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Foliation Cones

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Abstract David Gabai showed that disk decomposable knot and link complements carry taut foliations of depth one. In an arbitrary sutured 3–manifold M, such foliations $\mathcal{F}$, if they exist at all, are determined up to isotopy by an associated ray $[\mathcal{F}]$ issuing from the origin in $H^1(M;\mathbb{R})$ and meeting points of the integer lattice $H^1(M;\mathbb{Z})$. Here we show that there is a finite family of nonoverlapping, convex, polyhedral cones in $H^1(M;\mathbb{R})$ such that the rays meeting integer lattice points in the interiors of these cones are exactly the rays $[\mathcal{F}]$. In the irreducible case, each of these cones corresponds to a pseudo-Anosov flow and can be computed by a Markov matrix associated to the flow. Examples show that, in disk decomposable cases, these are effectively computable. Our result extends to depth one a well known theorem of Thurston for fibered 3-manifolds. The depth one theory applies to higher depth as well.

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1 Introduction

By theorems of Waldhausen [31] and Thurston [30], the classification of fibrations $\pi\colon M \to S^1$ which are transverse to ∂M is reduced to a finite problem for compact 3–manifolds. Indeed, the fibrations $\mathcal{F}$ correspond one–one, up to isotopy, to certain "fibered" rays $[\mathcal{F}] \subset H^1(M;\mathbb{R})$. More generally, the isotopy classes of C^2 foliations $\mathcal{F}$ without holonomy correspond one–one to "foliated" rays $[\mathcal{F}] = \{t[\omega]\}_{t\geq 0}$, where ω is a closed, nonsingular 1–form defining a foliation isotopic to $\mathcal{F}$ [19, 2]. These rays fill up the interiors of a finite family of convex, polyhedral cones subtended by certain top dimensional faces of the unit ball $B \subset H^1(M;\mathbb{R})$ of the Thurston norm. The foliated rays that meet nontrivial points of the integer lattice $H^1(M;\mathbb{Z})$ are the fibered rays.

In this paper, we consider sutured 3–manifolds (M,γ) that admit taut, transversely oriented depth one foliations $\mathcal{F}$. We will decompose $\partial M = \partial_\tau M \cup \partial_\pitchfork M$ so that the compact leaves of $\mathcal{F}$ are the components of the "tangential boundary" $\partial_\tau M$ and $\mathcal{F}$ is transverse to the "transverse boundary" $\partial_\pitchfork M$. In the language of sutured manifolds [12], $\partial_\pitchfork M = \gamma$ and $\partial_\tau M = R(\gamma)$. The depth one foliation fibers $M_0 = M \smallsetminus \partial_\tau M$ over S^1 with noncompact fibers. Typically, these sutured manifolds will result from cutting the complement $E(\kappa)$ of a k–component link κ along a Seifert surface S. In the resulting sutured manifold $(M_S(\kappa),\gamma)$, two copies of S make up $\partial_\tau M_S(\kappa)$ and $\partial_\pitchfork M_S(\kappa)$ consists of k annuli. In more general examples, $\partial_\pitchfork M$ has toral and/or annular components.

In [4], we showed that such depth one foliations correspond one–one, up to isotopy, to "depth one foliated" rays $[\mathcal{F}] \subset H^1(M;\mathbb{R})$. Here we will show that there is a finite family of convex, polyhedral cones in $H^1(M;\mathbb{R})$, with disjoint interiors, such that the rays through integer lattice points in the interiors of the cones are exactly the depth one foliated rays. Examples at the end of the paper will illustrate the fact that this cone structure is often effectively computable.

In [5], we exhibited families of depth one knot complements $E(\kappa)$ in which the foliation cones could be described by a norm on $H^1(M_S(\kappa);\mathbb{R})$, but our examples will show that such a description is generally impossible. Instead, the dynamical properties of flows transverse to the foliations will be exploited in analogy with Fried's determination of the fibered faces of the Thurston ball [11]. To show that the number of cones is finite, we will use branched surfaces in the spirit of Oertel's determination of the faces of the Thurston ball [21].

Again all rays in the interiors of these cones correspond to taut foliations $\mathcal{F}$ having holonomy only along $\partial_\tau M$, but those not meeting the integer lattice $H^1(M;\mathbb{Z})$ will have everywhere dense noncompact leaves. We conjecture that the isotopy class of such a foliation is also uniquely determined by the foliated ray $[\mathcal{F}]$.

The following theorem is meant to cover both the case of fibrations and that of foliations of depth one. Accordingly, the term "proper foliated ray" replaces the terms "fibered ray" and "depth one foliated ray" in the respective cases. Here and throughout the paper, $H^1(M)$ denotes de Rham cohomology and explicit reference to the coefficient ring $\mathbb{R}$ is omitted.

Theorem 1.1 *Let (M,γ) be a compact, connected, oriented, sutured 3–manifold. If there are taut, transversely oriented foliations $\mathcal{F}$ of M having holonomy (if at all) only on the leaves in $\partial_\tau M$, then there are finitely many closed, convex, polyhedral cones in $H^1(M)$, called foliation cones, having disjoint interiors and such that the foliated rays $[\mathcal{F}]$ are exactly those lying in the* *interiors of these cones. The proper foliated rays are exactly the foliated rays through points of the integer lattice and determine the corresponding foliations up to isotopy.*

In the fibered case, $\partial_\tau M = \emptyset$ and the theorem is due to Waldhausen and Thurston.

Remark It will be necessary to allow the possibility that the entire vector space $H^1(M;\mathbb{R})$ is a foliation cone, this happening if and only if $M = S \times I$ is the product of a compact surface S and a compact interval I (Proposition 3.7). This is the one case in which the vertex 0 of the cone lies in its interior. The foliated class 0 will correspond to the product foliation and $\{0\}$ will be a (degenerate) proper foliated ray.

The proof of the theorem is reduced to the hyperbolic case where the Handel–Miller theory of pseudo-Anosov endperiodic homeomorphisms pertains. (This theory is unpublished, but cf [9]). Determining the pseudo-Anosov monodromy for one foliation $\mathcal{F}$ gives rise to symbolic dynamics from which the parameters for the foliation cone containing $[\mathcal{F}]$ are easily read. In the case of foliations arising from disk decompositions [13], if the disks can be chosen in M from the start, this procedure is quite effective. Indeed, the disks of the decomposition typically split up in a natural way into the rectangles of a Markov partition associated to the pseudo-Anosov monodromy. This partition determines a finite set of "minimal loops" transverse to $\mathcal{F}$ which span the "tightest" cone of transverse cycles in $H_1(M)$. The dual of this cone is the maximal foliation cone containing $[\mathcal{F}]$.

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2 Higher depth foliations

Before proving Theorem 1.1, we indicate briefly its pertinence to taut foliations of finite depth $k > 1$ and smoothness class at least C^2. To avoid technical problems, we assume that $\partial_\pitchfork M = \emptyset$. The C^2 hypothesis guarantees that all junctures are compact, hence that $(M,\mathcal{F})$ is homeomorphic to a C^∞–foliated manifold [6, Main Theorem, page 4].

Remark The concept of a "juncture" is explained in [6, Section 2] and will have important use in this paper.

Let $\mathcal{S} \subset M$ be the compact lamination consisting of all leaves of $\mathcal{F}$ on which other leaves accumulate. While $\mathcal{S}$ can have infinitely many leaves, no real generality is lost by assuming it only has finitely many leaves. Indeed, it is possible to "blow down" finitely many foliated interval bundles in $\mathcal{F}$ to produce a foliation with $\mathcal{S}$ finite-leaved. The new foliation still has all junctures compact, hence can be taken to be of class C^∞.

There are infinitely many ways to complete $\mathcal{S}$ to a depth k foliation. More precisely, consider any one of the components U of $M \smallsetminus \mathcal{S}$, an open, connected, $\mathcal{F}$–saturated set which is fibered over S^1 by $\mathcal{F}|U$. The completion of U relative to a Riemannian metric on M is a (generally noncompact) manifold $\widehat{U}$ with boundary on which $\mathcal{F}$ induces a depth one foliation $\widehat{\mathcal{F}}$. As in [7, Theorem 1], we write

$$\widehat{U} = K \cup V_1 \cup \cdots \cup V_n,$$

where K is a compact, connected, foliated, sutured manifold (called the "nucleus") and $V_i \cong B_i \times I$ is a noncompact, connected, foliated interval bundle (called an "arm"), $1 \le i \le n$. Here, $\partial_\pitchfork K$ has exactly n components $A_i = \partial_\pitchfork V_i$, $1 \le i \le n$. The assumption that $\partial_\pitchfork M = \emptyset$ implies that these components are annuli. By choosing K sufficiently large, one guarantees that the foliation of each V_i is the product foliation. This last assertion is due to compactness of the junctures. Of course, $\widehat{\mathcal{F}}|K$ is of depth one. Each depth one foliation of K which is trivial (that is, a product) at $\partial_\pitchfork K$ determines a depth one foliation of $\widehat{U}$, trivial in the arms. The depth one foliations of K that are trivial at $\partial_\pitchfork K$ will be called $\partial_\pitchfork$–trivial.

In order to classify the $\partial_\pitchfork$–trivial depth one foliations of K, one first replaces K with the manifold K' obtained by gluing a copy of $D^2 \times I$ to each annular component of $\partial_\pitchfork K$. The $\partial_\pitchfork$–trivial depth one foliations of K correspond bijectively to the depth one foliations of K' with sole compact leaves the components of $\partial K'$. Furthermore, there is a canonical splitting

$$H^1_c(K \smallsetminus \partial_\pitchfork K) = H^1(K') \oplus V,$$

where V is spanned by the Poincaré duals of the components of $\partial_\pitchfork K$. Thus, the foliation cones in $H^1(K')$ can be viewed as "foliation cones" in $H^1_c(K \smallsetminus \partial_\pitchfork K)$. Obviously, these are not full dimensional in the latter space, but they classify the $\partial_\pitchfork$–trivial depth one foliations and will be called the $\partial_\pitchfork$–trivial foliation cones of K.

In order to classify all depth one foliations of $\widehat{U}$, one must allow an infinite exhaustion

$$K_0 \subset K_1 \subset \cdots \subset K_r \subset \cdots \subset \widehat{U}$$

by the potential nuclei. An inductive limit process then leads to a finite family of "foliation cones" $C \times \mathbb{R}^N \subset H^1_c(\widehat{U})$, where C ranges over the $\partial_\pitchfork$–trivial foliation cones of K_0 and $0 \leq N \leq \infty$. We omit the details.

Finally, under the assumption that $\mathcal{S}$ has finitely many leaves, this analysis only needs to be carried out for finitely many open, saturated sets U.

3 Reducing the sutured manifold

Let M be a compact, connected, sutured 3–manifold. A depth one foliation determines a foliated class $[\omega] \in H^1(M)$ which is represented by a *foliated form* $\omega \in A^1(M_0)$. This is to be a closed, nonsingular 1–form which blows up at $\partial_\tau M$ in such a way that the foliation $\mathcal{F}_0$, defined by ω on M_0, can be completed to a foliation $\mathcal{F}$ of M, integral to a C^0 plane field, by adjoining the components of $\partial_\tau M$ as leaves. We will say that ω "blows up nicely" at $\partial_\tau M$. The depth one condition implies that this form has period group of rank one. More generally, foliated forms of higher rank define foliations $\mathcal{F}$ tangent to $\partial_\tau M$ and such that each leaf of $\mathcal{F}|M_0$ is dense in M and is without holonomy. It can be shown that all smooth foliations of M having holonomy only along the boundary leaves are C^0 isotopic to foliations defined by foliated forms (Corollary 4.4). In case $M \cong S \times I$, we also allow exact foliated forms.

In order to prove Theorem 1.1, it will be necessary to "completely reduce" M. If $T \subset M$ is a compact, properly imbedded surface, let $N(T)$ be a closed, normal neighborhood of T in M and denote by $N_0(T)$ the corresponding open, normal neighborhood of T. If T_1 and T_2 are disjoint, properly imbedded surfaces, we always choose $N(T_1)$ and $N(T_2)$ to be disjoint.

