the cohomology of the moduli space which is constructed in the framework of the complex analytic Gel'fand–Fuks cohomology theory, whereas no other classes can be obtained in this way. Thirdly, in [76, 77] Kawazumi and the author showed that the image of the natural homomorphism

$$H^*(H_1(\mathcal{I}_g);\mathbb{Q})^{Sp} \longrightarrow H^*(\mathcal{M}_g;\mathbb{Q})$$

is exactly equal to the subalgebra generated by the classes e_i (see section 6.4 for more detailed survey of related works). Here Sp stands for $Sp(2g,\mathbb{Z})$. Finally, as is explained in a survey paper by Hain and Looijenga [43] and also in our paper [76], a combination of this result with Hain's fundamental work in [41] via Looijenga's idea to use Pikaart's purity theorem in [133] implies that there are no new classes in the continuous cohomology of $\mathcal{M}_g$, with respect to a certain natural filtration on it, in the stable range.

Now there seems to be a rather canonical way of realizing the homomorphism Φ of (3) at the space level. To describe this, we first recall the cohomological nature of the classical representation $\rho_0\colon \mathcal{M}_g \to Sp(2g,\mathbb{Z})$. The Siegel modular group $Sp(2g,\mathbb{Z})$ is a discrete subgroup of $Sp(2g,\mathbb{R})$ and the maximal compact subgroup of the latter group is isomorphic to the unitary group $U(g)$. Hence there exists a universal g–dimensional complex vector bundle on the classifying space of $Sp(2g,\mathbb{Z})$. Let η be the pull back, under ρ_0, of this bundle to the classifying space of $\mathcal{M}_g$. As was explained in [7] (see also [108]), the dual bundle η^* can be identified, on each family $\pi\colon E \to X$ of Riemann surfaces, as follows. Namely it is the vector bundle over the base space X whose fiber on $x \in X$ is the space of holomorphic differentials on the Riemann surface E_x. In the above paper, Atiyah used the Grothendieck Riemann–Roch theorem to deduce the relation

$$e_1 = 12c_1(\eta^*).$$

If we apply the above procedure to the universal family $\mathbf{C}_g \to \mathbf{M}_g$, then we obtain a complex vector bundle η^* (in the orbifold sense) over $\mathbf{M}_g$ (in fact, more generally, over the Deligne–Mumford compactification $\overline{\mathbf{M}}_g$) which is called the Hodge bundle. In [122], Mumford applied the Grothendieck Riemann–Roch theorem to the morphism $\overline{\mathbf{C}}_g \to \overline{\mathbf{M}}_g$ and obtained an identity, in the Chow algebra $A^*(\overline{\mathbf{M}}_g)$, which expresses the Chern classes of the Hodge bundle in terms of the tautological classes κ_{2i-1} with *odd* indices together with some canonical classes coming from the boundary. From this identity, we can deduce the relations

$$e_{2i-1} = \frac{2i}{B_{2i}}\, s_{2i-1}(\eta^*) \quad (i = 1, 2, \cdots) \tag{4}$$

in the rational cohomology of $\mathcal{M}_g$. Here B_{2i} denotes the $2i$-th Bernoulli number and $s_i(\eta^*)$ is the characteristic class of η^* corresponding to the formal sum $\sum_j t_j^i$ (sometimes called the i-th Newton class). We have also obtained the above relations in [107] by applying the Atiyah–Singer index theorem [9] for families of elliptic operators, along the lines of Atiyah's argument in [7]. Since η^* is *flat* as a real vector bundle, all of its Pontrjagin classes vanish so that we can conclude that the Chern classes of η^* can be expressed entirely in terms of the classes e_{2i-1}. Thus we can say that the totality of the classes e_{2i-1} of *odd* indices is equivalent to the total Chern class of the Hodge bundle which comes from the Siegel modular group.

Although the rational cohomology of $\mathcal{M}_g$ and $\mathbf{M}_g$ are canonically isomorphic to each other, there seems to be a big difference between the torsion cohomology of them. To be more precise, let

$$\mathrm{BDiff}_+\,\Sigma_g = K(\mathcal{M}_g, 1) \longrightarrow \mathbf{M}_g \quad (g \geq 2)$$

be the natural mapping which is uniquely defined up to homotopy, where the equality above is due to a result of Earle and Eells [20] as was already mentioned in the introduction. As is well known (see eg [46]), $\mathcal{M}_g$ is perfect for all $g \geq 3$ so that we can apply Quillen's plus construction on $K(\mathcal{M}_g, 1)$ to obtain a simply connected space $K(\mathcal{M}_g, 1)^+$ which has the same homology as that of $\mathcal{M}_g$. It is known that the moduli space $\mathbf{M}_g$ is simply connected. Hence, by the universal property of the plus construction, the above mapping factors through a mapping

$$K(\mathcal{M}_g, 1)^+ \longrightarrow \mathbf{M}_g.$$

Problem 3.2 Study the homotopy theoretical properties of the above mapping $K(\mathcal{M}_g, 1)^+ \longrightarrow \mathbf{M}_g$. In particular, what is its homotopy fiber ?

The classical representation $\rho_0\colon \mathcal{M}_g \to Sp(2g, \mathbb{Z})$ induces a mapping

$$K(\mathcal{M}_g, 1)^+ \longrightarrow K(Sp(2g, \mathbb{Z}), 1)^+ \tag{5}$$

because $Sp(2g, \mathbb{Z})$ is also perfect for $g \geq 3$. Homotopy theoretical properties of this map (or rather its direct limit as $g \to \infty$) have been studied by many authors and they produced interesting implications on the torsion cohomology of $\mathcal{M}_g$ (see [15, 16, 36, 138] as well as their references). A final result along these lines was obtained by Tillmann. This says that $K(\mathcal{M}_\infty, 1)^+$ is an infinite loop space and the natural map $K(\mathcal{M}_\infty, 1)^+ \to K(Sp(2\infty, \mathbb{Z}), 1)^+$ is that of infinite loop spaces (see [138] for details). See also [104] for a different feature of the above map, [142] for a homotopy theoretical implication of Conjecture 3.1 and [128] for the *etale* homotopy type of the moduli spaces.

Let F_g be the homotopy fiber of the above mapping (5). Then, we have a map

$$\mathbf{T}_g \longrightarrow F_g.$$

Using the fact that any class e_i is primitive with respect to Miller's loop space structure on $K(\mathcal{M}_\infty, 1)^+$, it is easy to see that the natural homomorphism

$$\mathbb{Q}[e_2, e_4, \cdots] \longrightarrow H^*(F_g; \mathbb{Q})$$

is *injective* in a certain stable range and we can ask how these cohomology classes behave on the Torelli space.

We would like to show that the classes e_{2i} of *even* indices are closely related to the Pontrjagin classes of the moduli space $\mathbf{M}_g$ and also of the Torelli space $\mathbf{T}_g$. To see this, recall that $\mathbf{T}_g$ is a complex manifold and $\mathbf{M}_g$ is *nearly* a complex manifold of dimension $3g-3$. More precisely, as is well known it has a finite ramified covering $\widetilde{\mathbf{M}}_g$ which is a complex manifold and we can write $\mathbf{M}_g = \widetilde{\mathbf{M}}_g / G$ where G is a suitable finite group acting holomorphically on $\widetilde{\mathbf{M}}_g$. Hence we have the Chern classes

$$c_i \in H^{2i}(\widetilde{\mathbf{M}}_g; \mathbb{Z}) \quad (i = 1, 2, \cdots)$$

of the tangent bundle of $\widetilde{\mathbf{M}}_g$ which is invariant under the action of G. Hence we have the rational cohomology classes

$$c_i^o \in H^{2i}(\mathbf{M}_g; \mathbb{Q})$$

which is easily seen to be independent of the choice of $\widetilde{\mathbf{M}}_g$. We may call them *orbifold* Chern classes of the moduli space. To identify these classes, we use the Grothendieck Riemann–Roch theorem applied to the morphism $\pi\colon \mathbf{C}_g \to \mathbf{M}_g$

$$\pi_*(ch(\xi) Td(\omega^*)) = ch(\pi_!(\xi))$$

where ω^* denotes the relative tangent bundle (in the orbifold sense) of π and ξ is a vector bundle over $\mathbf{C}_g$. If we take ξ to be the relative cotangent bundle ω

as in [122], then we obtain the relations (4) above. Instead of this, let us take ξ to be ω^*. Since $\pi_!(\omega^*) = -T\mathbf{M}_g$ by the Kodaira–Spencer theory, we have

$$\begin{aligned}
ch^o(\mathbf{M}_g) &= -\,\pi_*\big(ch(\omega^*)Td(\omega^*)\big)\\
&= -\,\pi_*\big(\exp\ e\ \frac{e}{1-\exp^{-e}}\big)\\
&= -\,\pi_*\Big\{\big(1+e+\cdots+\frac{1}{n!}e^n+\cdots\big)\big(1+\frac{1}{2}e+\sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}e^{2k}\big)\Big\}
\end{aligned}$$

where $e \in H^2(\mathbf{C}_g;\mathbb{Q})$ denotes the Euler class of ω^*. From this, we can conclude

$$\begin{aligned}
s^o_{2k-1}(\mathbf{M}_g) &= -\Big\{\frac{1}{(2k)!}+\frac{1}{(2k-1)!}\cdot\frac{1}{2}+\frac{1}{(2k-2)!}\ \frac{B_1}{2}+\frac{1}{(2k-4)!}\cdot-\frac{B_2}{4!}\\
&\qquad+\cdots+\frac{1}{2}\cdot(-1)^k\frac{B_{k-1}}{(2k-2)!}+(-1)^{k-1}\frac{B_k}{(2k)!}\Big\}e_{2k-1}\\
s^o_{2k}(\mathbf{M}_g) &= -\Big\{\frac{1}{(2k+1)!}+\frac{1}{(2k)!}\cdot\frac{1}{2}+\frac{1}{(2k-1)!}\ \frac{B_1}{2}+\frac{1}{(2k-3)!}\cdot-\frac{B_2}{4!}\\
&\qquad+\cdots+\frac{1}{6}\cdot(-1)^k\frac{B_{k-1}}{(2k-2)!}+(-1)^{k-1}\frac{B_k}{(2k)!}\Big\}e_{2k}.
\end{aligned}$$

The first few classes are given by

$$s^o_1(\mathbf{M}_g) = -\frac{13}{12}e_1,\quad s^o_2(\mathbf{M}_g) = -\frac{1}{2}e_2,\quad s^o_3(\mathbf{M}_g) = -\frac{119}{720}e_3.$$

Thus the orbifold Chern classes of $\mathbf{M}_g$ turn out to be, in some sense, independent of g. The pull back of these classes to the Torelli space $\mathbf{T}_g$ are equal to the (genuine) Chern classes of it because $\mathbf{T}_g$ is a complex manifold. Since the pull back of e_{2i-1} to $\mathbf{T}_g$ vanishes for all i, we can conclude that $s_{2i-1}(\mathbf{T}_g) = 0$ and only the classes $s_{2i}(\mathbf{T}_g)$ may remain to be non-trivial. As is well known, these classes are equivalent to the Pontrjagin classes of $\mathbf{T}_g$ as a differentiable manifold.

In view of the above facts, it may be said that the classifying map $\mathbf{M}_g \to BU(3g-3)$ of the holomorphic tangent bundle of $\mathbf{M}_g$ would realize the conjectural isomorphism (3) at the space level (rigorously speaking, we have to use some finite covering of $\mathbf{M}_g$). Alternatively we could use the map

$$\mathbf{M}_g \to \mathbf{A}_g \times BSO(6g-6)$$

where the second factor is the classifying map of the tangent bundle of $\mathbf{M}_g$ as a *real* vector bundle. In short, we can say that the odd classes e_{2i-1} serve as Chern classes of the Hodge bundle while the even classes e_{2i} embody the orbifold Pontrjagin classes of the moduli space.

According to Looijenga [91], the Deligne–Mumford compactification $\overline{\mathbf{M}}_g$ can also be described as a finite quotient of some compact complex manifold. Hence we have its orbifold Chern classes as well as orbifold Pontrjagin classes. On the other hand, since $\overline{\mathbf{M}}_g$ is a rational homology manifold, its combinatorial Pontrjagin classes in the sense of Thom are defined.

Problem 3.3 Study the relations between orbifold Chern classes, orbifold Pontrjagin classes and Thom's combinatorial Pontrjagin classes of $\overline{\mathbf{M}}_g$. In particular, study the relation between the corresponding charateristic numbers.

If we look at the basic extension (1) given in section 2, keeping in mind the above discussions together with the Borel vanishing theorem given in [13, 14] concerning the triviality of twisted cohomology of $Sp(2g,\mathbb{Z})$ with coefficients in non-trivial algebraic representations of $Sp(2g,\mathbb{Q})$, we arrive at the following conjecture.

Conjecture 3.4 Any class e_{2i} of even index is non-trivial in the rational cohomology of the Torelli group $\mathcal{I}_g$ for sifficiently large g. Moreover the Sp–invariant part of the rational cohomology of $\mathcal{I}_g$ stabilizes and we have an isomorphism

$$\lim_{g\to\infty} H^*(\mathcal{I}_g;\mathbb{Q})^{Sp} \cong \mathbb{Q}[e_2, e_4, \cdots].$$

At present, even the non-triviality of the first one e_2 is not known. One of the difficulties in proving this lies in the fact that the rational cohomology of $\mathcal{I}_g$ is *infinite* dimensional in general. Mess observed this fact for $g=2,3$ and recently Akita [1] proved that $H^*(\mathcal{I}_g;\mathbb{Q})$ is infinite dimensional for all $g \geq 7$. His argument can be roughly described as follows. He compares the orbifold Euler characteristic of $\mathbf{M}_g$ given by Harer–Zagier in [52] with that of $\mathbf{A}_g$ given by Harder [45] to conclude that the Euler number of $\mathbf{T}_g$, if defined, cannot be an integer because the latter number is much larger than the former one. On the other hand, it seems to be extremely difficult to construct a family of Riemann surfaces such that its monodromy does not act on the homology of the fiber wheras the moduli moves in such a way that the classes e_{2i} are non-trivial (see a recent result of I Smith described in [2] for example). Perhaps completely different approaches to this problem along the lines of works of Jekel [59] or Klein [82] might also be possible.

4 Symplectic representation theory

As was explained in section 2, it is an important method of studying the structure of the mapping class group to combine those of the Siegel modular group $Sp(2g,\mathbb{Z})$ and the Torelli group $\mathcal{I}_g$ together with the action of the former group on the structure of the latter group. More precisely, there arise various representations of the algebraic group $Sp(2g,\mathbb{Q})$ in the study of $\mathcal{M}_g$. For example, the rational homology group $H_\mathbb{Q} = H_1(\Sigma_g;\mathbb{Q})$ of the surface Σ_g is the fundamental representation of $Sp(2g,\mathbb{Q})$ and Johnson's result implies that $H_1(\mathcal{I}_g;\mathbb{Q}) \cong \Lambda^3 H_\mathbb{Q}/H_\mathbb{Q}$ is also a rational representation of it. Hereafter, the representation $\Lambda^3 H_\mathbb{Q}/H_\mathbb{Q}$ will be denoted by $U_\mathbb{Q}$. Thus the classical representation theory of $Sp(2g,\mathbb{Q})$ can play crucial roles.

On the other hand, as was already mentioned in the introduction, Kontsevich [85, 86] used Weyl's classical representation theory to describe invariant tensors of various representation spaces which appear in low dimensional topology in terms of graphs. In this section, we adopt this method to describe invariant tensors of various Sp–modules related to the mapping class group as well as the Torelli group.

As is well known, irreducible representations of $Sp(2g,\mathbb{Q})$ can be described as follows (see a book by Fulton and Harris [29]). Let $\mathfrak{sp}(2g,\mathbb{C})$ be the Lie algebra of $Sp(2g,\mathbb{C})$ and let $\mathfrak{h}$ be its Cartan subalgebra consisting of diagonal matrices. Choose a system of fundamental weights $L_i\colon\ \mathfrak{h}\to\mathbb{R}$ $(i=1,\cdots,g)$ as in [29]. Then for each g–tuple $(a_1,\cdots,a_g)$ of non-negative integers, there exists an irreducible representation with highest weight $(a_1+\cdots+a_g)L_1+(a_2+\cdots+a_g)L_2+\cdots+a_gL_g$. In [29], this representation is denoted by $\Gamma_{a_1,\cdots,a_g}$. In this paper, following [6] we use the notation $[a_1+\cdots+a_g, a_2+\cdots+a_g,\cdots,a_g]$ for it. In short, irreducible representations of $Sp(2g,\mathbb{C})$ are indexed by Young diagrams whose number of rows are less than or equal to g. These representations are all rational representations defined over $\mathbb{Q}$ so that we can consider them as irreducible representations of $Sp(2g,\mathbb{Q})$. For example $H_\mathbb{Q}=\Gamma_1=[1]$, $U_\mathbb{Q}=\Gamma_{0,0,1}=[111]$ (which will be abbreviated by $[1^3]$ and similarly for others with duplications) and $S^kH_\mathbb{Q}=\Gamma_k=[k]$ where $S^kH_\mathbb{Q}$ denotes the k-th symmetric power of $H_\mathbb{Q}$.

Recall from section 2 that $\omega_0\in H^{\otimes 2}$ denotes the symplectic class defined as $\omega_0=\sum_i(x_i\otimes y_i - y_i\otimes x_i)$ for any symplectic basis $x_1,\cdots,x_g,y_1,\cdots,y_g$ of H. As is well known, ω_0 is the generator of $(H_\mathbb{Q}^{\otimes 2})^{Sp}$. Also the intersection pairing $\mu\colon\ H\otimes H\to\mathbb{Q}$ serves as the generator of $\mathrm{Hom}(H_\mathbb{Q}^{\otimes 2},\mathbb{Q})^{Sp}$.

