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Smooth Euclidean 4–spaces with few symmetries

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Abstract

We say that a topologically embedded 3–sphere in a smoothing of Euclidean 4–space is a barrier provided, roughly, no diffeomorphism of the 4–manifold moves the 3–sphere off itself. In this paper we construct infinitely many one parameter families of distinct smoothings of 4–space with barrier 3–spheres.

The existence of barriers implies, amongst other things, that the isometry group of these manifolds, in any smooth metric, is finite. In particular, S^1 can not act smoothly and effectively on any smoothing of 4–space with barrier 3–spheres.

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We say that a smooth manifold has *few symmetries* provided that, for every choice of smooth (C^1 or better) metric, the isometry group for that metric is finite. Let E^4 be a smooth manifold homeomorphic to $\mathbb{R}^4$. We say that a flat embedding $S^3 \subset E$ is a *barrier* S^3 provided that, given any open set $U \subset E^4$ containing S^3 and any smooth embedding $e\colon U \to E^4$, then $e(S^3) \cap S^3 \neq \emptyset$. Given a barrier $S^3 \subset E^4$, the *inside* is the component of $E^4 - S^3$ whose closure is compact: note that it is a smoothing of $\mathbb{R}^4$.

Theorem *Let E^4 be a smoothing of $\mathbb{R}^4$ with a barrier S^3 whose inside does not smoothly embed in any integral homology 4–sphere. Then E has few symmetries.*

Remarks In [9], we have constructed many examples of smooth $\mathbb{R}^4$'s as in the theorem. See the discussion leading up to Theorem 3.1 below and the theorem itself. There definitely are examples for which the isometry group is not trivial: eg, the end-connected sum of E with itself supports an involution and any metric can be averaged so as to make the involution an isometry.

Proof By Theorem 1.1 below, the barrier forces the isometry group to be a compact Lie group. Myers and Steenrod [6] have already proved that the isometry group is a Lie group, so our contribution is that it must be compact. By Theorem 2.1 below, if S^1 acts effectively on E^4, then any compact subset of E^4 embeds smoothly in an integral homology 4–sphere. Hence, under our hypotheses, the isometry group is a compact Lie group with no S^1 subgroups, which implies that it is finite. □

1 Barriers and isometry groups.

The goal of this section is to prove:

Theorem 1.1 *Fix a C^1 metric on a smooth $\mathbb{R}^4$ with a barrier S^3. Then the isometry group in this metric is compact.*

Proof Myers and Steenrod [6] have proved that the isometry group is a Lie group. Hence it will suffice to show that any infinite set of isometries has a convergent subsequence. Let $\mathcal{I}$ denote any infinite set of isometries.

Fix $\epsilon > 0$ so that $C_\epsilon = \{x \in E \mid d(x, S^3) \leq \epsilon\}$ is compact. If the metric is complete, any finite number will do, but even if the metric is not complete, there are sufficiently small ϵ with this property. Cover S^3 with finitely many balls, $B(x_i, \epsilon/2)$ with $x_i \in S^3$, $i = 1, \ldots, r$. Let $\mathcal{I}_i = \{f \in \mathcal{I} \mid f\big(B(x_i, \epsilon/2)\big) \subset E^4 - S^3\}$. If f is an isometry and if $f \notin \mathcal{I}_i$, equivalently $f\big(B(x_i,\epsilon/2)\big) \cap S^3 \neq \emptyset$, then $f\big(B(x_i,\epsilon/2)\big) \subset C_\epsilon$.

There exists an i such that $\mathcal{I} - \mathcal{I}_i$ is infinite. To see this, observe that $\cap_{i=1}^r \mathcal{I}_i = \emptyset$ because S^3 is a barrier and $\cup_{i=1}^r \big(\mathcal{I} - \mathcal{I}_i\big) = \mathcal{I} - \cap_{i=1}^r \mathcal{I}_i = \mathcal{I}$. Since $\mathcal{I}$ is infinite, so is at least one $\mathcal{I} - \mathcal{I}_i$. Let $\mathcal{I}^0 = \mathcal{I} - \mathcal{I}_i$ for any i such that $\mathcal{I} - \mathcal{I}_i$ is infinite.