Definition 3.1 Let $T \subset M$ be a properly imbedded, incompressible torus or annulus. If T is an annulus, require that one component of ∂T lie on an inwardly oriented component of $\partial_\tau M$ and the other component of ∂T on an outwardly oriented one. Then T is a reducing surface if it is not isotopic through surfaces of the same type to a component of $\partial_\pitchfork M$.

If $T \subset M$ is a reducing surface, we regard $M' = M \smallsetminus N_0(T)$ as a (possibly disconnected) sutured manifold, $\partial_\pitchfork M'$ being the union of $\partial_\pitchfork M$ and the two copies of T in $\partial M'$.

Definition 3.2 If $T \subset M$ is a reducing surface, then $\{T\}$ is a reducing family. Inductively, a reducing family is a finite collection $\{T_1, \dots, T_r\}$ of disjoint reducing surfaces such that $\{T_1, \dots, T_{r-1}\}$ is a reducing family and T_r is a reducing surface in whichever component of $M \setminus \bigcup_{i=1}^{r-1} N_0(T_i)$ it lies. A maximal reducing family is called a completely reducing family. If M contains no reducing family, it is said to be completely reduced.

If M contains a reducing family, it contains a completely reducing family. In the following sections, we are going to prove Theorem 1.1 for the completely reduced case. Here, we will show that this is sufficient.

Theorem 3.3 *If the conclusion of* Theorem 1.1 *is true for completely reduced, sutured* 3*–manifolds, it is true for arbitrary connected, sutured* 3*–manifolds* M.

Theorem 3.3 is proven by induction on the number r of elements of a completely reducing family in M. If $r = 0$, there is nothing to prove. The inductive step is given by the following lemmas.

Lemma 3.4 *Let* $T \subset M$ *be a reducing surface,* $M' = M \setminus N_0(T)$. *If* M' *has two components and if the conclusion of* Theorem 1.1 *holds for each of these components, it holds for* M.

Proof Indeed, fix an identification $N(T) = T \times [-1, 1]$ and let M'_+ (respectively, M'_-) be the component of M' meeting $N(T)$ along $T \times \{1\}$ (respectively, $T \times \{-1\}$). Set $M_\pm = M'_\pm \cup N(T)$. Then

$$\begin{aligned} M_- \cup M_+ &= M \\ M_- \cap M_+ &= N(T) \end{aligned}$$

and Mayer–Vietoris gives an exact sequence

$$0 \to H^1(M; \mathbb{R}) \xrightarrow{i} H^1(M_-) \oplus H^1(M_+) \xrightarrow{j} H^1(N(T)).$$

Here, we use the conventions that

$$\begin{aligned} i([\omega]) &= ([\omega|M_-], [\omega|M_+]) \\ j([\alpha], [\beta]) &= [\alpha|N(T)] - [\beta|N(T)]. \end{aligned}$$

If ω is a foliated form, the fact that T is incompressible allows us to assume that $\omega \pitchfork T$ [24, 30] so that $i([\omega])$ is a pair of foliated classes. By the inductive hypothesis, $i[\omega] \in \operatorname{int}(\mathcal{C}_- \times \mathcal{C}_+)$ for foliation cones $\mathcal{C}_\pm$ in $H^1(M_\pm)$. Conversely, the Mayer–Vietoris sequence implies that every class $[\omega]$ carried into $\operatorname{int}(\mathcal{C}_- \times$

$\mathcal{C}_+$) is represented by a foliated form obtained by piecing together foliated forms $\omega_\pm$ representing classes in $\operatorname{int}(\mathcal{C}_\pm)$. In order to make the cohomologous forms $\omega_\pm|N(T)$ agree, one uses a theorem of Blank and Laudenbach [19]. It follows that the connected components of the set of foliated classes in $H^1(M;\mathbb{R})$ are exactly the interiors of a family of convex, polyhedral cones of the form $i^{-1}(\mathcal{C}_- \times \mathcal{C}_+)$. □

We turn to the case that M' is connected. Again identify $N(T) = N \times [-1,1]$, but realize M' as $M \smallsetminus \{T \times (-1/2, 1/2)\}$. Set

$$\begin{aligned} N_- &= T \times [-1,-1/2] \\ N_+ &= T \times [1/2, 1] \end{aligned}$$

and note that

$$\begin{aligned} M' \cup N(T) &= M \\ M' \cap N(T) &= N_- \cup N_+. \end{aligned}$$

The Mayer–Vietoris theorem gives the exact sequence

$$H^1(M;\mathbb{R}) \xrightarrow{i} H^1(M') \oplus H^1(N(T)) \xrightarrow{j} H^1(N_-) \oplus H^1(N_+).$$

One easily checks that the kernel of i is spanned by the class that is Poincaré dual to $[T] \in H_2(M, \partial M)$ and arguments analogous to the above prove the following.

Lemma 3.5 *If M' is connected and the conclusion of* Theorem 1.1 *holds for M', then it holds for M.*

Theorem 3.3 follows. We turn to some further simplifying conditions.

Lemma 3.6 *In* Theorem 1.1, *no generality is lost in assuming that no component of $\partial_\tau M$ is an annulus or torus.*

Proof Indeed, let $\mathcal{F}$ be defined by a closed, nonsingular 1–form ω which blows up nicely at $\partial_\tau M$. By the well understood structure of foliation germs along toral and annular leaves, any such leaves in the boundary can be perturbed inwardly by an arbitrarily small isotopy to become transverse to $\mathcal{F}$. Equivalently, ω is replaced by a nonsingular, cohomologous form, differing from ω only in small neighborhoods of these boundary leaves and transverse to them. The former toral leaves are now components of $\partial_\pitchfork M$ and the former annular leaves are incorporated into transverse boundary components. The foliated classes are unchanged. □

Proposition 3.7 *The sutured manifold M has the form $S \times I$ if and only if every class in $H^1(M)$ is a foliated class. That is, $H^1(M)$ is an entire foliation cone.*

Proof Suppose that $M = S \times I$ and identify the compact interval as $I = [-1,1]$. Let $\lambda\colon [-1,1] \to [0,1]$ be smooth, strictly positive on $(-1,1)$, and C^∞–tangent to 0 at ± 1. Each class $[\gamma] \in H^1(S \times I)$ can be represented by a closed form $\gamma = \omega + \lambda(t)^{-1} dt$ on $M_0 = S \times (-1,1)$, where ω is constant in the coordinate t of $(-1,1)$. This form is nonsingular and blows up at the boundary. The normalized form $\lambda(t)\omega + dt$, while not closed, is defined and integrable on all of M, determines the same foliation as γ on M_0 and has the two components of $S \times \{\pm 1\}$ as compact leaves. If ω is not exact, this foliation has nontrivial holonomy exactly on these boundary leaves, defining a foliation of the type we are studying. Note that γ is exact if and only if the form ω is exact, in which case the boundary leaves also have trivial holonomy. Thus, Reeb stability, coupled with Haefliger's theorem [15] that the union of compact leaves is compact, implies that the foliation is isomorphic to the product foliation. In any event, the entire vector space $H^1(M)$ is the unique foliation cone. Conversely, suppose that the entire vector space is a foliation cone. In particular, 0 is a foliated class. It is clear that the foliations having nontrivial holonomy exactly on $\partial_\tau M$ correspond to nontrivial foliated classes, so Reeb stability and Haefliger's theorem again imply that the foliation corresponding to 0 must be isomorphic to the product foliation on a manifold of the form $S \times I$. □

Proposition 3.8 *In* Theorem 1.1, *it can be assumed without loss of generality that $\partial_\tau M \neq \emptyset$.*

Indeed, if $\partial_\tau M = \emptyset$, the foliations we are studying are without holonomy, the case of Theorem 1.1 already covered by the results of Waldhausen and Thurston. In summary:

Theorem 3.9 *The proof of* Theorem 1.1 *is reduced to the case that M is completely reduced, $\partial_\tau M \neq \emptyset$ has no toral or annular components and M is not a product $S \times I$.*

The hypotheses in Theorem 3.9 will now be fixed as the ongoing hypotheses in this paper.

4 The transverse structure cycles

Let $\mathcal{L}$ be a 1–dimensional foliation of M, integral to a nonsingular C^0 vector field ("leafwise C^1") which is transverse to $\partial_\tau M$ and tangent to $\partial_\pitchfork M$. On the annular components of $\partial_\pitchfork M$, it is assumed that $\mathcal{L}$ induces the product foliation by compact intervals. Let $\mathcal{Z}$ be the union of those leaves of $\mathcal{L}$ which do not meet $\partial_\tau M$. It is evident that $\mathcal{Z}$ is a compact, 1–dimensional lamination of M_0. This lamination is nonempty. Indeed, if some leaf ℓ of $\mathcal{L}$ issues from one component of $\partial_\tau M$ but never reaches another, the asymptote of ℓ in M will be a nonempty subset of $\mathcal{Z}$. Thus, if $\mathcal{Z} = \emptyset$, every leaf of $\mathcal{L}$ issues from a component of $\partial_\tau M$ and ends at another, implying that $M \cong S \times I$. This contradicts one of our ongoing hypotheses (Theorem 3.9). We call $\mathcal{Z}$ the "core lamination" of $\mathcal{L}$. The following useful observation is left as an exercise.

Lemma 4.1 *The foliation $\mathcal{L}$ can be modified in a neighborhood of $\partial_\tau M$, leaving $\mathcal{Z}$ unchanged, so that $\mathcal{L}|M_0$ is integral to a continuous vector field v which is smooth near $\partial_\tau M$ and extends smoothly to $\partial_\tau M$ so as to vanish identically there.*

We will apply the Schwartzmann–Sullivan theory of asymptotic structure cycles [27, 29] to the core lamination $\mathcal{Z}$. For this, the fact that the leaves of $\mathcal{Z}$ are integral to a vector field which is at least continuous on M will be essential. It is not clear that we can significantly strengthen this regularity condition for the endperiodic, pseudo-Anosov flows that will be needed in the next section.

Let μ be a transverse, bounded, nontrivial, holonomy invariant measure on $\mathcal{Z}$. Since the 1–dimensional leaves of the core lamination $\mathcal{Z}$ have at most linear growth, such a measure exists by a theorem of Plante [22]. In standard fashion, 1–forms ω on M or M_0 can be integrated against μ in a well defined way. In local flow boxes, one integrates ω along the plaques of $\mathcal{Z}$ and then integrates the resulting plaque function against μ. Using a partition of unity, one assembles these local integrals into a global one which is well defined because of the holonomy invariance of μ. The resulting bounded linear functional

$$\mu\colon\ A^1(M) \to \mathbb{R}$$

is a "structure current" of $\mathcal{Z}$ in the sense of Sullivan [29]. The structure currents of $\mathcal{Z}$ form a closed, convex cone with compact base in the Montel space of all 1–currents on M.

For a current μ, defined as above by a transverse invariant measure, it is easily seen that $\mu(df) = 0$, for all smooth functions f. That is, μ is a *structure cycle* for $\mathcal{Z}$. The proof uses the fundamental theorem of calculus (a.k.a. Stokes's theorem) on plaques of $\mathcal{Z}$ and makes essential use of the fact that $\mathcal{Z}$ does not meet $\partial_\tau M$. Sullivan has proven (*op.cit.*) that the transverse invariant measures are exactly the structure cycles and form a closed, convex subcone $\mathbf{C}_\mathcal{Z}$ in the cone of structure currents of $\mathcal{Z}$. The natural map of closed currents to homology classes carries $\mathbf{C}_\mathcal{Z}$ onto a closed, convex cone $\mathcal{C}_\mathcal{Z} \subset H_1(M)$ with compact base. Each ray of the possibly infinite dimensional cone $\mathbf{C}_\mathcal{Z}$ is mapped one–one onto a ray of $\mathcal{C}_\mathcal{Z}$, but $\mathcal{C}_\mathcal{Z}$ is only finite dimensional.

