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Positive links are strongly quasipositive

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Abstract Let $S(\mathbf{D})$ be the surface produced by applying Seifert's algorithm to the oriented link diagram $\mathbf{D}$. I prove that if $\mathbf{D}$ has no negative crossings then $S(\mathbf{D})$ is a quasipositive Seifert surface, that is, $S(\mathbf{D})$ embeds incompressibly on a fiber surface plumbed from positive Hopf annuli. This result, combined with the truth of the "local Thom Conjecture", has various interesting consequences; for instance, it yields an easily-computed estimate for the slice euler characteristic of the link $L(\mathbf{D})$ (where $\mathbf{D}$ is arbitrary) that extends and often improves the "slice–Bennequin inequality" for closed-braid diagrams; and it leads to yet another proof of the chirality of positive and almost positive knots.

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For Rob Kirby

1 Introduction; statement of results

Given an oriented link diagram $\mathbf{D}$, let $\mathsf{X}_>(\mathbf{D})$ (resp. $\mathsf{X}_<(\mathbf{D})$) be the set of positive (resp. negative) crossings, and $\mathsf{O}_\geq(\mathbf{D})$ (resp. $\mathsf{O}_<(\mathbf{D})$) the set of Seifert circles adjacent to some $\mathsf{x} \in \mathsf{X}_>(\mathbf{D})$ (resp. to no $\mathsf{x} \in \mathsf{X}_>(\mathbf{D})$). Write $\#A$ for the number of elements of A. Let $S(\mathbf{D})$ be the surface produced by applying Seifert's algorithm to $\mathbf{D}$; let $L(\mathbf{D}) := \partial S(\mathbf{D})$. An oriented link L is *positive* if L is isotopic to $L(\mathbf{D})$ for some $\mathbf{D}$ with $\#\mathsf{X}_<(\mathbf{D}) = 0$, *almost positive* if L is not positive but L is isotopic to $L(\mathbf{D})$ for some $\mathbf{D}$ with $\#\mathsf{X}_<(\mathbf{D}) = 1$, and *strongly quasipositive* if L bounds a *quasipositive Seifert surface*, that is, a surface embedded incompressibly on a fiber surface plumbed from positive Hopf annuli.

Theorem *If $\#\mathsf{X}_<(\mathbf{D}) = 0$, then $S(\mathbf{D})$ is quasipositive, and so $L(\mathbf{D})$ is strongly quasipositive.*

Denote by $\chi_s(L)$ the greatest value of the euler characteristic $\chi(F)$ for $F \subset D^4$ a smooth, oriented 2–manifold such that $L = \partial F$ and F has no closed components. Let $D^1(K)$ be the untwisted positive Whitehead double of a knot K, $D^k(K) := D^1(D^{k-1}(K))$.

Corollary 1 *If* $\mathbf{D}$ *is any oriented link diagram, then*

$$(*[\mathbf{D}]) \qquad \chi_s(L(\mathbf{D})) \le (\#\mathsf{O}_{\ge}(\mathbf{D}) - \#\mathsf{O}_{<}(\mathbf{D})) - (\#\mathsf{X}_{>}(\mathbf{D}) - \#\mathsf{X}_{<}(\mathbf{D})).$$

Corollary 2 (A) *A non-trivial positive link is chiral.* (B) *An almost positive knot is chiral.*

This corollary is included chiefly for the novelty of the method; see Remark 4.

Corollary 3 *If* K *is a non-trivial positive knot, then* $\chi_s(D^k(K)) = -1$, $k > 0$.

This corollary, with "strongly quasipositive" in place of "positive", was proved in [14], so it is immediate from the Theorem. It is a partial extension of a result of Cochran and Gompf [1], which assumes much less than positivity of K, but concludes only that $\chi_s(D^k(K)) = -1$, $1 \le k \le 6$; see Remark 5.