Pick 5 points, $y_0, \ldots, y_4 \in \mathcal{I}^0$ in sufficiently general position as discussed by Myers and Steenrod in Theorem 3, [6]. Then $\mathcal{I}^0(y_t) = \{f(y_t) \mid \forall f \in \mathcal{I}^0\} \subset C_\epsilon$ so there is a subset of $\mathcal{I}^0$, $f_0, \ldots, f_r, \ldots$ so that for each t, the sequence $f_r(y_t)$ is a Cauchy sequence. As discussed by Myers and Steenrod [6] at the top of page 406, it follows from a theorem of van Danzig and van der Waerden [3] that there exists an isometry f such that a subsequence of the f_r converge to f. (A proof can also be found in [5], especially the proof of Theorem 4.7 starting on page 46.) Since every infinite subset of the isometry group has a convergent subsequence, it follows that the isometry group is compact. □

2 Effective S^1 actions on smoothings of $\mathbb{R}^4$'s.

In this section we will show:

Theorem 2.1 *If S^1 acts effectively on a smoothing E of $\mathbb{R}^4$ then any compact smooth submanifold of E embeds smoothly in an integral homology 4–sphere.*

The proof occupies the remainder of this section. We begin with some general results on S^1 actions. First recall that if S^1 acts effectively on a connected manifold M, then the dimension of each component of M^{S^1} is congruent mod 2 to the dimension of M. If $H \subset S^1$ is a proper subgroup, then it is finite cyclic. If a component of M^H contains a component of M^{S^1} then the dimension of that component of M^H is congruent mod 2 to the dimension of M. To see this, let C denote the component of M^H and let $\widehat{H}$ denote the subgroup for which $S^1/\widehat{H}$ acts effectively on C. Then $\big(C\big)^{S^1/\widehat{H}} = C \cap M^{S^1}$ so the dimension of C is congruent mod 2 to the dimension of a component of M^{S^1}.

Now suppose M is acyclic over the integers. Then so is M^{S^1}: in particular it is non-empty and connected. For any prime p, $M^{\mathbb{Z}/p\mathbb{Z}}$ is mod p acyclic, hence also non-empty and connected. If the codimension of M^{S^1} in M is 2, then the action must be semi-free since $M^{\mathbb{Z}/p\mathbb{Z}} = M^{S^1}$ for all primes p and hence $M^H = M^{S^1}$ for any non-trivial, proper subgroup $H \subset S^1$.

The only other case of relevance here is the case where M is still integrally acyclic, the dimension of M is 4, and the dimension of M^{S^1} is 0. Some of the $M^{\mathbb{Z}/p\mathbb{Z}}$ may have dimension 2, but there are only finitely many, say p_1, $\ldots$, p_r. Let $K(p_i)$ denote the subgroup so that $S^1/K(p_i)$ acts effectively on $M^{\mathbb{Z}/p_i\mathbb{Z}}$. Now $M^{\mathbb{Z}/p_i\mathbb{Z}}$ is a mod p_i acyclic, non-compact 2–manifold, hence $\mathbb{R}^2$. Conveniently, $M^{\mathbb{Z}/p_i\mathbb{Z}}$ remains integrally acyclic, so the $S^1/K(p_i)$ action must be semi-free. Hence either $M^{K(p_i)} \cap M^{K(p_j)} = M^{S^1}$ or $K(p_i) = K(p_j)$. By a theorem of Bochner's, the action in a neighborhood of M^{S^1} is linear, so the intersections are transverse as well.