Definition 4.2 The dual cone $\mathcal{D}_\mathcal{Z}$ to $\mathcal{C}_\mathcal{Z}$ consists of all $[\omega] \in H^1(M;\mathbb{R})$ such that $\zeta(\omega) \geq 0$, $\forall\, \zeta \in \mathbf{C}_\mathcal{Z}$.

Let $\mathcal{F}$ be a taut, transversely oriented foliation of M, smooth in M_0, integral to a C^0 2–plane field on M, having the components of $\partial_\tau M$ as sole compact leaves and having nontrivial holonomy exactly on these compact leaves. We say that the tautly foliated, sutured manifold $(M, \mathcal{F})$ is "almost without holonomy". Remark that foliations defined by foliated forms are of this type, but the converse is not quite true. By a theorem of Sacksteder [25], $\mathcal{F}|M_0$ admits a transverse, continuous, holonomy invariant measure ν which is finite on compact sets. By our hypothesis that the leaves in $\partial_\tau M$ are exactly the ones with nontrivial holonomy, such a nontrivial holonomy transformation must have no fixed points in M_0, hence ν becomes unbounded near $\partial_\tau M$. One should think of ν as a "C^0 foliated form".

Suppose that the one dimensional foliation $\mathcal{L}$ is transverse to $\mathcal{F}$. As is well known, ν has well defined line integrals in M_0 and $\int_\sigma d\nu$ depends only on the homology class of a loop σ. Thus, ν is a cocycle and can play a role analogous to that of a closed 1–form, allowing us to integrate ν against the structure cycle μ of $\mathcal{Z}$ in a well defined way. This integral is clearly positive, proving the "only if" part of the following.

Theorem 4.3 *The one dimensional foliation $\mathcal{L}$ is transverse to a foliation $\mathcal{F}$ which is almost without holonomy if and only if no nontrivial structure cycle of $\mathcal{Z}$ bounds. In this case, the dual cone $\mathcal{D}_\mathcal{Z}$ has nonempty interior and every element of* $\operatorname{int}\mathcal{D}_\mathcal{Z}$ *is a foliated class, represented by a foliated form which is transverse to $\mathcal{L}$.*

Proof We prove the "if" part and the subsequent assertions. By [29, Theorem I.7, part iv], the hypothesis that no nontrivial structure cycle bounds implies that $\operatorname{int}\mathcal{D}_\mathcal{Z} \neq \emptyset$ and that each class in this open cone has a representative form

$\omega \in A^1(M)$ which is transverse to $\mathcal{Z}$. We must replace ω with a form $\widetilde{\omega} = \omega + dg$ such that $g \in C^\infty(M_0)$, dg blows up nicely at $\partial_\tau M$ and $\widetilde{\omega} \in A^1(M_0)$ is transverse to $\mathcal{L}|M_0$. Remark that dg blows up nicely at $\partial_\tau M$ if dg is unbounded in a neighborhood V of $\partial_\tau M$ and the smooth foliation of $V \cap M_0$ by the level sets of g can be completed to a foliation of V integral to a C^0 plane field by adjoining the components of $\partial_\tau M$ as leaves. Since ω is bounded on the compact manifold M, it is clear that $\widetilde{\omega}$ will blow up nicely at the boundary, hence be a foliated class, and that $[\omega] = [\widetilde{\omega}]$.

Let v be a C^0 vector field to which $\mathcal{L}|M_0$ is integral and which extends by 0 to a continuous field on M. By Lemma 4.1, we assume that v is smooth in a neighborhood V of $\partial_\tau M$. Let S_- be the union of the inwardly oriented components of $\partial_\tau M$ and S_+ the union of the outwardly oriented ones. We can assume that the part of V bordering S_- is parametrized by the local v–flow as $S_- \times [-\infty, 0)$ and the part bordering S_+ as $S_+ \times (0, \infty]$. In particular, the fibers $\{x\} \times [-\infty, 0)$ (respectively, $\{x\} \times (0, \infty]$) are subarcs of leaves of $\mathcal{L}$, $\forall x \in S_-$ (respectively, $\forall x \in S_+$). Let $W \subset V$ be a neighborhood of $\partial_\tau M$ such that $\overline{W} \subset V$.

Define $f \in C^\infty(S_- \times (-\infty, 0))$ so that $f(x,t) = f(t)$ everywhere,

$$f(t) = \begin{cases} t, & t \leq -1, \\ 0, & -\frac{1}{2} \leq t, \end{cases}$$

and so that $f'(t) \geq 0$ on $(-\infty, 0)$. Define f analogously on $S_+ \times (0, \infty)$ and extend these definitions by 0 to a smooth function f on all of M_0. Remark that $df(v) \equiv 1$ in $V \cap M_0$, hence the form df clearly blows up nicely at $\partial_\tau M$.

There is an open neighborhood U of $\mathcal{Z}$ in M_0 such that $\omega(v) > 0$ on U. Let $z \in M_0 \smallsetminus \mathcal{Z}$ and let ℓ_z denote a compact subarc of the leaf of $\mathcal{L}$ through z which has z in its interior and exactly one end in $\partial_\tau M$. For definiteness, assume this end lies in S_-. There are open neighborhoods V_z and U_z of ℓ_z, $\overline{V}_z \subset U_z$, and a smooth field w_z on M which approximates v arbitrarily well on V_z, approximates the *direction* of v arbitrarily well on $U_z \cup V$, agrees with v on W and vanishes identically on $M \smallsetminus (U_z \cup V)$. Let Φ_t^z be the flow of w_z. If $x \in W \cap V_z$ near S_- and $\tau_z \in \mathbb{R}$ are such that the $z = \Phi^z_{\tau_z}(x)$, choose $t_z > \tau_z$ and let

$$df_z = \Phi^*_{-t_z}(df).$$

Thus, we can assume that $df_z(v) \geq 0$ on M_0 and $> 1/2$ on $(V_z \cup W) \cap M_0$. By compactness of M and boundedness of ω, find finitely many points $z_i \in M_0 \smallsetminus \mathcal{Z}$, $1 \leq i \leq r$, and a constant $c > 0$ such that $\{U, V_{z_i}\}_{i=1}^r$ is an open cover of M

and

$$\widetilde{\omega} = \omega + dg = \omega + c\sum_{i=1}^{r} df_{z_i}$$

is a closed form, strictly positive on $v|M_0$ which blows up at $\partial_\tau M$. In fact, we have guaranteed that dg is a constant multiple of df near $\partial_\tau M$, so $\widetilde{\omega}$ blows up nicely at $\partial_\tau M$. □

Examples show that, if $\mathcal{F}|M_0$ is dense-leaved, the invariant measure ν may not be absolutely continuous, let alone smooth, in which case $\mathcal{F}$ is not defined by a foliated form. Although these foliations are not the primary focus of this paper, we note the following.

Corollary 4.4 *The foliation $\mathcal{F}$ is C^0 isotopic to a foliation $\widetilde{\mathcal{F}}$ defined by a foliated form $\widetilde{\omega}$.*

Indeed, $[\nu] \in \mathcal{D}_{\mathcal{Z}}$, so Theorem 4.3 provides a foliated form $\widetilde{\omega}$ representing this same class. We take $\widetilde{\mathcal{F}}$ to be the foliation defined by $\widetilde{\omega}$ and find an isotopy of $\mathcal{F}$ along $\mathcal{L}$ to $\widetilde{\mathcal{F}}$, using the continuous measures on the leaves of $\mathcal{L}$ induced by ν and $\widetilde{\omega}$. For details, see [2, Section 2].

It will be important to characterize a particularly simple spanning set of $\mathbf{C}_{\mathcal{Z}}$, the so called "homology directions" of Fried [11, page 260]. Assuming that $\mathcal{L}|M_0$ has been parametrized as a nonsingular C^0 flow Φ_t, select a point $x \in \mathcal{Z}$ and let Γ denote the Φ–orbit of x. If this is a closed orbit, it defines a structure cycle which we will denote by $\overline{\Gamma}$. If it is not a closed orbit, let $\Gamma_\tau = \{\Phi_t(x) \mid 0 \le t \le \tau\}$. Let $\tau_k \uparrow \infty$ and set $\Gamma_k = \Gamma_{\tau_k}$. After passing to a subsequence, we obtain a structure current

$$\overline{\Gamma} = \lim_{k\to\infty} \frac{1}{\tau_k}\int_{\Gamma_k} .$$

Lemma 4.5 *A structure current $\overline{\Gamma}$, obtained as above, is a structure cycle.*

Proof By compactness of $\mathcal{Z}$, we can again pass to a subsequence so as to assume that the points $\Phi_{\tau_k}(x)$ all lie in the same flowbox B, $k \ge 1$. Thus, we can close up Γ_k to a loop Γ_k^* by adjoining an arc in B from x_k to x. These arcs can be kept uniformly bounded in length, hence the sequence of *singular cycles* $(1/\tau_k)\Gamma_k^*$ (generally not foliation cycles) also converges to $\overline{\Gamma}$. These approximating singular cycles are called "long, almost closed orbits" of Φ_t. Since the space of cycles is a closed subspace of the space of currents, this proves that $\overline{\Gamma}$ is a cycle. □

Definition 4.6 All structure cycles $\overline{\Gamma}$, obtained as above, and their homology classes are called homology directions of $\mathcal{Z}$.

An elementary application of ergodic theory proves the following (cf [29, Proposition II.25]).

Lemma 4.7 *Any structure cycle* $\mu \in \mathbf{C}_{\mathcal{Z}}$ *can be arbitrarily well approximated by finite linear combinations* $\sum_{i=1}^{r} a_i \overline{\Gamma}_i$ *of homology directions. If* $\mu \neq 0$, *the coefficients* a_i *are strictly positive and their sum is bounded below by a constant* $b_\mu > 0$ *depending only on* μ.

While much of the Schwartzmann–Sullivan theory requires that $\mathcal{L}$ be at least leafwise C^1, this lemma suggests how to define the cones $\mathcal{C}_{\mathcal{Z}}$ and $\mathcal{D}_{\mathcal{Z}}$, even for the case in which the transverse foliation $\mathcal{L}$ is only C^0. Indeed, the long, almost closed orbits (and the honest closed orbits) are defined, they are singular homology cycles, their classes in $H_1(M)$ form a bounded set and limit classes as above, taken over sequences for which $\tau_k \uparrow \infty$, are called homology directions. The closure of the set of positive linear combinations of homology directions forms a convex cone $\mathcal{C}_{\mathcal{Z}} \subset H_1(M)$ and $\mathcal{D}_{\mathcal{Z}} \subset H^1(M)$ is the dual cone. While results such as Theorem 4.3 become problematic in this context, we will always be in a position to identify these cones with ones coming from leafwise C^1 data.

In what follows, the foliation $\mathcal{F}$ will be of depth one and the transverse foliations $\mathcal{L}$ and $\mathcal{L}'$ may only be C^0.

Lemma 4.8 *Let* $\mathcal{L}$ *and* $\mathcal{L}'$ *be two* 1*–dimensional foliations transverse to* $\mathcal{F}$. *Suppose that the respective core laminations* $\mathcal{Z}$ *and* $\mathcal{Z}'$ *are* C^0*–isotopic by an isotopy* $\varphi_t\colon \mathcal{Z} \hookrightarrow M$, $\varphi_0|\mathcal{Z} = \mathrm{id}_{\mathcal{Z}}$ *and* $\varphi_1(\mathcal{Z}) = \mathcal{Z}'$, *such that* $\varphi_t(x)$ *lies in the same leaf of* $\mathcal{F}$ *for* $0 \leq t \leq 1$, $\forall\, x \in \mathcal{Z}$. *Then* $\mathcal{C}_{\mathcal{Z}} = \mathcal{C}_{\mathcal{Z}'}$.

Proof Parametrize the two foliations as flows using the same transverse invariant measure for $\mathcal{F}$. Since $\mathcal{F}$ is leafwise invariant under the isotopy, the flow parameter is preserved and the long, almost closed orbits of $\mathcal{Z}$ are isotoped to the long, almost closed orbits of $\mathcal{Z}'$. Homotopic singular cycles are homologous and the assertions follow. □

Note that we do not require this to be an ambient isotopy. The property that points of $\mathcal{Z}$ remain in the same leaf of $\mathcal{F}$ throughout the isotopy will be indicated, as above, by saying that $\mathcal{F}$ is leafwise invariant by φ_t.