Remarks (1) As defined above, strong quasipositivity is an intrinsic geometric property. Its original definition ([11]; cf [9]), like those of positivity and almost-positivity, was "diagrammatic": an oriented link L is strongly quasipositive if and only if, for some n, L can be represented as the closure $\widehat{\beta}$ of a braid $\beta \in B_n$ that is the product of "embedded positive bands" $\sigma_{i,j} := (\sigma_i \ldots \sigma_{j-2})\sigma_{j-1}(\sigma_i \ldots \sigma_{j-2})^{-1}$ (where $\sigma_1, \ldots, \sigma_{n-1}$ are the standard generators of B_n). The equivalence of this to the intrinsic definition follows by combining [12] and [16].

Question Can positive links be characterized as strongly quasipositive links that satisfy some extra geometric conditions?

(2) If Corollary 1 is weakened by restricting $\mathbf{D}$ to be a closed braid diagram and by replacing $(*[\mathbf{D}])$ by

$$(*'[\mathbf{D}]) \qquad \chi_s(L(\mathbf{D})) \le (\#\mathsf{O}_{\ge}(\mathbf{D}) + \#\mathsf{O}_{<}(\mathbf{D})) - (\#\mathsf{X}_{>}(\mathbf{D}) - \#\mathsf{X}_{<}(\mathbf{D})),$$

then it becomes the *slice–Bennequin inequality* sBi, [14]; like the proof of Corollary 1 below, the proof of sBi in [14] makes essential use of the truth of the "local Thom Conjecture", a result originally established using gauge theory for embedded surfaces [3] which now follows from more general results established using monopole methods [4], [5]. (In fact, $(*[\mathbf{D}])$ for a closed braid diagram $\mathbf{D}$ already follows from sBi, [16], Corollary 5.2.2.)

(3) A different weakening of Corollary 1—allowing $\mathbf{D}$ to be arbitrary, but concluding only $(*'[\mathbf{D}])$—follows easily by combining sBi and the (proof of the) main result of [20]. In fact, after posting the first version of the present article to the xxx Mathematics Archives, I received e-mail from Takuji Nakamura informing me that Nakamura's February 1998 master's thesis at Keio University (Japan) gives a proof of the Theorem using the techniques of [20]; and another reader has since kindly shown me how to use Vogel's algorithm [19] to give yet another proof.

(4) Chirality of non-trivial positive links was established well over a decade ago: the proof in [1] (which does not, itself, depend on "Donaldson's theorems", ie, gauge theory, but uses only a classical invariant, the signature) was described by Cochran and Gompf in talks at MSRI in 1985 and at the 1986 Santa Cruz Summer Research Conference on braids; other proofs using the signature were published independently by Przytycki [7] and Traczyk [18] (cf [10] for an earlier special case, with an unconventional choice of sign). Chirality of almost positive links (via negativity of the signature) was announced in a 1991 abstract [8] by Przytycki and Taniyama; Stoimenow has recently given a proof [17] (for knots) which uses a Vassiliev-style invariant due to Fiedler. Chirality of non-trivial strongly quasipositive knots follows from a calculation with the FLYPMOTH link polynomial ([6], remark following Problem 9), and chirality of non-trivial strongly quasipositive links is an easy consequence of sBi (cf [6], Problems 8.2 and 9.2). When I came across [17] while perusing `http://front.math.ucdavis.edu/math.GT/` preparatory to uploading what I had thought was the finished version of the present paper, I realized that an alternative proof of the chirality of almost positive links could be given using Corollary 1, and I revised this paper accordingly. Many thanks are due to the maintainers both of the xxx Mathematics Archives and of the Front for keeping us all on our toes. Thanks also to Tim Cochran, Jozef Przytycki, and others for helpful e-mail on the history of chirality results for positive and almost positive links.