4.1 Invariant tensors of $H_{\mathbb{Q}}^{\otimes 2k}$ and its dual

It is one of the classical results of Weyl that any invariant tensor of $H_{\mathbb{Q}}^{\otimes 2k}$, namely any element of $(H_{\mathbb{Q}}^{\otimes 2k})^{Sp}$ can be described as follows. A *linear chord diagram* C with $2k$ vertices is a decomposition of the set of labeled vertices $\{1,2,\cdots,2k-1,2k\}$ into pairs $\{(i_1,j_1),(i_2,j_2),\cdots,(i_k,j_k)\}$ such that $i_1 < j_1, i_2 < j_2, \cdots, i_k < j_k$ (cf Bar-Natan [10], see also [34]). We connect two vertices in each pair (i_s,j_s) by an edge so that C becomes a graph with k edges. We define $\operatorname{sgn} C$ by

$$\operatorname{sgn} C = \operatorname{sgn}\begin{pmatrix} 1 & 2 & \cdots & 2k-1 & 2k \\ i_1 & j_1 & \cdots & i_k & j_k \end{pmatrix}.$$

It is easy to see that there are exactly $(2k-1)!!$ linear chord diagrams with $2k$ vertices. For each linear chord diagram C, let

$$a_C \in (H_{\mathbb{Q}}^{\otimes 2k})^{Sp}$$

be the invariant tensor defined by permuting the tensor product $(\omega_0)^{\otimes k}$ in such a way that the s-th part $(\omega_0)_s$ goes to $(H_{\mathbb{Q}})_{i_s} \otimes (H_{\mathbb{Q}})_{j_s}$, where $(H_{\mathbb{Q}})_i$ denotes the i-th component of $H_{\mathbb{Q}}^{\otimes 2k}$, and multiplied by the factor $\operatorname{sgn} C$. We also consider the dual element

$$\alpha_C \in \operatorname{Hom}(H_{\mathbb{Q}}^{\otimes 2k}, \mathbb{Q})^{Sp}$$

which is defined by applying the intersection pairing μ on each two components corresponding to pairs (i_s,j_s) of C and multiplied by $\operatorname{sgn} C$. Namely we set

$$\alpha_C(u_1 \otimes \cdots \otimes u_{2k}) = \operatorname{sgn} C \prod_{s=1}^{k} u_{i_s} \cdot u_{j_s} \quad (u_i \in H_{\mathbb{Q}}).$$

Let us write

$$\mathcal{D}^\ell(2k) = \{C_i; i = 1,\cdots,(2k-1)!!\}$$

for the set of all linear chord diagrams with $2k$ vertices.

Lemma 4.1 $\dim(H_{\mathbb{Q}}^{\otimes 2k})^{Sp} = \dim \operatorname{Hom}(H_{\mathbb{Q}}^{\otimes 2k}, \mathbb{Q})^{Sp} = (2k-1)!!$ *for* $k \le g$.

Proof Let $x_1,\cdots,x_g,y_1,\cdots,y_g$ be a symplectic basis of H. There are $2g$ members in this basis while if $k \le g$, then there are only $2k$ $(\le 2g)$ positions in the tensor product $H_{\mathbb{Q}}^{2k}$. It is now a simple matter to construct $(2k-1)!!$ elements ξ_j in $H_{\mathbb{Q}}^{2k}$ such that $\big(\alpha_{C_i}(\xi_j)\big)$ $(C_i \in \mathcal{D}^\ell(2k))$ is the identity matrix. Hence the elements $\{\alpha_{C_i}\}_i$ are linearly independent. By the obvious duality, the Sp–invariant components of tensors $\{a_{C_i}\}_i$ are also linearly independent. □

Remark The stable range of the Sp–*invariant part* of $H_{\mathbb{Q}}^{\otimes 2k}$, which is $k \leq g$, is twice the stable range of the irreducible decomposition of it, which is $k \leq \frac{g}{2}$. A similar statement is true for other Sp–modules related to the mapping class group, eg, $\Lambda^*(\Lambda^3 H_{\mathbb{Q}})$ and $\Lambda^* U_{\mathbb{Q}}$ (see Remark at the end of section 4.2).

Let $C, C' \in \mathcal{D}^{\ell}(2k)$ be two linear chord diagrams with $2k$ vertices. Then the number $\alpha_C(a_{C'})$ is given by

$$\alpha_C(a_{C'}) = \mathrm{sgn}(C, C')(2g)^r$$

where r is the number of connected components of the graph $C \cup C'$ and $\mathrm{sgn}(C, C') = \pm 1$ is suitably defined. If $k \leq g$, then Lemma 4.1 above implies that the matrix $\left(\alpha_{C_i}(a_{C_j})\right)$ is non-singular. If we go into the unstable range, degenerations occur and it seems to be not so easy to analyze them. However, the first degeneration turns out to be remarkably simple and can be described as follows.

Proposition 4.2 *If $g = k - 1$, then the dimension of Sp–invariant part of $H_{\mathbb{Q}}^{\otimes 2k}$ is exactly one less than the stable dimension. Namely*

$$\dim(H_{\mathbb{Q}}^{\otimes 2k})^{Sp} = (2k - 1)!! - 1$$

and the unique linear relation between the elements a_C $(C \in \mathcal{D}^{\ell}(2k))$ is given by

$$\sum_{C \in \mathcal{D}^{\ell}(2k)} a_C = 0.$$

Sketch of proof For $k = 1$ the assertion is empty and for $k = 2$ we can check the assertion by a direct computation. Using the formula for the number $\alpha_C(a_{C'})$ given above, it can be shown that

$$\sum_{C \in \mathcal{D}^{\ell}(2k)} \alpha_{C'}(a_C) = 2^k g(g-1) \cdots (g - k + 1)$$

for any $C' \in \mathcal{D}^{\ell}(2k)$. Hence $\sum_{C \in \mathcal{D}^{\ell}(2k)} a_C = 0$ for $g \leq k - 1$. On the other hand, we can inductively construct $(2k - 1)!! - 1$ elements in $(H_{\mathbb{Q}}^{\otimes 2k})^{Sp}$ which are linearly independent for $g = k - 1$. □

Remark After we had obtained the above Proposition 4.2, a preprint by Mihailovs [101] appeared in which he gives a beautiful basis of $(H_{\mathbb{Q}}^{\otimes 2k})^{Sp}$ *for all genera g*. Members of his basis are linearly ordered and the above element

$\sum_{C\in\mathcal{D}^\ell(2k)} a_C$ appears as the last one for $g=k$. (More precisely, his last element ω^k in his notation is equal to $k!$ times our element above.) In particular, the dimension formula above follows immediately from his result. We expect that we can use his basis in our approach to the Faber's conjecture (see section 6.4 for more details).

4.2 Invariant tensors of $\Lambda^*(\Lambda^3 H_{\mathbb{Q}})$ and $\Lambda^* U_{\mathbb{Q}}$

In our paper [118], we described invariant tensors of $\Lambda^*(\Lambda^3 H_{\mathbb{Q}})$ and $\Lambda^* U_{\mathbb{Q}}$ (or rather those of their duals) in terms of trivalent graphs. It turns out that they are specific cases of Kontsevich's general framework given in [85, 86]. Here we briefly summarize them. These descriptions were utilized in [118, 76] to construct explicit group cocycles for the characteristic classes $e\in H^2(\mathcal{M}_{g,*};\mathbb{Q})$ and $e_i\in H^{2i}(\mathcal{M}_g;\mathbb{Q})$ (see section 6.4 for more details).

As is well known, $\Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}})$ can be considered as a natural quotient as well as a subspace of $H_{\mathbb{Q}}^{\otimes 6k}$. More precisely, let $p\colon H_{\mathbb{Q}}^{\otimes 6k}\to\Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}})$ be the natural projection and let $i\colon \Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}})\to H_{\mathbb{Q}}^{\otimes 6k}$ be the inclusion induced from the embedding

$$\Lambda^3 H_{\mathbb{Q}} \ni u_1\wedge u_2\wedge u_3 \mapsto \sum_\sigma \operatorname{sgn}\sigma\; u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes u_{\sigma(3)} \in H_{\mathbb{Q}}^{\otimes 3}$$

and the similar one $\Lambda^{2k} H_{\mathbb{Q}}^{\otimes 3}\subset H_{\mathbb{Q}}^{\otimes 6k}$, where σ runs through the symmetric group $\mathfrak{S}_3$ of degree 3. Then for each linear chord diagram $C\in\mathcal{D}^\ell(6k)$, we have the corresponding elements

$$p_*(a_C)\in(\Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}}))^{Sp},\quad i^*(\alpha_C)\in \operatorname{Hom}(\Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}}),\mathbb{Q})^{Sp}.$$

Out of each linear chord diagram $C\in\mathcal{D}^\ell(6k)$, let us construct a trivalent graph Γ_C having $2k$ vertices as follows. We group the labeled vertices $\{1,2,\cdots,6k\}$ of C into $2k$ classes $\{1,2,3\},\{4,5,6\},\cdots,\{6k-2,6k-1,6k\}$ and then join the three vertices belonging to each class to a single point. This yields a trivalent graph which we denote by Γ_C. It can be easily seen that if two linear chord diagrams C,C' yield isomorphic trivalent graphs $\Gamma_C,\Gamma_{C'}$, then the corresponding elements coincide

$$p_*(a_C)=p_*(a_{C'}),\quad i^*(\alpha_C)=i^*(\alpha_{C'}).$$

On the other hand, it is clear that we can lift any trivalent graph Γ with $2k$ vertices to a linear chord diagram C such that $\Gamma=\Gamma_C$. Hence to any such trivalent graph Γ, we can associate invariant tensors

$$a_\Gamma\in(\Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}}))^{Sp},\quad \alpha_\Gamma\in\operatorname{Hom}(\Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}}),\mathbb{Q})^{Sp}$$

by setting $a_\Gamma = p_*(a_C)$ and $\alpha_\Gamma = \frac{1}{(2k)!}\, i^*(\alpha_C)$ where $C \in \mathcal{D}^\ell(6k)$ is any lift of Γ.

Now let $\mathcal{G}_{2k}$ be the set of isomorphism classes of *connected* trivalent graphs with $2k$ vertices and let $\mathcal{G} = \coprod_{k\geq 1} \mathcal{G}_{2k}$ be the disjoint union of $\mathcal{G}_{2k}$ for $k \geq 1$. Let $\mathbb{Q}[a_\Gamma; \Gamma \in \mathcal{G}]$ be the polynomial algebra generated by the symbol a_Γ for each $\Gamma \in \mathcal{G}$.

Proposition 4.3 *The correspondence $\mathcal{G}_{2k} \ni \Gamma \mapsto a_\Gamma \in (\Lambda^{2k}(\Lambda^3 H_\mathbb{Q}))^{Sp}$ defines a surjective algebra homomorphism*

$$\mathbb{Q}[a_\Gamma; \Gamma \in \mathcal{G}] \longrightarrow (\Lambda^*(\Lambda^3 H_\mathbb{Q}))^{Sp}$$

which is an isomorphism in degrees $\leq \frac{2g}{3}$. Similarly the correspondence $\mathcal{G}_{2k} \ni \Gamma \mapsto \alpha_\Gamma \in \mathrm{Hom}(\Lambda^{2k}(\Lambda^3 H_\mathbb{Q}), \mathbb{Q})^{Sp}$ defines a surjective algebra homomorphism

$$\mathbb{Q}[\alpha_\Gamma; \Gamma \in \mathcal{G}] \longrightarrow \mathrm{Hom}(\Lambda^*(\Lambda^3 H_\mathbb{Q}), \mathbb{Q})^{Sp}$$

which is an isomorphism in degrees $\leq \frac{2g}{3}$. □

Next we consider invariant tensors of $\Lambda^* U_\mathbb{Q}$ and its dual. We have a natural surjection $p\colon \Lambda^3 H_\mathbb{Q} \to U_\mathbb{Q}$ and this induces a linear map $p_*\colon \Lambda^*(\Lambda^3 H_\mathbb{Q}) \to \Lambda^* U_\mathbb{Q}$. If a trivalent graph $\Gamma \in \mathcal{G}_{2k}$ has a *loop*, namely an edge whose two endpoints are the same, then clearly $p_*(a_\Gamma) = 0$. Thus let $\mathcal{G}^0_{2k}$ be the subset of $\mathcal{G}_{2k}$ consisiting of those graphs *without* loops and let $\mathcal{G}^0 = \coprod_k \mathcal{G}^0_{2k}$. For each element $\Gamma \in \mathcal{G}^0$, let $b_\Gamma = p_*(a_\Gamma)$. Also let $q\colon \Lambda^3 H_\mathbb{Q} \to \Lambda^3 H_\mathbb{Q}$ be the Sp–equivariant linear map defined by $q(\xi) = \xi - \frac{1}{2g-2} C\xi \wedge \omega_0 (\xi \in \Lambda^3 H_\mathbb{Q})$ where $C\colon \Lambda^3 H_\mathbb{Q} \to H_\mathbb{Q}$ is the contraction. Since $q(H_\mathbb{Q}) = 0$, it induces a homomorphism $q\colon U_\mathbb{Q} \to \Lambda^3 H_\mathbb{Q}$ and hence $q\colon \Lambda^{2k} U_\mathbb{Q} \to \Lambda^{2k}(\Lambda^3 H_\mathbb{Q})$. Now for each element $\Gamma \in \mathcal{G}^0_{2k}$, let $\beta_\Gamma\colon \Lambda^{2k} U_\mathbb{Q} \to \mathbb{Q}$ be defined by $\beta_\Gamma = \alpha_\Gamma \circ q$.

Proposition 4.4 *The correspondence $\mathcal{G}^0_{2k} \ni \Gamma \mapsto b_\Gamma \in (\Lambda^{2k} U_\mathbb{Q})^{Sp}$ defines a surjective algebra homomorphism*

$$\mathbb{Q}[b_\Gamma; \Gamma \in \mathcal{G}^0] \longrightarrow (\Lambda^* U_\mathbb{Q})^{Sp}$$

which is an isomorphism in degrees $\leq \frac{2g}{3}$. Similarly the correspondence $\mathcal{G}^0_{2k} \ni \Gamma \mapsto \beta_\Gamma \in \mathrm{Hom}(\Lambda^{2k} U_\mathbb{Q}, \mathbb{Q})^{Sp}$ defines a surjective algebra homomorphism

$$\mathbb{Q}[\beta_\Gamma; \Gamma \in \mathcal{G}] \longrightarrow \mathrm{Hom}(\Lambda^* U_\mathbb{Q}, \mathbb{Q})^{Sp}$$

which is an isomorphism in degrees $\leq \frac{2g}{3}$. □

Since $\Lambda^3 H_{\mathbb{Q}} \cong U_{\mathbb{Q}} \oplus H_{\mathbb{Q}}$, there is a natural decomposition

$$\Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}}) \cong \Lambda^{2k} U_{\mathbb{Q}} \oplus (\Lambda^{2k-1} U_{\mathbb{Q}} \otimes H_{\mathbb{Q}}) \oplus \cdots \oplus (U_{\mathbb{Q}} \otimes \Lambda^{2k-1} H_{\mathbb{Q}}) \oplus \Lambda^{2k} H_{\mathbb{Q}}$$

and it induces that of the corresponding Sp–invariant parts. Hence we can also decompose the space of invariant tensors of $\Lambda^*(\Lambda^3 H_{\mathbb{Q}})$ and its dual according to the above splitting. In fact, Proposition 4.4 gives the $\Lambda^* U_{\mathbb{Q}}$–part of Proposition 4.3. We can give formulas for other parts of the above decomposition which are described in terms of numbers of loops of trivalent graphs. We refer to [77] for details.

Remark As is described in the above propositions, the stable range of the Sp–*invariant part* of $\Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}})$ and $\Lambda^{2k} U_{\mathbb{Q}}$ is $2k \leq \frac{2g}{3}$. This range coincides with Harer's improved stability range of the homology of the mapping class group given in [50]. It turns out that this is far more than just an accident. In fact, this fact will play an essential role in our approach to the Faber's conjecture (see section 6.4 and [120] for details).

4.3 Invariant tensors of $\mathfrak{h}_{g,1}$

In this subsection, we fix a genus g and we write $\mathcal{L}_{g,1} = \oplus_k \mathcal{L}_{g,1}(k)$ for the free Lie algebra generated by H. Also we consider the module

$$\mathfrak{h}_{g,1}(k) = \mathrm{Ker}(H \otimes \mathcal{L}_{g,1}(k+1) \to \mathcal{L}_{g,1}(k+2))$$

which is the degree k summand of the Lie algebra consisting of derivations of $\mathcal{L}_{g,1}$ which kill the symplectic class $\omega_0 \in \mathcal{L}_{g,1}(2)$ (see the next section section 5 for details). We simply write $\mathcal{L}_{g,1}^{\mathbb{Q}}$ and $\mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$ for $\mathcal{L}_{g,1}(k) \otimes \mathbb{Q}$ and $\mathfrak{h}_{g,1}(k) \otimes \mathbb{Q}$ respectively. We show that invariant tensors of $\mathfrak{h}_{g,1}^{\mathbb{Q}}(2k)$ or its dual, namely any element of $\mathfrak{h}_{g,1}^{\mathbb{Q}}(2k)^{Sp}$ or $\mathrm{Hom}(\mathfrak{h}_{g,1}^{\mathbb{Q}}(2k), \mathbb{Q})^{Sp}$ can be represented by a linear combination of chord diagrams with $(2k+2)$ vertices. Here a *chord diagram* with $2k$ vertices is a partition of $2k$ vertices lying on a circle into k pairs where each pair is connected by a chord. Chord diagrams already appeared in the theory of Vassiliev knot invariants (see [10]) and they played an important role. In the following, we will see that they can play another important role also in our theory.

To show this, we recall a well known characterization of elements of $\mathcal{L}_{g,1}^{\mathbb{Q}}(k)$ in $H_{\mathbb{Q}}^{\otimes k}$. There are several such characterizations which are given in terms of various projections $H_{\mathbb{C}}^{\otimes k} \to \mathcal{L}_{g,1}(k) \otimes \mathbb{C}$ (see [135]). Here we adopt the following one.