The following is proven using the well understood structure of depth one foliations in neighborhoods of $\partial_\tau M$.

Lemma 4.9 *If $\mathcal{L}$ and $\mathcal{L}'$ are two transverse foliations which induce the same first return map f on a noncompact leaf L_0 of $\mathcal{F}$, then, without changing this property, one can modify $\mathcal{L}'$ in an arbitrarily small neighborhood of $\partial_\tau M$ to agree with $\mathcal{L}$ in a smaller neighborhood.*

Lemma 4.10 *Let $\mathcal{L}$ and $\mathcal{L}'$ be two 1–dimensional foliations transverse to $\mathcal{F}$ and inducing the same first return map $f\colon L_0 \to L_0$ on some leaf L_0 of $\mathcal{F}|M_0$. Then $\mathcal{Z}'$ is isotopic to $\mathcal{Z}$ by a C^0 isotopy leaving $\mathcal{F}$ leafwise invariant, hence $\mathcal{C}_{\mathcal{Z}} = \mathcal{C}_{\mathcal{Z}'}$ and $\mathcal{D}_{\mathcal{Z}} = \mathcal{D}_{\mathcal{Z}'}$.*

Proof Parametrize the two foliations, using the same transverse invariant measure for $\mathcal{F}$, so as to obtain flows Φ_t and Φ'_t, both carrying leaves L of $\mathcal{F}$ to leaves $\Phi_t(L) = \Phi'_t(L)$ of $\mathcal{F}$, and such that

$$\Phi_1|L_0 = f = \Phi'_1|L_0.$$

We choose a compact, connected submanifold $K \subset L_0$, separating all the ends of L_0, with $Z \subset \operatorname{int} K$. By Lemma 4.9, we lose no generality in assuming that $\mathcal{L}$ and $\mathcal{L}'$ coincide near $\partial_\tau M$, so we can choose K larger, if necessary, to guarantee that $\Phi_t(\partial K) = \Phi'_t(\partial K)$, $0 \le t \le 1$. These properties imply that $\Phi_t(K) = \Phi'_t(K)$, $0 \le t \le 1$. Define

$$\begin{aligned} \varphi_t\colon K \to K, &\quad 0 \le t \le 1, \\ \varphi_t = {\Phi'}_{-t} \circ \Phi_t|K, & \end{aligned}$$

a loop in $\operatorname{Homeo}_0(K)$ based at $\varphi_0 = \varphi_1 = \mathrm{id}_K$. Here, $\operatorname{Homeo}_0(K)$ denotes the identity component of the group of homeomorphisms of K (with the compact–open topology). Since no end of L_0 has a neighborhood of the form $\mathbb{R} \times I$ or $\mathbb{R} \times S^1$ (a consequence of our ongoing hypotheses), we can choose K to have negative Euler characteristic. Theorems of M. E. Hamstrom [16, 17, 18] then imply that the group $\operatorname{Homeo}_0(K)$ is simply connected (see the remark below), so there is a homotopy φ_t^s in $\operatorname{Homeo}_0(K)$, $0 \le s \le 1$, fixing the basepoint, with $\varphi_t^0 = \varphi_t$ and $\varphi_t^1 = \mathrm{id}_K$, $0 \le t \le 1$. This defines a continuous deformation of

$$\Phi_t|K = \Phi'_t \circ \varphi_t^0 \text{ to } \Phi'_t \circ \varphi_t^1 = \Phi'_t|K$$

which slides points along the leaves of $\mathcal{F}$. This restricts to a C^0 isotopy of

$$\mathcal{Z} \subset \bigcup_{0 \le t \le 1} \Phi_t(K) \text{ to } \mathcal{Z}' \subset \bigcup_{0 \le t \le 1} \Phi'_t(K)$$

and everything now follows by Lemma 4.8. □

Remark The theorem of Hamstrom, cited in the above proof, is that the group $\mathrm{Homeo}_0(K, \partial K)$ which fixes $\partial K \neq \emptyset$ pointwise is homotopically trivial. This is true whether or not $\chi(K)$ is negative. The assumption of negative Euler characteristic implies that $\pi_1(K, x)$ is free on at least two generators, in which case one shows that any loop φ on $\mathrm{Homeo}_0(K)$, based at the identity, is base point homotopic to a loop in the subgroup $\mathrm{Homeo}_0(K, \partial K)$. It follows that $\mathrm{Homeo}_0(K)$ is simply connected. Indeed, for each point $x \in K$, φ defines a loop φ_x on K based at x and the assignment $x \mapsto \varphi_x$ is continuous in the compact–open topology. It follows rather easily that, for each $x \in K$ and each loop σ_x on K based at x, the composed loop $\sigma_x \varphi_x \sigma_x^{-1}$ is base point homotopic to φ_x. This can only be true if φ_x is homotopically trivial, $\forall\, x \in K$. Using this fact for each $x \in \partial K$, one constructs the desired homotopy of φ in $\mathrm{Homeo}_0(K)$. We remark that the group $\mathrm{Diff}_0(K)$ is also known to be homotopically trivial [8] as is the group of piecewise linear homeomorphisms [28].

Because of this lemma, we may write $\mathcal{C}_f$ for $\mathcal{C}_{\mathcal{Z}}$ and $\mathcal{D}_f$ for $\mathcal{D}_{\mathcal{Z}}$, where f is the first return homeomorphism induced by $\mathcal{L}$ on a depth one leaf. We can also use $\mathcal{Z}_f$ to denote the isotopy class of the core laminations corresponding to f. Here, the isotopies should preserve each leaf of $\mathcal{F}$.

Lemma 4.11 *Let $f\colon L_0 \to L_0$ be the first return homeomorphism induced on a depth one leaf L_0 of $\mathcal{F}$ by a transverse, 1–dimensional foliation $\mathcal{L}$. If $g\colon L_0 \to L_0$ is a homeomorphism isotopic to the identity, then $\mathcal{C}_f = \mathcal{C}_{gfg^{-1}}$.*

Indeed, in standard fashion, the isotopy g_t, $g_0 = g$ and $g_1 = \mathrm{id}$, induces an isotopy of $\mathcal{L}$ to $\mathcal{L}'$, leaving $\mathcal{F}$ leafwise invariant, such that $\mathcal{L}'$ induces first return map gfg^{-1} on L_0.

The set $Z = \mathcal{Z} \cap L_0$ is exactly the set of points which never cluster at ends of L_0 under forward or backward iteration of the monodromy f. Assume that the dynamical system (Z, f) admits a Markov partition $\{R_1, \dots, R_n\}$ (in particular, these are imbedded rectangles in L_0 that cover Z and have disjoint interiors) and let (Σ_A, σ_A) be the associated symbolic dynamical system. Here, an $n \times n$ incidence matrix $A = [a_{ij}]$ of 0's and 1's determines a closed subset

$$\Sigma_A \subseteq \{1, 2, \dots, n\}^{\mathbb{Z}},$$

a sequence $\iota = \{i_k\}_{k=-\infty}^{\infty}$ being an element of Σ_A if and only if $a_{i_k i_{k+1}} = 1$, $\forall\, k$. This is a compact, metrizable, totally disconnected space and the shift map σ_A is a homeomorphism. In the usual scheme, there is a semiconjugacy

$$\varphi\colon (\Sigma_A, \sigma_A) \to (Z, f)$$

defined by

$$\varphi(\iota)=\varphi(\cdots,i_{-1},i_0,i_1,\cdots)=\bigcap_{k=-\infty}^{\infty} f^{-k}(R_{i_k})=R_\iota.$$

This assumes that the infinite intersection R_ι of rectangles $f^{-k}(R_{i_k})$ degenerates to a singleton, but we are going to allow this set to be either a singleton, a nondegenerate arc, or a nondegenerate rectangle. We will still require that

$$Z=\bigcup_{\iota\in\Sigma_A} R_\iota,$$

but each symbol sequence

$$\iota=(\cdots,i_{-1},i_0,i_1,\cdots)\in\Sigma_A$$

will represent all the points in R_ι. Remark that a boundary point of R_ι might be represented by distinct sequences in Σ_A.

The closed orbits Γ of Φ_t determine periodic orbits of f in Z, hence correspond to periodic orbits of σ_A. A point $\iota\in\Sigma_A$ has periodic σ_A–orbit if and only if ι itself breaks down into a bi-infinite sequence of a repeated finite string $i_0,\dots,i_{q-1}$, called a *period* of σ_A. In this case, ι is called a *periodic point.* Given a periodic point ι, the Brouwer fixed point theorem implies that there is at least one corresponding periodic f–orbit $\{x,f(x),\dots,f^q(x)=x\}$, $x\in R_\iota$, and a corresponding closed leaf $\Gamma_\iota=\{\Phi_t(x)\}_{t\in\mathbb{R}}$ of $\mathcal{Z}$.

Let $\iota=\{i_k\}_{k=-\infty}^{\infty}\in\Sigma_A$ and suppose that $i_q=i_0$ for some $q>0$. Let $x\in R_\iota$. Then there is a corresponding singular cycle Γ_q formed from the orbit segment $\gamma_q=\{\Phi_t(x)\}_{0\le t\le q}$ and an arc $\tau\subset R_{i_0}$ from $\Phi_q(x)=f^q(x)$ to x. Also, since $i_q=i_0$, there is a periodic element $\iota'\in\Sigma_A$ with period $i_0,\dots,i_{q-1}$ and a corresponding closed leaf $\Gamma_{\iota'}=\Gamma'$ of $\mathcal{Z}$.

Lemma 4.12 *The singular cycle Γ_q and closed leaf Γ', obtained as above, are homologous. In particular, the homology class of Γ_q depends only on the periodic element ι'.*

Proof The loop Γ' is the orbit segment $\{\Phi_t(x')\}_{0\le t\le q}$, for a periodic point

$$x'\in R_{i_0}\cap f^{-1}(R_{i_1})\cap\cdots\cap f^{-q}(R_{i_q})=R'.$$

Remark that $x\in R'$ also. Let τ' be an arc in the subrectangle $R'\subset R_{i_0}$ from x to x' and set $\tau''=f^q(\tau')$, an arc in $f^q(R')$ from $f^q(x)$ to x'. Since $i_q=i_0$, $f^q(R')\subset R_{i_0}$ and the cycle $\tau+\tau'-\tau''$ in the rectangle R_{i_0} is homologous to 0. That is, we can replace the cycle $\Gamma_q = \gamma_q + \tau$ by the homologous cycle $\gamma_q - \tau' + \tau''$. Finally, a homology between this cycle and Γ' is given by the map

$$H\colon\ [0,1] \times [0,q] \to M,$$

defined by parametrizing τ' on $[0,1]$ and setting

$$H(s,t) = \Phi_t(\tau'(s)). \qquad \square$$

If no proper, cyclicly consecutive substring of a σ_A–period $i_0, \dots, i_{q-1}$ also occurs as a period, we say that the period is *minimal.* It is elementary that there are only finitely many minimal periods. Those closed leaves Γ of $\mathcal{Z}$ that correspond to minimal periods in the symbolic system will be called minimal loops in $\mathcal{Z}$. The following is an easy consequence of Lemma 4.12.

Corollary 4.13 *Every closed leaf Γ of $\mathcal{Z}$ is homologous in M to a linear combination of the minimal loops in $\mathcal{Z}$ with non-negative integer coefficients. Furthermore, every homology direction can be arbitrarily well approximated by positive multiples of closed leaves of $\mathcal{Z}$.*

This corollary and Lemmas 4.7 and 4.12 give the following important result.