(5) A long-standing conjecture ([2], Problem 1.38) asserts that, for a knot K, $\chi_s(D^1(K)) = 1$ iff $\chi_s(K) = 1$. Since any Whitehead double of a knot bounds a punctured torus (already in S^3), this conjecture implies a second conjecture: if $\chi_s(K) < 1$, then $\chi_s(D^k(K)) = -1$ for $1 \leq k < \infty$. Cochran and Gompf [1] made some progress towards the second conjecture: they defined what it means for a knot K to be "greater than or equal to the positive trefoil" (briefly,

$K \geq \mathbb{T}$); they proved (by way of providing a large class of examples) that if $K \neq O$ is a positive knot (in particular, a closed positive braid) then $K \geq \mathbb{T}$; and they applied gauge theory to show that if $K \geq \mathbb{T}$ then $\chi_s(K) < 1$ and $\chi_s(D^k(K)) = -1$ for $1 \leq k \leq 6$. In [14], I proved the second conjecture if $K \neq O$ is a strongly quasipositive knot (in particular, a closed positive braid). Corollary 3 is the observation that the second conjecture is true for positive knots.

Questions Is the second conjecture true for all $K \geq \mathbb{T}$? Might it in fact be the case that $K \geq \mathbb{T}$ implies K is strongly quasipositive?

(6) If $L(\mathbf{D})$ is a knot K, then the estimate in [15] can be rewritten as

$$(*''[\mathbf{D}]) \qquad \chi_s(K) \leq m(\mathbf{D}) - (\#\mathsf{X}_>(\mathbf{D}) - \#\mathsf{X}_<(\mathbf{D}) - s_-(\mathbf{D})),$$

where $\mathbf{D} \subset \mathbb{C}$ is taken to be in general position with respect to $\Im\colon \mathbb{C} \to \mathbb{R} : z \mapsto (z - \bar{z})/2i$, $m(\mathbf{D})$ is the number of local maxima of $\Im|\mathbf{D}$, and $s_-(\mathbf{D})$ is the number of crossings of $\mathbf{D}$ which (disregarding their actual signs) are "locally negative" when oriented by $\Im$. This inequality resembles $(*'[\mathbf{D}])$ (or sBi).

Question Is there—in general, or in the special case that $s_-(\mathbf{D}) = 0$ (so that $\mathbf{D}$ is a *positive plat diagram* of K, [13])—a modification of $(*''[\mathbf{D}])$, analogous to $(*[\mathbf{D}])$, in which (for some $\mathbf{D}$) some local maxima of $\Im|\mathbf{D}$ contribute -1 rather than 1 to the right-hand side?

(7) Let $u(K)$ denote the *unknotting number* of the knot K. Of course for all K, $u(K) \geq (1 - \chi_s(K))/2$; there exist K such that the estimate for $u(K)$ based on $(*[\mathbf{D}])$ is sharper than the estimate based on sBi.

2 Proof that positive links are strongly quasipositive

In preparation for the proofs, recall Seifert's algorithm. Given an oriented link diagram $\mathbf{D} \subset \mathbb{C}$ (where $\mathbb{C}$ has its standard orientation), let $\mathsf{X}(\mathbf{D}) := \mathsf{X}_>(\mathbf{D}) \cup \mathsf{X}_<(\mathbf{D})$, $\mathsf{O}(\mathbf{D}) := \mathsf{O}_\geq(\mathbf{D}) \cup \mathsf{O}_<(\mathbf{D})$. The algorithm, given input $\mathbf{D}$, produces a piecewise-smooth oriented surface $S(\mathbf{D}) \subset \mathbb{C} \times \mathbb{R}$ equipped with a $(0,1)$–handle decomposition $S(\mathbf{D}) = \bigcup_{\mathsf{o} \in \mathsf{O}(\mathbf{D})} h^{(0)}_{\mathsf{o}} \cup \bigcup_{\mathsf{x} \in \mathsf{X}(\mathbf{D})} h^{(1)}_{\mathsf{x}}$ such that:

(1) $\mathrm{pr}_1(\partial S(\mathbf{D}))$ is the underlying oriented immersed 1–manifold of $\mathbf{D}$;

(2) for every $\mathsf{o} \in \mathsf{O}(\mathbf{D})$, $\mathrm{pr}_1|h^{(0)}_{\mathsf{o}}\colon h^{(0)}_{\mathsf{o}} \to \mathbb{C}$ is a (non-oriented) embedding with $\mathrm{pr}_1(\partial h^{(0)}_{\mathsf{o}}) = \mathsf{o}$;