Lemma 4.5 *Let $\mathfrak{S}_k$ be the symmetric group of degree k and let $\sigma_i = (12\cdots i) \in \mathfrak{S}_k$ be the cyclic permutation. Let $p_k = (1-\sigma_k)(1-\sigma_{k-1})\cdots(1-\sigma_2) \in \mathbb{Z}[\mathfrak{S}_k]$ which acts linearly on $H_{\mathbb{Q}}^{\otimes k}$. Then $p_k^2 = kp_k$ and an element $\xi \in H_{\mathbb{Q}}^{\otimes k}$ belongs to $\mathcal{L}_{g,1}^{\mathbb{Q}}(k)$ if and only if $p_k(\xi) = k\xi$. Moreover $\mathcal{L}_{g,1}^{\mathbb{Q}}(k) = \operatorname{Im} p_k$.* □

If we consider $\mathcal{L}_{g,1}^{\mathbb{Q}}(k+1)$ as a subspace of $H_{\mathbb{Q}}^{\otimes(k+1)}$, then the bracket operation

$$H_{\mathbb{Q}} \otimes \mathcal{L}_{g,1}^{\mathbb{Q}}(k+1) \longrightarrow \mathcal{L}_{g,1}^{\mathbb{Q}}(k+2)$$

is simply given by the correspondence $u \otimes \xi \mapsto u \otimes \xi - \xi \otimes u$ ($u \in H_{\mathbb{Q}}, \xi \in \mathcal{L}_{g,1}^{\mathbb{Q}}(k+1)$). Hence it is easy to deduce the following characterization of $\mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$ inside $H_{\mathbb{Q}}^{\otimes(k+2)}$.

Proposition 4.6 *An element $\xi \in H_{\mathbb{Q}}^{\otimes(k+2)}$ belongs to $\mathfrak{h}_{g,1}^{\mathbb{Q}}(k) \subset H_{\mathbb{Q}}^{\otimes(k+2)}$ if and only if the following two conditions are satisfied. (i) $(1 \otimes p_{k+1})\xi = (k+1)\xi$ and (ii) $\sigma_{k+2}\xi = \xi$.* □

We can construct a basis of $(H_{\mathbb{Q}} \otimes \mathcal{L}_{g,1}^{\mathbb{Q}}(2k+1))^{Sp}$ as follows. Recall that we write $\mathcal{D}^\ell(2k)$ for the set of linear chord diagrams with $2k$ vertices so that it gives a basis of $(H_{\mathbb{Q}}^{\otimes 2k})^{Sp}$ for $k \leq g$ (see Lemma 4.1). By Lemma 4.5, we have

$$H_{\mathbb{Q}} \otimes \mathcal{L}_{g,1}^{\mathbb{Q}}(2k+1) = \operatorname{Im}(1 \otimes p_{2k+1})$$

where we consider $1 \otimes p_{2k+1}$ as an endomorphism of $H_{\mathbb{Q}} \otimes (H_{\mathbb{Q}} \otimes H_{\mathbb{Q}}^{\otimes 2k})$. Let C_0 be the edge which connects the first two of the $(2k+2)$ vertices corresponding to $H_{\mathbb{Q}} \otimes (H_{\mathbb{Q}} \otimes H_{\mathbb{Q}}^{\otimes 2k})$. For each element $C \in \mathcal{D}^\ell(2k)$, consider the disjoint union $\widetilde{C} = C_0 \coprod C$ which is a linear chord diagram with $(2k+2)$ vertices. Hence we have the corresponding invariant tensor

$$a_{\widetilde{C}} \in (H_{\mathbb{Q}} \otimes (H_{\mathbb{Q}} \otimes H_{\mathbb{Q}}^{\otimes 2k}))^{Sp}.$$

Let $\ell_C = 1 \otimes p_{2k+1}(a_{\widetilde{C}})$. Then by Proposition 4.6, ℓ_C is an element of $(H_{\mathbb{Q}} \otimes \mathcal{L}_{g,1}^{\mathbb{Q}}(2k+1))^{Sp}$.

Proposition 4.7 *If $k \leq g$, then the set of elements $\{\ell_C; C \in \mathcal{D}^\ell(2k)\}$ forms a basis of invariant tensors of $H_{\mathbb{Q}} \otimes \mathcal{L}_{g,1}^{\mathbb{Q}}(2k+1)$. In particular*

$$\dim(H_{\mathbb{Q}} \otimes \mathcal{L}_{g,1}^{\mathbb{Q}}(2k+1))^{Sp} = (2k-1)!!$$

Sketch of Proof It can be shown that the elements in $\{\ell_C; C \in \mathcal{D}^\ell(2k)\}$ are linearly independent because if we express ℓ_C as a linear combination of the standard basis of $(H_{\mathbb{Q}}^{\otimes(2k+2)})^{Sp}$ given in Lemma 4.1, then we find $\ell_C = a_{\widetilde{C}} +$ other terms. On the other hand, we can show that the projection under $1 \otimes p_{2k+1}$ of any member of this standard basis can be expressed as a linear combination of ℓ_C. □

Now we consider invariant tensors of $\mathfrak{h}_{g,1}^{\mathbb{Q}}(2k)$. Associated to any element $C \in \mathcal{D}^\ell(2k)$ we have the corresponding invariant tensor $\ell_C \in (H_{\mathbb{Q}} \otimes \mathcal{L}_{g,1}^{\mathbb{Q}}(2k+1))^{Sp}$. Consider the element

$$\xi_C = \sum_{i=1}^{2k+2} \sigma_{2k+2}^i \ell_C.$$

By Proposition 4.6 (in particular condition (ii)), ξ_C belongs to $\mathfrak{h}_{g,1}^{\mathbb{Q}}(2k)$ and it is clear from the above argument that these elements span the whole invariant space $\mathfrak{h}_{g,1}^{\mathbb{Q}}(2k)^{Sp}$. More precisely the cyclic group $\mathbb{Z}/(2k+2)$ of order $2k+2$ acts naturally on $(H_{\mathbb{Q}} \otimes \mathcal{L}_{g,1}^{\mathbb{Q}}(2k+1))^{Sp}$ and $\mathfrak{h}_{g,1}^{\mathbb{Q}}(2k)^{Sp}$ is nothing but the invariant subspace of this action. Although there does not seem to exist any simple formula for the dimension of this invariant subspace, this procedure gives a method of enumerating all the elements of it.

Here is another approach to this problem which might be practically better than the above, in particular in the dual setting. We simply use two conditions (i), (ii) in Proposition 4.6 in the opposite way. Namely we first consider condition (ii). Recall that any linear chord diagram C with $(2k+2)$ vertices gives rise to an Sp–invariant map

$$\alpha_C\colon\ H_{\mathbb{Q}}^{\otimes(2k+2)} \longrightarrow \mathbb{Q}.$$

Let us write C_i for $\sigma_{2k+2}^i C$ $(i = 1, \cdots, 2k+2)$. Then, in view of condition (ii) above, the restrictions of α_{C_i} to the subspace $\mathfrak{h}_{g,1}^{\mathbb{Q}}(2k) \subset H_{\mathbb{Q}}^{\otimes(2k+2)}$ are equal to each other for all i. This means that, instead of linear chord diagram C, we may assume that all of the vertices of C are arranged on a circle. But then we obtain a usual chord diagram. Thus we can say that any chord diagram C with $(2k+2)$ vertices defines an element of $\mathrm{Hom}(\mathfrak{h}_{g,1}^{\mathbb{Q}}(2k), \mathbb{Q})^{Sp}$. In the case of Vassiliev knot invariant, the linear space spanned by chord diagrams with $2k$ vertices, which is denoted by $\mathcal{G}_k\mathcal{D}^c$ in [10], modulo the (4T) relation serves as the set of Vassiliev invariants of order k. In our case, the linear space spanned by chord diagrams with $(2k+2)$ vertices, namely $\mathcal{G}_{k+1}\mathcal{D}^c$, modulo the relations coming from condition (i) above can be identified with $\mathrm{Hom}(\mathfrak{h}_{g,1}^{\mathbb{Q}}(2k), \mathbb{Q})^{Sp}$ (and also its dual). In particular, we have a surjection

$$\mathcal{G}_{k+1}\mathcal{D}^c \longrightarrow \mathrm{Hom}(\mathfrak{h}_{g,1}^{\mathbb{Q}}(2k), \mathbb{Q})^{Sp}.$$

In section 6.2, we will show experimental results which have been obtained by explicit computations applying this method.

5 Graded Lie algebras related to the mapping class group

In this section, we introduce various graded Lie algebras which are related to the mapping class group. We begin by recalling the definition of the Johnson homomorphisms briefly (see [63, 115, 118] for details). Let $\mathcal{M}_{g,1}$ be the mapping class group of Σ_g relative to an embedded disk $D^2 \subset \Sigma_g$ and let $\mathcal{M}_{g,*}$ be the mapping class group of Σ_g relative to the base point $* \in D^2$. We write Σ_g^0 for $\Sigma_g \setminus \mathrm{Int}\, D^2$ so that $\pi_1\Sigma_g^0$ is a free group of rank $2g$. Let $\zeta \in \pi_1\Sigma_g^0$ be the element represented by a simple closed curve on Σ_g^0 which is parallel to the boundary. Then as is well known, we have natural isomorphisms due originally to Nielsen

$$\mathcal{M}_g \cong \mathrm{Out}_+\, \pi_1\Sigma_g, \quad \mathcal{M}_{g,*} \cong \mathrm{Aut}_+\, \pi_1\Sigma_g$$
$$\mathcal{M}_{g,1} \cong \{\varphi \in \mathrm{Aut}\, \pi_1\Sigma_g^0; \varphi(\zeta) = \zeta\}.$$

By virtue of this, any filtration on the fundamental groups of surfaces induces those of the corresponding mapping class groups. In particular, the lower central series induces natural filtrations. More precisely, let $\varGamma_k(G)$ denote the k-th term in the lower central series of a group G, where $\varGamma_0(G) = G$ and $\varGamma_k(G) = [G, \varGamma_{k-1}(G)]$ for $k \geq 1$. We set

$$\mathcal{M}_{g,1}(k) = \{\varphi \in \mathcal{M}_{g,1}; \varphi(\gamma)\gamma^{-1} \in \varGamma_k(\pi_1\Sigma_g^0) \text{ for any } \gamma \in \pi_1\Sigma_g^0\}$$
$$\mathcal{M}_{g,*}(k) = \{\varphi \in \mathcal{M}_{g,*}; \varphi(\gamma)\gamma^{-1} \in \varGamma_k(\pi_1\Sigma_g) \text{ for any } \gamma \in \pi_1\Sigma_g\}$$

and

$$\mathcal{M}_g(k) = \pi(\mathcal{M}_{g,*}(k))$$

where $\pi\colon \mathcal{M}_{g,*} \to \mathcal{M}_g$ is the natural projection. Thus we obtain a natural filtration $\{\mathcal{M}(k)\}_k$ on each of the three types of mapping class groups. It is easy to see that the first one $\mathcal{M}(1)$ is nothing but the Torelli group, namely $\mathcal{I}_{g,1}, \mathcal{I}_{g,*}$ or $\mathcal{I}_g$. Now we can say that the Johnson homomorphism is the one which describes the associated graded quotients of the mapping class groups, with respect to these fitrations, explicitly in terms of derivations of graded Lie algebras associated to the lower central series of the fundamental groups of surfaces.

Let us simply write H for $H_1(\Sigma_g;\mathbb{Z})$ as before and let $\mathcal{L}_{g,1}=\oplus_k\mathcal{L}_{g,1}(k)$ be the free graded Lie algebra generated by H. As is well known, $\mathcal{L}_{g,1}$ is the graded Lie algebra associated to the lower central series of $\pi_1\Sigma_g^0$, namely we have natural isomorphisms $\mathcal{L}_{g,1}(k)\cong\Gamma_{k-1}(\pi_1\Sigma_g^0)/\Gamma_k(\pi_1\Sigma_g^0)$ (see [95]). Let $\omega_0\in\Lambda^2H=\mathcal{L}_{g,1}(2)$ be the symplectic class and let $I=\oplus_{k\geq 2}I_k$ be the ideal of $\mathcal{L}_{g,1}$ generated by ω_0. Then a result of Labute [87] says that the quotient Lie algebra $\mathcal{L}_g=\mathcal{L}_{g,1}/I$ serves as the graded Lie algebra associated to the lower central series of $\pi_1\Sigma_g$. Now we define

$$\begin{aligned}\mathfrak{h}_{g,1}(k)&=\mathrm{Ker}(H\otimes\mathcal{L}_{g,1}(k+1)\to\mathcal{L}_{g,1}(k+2))\\&\cong\mathrm{Ker}(\mathrm{Hom}(H,\mathcal{L}_{g,1}(k+1))\to\mathcal{L}_{g,1}(k+2))\\\mathfrak{h}_{g,*}(k)&=\mathrm{Ker}(H\otimes\mathcal{L}_{g}(k+1)\to\mathcal{L}_{g}(k+2))\\&\cong\mathrm{Ker}(\mathrm{Hom}(H,\mathcal{L}_{g}(k+1))\to\mathcal{L}_{g}(k+2))\end{aligned}$$

where the second isomorphisms in each of the terms above are induced by the Poincaré duality $H^*\cong H$. In our previous papers [115, 118], $\mathcal{L}_{g,1},\mathcal{L}_g$ have been denoted by $\mathcal{L}^0,\mathcal{L}$ and also $\mathfrak{h}_{g,1}(k),\mathfrak{h}_{g,*}(k)$ have been denoted by $\mathcal{H}_k^0,\mathcal{H}_k$, respectively.

Then the k-th Johnson homomorphisms

$$\begin{aligned}\tau_{g,1}(k)&\colon\ \mathcal{M}_{g,1}(k)\longrightarrow\mathfrak{h}_{g,1}(k)\\\tau_{g,*}(k)&\colon\ \mathcal{M}_{g,*}(k)\longrightarrow\mathfrak{h}_{g,*}(k)\end{aligned}$$

are defined by the correspondence

$$\mathcal{M}_{g,1}(k)\ni\varphi\mapsto\tau_{g,1}(k)(\varphi)=\{[\gamma]\mapsto[\varphi(\gamma)\gamma^{-1}]\}\in\mathrm{Hom}(H,\mathcal{L}_{g,1}(k+1))$$

($\gamma\in\pi_1\Sigma_g^0$) for $\mathcal{M}_{g,1}$ and similarly for $\mathcal{M}_{g,*}$. Here $[\gamma]\in H$ denotes the homology class of $\gamma\in\pi_1\Sigma_g^0$ and $[\varphi(\gamma)\gamma^{-1}]$ denotes the class of $\varphi(\gamma)\gamma^{-1}\in\Gamma_k(\pi_1\Sigma_g^0)$ in $\mathcal{L}_{g,1}(k+1)=\Gamma_k(\pi_1\Sigma_g^0)/\Gamma_{k+1}(\pi_1\Sigma_g^0)$. It can be shown that

$$\mathrm{Ker}\,\tau_{g,1}(k)=\mathcal{M}_{g,1}(k+1),\quad\mathrm{Ker}\,\tau_{g,*}(k)=\mathcal{M}_{g,*}(k+1)$$

so that we have isomorphisms

$$\begin{aligned}\mathcal{M}_{g,1}(k)/\mathcal{M}_{g,1}(k+1)&\cong\mathrm{Im}\,\tau_{g,1}(k)\subset\mathfrak{h}_{g,1}(k)\\\mathcal{M}_{g,*}(k)/\mathcal{M}_{g,*}(k+1)&\cong\mathrm{Im}\,\tau_{g,*}(k)\subset\mathfrak{h}_{g,*}(k).\end{aligned}$$

Next we consider the filtration $\{\mathcal{M}_g(k)\}_k$ of the usual mapping class group $\mathcal{M}_g$. As is mentioned above, $\mathcal{M}_g(k)=\pi(\mathcal{M}_{g,*}(k))$ and it was proved in [5] that $\pi_1\Sigma_g \cap \mathcal{M}_{g,*}(k) = \Gamma_{k-1}(\pi_1\Sigma_g)$. Hence we have a natural injection $\mathcal{L}_g(k) \subset \mathfrak{h}_{g,*}(k)$. We define the k-th Johnson homomorphism

$$\tau_g(k)\colon\ \mathcal{M}_g(k) \longrightarrow \mathfrak{h}_g(k)$$

by setting $\tau_g(k)(\varphi) = \tau_{g,*}(k)(\tilde{\varphi}) \mod \mathcal{L}_g(k)$ $(\varphi \in \mathcal{M}_g(k))$ where $\mathfrak{h}_g(k) = \mathfrak{h}_{g,*}(k)/\mathcal{L}_g(k)$ and $\tilde{\varphi} \in \mathcal{M}_{g,*}(k)$ is any lift of φ.

Thus we obtain three graded modules

$$\mathfrak{h}_{g,1} = \bigoplus_{k=1}^{\infty} \mathfrak{h}_{g,1}(k), \quad \mathfrak{h}_{g,*} = \bigoplus_{k=1}^{\infty} \mathfrak{h}_{g,*}(k), \quad \mathfrak{h}_g = \bigoplus_{k=1}^{\infty} \mathfrak{h}_g(k)$$

and it turns out that they have natural structures of graded Lie algebras over $\mathbb{Z}$. The relations between these three graded Lie algebras $\mathfrak{h}_{g,1}, \mathfrak{h}_{g,*}, \mathfrak{h}_g$ are described simply by the following two short exact sequences

$$0 \longrightarrow \mathfrak{j}_{g,1} \longrightarrow \mathfrak{h}_{g,1} \longrightarrow \mathfrak{h}_{g,*} \longrightarrow 0$$
$$0 \longrightarrow \mathcal{L}_g \longrightarrow \mathfrak{h}_{g,*} \longrightarrow \mathfrak{h}_g \longrightarrow 0$$

where

$$\mathfrak{j}_{g,1} = \bigoplus_{k=2}^{\infty} \mathfrak{j}_{g,1}(k), \quad \mathfrak{j}_{g,1}(k) = \mathrm{Ker}(\mathrm{Hom}(H, I_{k+1}) \rightarrow I_{k+2}))$$

and $\mathcal{L}_g = \mathcal{L}_{g,1}/I$ is the graded Lie algebra associated to the lower central series of $\pi_1\Sigma_g$ as before.