Theorem 4.14 *Suppose that the dynamical system (Z, f) admits a Markov partition. Then the cone $\mathcal{C}_f \subset H_1(M)$ is the convex hull of finitely many rays through classes $[\Gamma_i]$, $1 \le i \le r$, where the structure cycles Γ_i are minimal loops in $\mathcal{Z}$. Consequently, the dual cone $\mathcal{D}_f$ is polyhedral and both $\mathcal{C}_f$ and $\mathcal{D}_f$ depend only on the symbolic dynamics.*

5 Pseudo-Anosov endperiodic maps

We continue with the hypotheses and notation of the preceding section. Fix a noncompact leaf L of $\mathcal{F}$ and let $f\colon L \to L$ be the first return map defined by a transverse 1–dimensional foliation $\mathcal{L}_f$. It is standard that f is an *endperiodic* homeomorphism [9]. Here, we use the well understood structure theory of depth one leaves, writing

$$L = K \cup U_+ \cup U_-,$$

where K is a compact, connected subsurface, called the *core* of L, $U_\pm$ falls into a disjoint union of finitely many closed neighborhoods of isolated ends of L and K meets $U_\pm$ only along common boundary components. The set U_+ is called the neighborhood of *attracting ends* and has the property that

$f(U_+) \subset U_+$. The neighborhood U_- of *repelling ends* has the property that $U_- \subset f(U_-)$. The core is not unique since it can always be made larger by adjoining a suitable piece of $U_\pm$. Set $K \cap U_\pm = \partial_\pm K$. While $f^n(\partial_+ K)$ suffers only bounded distortion as $n \to \infty$, it generally becomes unboundedly distorted as $n \to -\infty$, the situation being reversed for $f^n(\partial_- K)$.

The attracting ends $\{e_+^i\}_{i=1}^r$ of L are permuted by f, as are the repelling ends $\{e_-^j\}_{j=1}^k$. The set of cycles of these permutations corresponds one-to-one to the set of components of $\partial_\tau M$. Let $\{e_+^{i_1}, e_+^{i_2}, \dots, e_+^{i_q}\}$ be a cycle corresponding to the tangential boundary component F and let $U_+^{i_1}, U_+^{i_2}, \dots, U_+^{i_q}$ be the corresponding components of U_+ which are neighborhoods of these ends. One can choose the above data so that

$$f(U_+^{i_j}) = U_+^{i_{j+1}}, \quad 1 \le j < q,$$
$$f(U_+^{i_q}) \subset U_+^{i_1}.$$

There is a fundamental domain $F' \subset U_+^{i_1}$ for the action of the semigroup $\{f^n\}_{n=0}^\infty$ on the union of these neighborhoods. This domain F' is homeomorphic to a manifold obtained by cutting F along the juncture (cf [6, pages 3–4]) and it meets K in a union of common boundary components. A similar assertion holds for the repelling ends of L, the semigroup being $\{f^{-n}\}_{n=0}^\infty$. The way in which the manifolds $f^n(F')$ link together along parts of their boundaries can be surprisingly complicated.

As is well known, the depth one foliated manifold $(M, \mathcal{F})$ can be recovered, up to foliated homeomorphism, from the endperiodic map f. Indeed, the open, fibered manifold $(M_0, \mathcal{F}_0 = \mathcal{F}|M_0)$ is obtained, up to homeomorphism, by suspension of the homeomorphism f, while the completion

$$M = M_0 \cup \partial_\tau M$$

is determined by the endperiodic structure. For more details see, for example, [5, Lemma 2.3]. We further remark that the depth one foliated manifold is homeomorphic (indeed, isotopic) to one in which $\mathcal{F}$ is smooth, even at the boundary, so we assume smoothness.

The isotopy class $m(f)$ of f (also called the *mapping class* of f) is completely determined by the depth one foliation $\mathcal{F}$ and, in turn, $m(f)$ determines the fibered manifold $(M_0, \mathcal{F}_0)$. The transverse foliation $\mathcal{L}_f$ is not well defined by f, although the isotopy class of its core lamination $\mathcal{Z}_f$ is well defined (Lemma 4.10). This isotopy class varies, however, as f is varied through endperiodic elements of $m(f)$, so the cones $\mathcal{C}_f$ generally change as f is so varied. We want to choose f so that these cones are as "small" as possible. That is, we want the dual cone, $\mathcal{D}_f$ to be as large as possible. The tool for this is some unpublished work of Handel and Miller (see [9]) which generalizes the Nielsen–Thurston classification of homeomorphisms of compact surfaces [1]. In order to state this, some terminology is in order.

Denote by L^c the compactification of L obtained by adjoining its ends. By a *properly imbedded line* in L, we mean a topological imbedding

$$\sigma\colon\ [-\infty,\infty]\to L^c,$$

where $\{\sigma(\pm\infty)\}$ is a pair of ends of L and $\sigma(-\infty,+\infty)\subset L$. This will be distinguished from a *properly imbedded arc* which is an imbedding

$$\sigma\colon\ [-1,1]\to L,$$

where $\{\sigma(\pm 1)\}=\sigma[-1,1]\cap\partial L$. A *peripheral curve* in L is either a closed curve isotopic to a component of ∂L or a properly imbedded line, isotopic (with endpoints fixed) to the endpoint compactification of a noncompact component of ∂L. A *proper homotopy* between properly imbedded lines or arcs is a homotopy that fixes the endpoints. If an end e of L has a neighborhood that is homeomorphic either to $S^1\times[0,\infty)$ or $[0,1]\times[0,\infty)$, then e will be called a *trivial end.*

Definition 5.1 A closed, essential, nonperipheral curve $\gamma\subset L$ is a closed reducing curve if, for a sufficiently large integer $n>0$, $f^n(\gamma)\subset U_+$ and $f^{-n}(\gamma)\subset U_-$. A properly imbedded, nonperipheral line σ is a reducing line if one endpoint is an attracting end, the other a repelling end, and σ is periodic under f up to a proper homotopy. A periodic curve is a closed, nonperipheral curve which is periodic under f up to homotopy.

Definition 5.2 The endperiodic map $f\colon\ L\to L$ is periodic (or trivial) if every orbit $\{f^n(x)\}_{n=-\infty}^{\infty}$ has points in U_+ and points in U_-. The endperiodic map is irreducible if no end of L is trivial and there are no reducing lines, reducing curves, nor periodic curves. Otherwise, f is reducible.

Lemma 5.3 *If no component of $\partial_\tau M$ is an annulus or a torus, if M is completely reduced* (Definition 3.2), *and if L is a noncompact leaf of a taut, depth one foliation of M, then every endperiodic homeomorphism $f\colon\ L\to L$ that occurs as the first return map for a transverse foliation $\mathcal{L}_f$ is irreducible. The endperiodic homeomorphism is periodic if and only if $M\cong S\times I$ and $\mathcal{L}_f$ is a product I–bundle over S.*

The elementary proof is left to the reader. The point of this lemma is that our ongoing hypotheses (Theorem 3.9) imply that the endperiodic monodromy is irreducible and nonperiodic.

We can assume that the monodromy f and associated transverse foliation $\mathcal{L}_f$ are smooth. Since no component of $\partial_\tau M$ is an annulus or a torus, we will put a smooth Riemannian metric on M such that all the leaves are hyperbolic, chosen so that, in $U_\pm$, it is the lift of the metric on $\partial_\tau M$ via the projection $U_\pm \to \partial_\tau M$ along the leaves of $\mathcal{L}_f$ [3]. We can demand that ∂L be geodesic and choose the juncture in each boundary leaf to be geodesic. In particular, ∂K will consist of geodesic arcs and/or loops and f will be an isometry in the ends.

Theorem 5.4 (Handel–Miller) *Let $f\colon L \to L$ be an irreducible, endperiodic, nonperiodic monodromy diffeomorphism of a hyperbolic leaf as above. Then there is a pair of mutually transverse geodesic laminations $\Lambda_\pm$ of L and an endperiodic homeomorphism $h \in m(f)$ such that*

1. *The components of $h^n(\partial_\pm K)$ are geodesics, $\forall\, n \in \mathbb{Z}$, and*
   - (a) *$\Lambda_+ \subset K \cup U_+$ is the limit of the geodesic laminations $h^n(\partial_- K)$ as $n \to \infty$;*
   - (b) *$\Lambda_- \subset K \cup U_-$ is the limit of $h^n(\partial_+ K)$ as $n \to -\infty$;*
2. *$K \cap (\Lambda_+ \cup \Lambda_-)$ weakly binds K in the sense that complementary regions not meeting ∂K are simply connected;*
3. *$h|\partial L = f|\partial L$;*
4. *If $Z \subset K$ is the invariant set of h, the dynamical system $(Z, h|Z)$ admits a Markov partition.*

Finally, $m(f)$ together with the hyperbolic structure on L uniquely determines $\Lambda_\pm$ and $h|\Lambda_+ \cap \Lambda_-$.

Remark In (a), the assertion that

$$\lim_{n\to\infty} h^n(\partial_- K) = \Lambda_+$$

means that the leaves ℓ of Λ_+ are exactly the curves in L with the following property. For each compact subarc $J \subset \ell$, there is a sequence of subarcs $J_k \subset h^k(\partial_- K)$ converging uniformly to J as $k \uparrow \infty$. Equivalently, Λ_+ is the frontier of the open set

$$\mathcal{U}_- = \bigcup_{n\in\mathbb{Z}} h^n(U_-).$$

This is the weak interpretation of (a).

For the strong interpretation of (a), let $\widetilde{L}$ be the the universal cover of L and let $\overline{L}$ be the closure of $\widetilde{L}$ in the closed Poincaré disk $D \cup S_\infty$ (where S_∞ is the circle at infinity and D is the open unit disk). Either $\partial L = \emptyset$ and $\overline{L} = D \cup S_\infty$, or $\widetilde{L}$ has geodesic boundary in D and $\overline{L}$ is the union of $\widetilde{L}$ and a Cantor set $C \subset S_\infty$. Since everything in sight is a geodesic, the weak interpretation of (a) implies that the lifts $\widetilde{\ell} \subset \widetilde{L}$ of leaves of Λ_+ are exactly the uniform limits in the *Euclidean* metric of sequences $\widetilde{\sigma}_k$ of suitable lifts of components of $h^k(\partial_- K)$. When $\widetilde{L} = D$, this means that the endpoints in S_∞ of (the completion of) $\widetilde{\sigma}_k$ converge to the endpoints of $\widetilde{\ell}$. In general, the endpoints of $\widetilde{\sigma}_k$ may lie on geodesic boundary components of $\widetilde{L}$, but they still converge in the Euclidean metric to the endpoints of $\widetilde{\ell}$ in $C \subset S_\infty$. Similar remarks apply to (b).

We will say that $h \in m(f)$ is a pseudo-Anosov, endperiodic automorphism of L. Remark that h preserves the laminations, expanding the leaves of the unstable lamination Λ_+ and contracting those of the stable lamination Λ_-. Unlike the compact case, there may not be projectively invariant measures of full support.

Since $\bigcup_{n=1}^{\infty} h^n(\partial_- K)$ consists of disjoint geodesics and K is a compact surface, the intersection of this set with K consists of compact, properly imbedded geodesic arcs which fall into a finite number of isotopy classes rel $\partial_+ K$. The components of $\Lambda_+ \cap K$ all belong to these isotopy classes and every leaf of Λ_+ meets $\partial_+ K$, hence every leaf meets $h^n(\partial_+ K)$, $n \geq 1$. Evidently, none of these leaves can meet $\partial_- K$. Similar remarks hold for the stable lamination Λ_-.

Corollary 5.5 *Every leaf ℓ of Λ_+ contains points of Z and points going to infinity in both directions in ℓ which converge to an attracting end of L, but no points arbitrarily near repelling ends. The analogous assertions hold for Λ_-, with the roles of attracting and repelling ends interchanged.*

The points of $\Lambda_+ \cap \Lambda_-$ are contained in the core K and remain there under all forward and backward iterations of h. Generally, this intersection is not the entire invariant set Z, which may have nonempty interior [9, Proposition 2.12].