(3) for every $\mathsf{x} \in \mathsf{X}(\mathbf{D})$,

(3.1) there is one transverse arc $\tau(h^{(1)}_{\mathsf{x}}) \subset h^{(1)}_{\mathsf{x}}$ such that $\mathrm{pr}_1(\tau(h^{(1)}_{\mathsf{x}})) = \mathsf{x}$,

(3.2) $\mathrm{pr}_1 |(h^{(1)}_{\mathsf{x}} \setminus \tau(h^{(1)}_{\mathsf{x}}))$ is an embedding, and preserves orientation on precisely one component of $h^{(1)}_{\mathsf{x}} \setminus \tau(h^{(1)}_{\mathsf{x}})$,

(3.3) $h^{(1)}_{\mathsf{x}}$ is attached to $h^{(0)}_{\mathsf{o}}$ iff x is adjacent to o in $\mathbf{D}$, and

(3.4) the over-arc of $\mathbf{D}$ through x contains the image by pr_1 of the component of $h^{(1)}_{\mathsf{x}} \cap \partial S(\mathbf{D})$ which contains that endpoint of $\tau(h^{(1)}_{\mathsf{x}})$ at which pr_2 takes on the larger value.

Call $\mathsf{o} \in \mathsf{O}(\mathbf{D})$ *outermost* if $\mathsf{o} \cap \mathrm{pr}_1(h^{(0)}_{\mathsf{o}'}) = \varnothing$ for $\mathsf{o} \neq \mathsf{o}' \in \mathsf{O}(\mathbf{D})$; let $\mathsf{OO}(\mathbf{D})$ be the set of outermost Seifert circles of $\mathbf{D}$. Of course $\#\mathsf{OO}(\mathbf{D}) \geq 1$.

Proof of Theorem Let $\#\mathsf{X}_<(\mathbf{D}) = 0$. The split sum of quasipositive Seifert surfaces is quasipositive, so there is no loss of generality in assuming that $\mathbf{D}$ is connected. The proof proceeds by induction, first on $\#\mathsf{X}(\mathbf{D})$ and then on $|\#\mathsf{OO}(\mathbf{D}) - 2|$.

If $\#\mathsf{X}(\mathbf{D}) = 0$ then $\mathbf{D}$ is the trivial diagram of the trivial knot and the Theorem is trivially true.

If $\#\mathsf{X}(\mathbf{D}) > 0$ and $\#\mathsf{OO}(\mathbf{D}) = 1$, then $\mathbf{D}$ can be replaced by $\mathbf{D}'$ (isotopic to $\mathbf{D}$ on $S^2 = \mathbb{C} \cup \{\infty\}$) such that $\mathsf{O}(\mathbf{D}') = \mathsf{O}(\mathbf{D}) \cup \{\mathsf{o}'_0\} \setminus \{\mathsf{o}_0\}$, $\mathsf{X}(\mathbf{D}') = \mathsf{X}(\mathbf{D})$, and $S(\mathbf{D}')$ is ambient isotopic to $S(\mathbf{D})$ in $\mathbb{C} \times \mathbb{R}$. Thus it may be assumed that $\#\mathsf{OOD} \geq 2$. Since $\mathbf{D}$ is connected, there exist $\mathsf{o}_1, \mathsf{o}_2 \in \mathsf{OO}(\mathbf{D}))$ such that $\mathrm{pr}_1 |h^{(0)}_{\mathsf{o}}\colon h^{(0)}_{\mathsf{o}_1} \to \mathbb{C}$ preserves orientation and $\mathrm{pr}_1 |h^{(0)}_{\mathsf{o}}\colon h^{(0)}_{\mathsf{o}_2} \to \mathbb{C}$ reverses orientation.