Sometimes it is useful to consider the tensor products with $\mathbb{Q}$ of the modules appearing above. Let us denote them by attaching a superscript $\mathbb{Q}$ to the original $\mathbb{Z}$–form. For example $\mathcal{L}_{g,1}^{\mathbb{Q}} = \mathcal{L}_{g,1} \otimes \mathbb{Q}$ and $\mathfrak{h}_{g,1}^{\mathbb{Q}} = \mathfrak{h}_{g,1} \otimes \mathbb{Q}$ which we already used in section 4. In these terminologies, $\mathfrak{h}_{g,1}^{\mathbb{Q}}, \mathfrak{h}_{g,*}^{\mathbb{Q}}$ are nothing but the graded Lie algebras consisting of derivations of $\mathcal{L}_{g,1}^{\mathbb{Q}}, \mathcal{L}_g^{\mathbb{Q}}$ with positive degrees which kill the symplectic class ω_0 and $\mathfrak{h}_g^{\mathbb{Q}}$ is equal to the quotient of $\mathfrak{h}_{g,*}^{\mathbb{Q}}$ by inner derivations. We omit the degree 0 part $\mathfrak{h}_{g,1}^{\mathbb{Q}}(0) = \mathfrak{h}_{g,*}^{\mathbb{Q}}(0) = \mathfrak{h}_g^{\mathbb{Q}}(0) = \mathfrak{sp}(2g, \mathbb{Q})$ because it is the Lie algebra of the rational form of $\mathcal{M}_{g,1}/\mathcal{M}_{g,1}(1) = \mathcal{M}_{g,*}/\mathcal{M}_{g,*}(1) = \mathcal{M}_g/\mathcal{M}_g(1) = Sp(2g, \mathbb{Z})$.

Now consider projective limits of nilpotent groups

$$\mathcal{N}_{g,1} = \varprojlim_{k \geq 1} \mathcal{I}_{g,1}/\mathcal{M}_{g,1}(k), \quad \mathcal{N}_{g,*} = \varprojlim_{k \geq 1} \mathcal{I}_{g,*}/\mathcal{M}_{g,*}(k)$$
$$\mathcal{N}_g = \varprojlim_{k \geq 1} \mathcal{I}_g/\mathcal{M}_g(k)$$

which are associated to the filtrations $\mathcal{M}(k)$ on the corresponding mapping class groups. We can tensor these groups with $\mathbb{Q}$ to obtain pronilpotent Lie groups $\mathcal{N}_{g,1}^{\mathbb{Q}}, \mathcal{N}_{g,*}^{\mathbb{Q}}, \mathcal{N}_g^{\mathbb{Q}}$. Let us write $\mathfrak{n}_{g,1}, \mathfrak{n}_{g,*}, \mathfrak{n}_g$ for their Lie algebras and also let $\mathrm{Gr}\,\mathfrak{n}_{g,1}, \mathrm{Gr}\,\mathfrak{n}_{g,*}, \mathrm{Gr}\,\mathfrak{n}_g$ be their associated graded Lie algebras, respectively. Then the Johnson homomorphism induces embeddings of graded Lie algebras

$$\mathrm{Gr}\,\mathfrak{n}_{g,1} \subset \mathfrak{h}_{g,1}^{\mathbb{Q}}, \quad \mathrm{Gr}\,\mathfrak{n}_{g,*} \subset \mathfrak{h}_{g,*}^{\mathbb{Q}}, \quad \mathrm{Gr}\,\mathfrak{n}_g \subset \mathfrak{h}_g^{\mathbb{Q}}$$

(in fact, these embeddings are defined at the level of $\mathbb{Z}$–forms). More precisely, we can identify $\mathfrak{n}(k)$ with $\mathrm{Im}\,\tau^{\mathbb{Q}}(k)$ so that we can write $\mathrm{Gr}\,\mathfrak{n} = \mathrm{Im}\,\tau^{\mathbb{Q}} \subset \mathfrak{h}^{\mathbb{Q}}$ for any type of decorations $\{g,1\}, \{g,*\}, g$. It is a very important problem to idetify these Lie subalgebras inside the Lie algebras of derivations.

The cohomological structure of the first type $\mathfrak{h}_{g,1}^{\mathbb{Q}}$ of the above three graded Lie algebras was investigated by Kontsevich in his celebrated papers [85, 86] where he considered three types of Lie algebras ℓ_g, a_g, c_g which consist of derivations of certain Lie, associative, and commutative algebras. In fact our $\mathfrak{h}_{g,1}^{\mathbb{Q}}$ is nothing but the Lie subalgebra ℓ_g^+ of ℓ_g consisting of elements with positive degrees. There are natural injections $\mathfrak{h}_{g,1} \subset \mathfrak{h}_{g+1,1}$ so that we can make the direct limit

$$\mathfrak{h}_\infty^{\mathbb{Q}} = \lim_{g\to\infty} \mathfrak{h}_{g,1}^{\mathbb{Q}}$$

which is equal to the positive part ℓ_∞^+ of ℓ_∞ in Kontsevich's notation. In section 6.5, we will apply one of Kontsevich's results in the above cited papers to our $\mathfrak{h}_\infty^{\mathbb{Q}}$ and obtain definitions of certain (co)homology classes of outer automorphism groups $\mathrm{Out}\,F_n$ of free groups F_n of rank $n \geq 2$.

The second graded Lie algebra, which we consider in this paper, is the Torelli Lie algebra which is, by definition, the Malcev Lie algebra of the Torelli group. The structure of the Torelli Lie algebra has been extensively studied by Hain in [39, 41]. Here we summarize his results briefly for later use in section 6 (see the above papers for details). We write $\mathfrak{t}_{g,1}, \mathfrak{t}_{g,*}, \mathfrak{t}_g$ for the Torelli Lie algebras which correspond to three types of the Torelli groups $\mathcal{I}_{g,1}, \mathcal{I}_{g,*}, \mathcal{I}_g$, respectively (Hain uses the notation $\mathfrak{t}_g^1$ for $\mathfrak{t}_{g,*}$).

In the above, we considered certain surjective homomorphisms

$$\mathcal{I}_{g,1} \longrightarrow \mathcal{N}_{g,1}, \quad \mathcal{I}_{g,*} \longrightarrow \mathcal{N}_{g,*}, \quad \mathcal{I}_g \longrightarrow \mathcal{N}_g$$

from each type of the Torelli groups to a tower of torsion free nilpotent groups. Roughly speaking, the Malcev completion of the Torelli group (or more generally of any finitely generated group) is defined to be the projective limit of such homomorphisms. Since any finitely generated torsion free nilpotent group N can be canonically embedded into its Malcev completion $N \otimes \mathbb{Q}$ which is a Lie group over $\mathbb{Q}$, the Malcev completion of the Torelli groups can be described by certain homomorphisms

$$\mathcal{I}_{g,1} \longrightarrow \mathcal{T}_{g,1}, \quad \mathcal{I}_{g,*} \longrightarrow \mathcal{T}_{g,*}, \quad \mathcal{I}_g \longrightarrow \mathcal{T}_g$$

from the Torelli groups into pronilpotent Lie groups over $\mathbb{Q}$. They are characterized by the universal property that for any homomorphism $\rho\colon \mathcal{I}\to\mathcal{N}$ from the Torelli group into a pronilpotent group $\mathcal{N}$, there exists a unique homomorphism $\bar{\rho}\colon \mathcal{T}\to\mathcal{N}$ such that $\rho = \bar{\rho}\circ\mu$ where $\mu\colon \mathcal{I}\to\mathcal{T}$ is the homomorphism given above (we omit the subscripts). Since any nilpotent Lie group is determined by its Lie algebra, the pronilpotent group $\mathcal{T}$ is determined by its Lie algebra $\mathfrak{t}$ which is a pronilpotent Lie algebra over $\mathbb{Q}$. These are the definitions of the pronilpotent Lie algebras $\mathfrak{t}_{g,1}, \mathfrak{t}_{g,*}, \mathfrak{t}_g$ which we would like to call the Torelli Lie algebras. Let $\mathrm{Gr}\,\mathfrak{t} = \oplus_k \mathfrak{t}(k)$ be the graded Lie algebra associated to the lower central series of $\mathfrak{t}$ and also let $\mathrm{Gr}\,\mathcal{I}$ be the graded Lie algebra (over $\mathbb{Z}$) associated to the lower central series of the Torelli group $\mathcal{I}$. Then a general fact about the Malcev completion implies that there is a natural isomorphism $\mathrm{Gr}\,\mathfrak{t} \cong (\mathrm{Gr}\,\mathcal{I})\otimes\mathbb{Q}$. In particular, we have an isomorphism

$$\mathfrak{t}(k) \cong (\Gamma_{k-1}(\mathcal{I})/\Gamma_k(\mathcal{I}))\otimes\mathbb{Q}.$$

Now by the universal property of the Malcev completion, there is a uniquely defined homomorphism $\mathcal{T}\to\mathcal{N}$ which induces a morphism $\mathfrak{t}\to\mathfrak{n}$. This induces a homomorphism $\mathrm{Gr}\,\mathfrak{t}\to\mathrm{Gr}\,\mathfrak{n}$ of the associated graded Lie algebras. Thus we obtain homomorphisms

$$\mathfrak{t}_{g,1}(k)\to\mathfrak{n}_{g,1}(k)\subset\mathfrak{h}^{\mathbb{Q}}_{g,1}(k),\quad \mathfrak{t}_{g,*}(k)\to\mathfrak{n}_{g,*}(k)\subset\mathfrak{h}^{\mathbb{Q}}_{g,*}(k),$$
$$\mathfrak{t}_g(k)\to\mathfrak{n}_g(k)\subset\mathfrak{h}^{\mathbb{Q}}_g(k).$$

In fact, Johnson already observed in [63] that the $(k-1)$-th term $\Gamma_{k-1}(\mathcal{I}_{g,1})$ of the lower central series of $\mathcal{I}_{g,1}$ is contained in $\mathcal{M}_{g,1}(k)$ so that there is a natural homomorphism

$$\Gamma_{k-1}(\mathcal{I}_{g,1})/\Gamma_k(\mathcal{I}_{g,1})\longrightarrow\mathcal{M}_{g,1}(k)/\mathcal{M}_{g,1}(k+1)\subset\mathfrak{h}_{g,1}(k)$$

which is defined over $\mathbb{Z}$.

The third (and the final) Lie algebra, denoted by $\mathfrak{u}_g$, is the one introduced by Hain in [39]. It lies, in some sense, between the Torelli Lie algebra $\mathfrak{t}$ and $\mathfrak{n}$. We have the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{I}_g & \longrightarrow & \mathcal{M}_g & \longrightarrow & Sp(2g,\mathbb{Z}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{N}_g\otimes\mathbb{Q} & \longrightarrow & \mathcal{G}_g & \longrightarrow & Sp(2g,\mathbb{Q}) & \longrightarrow & 1 \end{array} \tag{6}$$

where $\mathcal{G}_g$ is the Zariski closure of the image of $\varprojlim \mathcal{M}_g/\mathcal{M}_g(k)$ in the automorphism group of the Malcev Lie algebra of $\pi_1\Sigma_g$. Hain applied the relative Malcev completion (see [42]), which is due to Deligne, to the mapping class group $\mathcal{M}_g$ relative to the classical representation $\mathcal{M}_g \to Sp(2g, \mathbb{Q})$. Roughly speaking, it is the projective limit of representations like in (6) where $\mathcal{G}_g$ is replaced by an algebraic group defined over $\mathbb{Q}$ such that the image of $\mathcal{M}_g$ there is Zariski dense and $N_g \otimes \mathbb{Q}$ is replaced by a unipotent subgroup. It has the form

$$1 \longrightarrow \mathcal{U}_g \longrightarrow \mathcal{E}_g \longrightarrow Sp(2g, \mathbb{Q}) \longrightarrow 1$$

where $\mathcal{E}_g$ is a proalgebraic group and $\mathcal{U}_g$ is a prounipotent group. $\mathcal{U}_g$ is called the prounipotent radical of the relative Malcev completion of the mapping class group. Let $\mathfrak{u}_g$ be the Lie algebra of $\mathcal{U}_g$. There are also decorated versions of these groups. By the definition, we have canonical homomorphisms

$$\mathcal{T}_g \longrightarrow \mathcal{U}_g \longrightarrow \mathcal{N} \otimes \mathbb{Q}$$

and their Lie algebra version

$$\mathfrak{t}_g \longrightarrow \mathfrak{u}_g \longrightarrow \mathfrak{n}_g.$$

Now we can state two fundamental results of Hain as follows. The first one gives a complete description of the relation between the Lie algebras $\mathfrak{t}$ and $\mathfrak{u}$.

Theorem 5.1 (Hain [39]) *Assume that $g \geq 3$. Then the natural homomorphisms $\mathcal{T}_{g,1} \to \mathcal{U}_{g,1}$, $\mathcal{T}_{g,*} \to \mathcal{U}_{g,*}$, $\mathcal{T}_g \to \mathcal{U}_g$ are all surjective and the kernel of each is a central subgroup isomorphic to $\mathbb{Q}$.* □

Equivalent statement can be given for the associated Lie algebras $\mathfrak{t}, \mathfrak{u}$. The second result is the explicit presentation of the Torelli Lie algebra $\mathfrak{t}$. Let $U = \Lambda^3 H/H$ and let $U_{\mathbb{Q}}$ be $U \otimes \mathbb{Q}$ as before. Then the Johnson homomorphism

$$\tau_g^{\mathbb{Q}}(1)\colon\ \mathcal{I}_g \longrightarrow U_{\mathbb{Q}}$$

induces a homomorphism between the second cohomology groups

$$\tau_g^{\mathbb{Q}}(1)^*\colon\ \Lambda^2 U_{\mathbb{Q}} \longrightarrow H^2(\mathcal{I}_g; \mathbb{Q})$$

and Hain determined the kernel of this homomorphism as follows.

Proposition 5.2 (Hain [41]) *Let $\tau_g^{\mathbb{Q}}(1)\colon \mathcal{I}_g \longrightarrow U_{\mathbb{Q}}$ be as above. Then*

$$\operatorname{Ker}\ \tau_g^{\mathbb{Q}}(1)^* = [2^2] + [0] \subset \Lambda^2 U_{\mathbb{Q}} = [1^6] + [1^4] + [1^2] + [2^2 1^2] + [2^2] + [0]$$

where the decomposition of $\Lambda^2 U_{\mathbb{Q}}$ is valid for $g \geq 6$. □

The fact that $[2^2]+[0]$ is contained in the kernel was essentially proved in our papers [111, 112], though in a more primitive way. In fact, we obtained the secondary invariant $d_1 \in H^1(\mathcal{K}_g;\mathbb{Q})$ (see section 6.6 below for details) from the vanishing of the trivial summand $[0]$ in $H^2(\mathcal{I}_g;\mathbb{Q})$ which turned out to be closely related to the Casson invariant. Hain proved the above result in a systematic way. In particular, the non-triviality in $H^2(\mathcal{I}_g;Q)$ of other summands than $[2^2]+[0]$ was shown by constructing explicit abelian cycles of $\mathcal{I}_g$ on which they take non-zero values.

Theorem 5.3 (Hain [41]) *If $g \geq 3$, then $\mathfrak{t}_g$ is isomorphic to the completion of its associated graded Lie algebra $\operatorname{Gr}\mathfrak{t}_g$. Moreover if $g \geq 6$, then it has a presentation*

$$\operatorname{Gr}\mathfrak{t}_g = \mathcal{L}(U_{\mathbb{Q}})/([1^6]+[1^4]+[1^2]+[2^2 1^2])$$

where $\mathcal{L}(U_{\mathbb{Q}})$ denotes the free Lie algebra generated by $U_{\mathbb{Q}}=[1^3]$. □

Hain obtained similar presentations for other Lie algebras $\mathfrak{t}_{g,1}, \mathfrak{t}_{g,*}, \mathfrak{u}_{g,1}, \mathfrak{u}_{g,*}, \mathfrak{u}_g$ as well. Although these presentations are very simple, the proof requires deep and powerful techniques in Hodge theory. One of the main ingredients is to put mixed Hodge structures on $\mathfrak{u}_g$ and then on $\mathfrak{t}_g$ which depend on a fixed complex structure on the reference surface. One important consequence of this is that after tensoring with $\mathbb{C}$, they are canonically isomorphic to their associated graded Lie algebras.

Thus for the study of the structure of the mapping class group, it is enough to investigate the morphisms

$$\mathfrak{t}_g(k) \longrightarrow \mathfrak{u}_g(k) \longrightarrow \operatorname{Im}\tau_g^{\mathbb{Q}}(k)$$

between the three graded Lie algebras $\operatorname{Gr}\mathfrak{t}_g, \operatorname{Gr}\mathfrak{u}_g$ and $\oplus_k \operatorname{Im}\tau_g^{\mathbb{Q}}(k) \subset \mathfrak{h}_g^{\mathbb{Q}}$. Another important consequence of Hain's work is that $\operatorname{Im}\tau_g^{\mathbb{Q}} = \oplus_k \operatorname{Im}\tau_g^{\mathbb{Q}}(k)$ is generated by degree one summand $\operatorname{Im}\tau_g^{\mathbb{Q}}(1)=[1^3]$.

6 A prospect on the structure of the mapping class group

In this section, we would like to describe some of the various aspects of the structure of the mapping class group which seem to deserve further investigation in the future. More precisely, we consider the following topics.

6.1 Structure of the graded Lie algebra $\mathfrak{h}_{g,1}$

We consider the structure of the graded Lie algebra

$$\mathfrak{h}_{g,1} = \bigoplus_{k=1}^{\infty} \mathfrak{h}_{g,1}(k)$$

(see section 5). Recall that this is the graded Lie algebra consisting of derivations, with positive degrees, of the free graded Lie algebra $\mathcal{L}_{g,1} = \oplus_k \mathcal{L}_{g,1}(k)$ which kill the symplectic class $\omega_0 \in \mathcal{L}_{g,1}(2) = \Lambda^2 H$. We omit the degree 0 part $\mathfrak{h}_{g,1}(0) \otimes \mathbb{Q}$ which is isomorphic to $\mathfrak{sp}(2g, \mathbb{Q})$.