The lamination Λ_+ can be augmented to an h–invariant geodesic lamination Γ_+ by adding on all $h^n(\partial_- K)$, $-\infty < n < \infty$, and a similar augmented lamination Γ_- is obtained by adding all $h^n(\partial_+ K)$ to Λ_-. Once the choice of K has been fixed, these augmented laminations are uniquely determined by $m(f)$ and the hyperbolic metric, they are mutually transverse and they weakly bind L. The automorphism h is unique on $\Gamma_+ \cap \Gamma_-$ and may be extended continuously in any convenient way on the complementary arcs of this set in $\Gamma_+ \cup \Gamma_-$ and on the components of the complement of $\Gamma_+ \cup \Gamma_-$ in L. It is not clear that this can be done so that h will be smooth, even though the original monodromy f was smooth.

The foliation $\mathcal{L}_h$, chosen to produce the pseudo-Anosov first return map h, is a bit problematic at $\partial_\tau M$. It happens that there will always be distinct h–orbits in $\Lambda_+ \cup \Lambda_-$ which cluster in M at the same point of $\partial_\tau M$ (see [9]). Each leaf of $\mathcal{L}_h$ which comes close enough to $\partial_\tau M$ actually limits on a unique point of $\partial_\tau M$, but the fact that distinct leaves can limit on the same point of $\partial_\tau M$ implies that the continuous extension of $\mathcal{L}_h$ to M will not be a foliation. We could remedy this by modifying $\mathcal{L}_h$ in arbitrarily small neighborhoods of $\partial_\tau M$ without affecting the core lamination $\mathcal{Z}_h$, thereby allowing the isotopy arguments of the previous section to be carried out. Since the conclusions of these arguments concern only the core lamination $\mathcal{Z}_h$, they remain true without making such modifications.

By this remark and in order to keep the full force of pseudo-Anosov monodromy, we agree that $\mathcal{L}_h$ be defined only in M_0, extending to M as a singular foliation.

Lemma 5.6 *There are mutually transverse, smooth leaved, 2–dimensional laminations $\Theta_\pm$ of M_0 which intersect the leaves of $\mathcal{F}_0$ transversely in the augmented geodesic laminations $\Gamma_\pm$ for the pseudo-Anosov monodromy of those leaves.*

Proof For simplicity of exposition, we consider the case that $\partial L = \emptyset$. This makes the universal cover of each leaf the full open unit disk D. Standard modifications of the following argument prove the general case. Let Φ_t be the leaf preserving flow in M_0 with flow lines the leaves of $\mathcal{L}_f|M_0$. Since $\mathcal{L}_f$ is smooth, so is this flow. Fix a leaf L_0 of $\mathcal{F}|M_0$, set $L_t = \Phi_t(L_0)$ and consider the open, $\mathcal{F}$–saturated set $V_\varepsilon = \bigcup_{-\varepsilon<t<\varepsilon} L_t$. Here $\varepsilon > 0$ is chosen small enough that $L_\varepsilon \neq L_{-\varepsilon}$. The universal cover of V_ε is of the form $D \times (-\varepsilon, \varepsilon)$, where the interval fibers are flow lines of the local flow $\widetilde{\Phi}_t$ obtained by lifting Φ_t, $-\varepsilon < t < \varepsilon$.

Write D_t for $D \times \{t\}$ and remark that the lifted metric gives a hyperbolic metric γ_t on D_t, $-\varepsilon < t < \varepsilon$. Projection along the interval fibers is smooth, but is not an isometry of these metrics. However, for $\varepsilon > 0$ sufficiently small, projection distorts the metrics only in a uniformly small way in the C^2 topology (indeed, this is true for Φ_t downstairs, which only fails to be an isometry on a compact neighborhood of K). In particular, each geodesic σ_0 in D_0 is carried to a curve σ_t in D_t with uniformly small geodesic curvature, $-\varepsilon < t < \varepsilon$.

An important property of hyperbolic geometry is that any curve with geodesic curvature < 1 remains uniformly near a unique geodesic, hence has well defined ends in the circle at infinity. In fact, any pencil of asymptotic geodesics limiting on the same point at infinity projects to a pencil of "almost geodesics" limiting on a common point at infinity. It follows that projections along the interval fibers extend to well defined projections on the circles at infinity. Thus, the completion $\mathcal{M}$ of $D \times (-\varepsilon, \varepsilon)$ by the circles at infinity of each D_t is canonically identified with $\overline{D} \times (-\varepsilon, \varepsilon)$, making $\mathcal{M}$ a smooth manifold.

Suppose that σ_0 is a geodesic covering a leaf ℓ_0 of $\Gamma_\pm$ in L_0 and consider the smooth surface $S = \bigcup_{-\varepsilon<t<\varepsilon} \widetilde{\Phi}_t(\sigma_0)$. If $z_{\pm\infty}$ are the endpoints of σ_0 at infinity, then S is continuously extended to $\overline{D} \times (-\varepsilon, \varepsilon)$ by adjoining $\{z_{\pm\infty}\} \times (-\varepsilon, \varepsilon)$. Fix some third arc $\{w\} \times (-\varepsilon, \varepsilon)$ at infinity. If one uses the product metric on $D \times (-\varepsilon, \varepsilon)$ instead of the lifted metric, each leaf D_t has metric γ_0 and $\sigma_t = S \cap D_t$ is a geodesic, $-\varepsilon < t < \varepsilon$. But there is a unique leafwise uniformizing diffeomorphism $\psi\colon D \times (-\varepsilon, \varepsilon) \to D \times (-\varepsilon, \varepsilon)$ which fixes pointwise the three transverse arcs at infinity and carries γ_0 to γ_t, carrying each σ_t to a γ_t–geodesic. Each of these geodesics in D_t covers the leaf ℓ_t of $\Gamma_\pm$ in L_t which is in the homotopy class of $\Phi_t(\ell)$. Since $\psi(S)$ is smooth, these leaves ℓ_t fit together to form a smooth piece of surface transverse to $\mathcal{F}$. Extending this local construction in the obvious manner, we sweep out the general leaf of the desired lamination $\Theta_\pm$. $\square$

There is a certain amount of freedom in the choice of h and $\mathcal{L}_h$. In order to use the Schwartzmann–Sullivan theory as in the previous section, we note that Lemma 5.6 implies the following.

Corollary 5.7 *The foliation $\mathcal{L}_h$, transverse to $\mathcal{F}_0 = \mathcal{F}|M_0$ and inducing pseudo–Anosov monodromy h on a leaf L, can be chosen to be integral to a C^0 vector field in M_0.*

Indeed, $\Theta_+ \cap \Theta_-$ is a smooth leaved, 1–dimensional lamination X and we choose a continuous unit tangent field to X. This then extends, first to a continuous unit field on $\Theta_+ \cup \Theta_-$, thence to a field on M_0, which is C^1 on each component of the complement of X and is transverse to the leaves of $\mathcal{F}_0$. Here we use the fact that $\Gamma_+ \cup \Gamma_-$ weakly binds L. The foliation integral to this field is the desired realization of $\mathcal{L}_h$.

We come to the key result of this section.

Theorem 5.8 *Let h be a pseudo-Anosov first return map for $\mathcal{F}$ and let $\mathcal{F}'$ be another depth one foliation transverse to $\mathcal{L}_h$, f' the first return map induced by $\mathcal{L}_h$ on the typical noncompact leaf L' of $\mathcal{F}'$. Then there is a homeomorphism $g\colon L' \to L'$ which is isotopic to the identity such that $h' = g \circ f' \circ g^{-1} \in m(f')$ is pseudo-Anosov.*

Corollary 5.9 *Under the hypotheses of* Theorem 5.8, $\mathcal{C}_h = \mathcal{C}_{h'}$ *and* $\mathcal{D}_h = \mathcal{D}_{h'}$ *and these cones are independent of the choice of h off of the set $\Lambda_+ \cap \Lambda_-$.*

Indeed, the first assertion is immediate by Lemma 4.11 and the second by Theorem 4.14.

The proof of Theorem 5.8 will be broken down into a series of lemmas. Again, for simplicity of exposition, we attend mainly to the case that $\partial L = \emptyset = \partial L'$ (equivalently, $\partial_{\pitchfork} M = \emptyset$), frequently leaving it to the reader to adapt arguments to the general case.

To begin with, note that $\mathcal{F}'$ will be transverse to $\mathcal{L}_f$ outside of a compact subset of M_0, so we fix a leafwise hyperbolic metric for $\mathcal{F}'$ such that, in neighborhoods of the ends of depth one leaves, the metric is lifted from $\partial_\tau M$ by projection along $\mathcal{L}_f$.

Fix the noncompact leaves L of $\mathcal{F}$ and L' of $\mathcal{F}'$. Projection along the leaves of $\mathcal{L}_h$ defines local homeomorphisms between L and L', but one cannot expect to piece these together to a well defined covering map. Nonetheless, if $s\colon J \to L$ is a curve, $t_0 \in J$, one can project s along $\mathcal{L}_h$ to a curve $s'\colon J \to L'$, this projection being completely determined by the choice of $s'(t_0)$. Similarly, curves s' on L' project to curves s on L. The analysis in [9, Section 4] implies the following.

Lemma 5.10 *Deep in the ends of L and L' (that is, arbitrarily near $\partial_\tau M$ in the topology of M), the projections along $\mathcal{L}_h$ can be arbitrarily well approximated by projections along $\mathcal{L}_f$.*

It is sometimes helpful to lift the picture of this projection operation to the universal cover $\widetilde{M_0}$ of M_0. The foliations $\mathcal{F}_0$ and $\mathcal{F}'_0$ lift to foliations $\widetilde{\mathcal{F}}_0$ and $\widetilde{\mathcal{F}}'_0$ and, since the foliations downstairs are taut, the lifted foliations have simply connected leaves. The lift $\widetilde{\mathcal{L}}_h$ of $\mathcal{L}_h$ is transverse to both lifted foliations. Thus, we may view

$$\widetilde{M_0} \cong \widetilde{L} \times \mathbb{R} \cong \widetilde{L}' \times \mathbb{R},$$

where $\widetilde{L}$ is a leaf of $\widetilde{\mathcal{F}}_0$ covering L, $\widetilde{L}'$ is a leaf of $\widetilde{\mathcal{F}}'_0$ covering L', and the $\mathbb{R}$ factors are leaves of $\widetilde{\mathcal{L}}_h$. Projection along the leaves of this 1–dimensional foliation carries $\widetilde{L}$ homeomorphically onto $\widetilde{L}'$ and *vice versa.* This projection, restricted to a lift of a curve in L is a lift of a projection along $\mathcal{L}_h$ as described above. This lifted picture of the projection is often useful. For instance, the following should be obvious.

Lemma 5.11 *Let s and s' be mutual projections along $\mathcal{L}_h$. Then s' is a nullhomotopic loop in L' if and only if s is a nullhomotopic loop in L.*

Let $\Lambda_\pm$ be the stable and unstable laminations preserved by h in the leaf L of $\mathcal{F}$. By taking all projections along $\mathcal{L}_h$ of the leaves of these laminations, we produce a pair of laminations $\Lambda'_\pm$ on L'. It is clear that these will be closed subsets of L', but we cannot hope that they will be geodesic laminations. We will show, however, that they have all the qualitative properties that the pseudo-Anosov laminations should have. It will then be possible to construct a homeomorphism $g\colon L' \to L'$ which is isotopic to the identity and simultaneously conjugates $\Lambda'_\pm$ to the geodesic laminations of the Handel–Miller theory. This will prove Theorem 5.8. Here again it will often be useful to lift the data to the universal cover.

We begin with the analogue of Corollary 5.5.

Lemma 5.12 *Let $Z' \subset L'$ be the invariant set for f'. Then every leaf ℓ of Λ'_+ contains points of Z' and points going to infinity in both directions in ℓ which converge to an attracting end of L', but no points in a suitable neighborhood of the repelling ends. The analogous assertions hold for Λ'_-, with the roles of attracting and repelling ends interchanged.*

Proof Indeed, by Corollary 5.5, every leaf ℓ of Λ_+ contains points of Z, hence meets $\mathcal{Z}_h$. It follows immediately that every leaf of Λ'_+ meets $\mathcal{Z}_h$, hence contains points of Z'. Similarly, no leaf of Λ_+ meets leaves of $\mathcal{L}_h$ which approach inwardly oriented components of $\partial_\tau M$, so the same holds for leaves of Λ'_+. That is, these leaves do not enter a periodic neighborhood U'_- of repelling ends. Analogous remarks hold for the leaves of Λ'_-.