If $\#\mathsf{OO}(\mathbf{D}) = 2$, then the union F of $h^{(0)}_{\mathsf{o}_1}$ and $h^{(0)}_{\mathsf{o}_2}$, together with all the 1–handles $h^{(1)}_{\mathsf{x}}$ such that x is adjacent to both o_1 and o_2, is ambient isotopic to $S(\mathbf{D}(\sigma_1^k))$, where $k \geq 1$ is the number of those 1–handles and $\mathbf{D}(\sigma_1^k)$ is the positive closed braid diagram of $\sigma_1^k \in B_2$; moreover, for $i = 1, 2$, the union G_i of $h^{(0)}_{\mathsf{o}_i}$ with all the 0–handles $h^{(0)}_{\mathsf{o}}$ such that $\mathsf{o} \subset \mathrm{pr}_1(h^{(0)}_{\mathsf{o}_i})$, together with all the 1–handles $h^{(1)}_{\mathsf{x}}$ to which any of these 0–handles is attached, is ambient isotopic to $S(\mathbf{D}_i)$ for an appropriate positive oriented link diagram; finally, $S(\mathbf{D})$ is an iterated Murasugi sum (Stallings plumbing) $G_1 * F * G_2$. Now, $S(\mathbf{D}(\sigma_1^k))$, the fiber surface of the $(2, k)$ torus link, is well known to be quasipositive (an explicit plumbing from positive Hopf annuli can be extracted from [12]), while $S(\mathbf{D}_1)$ and $S(\mathbf{D}_2)$ are quasipositive by induction on the number of crossings. Since, by [16], any plumbing of quasipositive Seifert surfaces is quasipositive, $S(\mathbf{D})$ is quasipositive.

If $\#\mathsf{OO}(\mathbf{D}) > 2$, then there exist $\mathsf{o}', \mathsf{o}'' \in \mathsf{OO}(\mathbf{D})$, $\mathsf{o}' \neq \mathsf{o}''$, such that $\mathrm{pr}_1 |h^{(0)}_{\mathsf{o}'}$ and $\mathrm{pr}_1 |h^{(0)}_{\mathsf{o}''}$ have the same orientation type (both preserve, or both reverse, orientation); a little thought about planar embeddings of bipartite graphs shows that such $\mathsf{o}', \mathsf{o}''$ may be chosen to have the further property that there is an arc $\alpha \subset \mathbb{C}$ with one endpoint on o' and the other endpoint on o'', which is otherwise disjoint from $\mathbf{D}$. Let $\mathbf{D}_\alpha$ be the positive link diagram with $\mathsf{X}(\mathbf{D}_\alpha) = \mathsf{X}(\mathbf{D})$ and $\mathsf{O}(\mathbf{D}_\alpha) = \mathsf{O}(\mathbf{D}) \cup \{\mathsf{o}' \#_\alpha \mathsf{o}''\} \setminus \{\mathsf{o}', \mathsf{o}''\}$, where $\mathsf{o}' \#_\alpha \mathsf{o}'' := \partial(\mathrm{pr}_1(h^{(0)}_{\mathsf{o}'}) \cup N(\alpha) \cup \mathrm{pr}_1(h^{(0)}_{\mathsf{o}''}))$ for a suitable relative regular neighborhood $N(\alpha)$ of α in $\mathbb{C} \setminus \mathrm{Int}(\mathrm{pr}_1(h^{(0)}_{\mathsf{o}'} \cup h^{(0)}_{\mathsf{o}''}))$. Evidently $S(\mathbf{D}_\alpha)$ may be constructed to contain $S(\mathbf{D})$, and then $S(\mathbf{D})$ is clearly embedded incompressibly on $S(\mathbf{D}_\alpha)$. By induction on the number of outermost Seifert circles, $S(\mathbf{D}_\alpha)$ is quasipositive. Since, by [12], an incompressible subsurface of a quasipositive Seifert surface is quasipositive, it follows that $S(\mathbf{D})$ is quasipositive. □