As was explained in section 5, this Lie algebra serves as the target of the Johnson homomorphisms

$$\tau_{g,1}(k)\colon\ \mathcal{M}_{g,1}(k)/\mathcal{M}_{g,1}(k+1) \longrightarrow \mathfrak{h}_{g,1}(k)$$

so that the main problem is to describe the image

$$\operatorname{Im}\tau_{g,1} \subset \mathfrak{h}_{g,1}$$

as a Lie subalgebra of $\mathfrak{h}_{g,1}$. It is one of the basic results of Johnson that

$$\operatorname{Im}\tau_{g,1}(1) = \mathfrak{h}_{g,1}(1) = \Lambda^3 H.$$

Here we mention two important topological applications obtained by Hain in his fundamental work in [41]. One is that the graded Lie algebra $\operatorname{Im}\tau_{g,1}^{\mathbb{Q}}$ (the superscript $\mathbb{Q}$ will mean that we take the operation $\otimes\mathbb{Q}$ as before) is generated by the degree one summand $\operatorname{Im}\tau_{g,1}^{\mathbb{Q}}(1) = \Lambda^3 H_{\mathbb{Q}}$, which was already mentioned in section 5. The other is that the natural map $\mathcal{M}_{g,1}(k)/\mathcal{M}_{g,1}(k+1) \to \mathcal{M}_{g,*}(k)/\mathcal{M}_{g,*}(k+1)$ is surjective after tensoring with $\mathbb{Q}$ (and hence is an isomorphism for $k \neq 2$). As is mentioned in [41], this is an answer to a problem raised by Asada–Nakamura in [6] where they pointed out the possibility that $\mathfrak{j}_{g,1}^{\mathbb{Q}}(4) = [2]$ might be outside of $\operatorname{Im}\tau_{g,1}^{\mathbb{Q}}(4)$. This possibility is now settled by Hain to be true as follows. It is easy to see that $\mathfrak{j}_{g,1}(2) = \mathbb{Z}$ and it is contained in $\operatorname{Im}\tau_{g,1}(2)$. However this is the whole of the intersection of $\operatorname{Im}\tau_{g,1}$ with the ideal $\mathfrak{j}_{g,1}$. Namely we have $\mathfrak{j}_{g,1}(k) \cap \operatorname{Im}\tau_{g,1}(k) = \{0\}$ for all $k \geq 3$. Note that $\operatorname{Im}\tau_{g,*}$ contains $\mathcal{L}_g$ as a natural Lie subalgebra which corresponds to the inner derivations of $\mathcal{L}_g$, as was already mentioned in section 5.

In our paper [115], we introduced certain Sp–equivariant mappings

$$\mathrm{Tr}(2k+1)\colon\ \mathfrak{h}_{g,1}(2k+1) \longrightarrow S^{2k+1}H \quad (k = 1, 2, \cdots)$$

such that they are surjective (after tensoring with $\mathbb{Q}$) and vanish on $\operatorname{Im}\tau_{g,1}$ as well as on $[\mathfrak{h}_{g,1}, \mathfrak{h}_{g,1}]$. We can also show that it vanishes on the ideal $\mathfrak{j}_{g,1}$. Thus the traces descend to Sp–equivariant mappings

$$\mathrm{Tr}(2k+1)\colon\ \mathfrak{h}_{g}(2k+1) \longrightarrow S^{2k+1}H \quad (k = 1, 2, \cdots).$$

These mappings are called *traces* because they were defined using the trace of some matrix representation of $\mathfrak{h}_{g,1}(2k+1)$.

We can summarize the above as follows. The ideal $\mathfrak{j}_{g,1}$ and the image $\operatorname{Im}\tau_{g,1}$ of the Johnson homomorphism are almost disjoint from each other and both are contained in the kernel of the traces. Thus the next problem is to determine how large is the remaining part of the Lie algebra $\mathfrak{h}_{g,1}$.

After we had found the traces, Nakamura [126] discovered another obstruction to the surjectivity of $\tau_{g,1}$ which has its origin in number theory. More precisely, it is related to theories of outer Galois representations of the absolute Galois group over $\mathbb{Q}$ initiated by Grothendieck, Ihara and Deligne and developed by many people (see [38, 57, 18, 19, 127] and references in them). Roughly speaking, Nakamura proved that certain graded Lie algebra, which arises in the above theories, appears in the cokernel of Johnson homomorphisms. In particular, he concluded that the dimension of the cokernel of the Johnson homomorphism

$$\tau_{g,*}^{\mathbb{Q}}\colon\ (\mathcal{M}_{g,*}(2k)/\mathcal{M}_{g,*}(2k+1)) \otimes \mathbb{Q} \longrightarrow \mathfrak{h}_{g,*}^{\mathbb{Q}}(2k)$$

is greater than or equal to a number r_k which is the free $\mathbb{Z}_\ell$–rank of a certain Galois group $\mathrm{Gal}(\mathbb{Q}(k+1)/\mathbb{Q}(k))$ associated to the outer representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on some nilpotent quotient of the pro-ℓ fundamental group of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ (see [126] for details). Thus it may depend on the prime ℓ. However it is conjectured that r_k is the dimension of the degree k part of the free graded Lie algebra over $\mathbb{Q}$ which is generated by one free generator σ_k of degree k for each odd $k > 1$ (Deligne's motivic conjecture). As Nakamura mentions in his paper cited above, the appearance of these Galois obstructions, in its primitive form, was predicted by Takayuki Oda and some closely related work was done by M. Matsumoto. Certain estimates of these numbers have been obtained by Ihara [56], M. Matsumoto [96], Tsunogai [141] and others.

Next we consider the images of the Johnson homomorphisms

$$\tau_g(k)\colon\ \mathcal{M}_g(k)/\mathcal{M}_g(k+1)\to\mathfrak{h}_g(k)$$

for low degrees. The following results have been obtained.

$$\begin{array}{ll} \operatorname{Im}\tau_g^{\mathbb{Q}}(1)=[1^3] & ([61]) \\ \operatorname{Im}\tau_g^{\mathbb{Q}}(2)=[2^2] & ([111,\ 41]) \\ \operatorname{Im}\tau_g^{\mathbb{Q}}(3)=[31^2] & ([6,\ 41]) \end{array}$$

Nakamura made a list of irreducible decompositions of the Sp–modules $\mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$ for small k by a computer calculation. Utilizing it, we made rather long explicit computations of the Lie bracket $\operatorname{Im}\tau_{g,1}^{\mathbb{Q}}(1)\otimes\operatorname{Im}\tau_{g,1}^{\mathbb{Q}}(3)\to\mathfrak{h}_{g,1}^{\mathbb{Q}}(4)$, in the framework of symplectic representation theory, and obtained the following result.

Proposition 6.1 *We have*

$$\operatorname{Im}\tau_g^{\mathbb{Q}}(4)=[42]+[31^3]+[2^3]+[31]+[2]\quad(g\geq 4)$$

and the cokernel of the homomorphism $\tau_{g,1}^{\mathbb{Q}}(4)\colon\ (\mathcal{M}_{g,1}(4)/\mathcal{M}_{g,1}(5))\otimes\mathbb{Q}\to\mathfrak{h}_{g,1}^{\mathbb{Q}}(4)$ *is isomorphic to* $[2]+[21^2]$ *where* $[2]=\mathfrak{j}_{g,1}^{\mathbb{Q}}(4)$. □

The following list describes the structure of $\mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$ for degrees $k\leq 4$.

k	$\mathfrak{j}_{g,1}^{\mathbb{Q}}(k)$	$\mathcal{L}_g^{\mathbb{Q}}(k)$	$\operatorname{Im}\tau_g^{\mathbb{Q}}(k)$	$\operatorname{Cok}\tau_g^{\mathbb{Q}}(k)\setminus\mathrm{Tr}$	Tr
1		$[1]$	$[1^3]$		
2	$[0]$	$[1^2]$	$[2^2]$		
3		$[21]$	$[31^2]$		$[3]$
4	$[2]$	$[31][21^2][2]$	$[42][31^3][2^3][31][2]$	$[21^2]$	

Here the term $\operatorname{Cok}\tau_g^{\mathbb{Q}}(k)\setminus\mathrm{Tr}$ means that we exclude the trace component from the cokernel of the homomorphism $\tau_g^{\mathbb{Q}}(k)$. Thus the summand $[21^2]$ in $\mathfrak{h}_{g,1}^{\mathbb{Q}}(4)$ is not contained in $\operatorname{Im}\tau_g^{\mathbb{Q}}(4)$ and it should be considered as a new type of obstruction for the surjectivity of $\tau_{g,1}^{\mathbb{Q}}$ other than the ideal $\mathfrak{j}_{g,1}$, the traces and the Golois obstructions. In the next degree 5, we found yet another new obstruction. More precisely, at least one copy of $[1^3] \subset \mathfrak{h}_{g,1}^{\mathbb{Q}}(5)$ cannot be hit by $\tau_{g,1}^{\mathbb{Q}}(5)$ and this leads us to the non-triviality of $H_4(\operatorname{Out} F_4;\mathbb{Q})$ as will be shown in section 6.5.

Thus the cokernels of the Johnson homomorphisms seem to grow rapidly according as the degree increases and it will require many more studies before we can figure out the precise form of $\operatorname{Im} \tau_{g,1}^{\mathbb{Q}}$ inside $\mathfrak{h}_{g,1}^{\mathbb{Q}}$. We mention that Kontsevich gave the irreducible decomposition of $\mathfrak{h}_{g,1}^{\mathbb{Q}}$ as an Sp–module in [85, 86] and it would be desirable to identify $\operatorname{Im} \tau_{g,1}^{\mathbb{Q}}$ as an explicit Sp–submodule of it.

6.2 Description of Sp–invariant part of $\mathfrak{h}_{g,1}^{\mathbb{Q}}(6)$

In this subsection, we describe the Sp–invariant part $\mathfrak{h}_{g,1}^{\mathbb{Q}}(6)^{Sp}$ of the degree 6 summand of the graded Lie algebra $\mathfrak{h}_{g,1}^{\mathbb{Q}}$ to illustrate our method. Here we only give an outline of our computation rather than the details.

We have an exact sequence

$$0 \longrightarrow \mathfrak{h}_{g,1}(6) \longrightarrow H \otimes \mathcal{L}_{g,1}(7) \longrightarrow \mathcal{L}_{g,1}(8) \longrightarrow 0.$$

By Proposition 4.7, we know that $\dim(H_{\mathbb{Q}} \otimes \mathcal{L}_{g,1}^{\mathbb{Q}}(7))^{Sp} = 15$ and it can be shown that $\dim \mathcal{L}_{g,1}^{\mathbb{Q}}(8)^{Sp} = 10$. Hence $\dim \mathfrak{h}_{g,1}^{\mathbb{Q}}(6)^{Sp} = 5$ in the stable range. Also it can be shown that $\dim \mathfrak{j}_{g,1}^{\mathbb{Q}}(6)^{Sp} = 2$. Alternatively, we can use the method described after Proposition 4.7 as follows. The set of chord diagrams with 4 chords has 18 elements. If we divide the vector space $\mathcal{G}_4\mathcal{D}^c$ spanned by it by the (4T) relation, then the dimension reduces to 6 (see [10]). For our purpose, instead of (4T) relation, we have to put the relation coming from condition (i) of Proposition 4.6. Then we find by an explicit computation that

$$\dim \mathfrak{h}_{g,1}^{\mathbb{Q}}(6)^{Sp} = \begin{cases} 5 & (g \geq 3) \\ 4 & (g = 2) \\ 1 & (g = 1). \end{cases}$$

We can also give a basis $\{\alpha_i; i = 1, \cdots, 5\}$ of $\operatorname{Hom}(\mathfrak{h}_{g,1}^{\mathbb{Q}}(6), \mathbb{Q})^{Sp}$ in terms of linear combinations of chord diagrams. Now we consider the following 5 elements $\{\xi_i; i = 1, \cdots, 5\}$ in $\mathfrak{h}_{g,1}^{\mathbb{Q}}(6)^{Sp}$. We choose the first two elements ξ_1, ξ_2 to be a basis of $\mathfrak{j}_{g,1}^{\mathbb{Q}}(6)^{Sp}$. On the other hand, we know that $\mathfrak{h}_{g,1}^{\mathbb{Q}}(3) = [21] + [31^2] + [3]$ where $[21] = \mathcal{L}_g^{\mathbb{Q}}(3)$, $[31^2] = \operatorname{Im} \tau_g^{\mathbb{Q}}(3)$ and $[3]$ is the trace component (see the table in section 6.1). It turns out that the unique trivial summand $[0]$ in each of $\Lambda^2[21], \Lambda^2[31^2]$ and $\Lambda^2[3]$ survives in $\mathfrak{h}_{g,1}^{\mathbb{Q}}(6)^{Sp}$ under the bracket operation

$\Lambda^2\mathfrak{h}_{g,1}^{\mathbb{Q}}(3)\rightarrow\mathfrak{h}_{g,1}^{\mathbb{Q}}(6)$. We set ξ_3,ξ_4,ξ_5 to be the images in $\mathfrak{h}_{g,1}^{\mathbb{Q}}(6)^{Sp}$ of these elements in the above order. We see that ξ_3 is a generator of $\mathcal{L}_g^{\mathbb{Q}}(6)^{Sp}\cong\mathbb{Q}$. Clearly ξ_4 is contained in $\operatorname{Im}\tau_g^{\mathbb{Q}}(6)^{Sp}$. Also it turns out that we can construct the first two elements ξ_1,ξ_2 by taking brackets of suitable elements from $\mathfrak{h}_{g,1}^{\mathbb{Q}}(1)$ and $[3]\subset\mathfrak{h}_{g,1}^{\mathbb{Q}}(3)$. This is one of the supporting pieces of evidence for Conjecture 6.10 given in section 6.5 below. Now explicit computation shows that $\det(\alpha_i(\xi_j))\neq 0$. Hence $\{\xi_i\}$ forms a basis of $\mathfrak{h}_{g,1}^{\mathbb{Q}}(6)^{Sp}$.

Summing up, we have the following table for the dimensions of Sp–invariant part of $\mathfrak{h}_{g,1}^{\mathbb{Q}}(6)$ corresponding to the ideal $\mathfrak{j}_{g,1}^{\mathbb{Q}}$, inner derivations $\mathcal{L}_g^{\mathbb{Q}}$, Johnson image $\operatorname{Im}\tau_g^{\mathbb{Q}}$ and the remaining part $\operatorname{Cok}\tau_g^{\mathbb{Q}}$ (for $g\geq 3$).

$\mathfrak{j}_{g,1}^{\mathbb{Q}}(6)^{Sp}$	$\mathcal{L}_g^{\mathbb{Q}}(6)^{Sp}$	$\operatorname{Im}\tau_g^{\mathbb{Q}}(6)^{Sp}$	$\operatorname{Cok}\tau_g^{\mathbb{Q}}(6)^{Sp}$	total
2	1	1	1	5

We can also see from the above table that the dimension of $\operatorname{Cok}\tau_g^{\mathbb{Q}}(6)$ is exactly equal to one. Hence it should be equal to Nakamura's Galois obstruction given in [126] which is in fact independent of ℓ.

The way of degeneration is interesting also here. More precisely, if we set $g=1$, then it turns out that any of the three elements ξ_3,ξ_4,ξ_5 goes to a unique non-trivial element in $\mathfrak{h}_{1,1}^{\mathbb{Q}}(6)^{Sp}\cong\mathbb{Q}$ (up to non-zero scalars). Observe here that if $g=1$, then the Torelli group is trivial, the ideal $\mathfrak{j}_{1,1}$ coincides with the whole of $\mathfrak{h}_{1,1}$ and the traces are all trivial. However the forgetful mapping $\mathfrak{h}_{g,1}\rightarrow\mathfrak{h}_{1,1}$ is *not* a Lie algebra homomorphism so that there is no contradiction here. It may be said that, in genus one, topology disappears and only arithmetic remains.

The above is only a special case of degree 6 summand. However we can see already here a glimpse of some general phenomena. Namely the elements ξ_3,ξ_4,ξ_5 can be defined for *all* degrees $4k+2$ and turn out to be non-trivial in $\mathfrak{h}_{g,1}^{\mathbb{Q}}(4k+2)^{Sp}$. For example ξ_4 is defined to be the image of the Sp–invariant part of $\Lambda^2[2k+1,1^2]$ in $\operatorname{Im}\tau_{g,1}^{\mathbb{Q}}(4k+2)$ where $[2k+1,1^2]\subset\operatorname{Im}\tau_{g,1}^{\mathbb{Q}}(2k+1)$ is the summand given in [6]. The elements ξ_5 for all $k>1$ were constructed in a joint work with Nakamura in which we are trying to understand the Galois obstructions topologically.

6.3 Kernel of $\mathfrak{t}_g\rightarrow\operatorname{Im}\tau_g^{\mathbb{Q}}$ and invariants of 3–manifolds

As is well known, there are close connections between the mapping class group and 3 dimensional manifolds. More precisely, the Heegaard decomposition of

3–manifolds gives rise to a direct correspondence between the two objects and the construction of surface bundles over the circle with a given monodromy from the mapping class group yields another such relation. Also the genus 1 mapping class group plays a crucial role in the theory of Dehn surgery along framed links in 3–manifolds, particularly in S^3 by virtue of Kirby's fundamental result [79]. Hence any invariant of 3–manifolds naturally has certain effect on the structure of the mapping class group.

Until the discovery of the Casson invariant for homology 3–spheres in 1985, the Rohlin invariant was almost the unique invariant for 3–manifolds. Motivated mainly by Witten's influential work in [144], the theory of topological invariants of 3–manifolds has been continually and rapidly developing. See [134, 83, 80, 123, 130, 88] as well as their references. It is beyond the scope of this article to review these developments. In the following, we would like to focus on those results which have direct relation with the structure of the mapping class group.