Suppose that ℓ is a leaf of (say) Λ'_+ such that an end α of ℓ has a neighborhood which does not enter a neighborhood of the attracting ends of L'. The asymptote of α in L' is therefore a nonempty, compact sublamination λ_α of Λ'_+. Evidently, the closure λ_n in L' of

$$\bigcup_{k=n}^{\infty} (f')^{-k}(\lambda_\alpha)$$

defines a descending nest $\{\lambda_n\}_{n=1}^{\infty}$ of nonempty, compact sublaminations of Λ'_+ with intersection a compact, nonempty, f'–invariant sublamination λ. Thus, $\lambda \subset Z'$, implying that corresponding leaves of Λ_+ are contained in Z. This contradicts Corollary 5.5. □

Let $\widetilde{\Lambda}_\pm$ and $\widetilde{\Lambda}'_\pm$ be the respective lifts of our laminations to $\widetilde{L}$ and $\widetilde{L}'$. Projection of $\widetilde{L}'$ onto $\widetilde{L}$ along the leaves of $\widetilde{\mathcal{L}}_h$ is a homeomorphism carrying $\widetilde{\Lambda}'_\pm$ onto $\widetilde{\Lambda}_\pm$. Since the latter are mutually transverse geodesic laminations, the following is immediate.

Lemma 5.13 *Every leaf of $\widetilde{\Lambda}'_+$ meets every leaf of $\widetilde{\Lambda}'_-$ in a single point.*

It is somewhat touchier to prove that the leaves of $\widetilde{\Lambda}'_\pm$ all limit nicely to endpoints at infinity. There is no guarantee that $\mathcal{F}'$ is tangentially close to $\mathcal{F}$, so there is no way to guarantee that the laminations in L' are leafwise close to geodesic ones.

Consider an end e of L' and a neighborhood W of e in L' which spirals on a leaf $F \subset \partial_\tau M$ over the juncture $N' \subset F$. If $\partial F = \emptyset$, this juncture can be taken to be a simple closed geodesic. In general, $\partial F \neq \emptyset$ and N' will be a disjoint union of simple closed geodesics and/or properly imbedded geodesic arcs. Similarly, there is a juncture $N \subset F$ for the spiraling ends of the leaf L of $\mathcal{F}$ which are asymptotic to F. We work entirely in a normal neighborhood $V = F \times [0, \varepsilon)$ of F in M with normal fibers subarcs of leaves of $\mathcal{L}_f$, assuming this neighborhood to be small enough that $\mathcal{F}'|V$ is transverse to these normal fibers. Thus, curves in $L \cap V$ can be projected both by $\mathcal{L}_f$ and $\mathcal{L}_h$ to curves in $L' \cap V$ and *vice versa.*

Let σ be a component of N' and fix a lift $\overline{\sigma}$ of σ to $L \cap V$ via projection along $\mathcal{L}_f$. We consider three cases.

Case 1 If σ is a properly imbedded geodesic arc in F, so is $\overline{\sigma}$ in L. Suitable projections σ_k of $\overline{\sigma}$ into L' along $\mathcal{L}_h$ will be boundary components of complete submanifolds $W_k \subset L'$ which form a fundamental neighborhood system of e. The reader can check that, in the upcoming arguments, this case can be handled in close analogy with the next case. In this way, we continue to focus on the case in which the leaves have empty boundary.

Case 2 If σ is a closed geodesic with homological intersection number $\sigma * N = 0$, then $\overline{\sigma}$ will be a closed geodesic in L. Again, suitable projections σ_k of $\overline{\sigma}$ into L' along $\mathcal{L}_h$ will be boundary components of complete submanifolds $W_k \subset L'$

which form a fundamental neighborhood system of e. Note that each σ_k has an $\mathcal{L}_h$ projection to the geodesic $\overline{\sigma}$ in L.

Case 3 If σ is a closed geodesic and $\sigma * N \neq 0$, then $\overline{\sigma}$ will be a half infinite geodesic. We parametrize it as $\overline{\sigma}(t)$, $0 \leq t < \infty$, remarking that, as $t \uparrow \infty$, $\overline{\sigma}(t)$ moves arbitrarily far into an end of L. Let $\overline{\sigma}_\tau$ denote the restriction of this geodesic ray to $[\tau, \infty)$. For each $\tau \geq 0$, suitable $\mathcal{L}_f$ projections σ_k of $\overline{\sigma}_\tau$ are closed geodesic boundary components of complete submanifolds $W_k \subset L'$ which form a fundamental neighborhood system of e. For suitably large values of τ and large enough values of k, a suitable projection σ'_k of $\overline{\sigma}_\tau$ into L' along $\mathcal{L}_h$ will wrap around σ_k, staying as close to σ_k as desired. This assertion follows easily from Lemma 5.10. We emphasize that σ'_k has an $\mathcal{L}_h$ projection to the geodesic ray $\overline{\sigma}_\tau$ in L.

In Cases 2 and 3, let $\widetilde{\sigma}_k$ be a lift of σ_k to the universal cover, remarking that this lift limits on two distinct points $a_k^\pm \in S_\infty$. Suppose the lifts have been chosen for all $k \geq K$ ($K > 0$ fixed) so that there is a nested sequence

$$J_K \supseteq J_{K+1} \supseteq \cdots \supseteq J_k \supseteq \cdots$$

of arcs in S_∞ with $\partial J_k = \{a_k^+, a_k^-\}$. Thus, the simple closed curves $\widetilde{\sigma}_k \cup J_k$ cut off a nested family $\{D_k\}_{k \geq K}$ of closed disks in the closed Poincaré disk.

Lemma 5.14 *The nested sequence $\{D_k\}_{k \geq K}$ has intersection a singleton $\{x\}$ in S_∞.*

Proof In both cases, the sequence $\{\sigma_k\}_{k \geq K}$ converges to e. Thus, for every compact subset $X \subset L'$, only finitely many terms of the sequence meet X. The sequence of lifts $\{\widetilde{\sigma}_k\}_{k \geq K}$ must also have the property that only finitely many terms meet any given compact set $Y \subset \widetilde{L}'$. In Case 3, σ_k is a closed geodesic and, in Case 2, the closed geodesic σ'_k freely homotopic to σ_k stays uniformly close to σ_k (again one uses Lemma 5.10). In either case, the lifts of these geodesics are complete geodesics in the hyperbolic plane $\widetilde{L}'$ having endpoints $a_k^\pm$ in the circle at infinity. This sequence must also have the property that only finitely many terms meet any given compact subset of $\widetilde{L}'$ and the assertion follows. □

In Case 1, The properly imbedded arcs σ_k are homotopic (with endpoints fixed) to properly imbedded geodesic arcs and the reader can formulate and prove the appropriate analogue of Lemma 5.14.

Lemma 5.15 *If $\varepsilon > 0$, then there is an integer K so large that, if $k \geq K$ and $N_{k,\varepsilon}$ is the normal neighborhood of hyperbolic radius ε of a lift $\widetilde{\sigma}_k$ of σ_k to $\widetilde{L}'$, then no leaf of $\widetilde{\Lambda}'_\pm$ can properly cross $N_{k,\varepsilon}$ twice.*

Proof Case 2 is the easier one. In this case we can take $K = 1$. If $\widetilde{\ell}$ is a leaf of $\widetilde{\Lambda}'_\pm$ which properly crosses $\widetilde{\sigma}_k$ twice, there is a closed loop in $\widetilde{L}'$ of the form $\widetilde{\tau} + \widetilde{\sigma}$, where $\widetilde{\tau}$ is an arc in $\widetilde{\ell}$ and $\widetilde{\sigma}$ an arc in $\widetilde{\sigma}_k$. The covering map carries this to a nullhomotopic loop $\tau + \sigma$ in L', where τ is a subarc of a leaf ℓ of $\Lambda'_\pm$ and σ is a path running around σ_k. A suitable projection along $\mathcal{L}_h$ carries this to a nullhomotopic loop in L (Lemma 5.11) running around the closed geodesic $\overline{\sigma}$ and along a subarc of a geodesic leaf of $\Lambda_\pm$. Evidently, these two geodesic segments do not coincide, contradicting a well known property of hyperbolic surfaces.

In Case 3, one chooses $\tau > 0$ and $K > 0$ so large that, if $k \geq K$, a suitable $\mathcal{L}_h$ projection σ'_k of the geodesic ray $\overline{\sigma}_\tau$ stays inside the ε–neighborhood of the closed geodesic σ_k. In the universal cover $\widetilde{L}'$, suitable lifts $\widetilde{\sigma}'_k$ lie in $N_{\varepsilon,k}$. If g is the hyperbolic transformation with axis the complete geodesic $\widetilde{\sigma}_k$, then $g^n(\widetilde{\sigma}'_k) \subset N_{\varepsilon,k}$, $\forall n \in \mathbb{Z}$. We suppose that $\widetilde{\ell}$ is a leaf of $\widetilde{\Lambda}'_\pm$ properly crossing $N_{\varepsilon,k}$ twice and note that, for suitable integers n, $\widetilde{\ell}$ properly crosses $g^n(\widetilde{\sigma}'_k)$ in two points. Again we obtain a nullhomotopic loop $\tau + \sigma$ in L', where τ is a subarc of a leaf ℓ of $\Lambda'_\pm$ and σ is a path running along σ'_k. A suitable $\mathcal{L}_h$ projection carries this to a nullhomotopic loop in L made up of two distinct geodesic segments, one a subarc of a leaf of $\Lambda_\pm$ and one a subarc of the geodesic ray $\overline{\sigma}_\tau$. This is the desired contradiction. □

Corollary 5.16 *Each leaf of $\widetilde{\Lambda}'_\pm$ limits on two distinct points in the circle at infinity.*

Proof Let $\widetilde{\ell}(t)$ cover the parametrized leaf $\ell(t)$ of $\Lambda'_\pm$, $-\infty < t < \infty$ in $\widetilde{L}'$. By Lemma 5.12, there is an end e of L' and a sequence $t_k \uparrow \infty$ such that $\{\ell(t_k\}_{k=1}^\infty$ clusters at e in L'. We can assume that $\ell(t_k) \in \partial W_k$ as above. Indeed, passing to a subsequence, if necessary, and choosing the component σ of N' appropriately, we can assume that $\ell(t_k) \in \sigma_k$ as in the above discussion. Furthermore, as a little thought shows, we can assume *wlog* that, for infinitely many values of k, the segment $\{\ell(t) \mid t_{k-1} < t < t_{k+1}\}$ does not meet σ_{k-1}. Thus, for all $k \geq 1$, let $\widetilde{\sigma}_k$ be the lift of σ_k passing through the point $\widetilde{\ell}(t_k)$. In the case on which we are focusing, this lift will be an imbedded arc in D limiting on two distinct points $a_k^\pm \in S_\infty$. In Lemma 5.15, choose $\varepsilon > 0$ so small that $N_{k,\varepsilon}$ cannot meet the lifts $\widetilde{\sigma}_{k\pm1}$. Thus, for suitable large values of k, there is no parameter value $t > t_{k-1}$ at which $\widetilde{\ell}$ again meets $\widetilde{\sigma}_{k-1}$. This traps $\widetilde{\ell}(t)$, $t > t_{k-1}$, in the closed disk D_{k-1} of Lemma 5.14, so

$$\lim_{t\to\infty} \widetilde{\ell}(t) = x$$

(see Figure 1, where we use the upper half plane model). Similarly the other end of $\widetilde{\ell}$ limits on a well defined point y in the circle at infinity. If $x = y$, $\widetilde{\ell}$ would properly cross some $N_{k,\varepsilon}$ twice, again contradicting Lemma 5.15. □

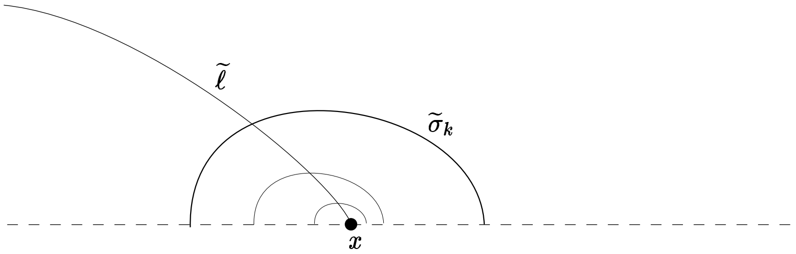


Figure 1: Each end of $\widetilde{\ell}$ limits to a point at infinity in $\widetilde{L}'$

Let $K' \subset L'$ be a choice of core and $U'_\pm$ the corresponding neighborhoods of attracting and repelling ends. Let $\mathcal{U}_\pm \subset L$ be the sets defined in the remark after Theorem 5.4 and set

$$\mathcal{U}'_\pm = \bigcup_{n\in\mathbb{Z}} (f')^n(U'_\pm).$$

Then $\mathcal{U}'_-$ is the set of points of intersection of L' with the leaves of $\mathcal{L}_h$ which limit on inwardly oriented components of $\partial_\tau M$ and $\mathcal{U}_-$ is the corresponding set in L. The sets $\mathcal{U}'_+$ and $\mathcal{U}_+$ have an analogous description. Since $\Lambda_\pm$ is the set theoretic boundary in L of $\mathcal{U}_\pm$, the following is immediate.