3 Proofs of the corollaries

Proof of Corollary 1 Let $Q(\mathbf{D}) := \bigcup_{\mathsf{o} \in \mathsf{O}_{\geq}(\mathbf{D})} h^{(0)}_{\mathsf{o}} \cup \bigcup_{\mathsf{x} \in \mathsf{X}_{>}(\mathbf{D})} h^{(1)}_{\mathsf{x}} \subset S(\mathbf{D})$. Then $Q(\mathbf{D})$ is clearly $S(\mathbf{D}^+)$, where $\mathrm{pr}_1(\partial Q(\mathbf{D}))$ is the underlying oriented immersed 1–manifold of $\mathbf{D}^+$, $\mathsf{O}(\mathbf{D}^+) = \mathsf{O}_{\geq}(\mathbf{D}^+) = \mathsf{O}_{\geq}(\mathbf{D})$, and $\mathsf{X}(\mathbf{D}^+) = \mathsf{X}_{>}(\mathbf{D}^+) = \mathsf{X}_{>}(\mathbf{D})$. By the Theorem, $Q(\mathbf{D})$ is quasipositive, so

$$\begin{aligned}\chi_s(L(\mathbf{D})) &\leq 2\chi(Q(\mathbf{D})) - \chi(S(\mathbf{D})) \\ &= 2(\#\mathsf{O}_{\geq}(\mathbf{D}) - \#\mathsf{X}_{>}(\mathbf{D})) - (\#\mathsf{O}(\mathbf{D}) - \#\mathsf{X}(\mathbf{D})) \\ &= (\#\mathsf{O}_{\geq}(\mathbf{D}) - \#\mathsf{O}_{<}(\mathbf{D})) - (\#\mathsf{X}_{>}(\mathbf{D}) - \#\mathsf{X}_{<}(\mathbf{D}))\end{aligned}$$

by [16], Corollary 5.2.2. □

Proof of Corollary 2 (A) As noted in Remark 3, chirality of non-trivial strongly quasipositive links is a consequence of sBi, so by the Theorem, if L is a non-trivial positive link then L is chiral.

(B) Let $\mathbf{D}$ be an oriented link diagram such that $L(\mathbf{D}) = K$ is a knot and $\#\mathsf{X}_{<}(\mathbf{D}) = 1$. Then $\#\mathsf{O}_{<}(\mathbf{D}) \leq 1$. The following case analysis shows that either K is positive (so that K is not almost positive) or $\chi_s(K \# K) \leq -1$ (so that K is not amphicheiral).

(1) If $\#\mathsf{O}_{<}(\mathbf{D}) = 1$ then a single Reidemeister move of type 1 replaces $\mathbf{D}$ by $\mathbf{D}_0$ with $L(\mathbf{D}_0) = K$ and $\#\mathsf{X}_{<}(\mathbf{D}) = 0$, so K is positive.

(2) If $\#\mathsf{O}_{<}(\mathbf{D}) = 0$ and $\chi(S(\mathbf{D})) = \#\mathsf{O}_{>}(\mathbf{D}) - (\#\mathsf{X}_{>} + 1) < -1$, then Corollary 1, applied to the connected sum $\mathbf{D} \# \mathbf{D}$, implies that $\chi_s(K \# K) \leq -1$.

(3) If $\chi(S(\mathbf{D})) = -1$, then a simple analysis of the possible configurations of the Seifert circles shows (without using any information about the signs of the crossings) that the punctured torus $S(\mathbf{D})$ is isotopic to either

(i) a plumbing $A(O,p) * A(O,q)$ of two unknotted twisted annuli, or

(ii) a pretzel surface $P(2a+1, 2b+1, 2c+1)$; if, further, $\#\mathsf{X}_{<}(\mathbf{D}) = 1$, then in case (i) p and q are non-positive so K is positive by inspection, while in case (ii) two of $2a+1, 2b+1, 2c+1$ are negative and the third is 1, whence $P(2a+1, 2b+1, 2c+1)$ is again isotopic to a plumbing $A(O,p) * A(O,q)$ with p and q non-positive (if $a, b < -1$, $c = 1$, then $p = a+1$, $q = b+1$), so again K is positive. □

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