If we are given a 3–manifold M together with an embedded oriented surface $\Sigma_g \subset M$, then we can associate a new manifold M_φ to each element $\varphi \in \mathcal{M}_g$ by cutting M along Σ_g and pasting it back together by φ. In the cases where M is an integral homology sphere and φ belongs to the Torelli group $\mathcal{I}_g$, the resultant manifold M_φ is again an integral homology sphere. Hence, given any topological invariant α of homology 3–spheres, with values in a module A, we can define a mapping $\alpha\colon \mathcal{I}_g \to A$ by setting $\alpha(\varphi) = \alpha(M_\varphi)$. Birman and Craggs [12] first studied such mappings for the case of the Rohlin invariant μ. Johnson [60] extended their results to obtain a complete enumeration of so-called Birman–Craggs homomorphisms. This result played an important role in his determination of the abelianization of the Torelli group in [65]. In our papers [111, 112], we studied the case of the Casson invariant λ and in particular we obtained an interpretation of λ in terms of the secondary characteristic classes of surface bundles (see [119] and section 6.6 below). This work has been generalized in two different ways. One is due to Lescop [89] where she obtained, among other things, a closed formula which expresses how the Casson–Walker–Lescop invariant behaves under the cut and paste operation on 3–manifolds. The other is given by a series of works of Garoufalidis and Levine [32, 33] where they generalized our results cited above extensively. More precisely, we considered the effect of Casson invariant on the structure of the Torelli group $\mathcal{I}_g$ while they considered all of the finite type invariants of homology 3–spheres introduced by Ohtsuki [129, 130]. In particular, they proved that Ohtsuki's filtration of the space of homology 3–spheres can be described in terms of the lower central series of the Torelli group. As a corollary to this statement, they proved that any primitive type $3k$ invariant α gives rise to a non-trivial homomorphism

$$\alpha\colon\ \mathfrak{t}_g(2k)\longrightarrow\mathbb{Q}.$$

Recall from section 5 that we have a series of homomorphisms

$$\mathfrak{t}_g(k)\longrightarrow\mathfrak{u}_g(k)\longrightarrow\operatorname{Im}\tau_g^{\mathbb{Q}}(k)$$

and we know by Hain [39] that $\mathfrak{t}_g(k)\cong\mathfrak{u}_g(k)$ for all $k\neq 2$ and that $\mathfrak{t}_g(2)=\mathfrak{u}_g(2)+[0]$. Thus the main problem concerning the above series is the following.

Problem 6.2 Determine whether the homomorphism $\mathfrak{t}_g(k)\to\operatorname{Im}\tau_g^{\mathbb{Q}}(k)$ is injective (and hence an isomorphism) for $k\neq 2$ or not.

The results of Garoufalidis and Levine mentioned above show that the above problem is crucial also from the point of view of the theory of invariants of 3–manifolds. We mention that, extending earlier results of Johnson [63], Kitano [81] proved that the k-th Johnson homomorphism $\tau_{g,1}(k)\colon\ \mathcal{M}_{g,1}(k)\to\mathfrak{h}_{g,1}(k)$ exactly mesures the higher Massey products of mapping tori which are associated to elements of the subgroup $\mathcal{M}_{g,1}(k)$. Hence if Problem 6.2 will be affirmatively solved, it would imply that the finite type invariants can be described in terms of the original Casson invariant together with the Massey products. This might sound unlikely from the point of view of 3–manifolds invariants. Also the author learned from J Murakami that the restriction, to the Torelli group, of the projective representation of $\mathcal{M}_g$ associated to the LMO invariant given in [88, 124] gives rise to a unipotent representation (after suitable truncations). Hence it should be described by the Torelli Lie algebra $\mathfrak{t}_g$. This also increases the importance of Problem 6.2.

Although Hain [41] obtained a presentation of the Torelli Lie algebra $\mathfrak{t}_g$ (see section 5), it is by no means easy to determine $\mathfrak{t}_g(k)$ $(k=1,2,\cdots)$ explicitly by using it. One way to compute them is to apply Sullivan's theory [137] of 1–minimal models to the Torelli group $\mathcal{I}_g$ which can be described as follows. First of all we know by Johnson [61] that $\mathfrak{t}_g(1)=U_{\mathbb{Q}}=[1^3]$. It was a consequence of the results of [111, 112] that $\mathfrak{t}_g(2)$ contains at least $[2^2]+[0]$ where the trivial summand $[0]$ reflects the influence of the Casson invariant on the structure of the Torelli group. We have an isomorphism

$$\mathfrak{t}_g(2)^*\cong\operatorname{Ker}(\Lambda^2\mathfrak{t}_g(1)^*\to H^2(\mathcal{I}_g)).$$

As was already mentioned in section 5 (Proposition 5.2), Hain [41] determined the right hand side to be precisely equal to $[2^2]+[0]$ and he concluded that

$$\mathfrak{t}_g(2)=[2^2]+[0]\quad(g\geq 3).$$

The next case, namely the case of degree 3 is given by the following result.

Proposition 6.3 *We have isomorphisms*

$$\mathfrak{t}_g(3) \cong \mathfrak{u}_g(3) \cong \operatorname{Im}\tau_g(3) = [31^2] \quad (g \geq 6).$$

Sketch of Proof Sullivan's theory [137] implies that $\mathfrak{t}_g(3)$ can be identified with the kernel of the following homomorphism

$$\operatorname{Ker}\left\{\begin{pmatrix} \mathfrak{t}_g(1)^* \otimes \mathfrak{t}_g(2)^* \\ \Lambda^2\mathfrak{t}_g(2)^* \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \Lambda^3\mathfrak{t}_g(1)^* \\ \Lambda^2\mathfrak{t}_g(1)^* \otimes \mathfrak{t}_g(2)^* \end{pmatrix}\right\} \longrightarrow H^2(\mathcal{I}_g)$$

where $d \equiv 0$ on $\mathfrak{t}_g(1)^*$ and $d\colon \mathfrak{t}_g(2)^* \to \Lambda^2\mathfrak{t}_g(1)^*$ is the natural injection. The differential $d\colon \Lambda^2\mathfrak{t}_g(2)^* \to \Lambda^2\mathfrak{t}_g(1)^* \otimes \mathfrak{t}_g(2)^*$ is given by $d(\alpha\wedge\beta) = d\alpha\otimes\beta - d\beta\otimes\alpha$ $(\alpha,\beta \in t_g(2)^*)$. It is easy to deduce from this fact that it is injective. Therefore $\mathfrak{t}_g(3)^*$ is isomorphic to the kernel of the following mapping

$$\operatorname{Ker}(\mathfrak{t}_g(1)^* \otimes \mathfrak{t}_g(2)^* \to \Lambda^3\mathfrak{t}_g(1)^*) \longrightarrow H^2(\mathcal{I}_g)$$

which is given by the Massey triple product. Hain proved in [41] that all of the higher Massey products of $\mathcal{I}_g$ vanishes for $g \geq 6$. Hence passsing to the dual, we see that $\mathfrak{t}_g(3)$ is determined by the following exact sequence

$$\Lambda^3\mathfrak{t}_g(1) \longrightarrow \mathfrak{t}_g(1) \otimes \mathfrak{t}_g(2) \xrightarrow{[\ ,\ ]} \mathfrak{t}_g(3) \longrightarrow 0.$$

Here the first mapping is given by

$$\Lambda^3\mathfrak{t}_g(1) \ni u\wedge v\wedge w \mapsto u\otimes[v,w] + v\otimes[w,u] + w\otimes[u,v] \quad (u,v,w \in \mathfrak{t}_g(1))$$

and the image of which in $\mathfrak{t}_g(3)$ under the bracket operation is trivial because of the Jacobi identity. The irreducible decomposition of $\mathfrak{t}_g(1)\otimes\mathfrak{t}_g(2) = [1^3] \otimes ([2^2]+[0])$ is given by

$$\mathfrak{t}_g(1)\otimes\mathfrak{t}_g(2) = ([3^21]+[321^2]+[2^21^3]+[31^2]+[32]+[21^3]+[2^21]+[21]+[1^3])+[1^3].$$

Explicit computations, corresponding to each summand in the above decomposition, show that the mapping $\Lambda^3\mathfrak{t}_g(1) \to \mathfrak{t}_g(1)\otimes\mathfrak{t}_g(2)$ hits any summand except $[31^2]$. Since we already know that $\operatorname{Im}\tau_g^{\mathbb{Q}}(3) = [31^2]$, the result follows. □

If we inspect the above proof carefully, we find that it was not necessary to use Hain's vanishing of Massey triple products of $\mathcal{I}_g$ for $g \geq 6$. It turns out that the computation itself contains a proof of the vanishing of them. However for higher degrees $k = 4, 5, \cdots$, Hain's result considerably simplifies the computations. By using this method, we can continue the computation for higher degrees. For example the degree 4 summand $\mathfrak{t}_g(4)$ can be determined by the following exact sequence

$$\Lambda^2\mathfrak{t}_g(1)\otimes\mathfrak{t}_g(2) \longrightarrow \big(\mathfrak{t}_g(1)\otimes\mathfrak{t}_g(3)\big) \oplus \Lambda^2\mathfrak{t}_g(2) \xrightarrow{[\ ,\ ]} \mathfrak{t}_g(4) \longrightarrow 0.$$

Here the first mapping is given by

$$(u \wedge v) \otimes w \mapsto u \otimes [v,w] - v \otimes [u,w] - [u,v] \wedge w \in \big(\mathfrak{t}_g(1) \otimes \mathfrak{t}_g(3)\big) \oplus \Lambda^2 \mathfrak{t}_g(2)$$

where $(u, v \in \mathfrak{t}_g(1), w \in \mathfrak{t}_g(2))$ and the above element vanishes in $\mathfrak{t}_g(4)$ again by the Jacobi identity. Although our computation is not finished yet, we see no signs of non-trivial kernel for $\mathfrak{t}_g(k) \to \operatorname{Im} \tau_g^{\mathbb{Q}}(k)$ $(k = 4, 5, 6)$ so far.

We can also ask how other invariants of homology 3–spheres, for example various Betti numbers of Floer homology [28] or infinitely many homology cobordism invariants the existence of which is guaranteed by a remarkable result of Furuta [31], will influence the structure of the Torelli group. Also it seems to be a challenging problem to seek those invariants of homology 3–spheres which reflect semi-simple informations, rather than the nilpotent ones, of the Torelli group.

6.4 Cocycles for the Mumford–Morita–Miller classes

In our paper [116], we constructed certain representations ρ_1 of the mapping class groups $\mathcal{M}_{g,*}, \mathcal{M}_g$ and obtained the following commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{g,*} & \xrightarrow{\rho_1} & \frac{1}{2}\Lambda^3 H \rtimes Sp(2g,\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathcal{M}_g & \xrightarrow[\rho_1]{} & \frac{1}{2}\Lambda^3 H/H \rtimes Sp(2g,\mathbb{Z}). \end{array}$$

By making use of a standard fact concerning the cohomology of a group which is a semi-direct product, we deduced the existence of the following diagram

$$\begin{array}{ccc} \operatorname{Hom}(\Lambda^*(\Lambda^3 H_{\mathbb{Q}}), \mathbb{Q})^{Sp} & \xrightarrow{\rho_1^*} & H^*(\mathcal{M}_{g,*};\mathbb{Q}) \\ \uparrow & & \uparrow \\ \operatorname{Hom}(\Lambda^* U_{\mathbb{Q}}, \mathbb{Q})^{Sp} & \xrightarrow[\rho_1^*]{} & H^*(\mathcal{M}_g;\mathbb{Q}). \end{array} \tag{7}$$

If we combine this with Proposition 4.3 and Proposition 4.4 (see section 4.2), then we obtain the following commutative diagram (which is defined at the cocycle level)

$$\begin{array}{ccc} \mathbb{Q}[\alpha_\Gamma; \Gamma \in \mathcal{G}] & \xrightarrow{\rho_1^*} & H^*(\mathcal{M}_{g,*};\mathbb{Q}) \\ \uparrow & & \uparrow \\ \mathbb{Q}[\beta_\Gamma; \Gamma \in \mathcal{G}^0] & \xrightarrow[\rho_1^*]{} & H^*(\mathcal{M}_g;\mathbb{Q}). \end{array} \tag{8}$$

It was proved in [118] that the images of ρ_1^* in (8) contain all of the characteristic classes e, e_i $(i = 1, 2, \cdots)$. However the problem of determining whether the images contain new classes or not remained open. This problem was soon solved by the introduction of generalized Mumford–Morita–Miller classes due to Kawazumi in [75]. There is a remarkable work of Looijenga [93] in which he determined the stable cohomology of $\mathcal{M}_g$ with coefficients in any finite dimensional irreducible representation of $Sp(2g, \mathbb{Q})$. In [74], Kawazumi used the above generalized classes to give a different basis for some of these twisted cohomology groups, thereby adding a topological flavor to Looijenga's result. We also mention that Ivanov [58] obtained a stability theorem for twisted cohomology groups of mapping class groups (for a surface with at least one boundary component).

Let us define $\mathcal{R}^*(\mathcal{M}_g)$ to be the subalgebra of $H^*(\mathcal{M}_g; \mathbb{Q})$ generated by the classes e_i. Similarly we define $\mathcal{R}^*(\mathcal{M}_{g,*})$ to be the subalgebra of $H^*(\mathcal{M}_{g,*}; \mathbb{Q})$ generated by the classes e, e_i. We may call them the *tautological algebra* of the mapping class groups. These are just the reduction to the rational cohomology of the original tautological algebras $\mathcal{R}^*(\mathbf{M}_g), \mathcal{R}^*(\mathbf{C}_g)$ of the moduli spaces which are defined to be subalgebras of the individual Chow algebras generated by the classes κ_i and $c_1(\omega)$ (see [26, 43, 53]).

Theorem 6.4 (Kawazumi–Morita [76, 77]) *For any g, the images of ρ_1^* in (8) coincide with the tautological algebras $\mathcal{R}^*(\mathcal{M}_{g,*})$ and $\mathcal{R}^*(\mathcal{M}_g)$ of the mapping class groups.* □

Thus the homomorphisms ρ_1^* in (7) have rather big kernel. For the lower homomorphism ρ_1^*, Garoufalidis and Nakamura [34] have given an interpretation of this fact in the framework of symplectic representation theory by showing an isomorphism

$$\left(\Lambda^* U_{\mathbb{Q}}^* / ([2^2])\right)^{Sp} \cong \mathbb{Q}[e_1, e_2, \cdots]$$

which holds in the stable range, where $([2^2])$ denotes the ideal of $\Lambda^* U_{\mathbb{Q}}^*$ generated by $[2^2] \subset \Lambda^2 U_{\mathbb{Q}}^*$. Their result can be generalized to the case of the upper ρ_1^* so that we have an isomorphism

$$\left(\Lambda^*(\Lambda^3 H_{\mathbb{Q}}^*)/([2^2] + [1^2])\right)^{Sp} \cong \mathbb{Q}[e, e_1, e_2, \cdots]$$

which holds also in the stable range. Here $[1^2]$ is a certain *diagonal* summand in $\Lambda^2(\Lambda^3 H_{\mathbb{Q}}^*)$, which has three copies of $[1^2]$, described explicitly in [118]. However these results are, at present, valid only in the stable range while Theorem 6.4 is true in all degrees. If we pass to the dual context, namely if we consider the homomorphism

$$(\rho_1)_* \colon\ H_*(\mathcal{M}_g;\mathbb{Q}) \longrightarrow H_*(\frac{1}{2}\Lambda^3 H/H \rtimes Sp(2g,\mathbb{Z});\mathbb{Q})$$

induced by ρ_1 on homology, then we find that those cycles in $(\Lambda^* U_{\mathbb{Q}})^{Sp}$ which come from the moduli space must have fairly restricted types. In particular, Faber's result below (Theorem 6.7) implies that the subspace of $(\Lambda^{2g-4}U_{\mathbb{Q}})^{Sp}$ generated by the *moduli cycles* is one dimensional and a similar statement is valid for $(\Lambda^{2g-2}(\Lambda^3 H_{\mathbb{Q}}))^{Sp}$. Hence we can define *fundamental cycles*

$$\mu_{g,*} \in (\Lambda^{2g-2}(\Lambda^3 H_{\mathbb{Q}}))^{Sp}, \quad \mu_g \in (\Lambda^{2g-4}U_{\mathbb{Q}})^{Sp}$$

which are well defined up to scalars and which should be expressed in terms of certain linear combinations of Sp–invariant tensors a_Γ, b_Γ described in section 4.

Starting from basic works on the Chow algebras of the moduli spaces $\mathbf{M}_3, \mathbf{M}_4$ in [23, 24], Faber made numerous explicit computations concerning the tautological algebra of the moduli spaces. Based on them, he proposed the following beautiful conjecture about the structure of the tautological algebra $\mathcal{R}^*(\mathbf{M}_g)$ (see [26] for details, in particular for the precise description of part (3) below).

Conjecture 6.5 (Faber [26])

(1) The tautological algebra $\mathcal{R}^*(\mathbf{M}_g)$ of the moduli space $\mathbf{M}_g$ "behaves like" the cohomology algebra of a nonsingular projective variety of dimension $g-2$. More precisely, it vanishes in degrees $> g-2$, is one dimensional in degree $g-2$ and the natural pairing $\mathcal{R}^i(\mathbf{M}_g) \times \mathcal{R}^{g-2-i}(\mathbf{M}_g) \to \mathcal{R}^{g-2}(\mathbf{M}_g)$ is perfect. It also satisfies the Hard Lefschetz and the Hodge Positivity properties with respect to the class κ_1.

(2) The $[\frac{g}{3}]$ classes $\kappa_1, \cdots, \kappa_{[\frac{g}{3}]}$ generate the algebra with no relations in degrees $\leq [\frac{g}{3}]$.

(3) There exist explicit formulas for the proportionalities in degree $g-2$.

Several supporting pieces of evidence for this conjecture have been obtained.