If M^4 is actually contractible, then so is the orbit space M^* [7; page 644 Theorem 5]. By [2; page 189, 4.6], M^* is a 3–manifold, with boundary if $M^{S^1} = \mathbb{R}^2$ and without boundary if M^{S^1} is a point. The image of M^{S^1} in M^* is the boundary if $M^{S^1} = \mathbb{R}^2$. Let $P \subset M$ be the set of principal orbits: P is an open dense set. If $P^* = \pi(P)$, then P^* is an open dense set and the map $\pi\colon P \to P^*$ is a submersion. Since M^* is orientable, for any any embedded $S^1 \subset P^*$, $\pi^{-1}(S^1)$ is a torus, not a Klein bottle: ie, for any $S^1 \subset P^*$, the circle action on $\pi^{-1}(S^1)$ is trivial.

Let us now restrict attention to the case $M = E$ is homeomorphic to $\mathbb{R}^4$. Let U be the interior of a topological ball in E which contains our compact submanifold. It will suffice to embed U smoothly in an integral homology 4–sphere. Let U^* denote the image of U in E^*.

Pick a point $p \in E^{S^1}$ and let $V \in E$ be a smooth linear 4–ball centered at p. Let V^* denote the image of V in E^* and note that V^* is a 3–ball. If $E^{S^1} = \mathbb{R}^2$ then $E^{S^1} \cap V^*$ is a 2–ball: if E^{S^1} is a point, then $E^{S^1} \cap V^*$ is a point in the interior of the 3–ball.

In case $E^{S^1} = \mathbb{R}^2$, choose a smoothly embedded, compact, closed surface $Q \subset E^*$, so that $Q \cap \partial E^*$ is a 2–ball which contains both $V^* \cap \partial E^*$ and $U^* \cap \partial E^*$ and so that the compact component of $E^* - Q$ contains both U^* and V^*. In case E^{S^1} is a point, the image of the singular set has the following description. There is one point, for the image of E^{S^1}, together with a finite number of proper rays, the images of the various $E^{K(p_i)}$'s. Each ray crosses $\partial V^* = S^2$ transversely in one point. Choose $Q \subset E^*$ to be a smoothly embedded, compact, closed surface so that the compact component of $E^* - Q$ contains both U^* and V^*. Further require that Q intersects each of the rays transversely in a single point.

Now write $E^* = N \cup W$ where $\partial N = \partial W = Q$ and N is compact. It follows from the Mayer–Vietoris theorem and intersection theory that $H_1(Q;\mathbb{Z}) = H_1(N;\mathbb{Z}) \oplus H_1(W;\mathbb{Z})$ and that there is a symplectic basis for $H_1(Q;\mathbb{Z})$: x_1, $\ldots$, x_g, y_1, $\ldots$, y_g, so that $x_i \cap x_j = y_i \cap y_j = 0$ and $x_i \cap y_j = \delta_{ij}$. Furthermore, the x_i generate $H_1(N;\mathbb{Z})$ and the y_i generate $H_1(W;\mathbb{Z})$ under the decomposition.

Represent the elements y_i by disjoint embedded circles in Q: in case E^{S^1} is a point, arrange for these circles to miss the points where the rays cross. Let $\hat{N}$ be the result of doing surgery on these circles. Each of these circles lies in the image of the principal orbit, so the circle action over them is trivial. Hence we can construct a compact, smooth 4–manifold X^4 with boundary which supports a smooth S^1 action and so that $\pi^{-1}(N)$ is a smooth, equivariant submanifold of X^4. The orbit space of the S^1 action on X is just $\hat{N}$. Let $J = \overline{X - V}$ and let J^* be the image of J in $\hat{N}$. Let P denote the open dense subset of principal orbits in J. If $E^{S^1} = \mathbb{R}^2$, $P = J - A$ where $A = J^{S^1}$ is an annulus. If E^{S^1} is a point, $P = X - \coprod_{i=1}^{r} A_i$, where each $A_i = J^{K(p_i)}$ is an annulus.