Theorem 6.6 (Looijenga [92]) *The tautological algebra $\mathcal{R}^*(\mathbf{M}_g)$ is trivial in degrees $> g-2$ and $\mathcal{R}^{g-2}(\mathbf{M}_g)$ is at most one dimensional. Similarly $\mathcal{R}^*(\mathbf{C}_g)$ is trivial for $* > g-1$ and $\mathcal{R}^{g-1}(\mathbf{C}_g)$ is at most one dimensional.* □

We mention that Jekel [59] proved the vanishing $e^g = 0 \in H^{2g}(\mathcal{M}_{g,*};\mathbb{Q})$ by making use of a certain representation $\mathcal{M}_{g,*} \to \mathrm{Homeo}_+ S^1$ (cf [109]). This gives a purely topological proof of a part of Looijenga's result above. Using the results of Mumford in [122] as well as those of Witten [145] and Kontsevich [84] combined with Looijenga's theorem above, Faber proved the following.

Theorem 6.7 (Faber [25]) *κ_{g-2} is non-zero on $\mathbf{M}_g$ so that $\mathcal{R}^{g-2}(\mathbf{M}_g)$ is one dimensional.* □

It follows immediately that $\mathcal{R}^{g-1}(\mathbf{C}_g)$ is also one dimensional.

The above results are obtained mainly in the framework of algebraic geometry. Here we would like to describe a topological approach to Faber's conjecture which has fairly different feature. Naturally we can obtain relations only in the rational cohomology algebra rather than the Chow algebra. However we hope that this approach would have its own meaning.

Our method is very simple. Namely, associated to any unstable relation in $H^*(\Lambda^3 H_{\mathbb{Q}})^{Sp}$ or in $H^*(U_{\mathbb{Q}})^{Sp}$, we can obtain a polynomial relation in the tautological algebra $\mathcal{R}^*(\mathcal{M}_{g,*})$ or $\mathcal{R}^*(\mathcal{M}_g)$ by applying Theorem 6.4. For example, we can apply Proposition 4.2 to obtain a series of non-trivial relations as follows. We know by this proposition that there is a unique relation

$$\sum_{C\in\mathcal{D}^\ell(6k)} a_C = 0 \tag{9}$$

in $(H_{\mathbb{Q}}^{6k})^{Sp}$ for $g = 3k-1$. Fortunately, this relation survives in $(\Lambda^{2k}U_{\mathbb{Q}})^{Sp}$ under the natural projection $H_{\mathbb{Q}}^{6k}\to\Lambda^{2k}U_{\mathbb{Q}}$. Hence passing to the dual, the relation $\sum_C \alpha_C = 0$ gives rise to a polynomial relation in $\mathcal{R}^{2k}(\mathcal{M}_{3k-1})$ which expresses the class e_k as a polynomial in lower e_i's. It turns out that this relation is exactly the same as Faber's relation mentioned in [26] up to a factor of some powers of 2. The associated generating function appears, in our context, as a result of enumeration of certain trivalent graphs. One of the merits of our method is that once we obtain a relation in $\mathcal{R}^*(\mathcal{M}_g)$ for some g, we can obtain associated relations for all genera $< g$. This is because of the following reason. Although the mapping $H_1(\Sigma_g;\mathbb{Q})\to H_1(\Sigma_{g-1};\mathbb{Q})$, which is induced by collapsing the last handle, is not very natural from the point of view of algebraic geometry, it does induce a natural mapping

$$(H_g^{\otimes 2k})^{Sp(2g,\mathbb{Q})}\longrightarrow(H_{g-1}^{\otimes 2k})^{Sp(2g-2,\mathbb{Q})}$$

where H_g and H_{g-1} stand for $H_1(\Sigma_g;\mathbb{Q})$ and $H_1(\Sigma_{g-1};\mathbb{Q})$ respectively. For example, the relation (9) which is the unique relation for $g=3k-1$ continues to hold for all $g<3k-1$. In this way, using the unique relation (9) above, we can prove the following theorem.

Theorem 6.8 *The tautological algebra $\mathcal{R}^*(\mathcal{M}_g)$ of $\mathcal{M}_g$ is generated by the first $[\frac{g}{3}]$ Mumford–Morita–Miller classes*

$$e_1, e_2, \cdots, e_{[\frac{g}{3}]}.$$

Moreover there are explicit formulas which express any class e_j ($j > [\frac{g}{3}]$) as a polynomial in the above classes. □

This theorem gives an affirmative solution (at the level of rational cohomology) to a part of Faber's conjecture (Conjecture 6.5). Details will be given in a forthcoming paper [120]. We expect that we can obtain further relations in $\mathcal{R}^*(\mathcal{M}_g)$ as well as in $\mathcal{R}^*(\mathcal{M}_{g,*})$ by this method. We also expect that the explicit form of the fundamental cycle μ_g mentioned above should be closely related to part (3) of Conjecture 6.5.

Concerning the (co)homology of the moduli space $\mathbf{M}_g$, there is another well known conjecture due to Witten and Kontsevich (see [84]) which says that certain natural cycles of $\mathbf{M}_g$ constructed by them should be expressed in terms of the Mumford–Morita–Miller classes. The first case was affirmatively solved by Penner in [131] and Arbarello and Cornalba made considerable progress on this problem in [3]. We expect that there would exist certain connection between their method in the framework of algebraic geometry with our approach given above which uses symplectic representation theory. Penner [132] described a related conjecture in the context of his new model of a universal Teichmüller space.

Also there is a problem concerning the *unstable* cohomology of $\mathcal{M}_g$ or $\mathbf{M}_g$. Harer and Zagier pointed out in [52] that their determination of the orbifold Euler characteristic of $\mathbf{M}_g$ implies that there must exist many unstable cohomology classes. However it seems that the unstable class constructed by Looijenga [90] for genus 3 moduli space is the only known explicit example.

Problem 6.9 Construct unstable cohomology classes of $\mathcal{M}_g$ explicitly.

6.5 Cohomology of the graded Lie algebra $\mathfrak{h}_\infty^{\mathbb{Q}}$ and $\operatorname{Out} F_n$

Here we consider the (co)homology of the graded Lie algebra

$$\mathfrak{h}_{g,1} = \bigoplus_{k=1}^{\infty} \mathfrak{h}_{g,1}(k).$$

First we would like to know the abelianization of $\mathfrak{h}_{g,1}$. The first Johnson homomorphism together with the traces induce a graded Lie algebra homomorphism

$$(\tau_{g,1}(1), \oplus_k \operatorname{Tr}(2k+1))\colon\ \mathfrak{h}_{g,1} \longrightarrow \Lambda^3 H \oplus \big(\bigoplus_{k\geq 1} S^{2k+1} H\big)$$

which is surjective after tensoring with $\mathbb{Q}$ where the target is considered as an *abelian* Lie algebra. As in section 5, let us consider the direct limit

$$\mathfrak{h}_\infty = \lim_{g\to\infty} \mathfrak{h}_{g,1}, \quad \mathfrak{h}_\infty^{\mathbb{Q}} = \lim_{g\to\infty} \mathfrak{h}_{g,1}^{\mathbb{Q}}.$$

It is easy to see that the Johnson homomorphism $\tau_{g,1}(1)$ and the traces $\operatorname{Tr}(2k+1)$ are all compatible with respect to the inclusions $\mathfrak{h}_{g,1} \subset \mathfrak{h}_{g+1,1}$. Namely the following diagram is commutative

$$\begin{array}{ccc}
\mathfrak{h}_{g,1} & \longrightarrow & \Lambda^3 H_g \oplus \big(\bigoplus_{k\geq 1} S^{2k+1} H_g\big) \\
\downarrow & & \downarrow \\
\mathfrak{h}_{g+1,1} & \longrightarrow & \Lambda^3 H_{g+1} \oplus \big(\bigoplus_{k\geq 1} S^{2k+1} H_{g+1}\big)
\end{array}$$

where H_g denotes the module H corresponding to the genus g. In view of the explicit computations in low degrees we have done so far, it seems to be reasonable to make the following conjecture.

Conjecture 6.10 The abelianization of the Lie algebra $\mathfrak{h}_{g,1}^{\mathbb{Q}}$ is given by $\tau_{g,1}(1)$ and the traces, so that we have an isomorphism

$$H_1(\mathfrak{h}_{g,1}^{\mathbb{Q}}) = \mathfrak{h}_{g,1}^{\mathbb{Q}}/[\mathfrak{h}_{g,1}^{\mathbb{Q}}, \mathfrak{h}_{g,1}^{\mathbb{Q}}] \cong \Lambda^3 H_{\mathbb{Q}} \oplus \big(\bigoplus_{k\geq 1} S^{2k+1} H_{\mathbb{Q}}\big).$$

A similar statement is true for $\mathfrak{h}_\infty^{\mathbb{Q}}$.

If this conjecture were true, then any element in $\mathfrak{h}_{g,1}^{\mathbb{Q}}$, including the Galois obstructions, can be described by taking brackets of suitable elements of $\Lambda^3 H_{\mathbb{Q}}$ and $S^{2k+1} H_{\mathbb{Q}}$. Regardless of whether the above conjecture is true or not, we have a homomorphism

$$H_c^*\big(\Lambda^3 H_{\mathbb{Q}} \oplus \big(\bigoplus_{k\geq 1} S^{2k+1} H_{\mathbb{Q}}\big)\big) \longrightarrow H_c^*(\mathfrak{h}_{g,1}^{\mathbb{Q}})$$

where H_c^* denotes the continuous cohomology, in the usual sense (cf [85]), of graded Lie algebras which are infinite dimensional but each degree k summand is finite dimensional for all k . The Sp–invariant part of the above homomorphism can be written as

$$\begin{aligned} &H_c^*\big(\Lambda^3 H_{\mathbb{Q}} \oplus \big(\bigoplus_{k\geq 1} S^{2k+1} H_{\mathbb{Q}}\big)\big)^{Sp} \\ &\cong \big(\Lambda^*(\Lambda^3 H_{\mathbb{Q}}^*) \otimes \Lambda^*(S^3 H_{\mathbb{Q}}^*) \otimes \Lambda^*(S^5 H_{\mathbb{Q}}^*) \otimes \cdots\big)^{Sp} \longrightarrow H_c^*(\mathfrak{h}_\infty^{\mathbb{Q}})^{Sp}. \end{aligned} \tag{10}$$

where $H_c^*(\mathfrak{h}_\infty^{\mathbb{Q}})^{Sp}$ denotes the inverse limit of the Sp–invariant part of the continuous cohomology of the graded Lie algebras $\mathfrak{h}_{g,1}^{\mathbb{Q}}$ which stabilizes. The Lie algebra $\mathfrak{h}_{g,1}^{\mathbb{Q}}$ contains $\operatorname{Im}\tau_{g,1}^{\mathbb{Q}}$ as a natural Lie subalgebra and by restriction and passing to the limit, we obtain a series of homomorphisms

$$\begin{aligned} H_c^*(\mathfrak{h}_\infty^{\mathbb{Q}})^{Sp} \longrightarrow H_c^*(\operatorname{Gr}\mathfrak{n}_\infty)^{Sp} \cong H_c^*(\mathfrak{n}_\infty)^{Sp} \cong \lim_{g\to\infty} H^*(\mathcal{N}_{g,1})^{Sp} \\ \longrightarrow \lim_{g\to\infty} H^*(\mathcal{M}_{g,1}). \end{aligned} \tag{11}$$

Here $\mathfrak{n}_\infty = \lim_{g\to\infty} \mathfrak{n}_{g,1}$ $(\mathfrak{n}_{g,1}(k) \cong \operatorname{Im}\tau_{g,1}^{\mathbb{Q}}(k) \subset \mathfrak{h}_{g,1}^{\mathbb{Q}}(k))$ and the second isomorphism is due to Hain [41] where he showed that $\mathfrak{n}_{g,1}$ has a mixed Hodge structure (depending on a fixed complex structure on the reference surface) so that there is a canonical isomorphism of $\mathfrak{n}_{g,1}$ with $\operatorname{Gr}\mathfrak{n}_{g,1}$ after tensoring with $\mathbb{C}$. For the last homomorphism, see [118]. Since $\operatorname{Tr}(2k+1)$ is trivial on $\operatorname{Im}\tau_{g,1}(2k+1)$ for any k (see [115]), the composition of (10) with (11) is trivial on any Sp–invariant which contains the trace component $S^{2k+1}H_{\mathbb{Q}}^*$. Hence, the *moduli part* of the homomorphism (10) is given by

$$\Lambda^*(\Lambda^3 H_{\mathbb{Q}}^*)^{Sp} \longrightarrow H_c^*(\mathfrak{h}_\infty^{\mathbb{Q}})^{Sp}.$$

As was explained in [43, 76] (see also section 2), a combination of our result [76] with that of [41] implies that the image of the above homomorphism can be identified with the polynomial algebra

$$\mathbb{Q}[e_1, e_2, \cdots] \subset \lim_{g\to\infty} H^*(\mathcal{M}_{g,1};\mathbb{Q}).$$

Then it is a natural question to ask the geometric meaning of the remaining part of the homomorphism (10). This can be answered by invoking important work of Kontsevich [85, 86], which we now briefly review.

As was mentioned in section 5, our $\mathfrak{h}_\infty^{\mathbb{Q}}$ is the degree positive part of his Lie algebra ℓ_∞ described in the above cited papers. The homology group $H_*(\ell_\infty)$ has a natural structure of a commutative and cocommutative Hopf algebra, where the multiplication comes from the sum operation $\ell_\infty \oplus \ell_\infty \to \ell_\infty$. Also it can be decomposed as

$$H_*(\ell_\infty) \cong H_*(\mathfrak{sp}(2\infty;\mathbb{Q})) \otimes H_*(\mathfrak{h}_\infty^{\mathbb{Q}})_{Sp}.$$

By making use of his theory of graph cohomology together with a result of Culler–Vogtmann [17], he proved the following remarkable theorem.

Theorem 6.11 (Kontsevich [85, 86]) *There exists an isomorphism*

$$PH_k(\ell_\infty) \cong PH_k(\mathfrak{sp}(2\infty;\mathbb{Q})) \oplus \bigoplus_{n\geq 2} H^{2n-2-k}(\mathrm{Out}\,F_n;\mathbb{Q})$$

where PH_k denotes the primitive part of the k–dimensional homology and $\mathrm{Out}\,F_n$ denotes the outer automorphism group of the free group of rank n. More precisely, for each even degree $2n$ $(n>0)$ with respect to the grading of ℓ_∞, there exists an isomorphism

$$PH_k(\ell_\infty)_{2n} \cong H^{2n-k}(\mathrm{Out}\,F_{n+1};\mathbb{Q}).$$

Passing to the dual, we also have an isomorphism

$$PH^k_c(\mathfrak{h}^{\mathbb{Q}}_\infty)^{Sp}_{2n} \cong H_{2n-k}(\mathrm{Out}\,F_{n+1};\mathbb{Q}).$$ □

Let us observe here that if Conjecture 6.10 were true, then $H_1(\mathfrak{h}^{\mathbb{Q}}_\infty)_{Sp} = 0$ so that we can conclude $H^{2n-3}(\mathrm{Out}\,F_n;\mathbb{Q}) = 0$ for any $n \geq 2$ by the above theorem of Kontsevich. We have checked that this is the case up to $n=4$. We mention that Culler–Vogtmann proved, in the above paper, that the virtual cohomological dimension of $\mathrm{Out}\,F_n$ is equal to $2n-3$.

Now we apply Theorem 6.11 to the homomorphism (10). We know that there is a copy $S^3H_{\mathbb{Q}} = [3] \subset \mathfrak{h}^{\mathbb{Q}}_{g,1}(3)$ which goes to $S^3H_{\mathbb{Q}}$ bijectively by the trace $\mathrm{Tr}(3)$.

Proposition 6.12 *The homomorphism*

$$\Lambda^2 S^3 H_{\mathbb{Q}} \ni \xi\wedge\eta \longmapsto [\xi,\eta] \in \mathfrak{h}^{\mathbb{Q}}_{g,1}(6) \quad (\xi,\eta\in S^3H_{\mathbb{Q}})$$

is injective.

Sketch of Proof It is easy to see that the irreducible decomposition of the module $\Lambda^2 S^3 H_{\mathbb{Q}} = \Lambda^2[3]$ is given by

$$\Lambda^2 S^3 H_{\mathbb{Q}} = [51] + [4] + [3^2] + [2^2] + [1^2] + [0].$$

Then the result follows from rather long explicit computations, in the framework of symplectic representation theory, of the Lie bracket $\Lambda^2[3] \to \mathfrak{h}^{\mathbb{Q}}_{g,1}(6)$. □

Recall that similar homomorphism

$$\Lambda^2(\Lambda^3 H_{\mathbb{Q}}) \longrightarrow \mathfrak{h}_{g,1}^{\mathbb{Q}}(2)$$

has a big kernel (Hain [41], see also [118] for the case of one boundary component). Thus we find that the behaviour of the trace component is fairly different from that of the moduli part. In particular, the pull back of any class in $H^2(S^3 H_{\mathbb{Q}})$ to the subalgebra of $\mathfrak{h}_{g,1}^{\mathbb{Q}}$ generated by $S^3 H_{\mathbb{Q}}$ is trivial. However if we consider the interaction of the trace component with the moduli part, the situation changes. In fact, we obtain the following result.

Proposition 6.13 *The pull back of the Sp–invariant part $H^2(S^3 H_{\mathbb{Q}})^{Sp} \cong \mathbb{Q}$ to $H_c^2(\mathfrak{h}_\infty^{\mathbb{Q}})^{Sp}$ by the trace* $\mathrm{Tr}(3)$ *is a non-trivial primitive element.*

Sketch of Proof By Proposition 6.12, there is a chain $u = \sum_i (a_i, b_i) \in C_2(\mathfrak{h}_{g,1}^{\mathbb{Q}})$ such that $a_i, b_i \in S^3 H_{\mathbb{Q}} \subset \mathfrak{h}_{g,1}^{\mathbb{Q}}(3)$ and ∂u is a non-zero element of $\mathfrak{h}_{g,1}^{\mathbb{Q}}(6)^{Sp}$. On the other hand, explicit computation shows that there is a chain $v = \sum_j (a'_j, b'_j) \in C_2(\mathfrak{h}_{g,1}^{\mathbb{Q}})$ such that $a'_j \in \mathfrak{h}_{g,1}^{\mathbb{Q}}(1), b'_j \in \mathfrak{h}_{g,1}^{\mathbb{Q}}(5)$ and $\partial v = \partial u$. We can arrange the above elements so that the 2–cycle $u - v$ is in fact an Sp–invariant one. Thus it defines an element of $H_2(\mathfrak{h}_\infty^{\mathbb{Q}})_6^{Sp}$ on which the cohomology class in question takes a non-zero value. The primitivity follows from the property of the trace. □

If we combine the above result with Thoerem 6.11, we can conclude the non-triviality of $H_4(\mathrm{Out}\, F_4; \mathbb{Q})$. This seems to be consistent with a recent result of Hatcher–Vogtmann [54] in which they proved that $H_4(\mathrm{Aut}\, F_4; \mathbb{Q}) \cong \mathbb{Q}$ (Vogtmann informed us that she obtained an isomorphism $H_4(\mathrm{Aut}\, F_4; \mathbb{Q}) \cong H_4(\mathrm{Out}\, F_4; \mathbb{Q})$ by a computer calculation).

If we use higher traces $\mathrm{Tr}(2k+1)$ in the above consideration, we obtain infinitely many classes in $H_c^2(\mathfrak{h}_\infty^{\mathbb{Q}})^{Sp}$ and these in turn give rise to a series of certain homology classes in

$$H_{4k}(\mathrm{Out}\, F_{2k+2}; \mathbb{Q}) \quad (k = 1, 2, \cdots).$$

It seems highly likely that all of these classes are non-trivial. Also by combining various trace components as well as the moduli part at the same time, we can define many primitive (co)homology classes of the Lie algebra ℓ_∞.

6.6 Secondary characteristic classes of surface bundles

Chern classes and Pontrjagin classes are representatives of characteristic classes of vector bundles and they play fundamental roles in diverse branches of mathematics. We may call these classes *primary* characteristic classes. In some cases where these primary classes vanish, there arise various theories of so-called *secondary* characteristic classes. For example, we have the theory of characteristic classes of foliations, characteristic classes of flat bundles, the theory of Chern and Simons and also that of Cheeger and Simons and so on.

In the case of surface bundles, it is natural to call the classes e_i the primary characteristic classes. As was mentioned in section 3, the odd classes e_{2i-1} come from the Siegel modular group $Sp(2g,\mathbb{Z})$ via the classical representation $\rho_0\colon \mathcal{M}_g \to Sp(2g,\mathbb{Z})$. Hence these classes (with rational coefficients) vanish on the Torelli group $\mathcal{I}_g$. It is a fundamental question concerning the cohomology of $\mathcal{I}_g$ whether even classes e_{2i} are non-trivial on it or not (see Conjecture 3.4). On the other hand, it was proved in [118] that any class e_i, including the cases of even i, vanish on the subgroup $\mathcal{K}_g$ of the mapping class group $\mathcal{M}_g$, which is the subgroup generated by all Dehn twists along *separating* simple closed curves on Σ_g (see section 2). The proof of this fact goes roughly as follows. As was recalled in section 6.4, in our paper [118] we have constructed certain natural cocycles for e_i and by the very definition they vanish on $\mathcal{K}_g$. Thus there are *two* reasons with different sources that the odd classes e_{2i-1} vanish on $\mathcal{K}_g$.

By making use of this fact, we can define the *secondary* characteristic classes of surface bundles. One way to do so can be described as follows. For each e_{2i-1}, choose two cocycles c, c' both of which represent e_{2i-1} such that c comes from the Siegel modular group while c' is a cocycle constructed in [118] using the linear representation ρ_1 described there. Since these two cocycles are cohomologous to each other, there exists a cochain

$$d \in C^{4i-3}(\mathcal{M}_g;\mathbb{Q})$$

such that $\delta d = c - c'$. Now both of c, c' are 0 on $\mathcal{K}_g$ so that if we restrict d to $\mathcal{K}_g$, it is a cocycle. Hence we can define a cohomology class

$$d_i \in H^{4i-3}(\mathcal{K}_g;\mathbb{Q})$$

to be the class of the cocycle $d \in Z^{4i-3}(\mathcal{K}_g;\mathbb{Q})$. It can be shown that the cohomology class of $[d]$ does not depend on the choices of the cocycles c, c' modulo the indeterminacy

$$\mathrm{Im}\big(H^{4i-3}(\mathcal{M}_g;\mathbb{Q}) \longrightarrow H^{4i-3}(\mathcal{K}_g;\mathbb{Q})\big).$$

We remark that if Conjecture 3.1 were true, then the above indeterminacy vanishes, at least in the stable range, so that d_i would be uniquely defined.

We know that the first one

$$d_1\colon\ \mathcal{K}_g \longrightarrow \mathbb{Q}$$

can be defined uniquely. Here we describe the precise formula for it. Let $\tau \in Z^2(\mathcal{M}_g;\mathbb{Z})$ be Meyer's signature cocyle given in [100] which is in fact defined on $Sp(2g,\mathbb{Z})$. It represents $-\frac{1}{3}e_1$. On the other hand in [110, 114] we gave another cocycle for e_1. More precisely, in the notation of section 6.4 the cocycle

$$c' = \frac{1}{2g+1}\,(-3\alpha_{\Gamma_1} + (2g-2)\alpha_{\Gamma_2})$$

represents e_1 (see [44] for an interpretation of this result as well as others from the point of view of algebraic geometry). Here Γ_1 is the trivalent graph which has 2 vertices and 2 loops while Γ_2 is the trivalent graph with 2 vertices and without loops (namely the *theta* graph). The above two cocycles are cohomologous to each other so that there exists a mapping

$$d_1\colon\ \mathcal{M}_g \longrightarrow \mathbb{Q} \quad (g \geq 2)$$

such that $\delta d_1 = -3\tau - c'$. Since $\mathcal{M}_g$ is perfect for $g \geq 3$ and $H_1(\mathcal{M}_2) = \mathbb{Z}/10$, the above map d_1 is uniquely defined. Also since the restriction of both of τ and c' to the subgroup $\mathcal{K}_g$ is trivial, we obtain a homomorphism

$$d_1\colon\ \mathcal{K}_g \longrightarrow \mathbb{Q}.$$

This is the definition of our secondary class $d_1 \in H^1(\mathcal{K}_g;\mathbb{Q})$.

Theorem 6.14 (Morita [112]) *Let $\varphi \in \mathcal{K}_g$ be a Dehn twist along a separating simple closed curve on Σ_g such that it divides Σ_g into two compact surfaces of genus h and $g-h$. Then the value of the secondary class $d_1 \in H^1(\mathcal{K}_g;\mathbb{Q})$ on it is given by*

$$d_1(\varphi) = \frac{12}{2g+1}\, h(g-h).$$

Moreover d_1 is the generator of $H^1(\mathcal{K}_g;\mathbb{Q})^{\mathcal{M}_g} \cong \mathbb{Q}$ for all $g \geq 2$. □

In our paper [112], the coefficient of $h(g-h)$ in the above formula for $d_1(\varphi)$ was not mentioned. However it is easy to deduce it from the results obtained there. In [119] we gave an interpretation of d_1 in terms of Hirzebruch's signature defect of certain framed 3–manifolds. Generalizing this, we obtained another more geometrical definition of higher secondary classes $d_i \in H^{4i-3}(\mathcal{K}_{g,1};\mathbb{Q})$. We expect that these two definitions of d_i would coincide for all i.

Remark Hain informed us the following interesting facts. In the above theorem, the number $h(g-h)$ appeared in relation to the Dehn twist φ. About the same time, the same number appeared in the works of Jorgenson [67] and Wentworth [143]. In fact, it plays the role of the principal factor in their asymptotic formula of Faltings delta function around the divisor Δ_h, which is associated to φ, of the Deligne–Mumford compactification $\overline{\mathbf{M}}_g$. This number plays one more important role also in a recent work of Moriwaki [121] where he obtained a certain inequality related to the cone of positive divisors on $\overline{\mathbf{M}}_g$. In fact, Hain has an interpretation that these phenomena are more than just a coincidence. Kawazumi is also trying to develop a theory related to d_1. From the topological point of view, we may say that they are the manifestation of the Casson invariant in the geometry of the moduli space of curves. We expect that there should exist very rich structures here which deserve future investigations.

Conjecture 6.15 All of the secondary classes $d_i \in H^{4i-3}(\mathcal{K}_g;\mathbb{Q})$ are uniquely defined and non-trivial if g is sufficiently large. Moreover we have an isomorphism

$$\lim_{g\to\infty} H^*(\mathcal{K}_g;\mathbb{Q})^{\mathcal{M}_g} \cong E_{\mathbb{Q}}(d_1, d_2, d_3, \cdots).$$

There has been rapid progress in the theory of the moduli space $\overline{\mathbf{M}}_g$ related to the primary charactersitic classes κ_i or e_i, namely from the celebrated works of Witten [145] and Kontsevich [84] to the recent developments reaching to the Gromov–Witten invariants (see [27, 35] and references in them). In contrast with this, the secondary classes d_i seem to be beyond the reach of any explicit study at present, except for the first one. In relation to the first class d_1, it seems to be a challenging problem to try to generalize the genus one story given in Atiyah's paper [8] to the cases of higher genera in various ways. We believe that the higher classes d_i will also eventually play an important role in hopefully deeper geometrical study of the moduli space.

6.7 Bounded cohomology of $\mathcal{M}_g$

In this subsection we consider the mapping class group from the viewpoint of Gromov's bounded cohomology (see [37]). Recall that the bounded cohomology (with coefficients in $\mathbb{R}$) of a group Γ, denoted by $H_b^*(\Gamma)$, is defined to be the cohomology of the subcomplex of the ordinary $\mathbb{R}$–valued cochain complex $C^*(\Gamma;\mathbb{R})$ consisting of all cochains

$$c\colon\ \Gamma\times\cdots\times\Gamma\longrightarrow\mathbb{R}$$

which are bounded as functions. We have a natural homomorphism

$$H_b^*(\Gamma) \longrightarrow H^*(\Gamma; \mathbb{R})$$

and it reflects algebraic as well as geometric properties of the group Γ rather closely. For example if Γ is amenable, then $H_b^*(\Gamma)$ is trivial in positive degrees while if Γ is the fundamental group of a closed negatively curved manifold, then the above map (except for degree one part) is known to be surjective by an argument due to Thurston (see [37]).

Problem 6.16 Study the bounded cohomology of the mapping class group $\mathcal{M}_g$. More precisely, determine the kernel as well as the image of the natural map $H_b^*(\mathcal{M}_g) \to H^*(\mathcal{M}_g; \mathbb{R})$.

In particular, we may ask whether the characteristic class $e_i \in H^{2i}(\mathcal{M}_g)$ can be represented by a bounded cocycle or not. It was remarked in [109] that Gromov's general result in [37] implies that any odd class e_{2i-1} can be represented by a bounded cocycle. This is because, as mentioned in section 3, these classes are pull backs of some classical characteristic classes of $Sp(2g, \mathbb{Z})$ which is a discrete subgroup of $Sp(2g, \mathbb{R})$. It seems to be natural to conjecture that even classes e_{2i} can also be represented by bounded cocycles. One evidence for this was given in [109] where we proved that any surface bundle with amenable monodromy group has trivial characteristic classes as it should be if e_i were all bounded cohomology classes. Meyer's signature cocycle given in [100] is an explicit bounded cocycle which represents $-\frac{1}{3}e_1$ (see [111]) but no other explicit bounded cocycle has been constructed for higher odd classes. We mention here that the cocycles for e_i constructed in [118, 76] using trivalent graphs are far from being bounded.

Problem 6.17 Construct explicit *bounded* cocycles of $\mathcal{M}_g$ which represent the characteristic classes e_i for $i > 1$.

The particular case of degree 2, namely the map

$$H_b^2(\mathcal{M}_g) \longrightarrow H^2(\mathcal{M}_g; \mathbb{R})$$

already deserves further investigation. Harer's determination of $H^2(\mathcal{M}_g)$ in [46] together with Meyer's result [100] mentioned above implies that the above map is surjective. If $g = 1$, then $\mathcal{M}_1 = SL(2, \mathbb{Z})$ so that $H^2(SL(2, \mathbb{Z}); \mathbb{R}) = 0$ while it is well known that $H_b^2(SL(2, \mathbb{Z}))$ is infinite dimensional. If $g = 2$, then we know that $H^2(\mathcal{M}_2; \mathbb{R}) = 0$ because $\mathbf{M}_2$ is contractible by a result of Igusa [55].

Proposition 6.18 *The two dimensional bounded cohomology $H_b^2(\mathcal{M}_2)$ of the genus 2 mapping class group is non-trivial.*

Proof We consider the cochain $d_1 \in C^1(\mathcal{M}_g;\mathbb{Q})$ described in section 6.6. If $g=2$, then $U=0$ so that $\delta d_1 = -3\tau$ where $\tau \in Z^2(\mathcal{M}_2)$ is Meyer's signature 2–cocycle for the genus 2 mapping class group. Since $H^1(\mathcal{M}_2;\mathbb{Q}) = 0$, d_1 is uniquely determined by the above equality. Observe that d_1 is not a bounded cochain because $d_1\colon \mathcal{K}_2 \to \mathbb{Q}$ is a non-trivial homomorphism. We can now conclude that the bounded cohomology class $[\tau] \in H_b^2(\mathcal{M}_2)$ is non-trivial. □

Remark Meyer's signature cocycle has been investigated from various points of view, see work of Y. Matsumoto [98] for the case of $g=2$, Endo [21] and Morifuji [106] for the cases of hyperelliptic mapping class groups and Morifuji [105] and Kasagawa [69, 70] for certain geometric aspects of it.

Epstein and Fujiwara [22] proved that the second bounded cohomology of any Gromov hyperbolic group is infinite dimensional. Although the mapping class group is not a Gromov hyperbolic group, because it contains many free abelian subgroup of rather high ranks, it was proved by Tromba [139] and Wolpert [146] (see also [147]) that the sectional curvature of the Teichmüller space with respect to the Weil–Petersson metric is negative.

Conjecture 6.19 The mapping $H_b^2(\mathcal{M}_g) \to H^2(\mathcal{M}_g;\mathbb{R})$ is not injective for all g. More strongly, $H_b^2(\mathcal{M}_g)$ would be infinite dimensional.

We recall the following definition which is relevant to the above problem.

Definition 6.20 ([97]). A group Γ is said to be *uniformly perfect* if there exists a natural number N such that any element $\gamma \in \Gamma$ can be expressed as a product of at most N commutators.

Recall that $\mathcal{M}_g$ is known to be perfect for all $g \geq 3$ (see [46]).

Conjecture 6.21 The mapping class group $\mathcal{M}_g$ is *not* uniformly perfect for all $g \geq 3$.

It was proved in [97] that if Γ is a uniformly perfect group, then the mapping $H_b^2(\Gamma) \to H^2(\Gamma;\mathbb{R})$ is injective. Hence if the former part of Conjecture 6.19 were true, then Conjecture 6.21 is also true. In the case of $g=2$, $\mathcal{M}_2$ is not perfect. However it is also known that its abelianization is finite, namely we have $H_1(\mathcal{M}_2) \cong \mathbb{Z}/10$. The following result is a companion of Proposition 6.18.

Proposition 6.22 *There is no natural number N such that any element in the commutator subgroup of the genus 2 mapping class group can be expressed as a product of at most N commutators.*

Proof If we assume the contrary, then it would follow that the value of the cochain d_1 is bounded. But as was mentioned above, this is not the case. $\square$

6.8 Representations of the mapping class group

In this subsection, we consider various representations of the mapping class group. First we mention the Magnus representation of the Torelli group. Let $\mathcal{M}_{g,1}$ be the mapping class group of Σ_g *relative* to an embedded disk $D \subset \Sigma_g$ as before and let $\mathbb{Z}[\Gamma]$ be the integral group ring of $\Gamma = \pi_1(\Sigma_g \setminus \operatorname{Int} D)$. Following a general theory of so-called Magnus representation described in Birman's book [11], the author defined in [115] a mapping

$$\rho\colon\ \mathcal{M}_{g,1} \longrightarrow GL(2g, \mathbb{Z}[\Gamma])$$

and considered various properties of it. It is easy to see that this mapping is injective. However it is not a homomorphism in the usual sense but is rather a crossed homomorphism. To obtain a genuine homomorphism, we have to restrict this mapping to the Torelli group $\mathcal{I}_{g,1} \subset \mathcal{M}_{g,1}$ and reduce the coefficients to $\mathbb{Z}[H]$ which is induced by the abelianization $\Gamma \to H$. Then we obtain a homomorphism

$$\rho\colon\ \mathcal{I}_{g,1} \longrightarrow GL(2g, \mathbb{Z}[H])$$

(see Corollary 5.4 of [115]).

Problem 6.23 Determine whether the representaion $\rho\colon\ \mathcal{I}_{g,1} \longrightarrow GL(2g, \mathbb{Z}[H])$ described above is injective or not.

Here we would like to mention that Moody proved in [103] that the Burau representation of the braid group B_n is not faithful for sufficiently large n while it seems to be still unknown whether the Gassner representation of the pure braid group is faithful or not.

The augmentation ideal of $\mathbb{Z}[H]$ induces a filtration of $\mathcal{I}_{g,1}$. It is easy to see that this filtration is strictly coarser than $\{\mathcal{M}_{g,1}(k)\}_{k \geq 1}$ which was described in section 5. It would be interesting to study how they differ from each other.

Besides the Magnus representation, we have now various representations of the mapping class group associated to newly developed theories which are related to low dimensional topology. For example, we have projective representations of $\mathcal{M}_g$ arising from the conformal field thoery (see [144, 140, 83], see also [148, 30] for certain explicit studies of them) or from the theory of universal perturbative invariants of 3–manifolds (see [88, 124, 125]). We also have Jones representations [66] of the hyperelliptic mapping class group and the Prym representations of certain subgroups of $\mathcal{M}_g$ given by Looijenga [94]. It seems that there are only a few results which clarify how these representations are related to the structure of the mapping class group.

Problem 6.24 Study various properties of the above representations of the mapping class group. In particular, determine the kernel as well as the image of them.

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