Consider the pair $(P, \partial V \cap P)$. The S^1 action on this pair is free, and for the orbit space pair $H^*(P^*, \partial V^* \cap P^*; \mathbb{Z}) = 0$. Hence $H^*(P, \partial V \cap P; \mathbb{Z}) = 0$ by a spectral sequence argument. From the description of P in the last paragraph and the Mayer–Vietoris sequence $H^*(J, \partial V; \mathbb{Z}) = 0$. Since V is a 4–ball, $X = V \cup J$ is an integral homology 4–ball. The double of X is the required integral homology 4–sphere. □

3 A construction of $\mathbb{R}^4$'s with barrier S^3's.

In [9] we defined an invariant of smooth $\mathbb{R}^4$'s, γ, which takes on integer values greater than or equal to 0 and $+\infty$. We can only prove the existence of barriers in certain smoothings: we call a smoothing *definite* provided it is diffeomorphic at ∞ to the end of some smoothing $M^4 - pt$ where M^4 is a simply-connected, compact topological manifold with a definite intersection form which can not be diagonalized over the integers. In [9; eg 5.6] we constructed smoothings of $\mathbb{R}^4$, E_n, which are definite and which satisfy $\gamma(E_n) = n$, $0 < n \leq \infty$: indeed for each n, $0 < n \leq \infty$, we construct a one parameter family of them.

Theorem 3.1 *Let E be a definite smoothing of $\mathbb{R}^4$ with $0 < \gamma(E) < \infty$. Then there exists a compact set $K \subset E$, such that any flat $S^3 \subset E$ with K on the inside of S^3 is a barrier. Furthermore, $\gamma(inside) = \gamma(E)$ and the inside is definite.*

Remarks We know of no example of a smoothing E with $0 < \gamma(E) \leq \infty$ which is not definite. There are examples of E with $\gamma(E) = \infty$ which are definite but do not have barrier S^3's, for example the universal $\mathbb{R}^4$ of [4]. Any E which embeds in the standard $\mathbb{R}^4$ has no barriers.

Proof From [9; Theorem 5.1] we see that if $E_0 \subset E_1$ are smoothings of $\mathbb{R}^4$, $\gamma(E_0) \leq \gamma(E_1)$. It follows from the definition that if E_0 is definite so is E_1.

Call a neighborhood U of S^3 a *0-neighborhood* if U is open and $U - S^3$ has two components. We label the component which intersects the inside of $E - S^3$ the *inside* and the other component the *outside*. Given any neighborhood U of a flat S^3 in E^4, we can find a smaller neighborhood homeomorphic to $S^3 \times (-\infty, \infty)$ and this is a 0-neighborhood. Hence, to prove S^3 is a barrier, it suffices to prove $e(S^3) \cap S^3 \neq \emptyset$ for all smooth embeddings $e: U \to E$ where U is a 0-neighborhood.

Assume we have a flat $S^3 \subset E$ and let U be a 0-neighborhood of S^3. Let $e: U \to E_1$ be a smooth embedding of U into any smoothing of $\mathbb{R}^4$. Note $e(U)$ is a 0-neighborhood of $e(S^3)$.

Lemma 3.2 *Let I denote the smoothing on the inside of $E - S^3$. If $\gamma(I) > 0$, e takes the inside of U to the inside of $e(U)$.*

Proof If not, one can construct a smooth homotopy 4–sphere with I smoothly embedded. But this contradicts $\gamma(I) > 0$. □

Again assume we have a flat $S^3 \subset E$ and let U be a 0-neighborhood of S^3. Let e_1, $e_2: U \to E_1$ be smooth embeddings of U into the same smoothing of $\mathbb{R}^4$.

Lemma 3.3 *Let I again denote the smoothing on the inside of $E - S^3$. Suppose I is definite and $e_1(S^3) \cap e_2(S^3) = \emptyset$. Then the inside of $e_1(S^3)$ and the inside of $e_2(S^3)$ are disjoint.*

Proof Note $\gamma(I) > 0$ for any definite $\mathbb{R}^4$, so the "inside" is well–defined by Lemma 3.2. If 3.3 were false, then we could construct a new smoothing of $M^4 - pt$ with a periodic end in the sense of Taubes, [8]. But this is precisely what the main theorem of [8] forbids. □

We now return to the proof of the existence of barriers. It follows from [9] that we can find an $S^3 \subset E$ so that $\gamma(I) = n$ and I is definite, where I is the inside of $E - S^3$. We can further assume that $S^3 \subset E$ has a smooth point. Let K denote the closure of I in E. Of course it is homeomorphic to a 4–ball. Since I is definite, it follows from [9; Theorem 5.3] that we can find an integer $N > 0$ such that $\gamma(\natural^{2^N} I) > \gamma(E)$, where $\natural^{2^N} I$ denotes the end-connected sum of I with itself 2^N times.

The proof proceeds by constructing successively larger compact sets until the conclusion of the theorem holds. We introduce some notation for the proof. First we label the embedding $S^3 \subset E$ by $e_0^{(0)}: S^3 \subset E$. Then we label I as $I^{(0)}$ and K as $K^{(0)}$. We will construct a sequence of embeddings, $e_0^{(j)}: S^3 \subset E$ starting with the $j = 0$ we have just exhibited. Let $I^{(j)}$ be the inside of $E - e_0^{(j)}(S^3)$ and let $K^{(j)}$ denote the closure of $I^{(j)}$. As part of the construction, we will have $K^{(j-1)} \subset I^{(j)}$. We will continue the construction until the conclusion of the theorem holds for $K^{(j)}$: this must happen for some $j < N$ as we shall see. Since $I^{(0)}$ is definite, $I^{(j)}$ is definite, and $0 < \gamma(I^{(j)}) = \gamma(E) < \infty$. A second part of the construction guarantees that $\gamma(\natural^{2^{N-j}} I^{(j)}) > \gamma(E)$.

- *Suppose we have constructed $e_0^{(j-1)}$.*

If the conclusion of the theorem holds with $K = K^{(j-1)}$ we are done. If not, there exists an $e_1: S^3 \subset E$ with $K^{(j-1)}$ on the inside and a 0-neighborhood of $e_1(S^3)$, say U, which we may take to miss $K^{(j-1)}$, so that there exists a smooth embedding $e: U \to E$ such that $e_1(S^3) \cap e(S^3) = \emptyset$. By Lemmas 3.2 and 3.3, the insides of $E - e_1(S^3)$ and $E - e(S^3)$ are disjoint.

Take a flat S^3 whose inside contains both $e_1(S^3)$ and $e(S^3)$ and whose boundary has a smooth point. Denote the embedding by $e_0^{(j)}: S^3 \subset E$ and let $I^{(j)}$ denote the inside of $E - e_0^{(j)}(S^3)$. By the definition of γ, $I^{(j)}$ smoothly embeds in X^4, X^4 a closed, smooth, compact, Spin 4–manifold whose rational intersection form is hyperbolic with $\gamma(E)$ hyperbolic summands. Let Y^4 denote the result of removing the inside of $E - e(S^3)$ from X and replacing it by the inside of $E - e_1(S^3)$: Y remains a closed, smooth, compact, Spin 4–manifold whose rational intersection form is hyperbolic with $\gamma(E)$ hyperbolic summands and two disjoint copies of $I^{(j-1)}$ embed smoothly in it, each copy having a smooth boundary point. If $N-(j-1) = 1$, then $\gamma(\natural^2 I^{(j-1)}) > \gamma(I)$ and this contradicitions the embedding of two copies of $I^{(j-1)}$ in Y. Hence if $N-(j-1) = 1$ the conclusion of the theorem must have held and we are done.

If $N-(j-1) > 1$, note the following. By construction $I^{(j)}$ has 2 copies of $I^{(j-1)}$ embedded in it, each with a smooth boundary point. Hence $\natural^{2^{N-(j-1)}} I^{(j-1)} \subset \natural^{2^{N-j}} I^{(j)}$, so $\gamma(\natural^{2^{N-j}} I^{(j)}) > \gamma(E)$.

Now repeat the argument starting at $\bullet$ above with $j = j+1$. Eventually $N-(j-1) = 1$ and the process halts if it has not halted earlier. □

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