Lemma 5.17 *The laminations $\Lambda'_\pm$ are the respective set theoretic boundaries in L' of $\mathcal{U}'_\pm$.*

Thus,

$$\Lambda'_+ = \lim_{n\to\infty} (f')^n(\partial_- K')$$
$$\Lambda'_- = \lim_{n\to\infty} (f')^{-n}(\partial_+ K')$$

in the weak sense. It will be necessary to prove convergence in the strong sense described in the remark following Theorem 5.4.

Denote by C the set of points at infinity of $\widetilde{L}'$. We are focusing on the case that $C = S_\infty$, but alternatively, it is a Cantor set. The following fact is well known for the universal covering of a compact, hyperbolic surface, but is not generally true for the noncompact case. The proof for endperiodic surfaces was communicated to us by S Fenley.

Lemma 5.18 *Let G be the group of covering transformations on $\widetilde{L}'$. For each point $x \in \widetilde{L}'$, the orbit $G(x)$ accumulates in $\widetilde{L}' \cup C$ exactly on the set C. In particular, C is minimal under the induced action of G.*

Proof Let $x \in \widetilde{L}'$ and $z \in C$. It is clear that $G(x)$ cannot accumulate in $\widetilde{L}'$ and we will show that $G(x)$ contains a sequence converging to z in the Euclidean metric. Let γ denote the unique unit speed geodesic ray in $\widetilde{L}'$ from x to z. This projects to a geodesic ray or loop ρ in L', parametrized on $[0, \infty)$, and we consider two cases.

Case I For a suitable sequence $t_k \uparrow \infty$, the hyperbolic distance between $\rho(0)$ and $\rho(t_k)$ is bounded by a finite positive constant B. Let $\rho_k = \rho|[0, t_k]$ and choose a path τ_k in L' from $\rho(t_k)$ to $\rho(0)$ and having length at most B. Let $g_k \in G$ correspond to the lift of the loop $\rho_k + \tau_k$ to a path starting at x. Thus, this path ends at the point $g_k(x)$ and is written as $\gamma_k + \widetilde{\tau}_k$, where $\gamma_k = \gamma|[0, t_k]$ and $\widetilde{\tau}_k$ is the lift of τ_k starting at $\gamma(t_k)$. The points $\gamma(t_k)$ converge to z in the Euclidean metric, while the Euclidean length of $\widetilde{\tau}_k$ converges to 0. It follows that

$$\lim_{k \to \infty} g_k(x) = z.$$

Case II The hyperbolic distance between $\rho(0)$ and $\rho(t)$ goes to infinity with t. We can choose the sequence $t_k \uparrow \infty$ so that $\rho(t_k)$ converges to an end e. We set $\rho_k = \rho|[0, t_k]$ and $\gamma_k = \gamma|[0, t_k]$. We can choose a fundamental neighborhood system $\{W_k\}_{k=1}^\infty$ of e with ∂W_k geodesic and assume that $\rho(t_k) \in \sigma_k$, a component of ∂W_k, $k \geq 1$. The lifts $\widetilde{\sigma}_k$ passing through $\gamma(t_k)$ cut off a fundamental neighborhood system $\{D_k\}_{k=1}^\infty$ of z in $\widetilde{L}' \cup C$. Since the end e is nontrivial, there is a properly imbedded arc τ_k in W_k with endpoints in σ_k which cannot be deformed into σ_k while keeping the endpoints in σ_k. We can deform τ_k, keeping the endpoints in σ_k, to a loop (again denoted by τ_k) based at $\rho(t_k)$ and let $g_k \in G$ correspond to the lift of $\rho_k + \tau_k - \rho_k$ starting at x. This lift has the form $\gamma_k + \widetilde{\tau}_k - \widetilde{\rho}_k$, where $\widetilde{\tau}_k$ is the lift of τ_k starting at $\gamma(t_k) \in \widetilde{\sigma}_k$ and $-\widetilde{\rho}_k$ is the lift of $-\rho_k$ starting at the terminal point of $\widetilde{\tau}_k$. By the assumption on τ_k this terminal point cannot lie on $\widetilde{\sigma}_k$ and it follows that $g_k(x) \in D_k$. □

Consider a leaf ℓ of $\Lambda'_\pm$ and a lift $\widetilde{\ell}$ of this leaf to $\widetilde{L}'$. For definiteness, assume ℓ to be a leaf of Λ'_+. Since $(f')^k(\partial_- K')$ converges to Λ'_+ in the weak sense, we can find a component σ of a juncture arbitrarily deep in a repelling end and a sequence of positive integers $n_k \uparrow \infty$ such that arbitrarily long subarcs of $\sigma_k = (f')^{n_k}(\sigma)$ uniformly well approximate arbitrarily long subarcs of ℓ as $k \uparrow \infty$. This likewise holds for the lift $\widetilde{\ell}$ and for suitable choices of the lifts $\widetilde{\sigma}_k$. Note that $\widetilde{\sigma}_k$ has well defined endpoints $a_k^\pm$, either in $\partial\widetilde{L}'$ or in C and that Corollary 5.16 guarantees that $\widetilde{\ell}$ has well defined endpoints in C. We can assume that there are well defined limits

$$a^\pm = \lim_{k\to\infty} a_k^\pm .$$

In order to prove the strong convergence, we need only show that $a^\pm \in C$ are the endpoints of $\widetilde{\ell}$. We suppose not and reach a contradiction.

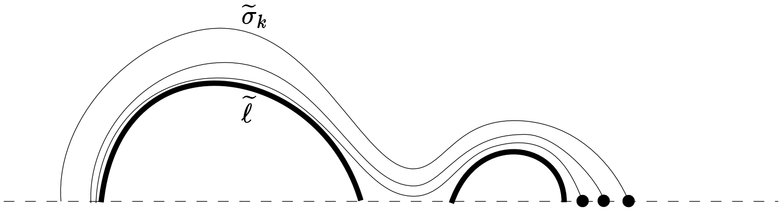


Figure 2: Nonhausdorff accumulation of $\{\widetilde{\sigma}_k\}_{k=1}^\infty$ on leaves of $\widetilde{\Lambda}'_+$

No generality is lost in assuming that the component σ of juncture is chosen as in one of the three cases discussed in the remarks preceding Lemma 5.14. There, σ denoted a component of juncture in a boundary leaf F, but here we fix a homeomorphic lift deep in a repelling end of L' and denote that lift by σ. By Lemma 5.18 and the invariance of $\widetilde{\Lambda}'_+$ under the covering group G, we conclude that the endpoints of the leaves of this lamination are dense in C. Since $\widetilde{\sigma}_k$ cannot meet these leaves, we easily conclude that the sequence $\{\widetilde{\sigma}_k\}_{k=1}^\infty$ has nonhausdorff accumulation on more than one leaf of $\widetilde{\Lambda}'_+$ (actually, on infinitely many) as indicated in Figure 2. The following, therefore, completes the proof of convergence in the strong sense. The idea for this proof was suggested to us by S Fenley.

Lemma 5.19 *The sequence $\widetilde{\sigma}_k$ converges uniformly to $\widetilde{\ell}$ in the Euclidean metric on D.*

Proof We suppose nonhausdorff accumulation as in Figure 2 and deduce a contradiction. Let $\pi\colon \widetilde{L}' \to \widetilde{L}$ denote the projection along the leaves of $\widetilde{\mathcal{L}}_h$.

This is a homeomorphism. It will preserve the nonhausdorff accumulation, although we are no longer assured that $\pi(\widetilde{\sigma}_k)$ has well defined endpoints at infinity. Since $\pi(\widetilde{\ell})$ is a geodesic leaf of $\widetilde{\Lambda}_+$, it does have well defined endpoints. Let z be one of these endpoints, remarking that $\pi(\widetilde{\sigma}_k)$ passes arbitrarily near z (in the Euclidean metric) as $k \to \infty$.

Let $\overline{\ell}$ denote the leaf of Λ_+ covered by $\pi(\widetilde{\ell})$. By Corollary 5.5, there is a sequence $\{W_k\}_{k=1}^{\infty}$ of closed neighborhoods of an attracting end of L such that ∂W_k is geodesic and such that $\overline{\ell}(t_k) \in \partial W_k$, where $\pi(\widetilde{\ell}(t_k)) \to z$ as $k \to \infty$. Thus, we can fix a fundamental system $\{D_k\}_{k=1}^{\infty}$ of closed neighborhoods of z in $\widetilde{L}$, each bounded by a geodesic (or geodesic arc) $\widetilde{\tau}_k$ which is a lift of the component τ_k of ∂W_k which passes through $\overline{\ell}(t_k)$.

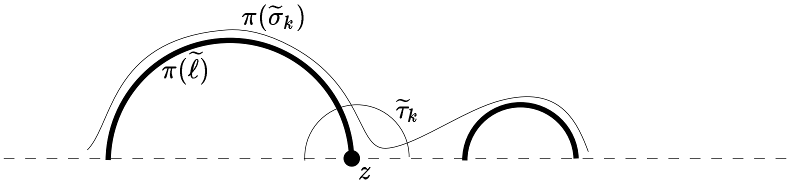


Figure 3: $\pi(\widetilde{\sigma}_k)$ crosses the geodesic $\widetilde{\tau}_k$ twice

Since the curves $\pi(\widetilde{\sigma}_k)$ also accumulate on other leaves of $\widetilde{\Lambda}_+$, there will be large enough values of k so that $\pi(\widetilde{\sigma}_k)$ crosses $\widetilde{\tau}_k$ twice (Figure 3). Thus, we obtain a nullhomotopic loop in $\widetilde{L}$ made up of a segment of $\pi(\widetilde{\sigma}_k)$ and a segment of the geodesic $\widetilde{\tau}_k$. If $p\colon \widetilde{L} \to L$ denotes the covering projection, we obtain a nullhomotopic loop ρ_k in L made up of a segment of the geodesic τ_k and of the curve $p(\pi(\widetilde{\sigma}_k))$.

If σ is as in Case 2, then there is an $\mathcal{L}_h$ projection of σ to a geodesic loop $\overline{\sigma}$ deep in a repelling end of L. Since f' is everywhere defined by projection along $\mathcal{L}_h$, every σ_k has an $\mathcal{L}_h$ projection to the closed geodesic $\overline{\sigma}$. Since h is also defined by projection along $\mathcal{L}_h$, it follows that a suitable negative iterate h^{-n_k} carries $p(\pi(\widetilde{\sigma}_k))$ onto the geodesic $\overline{\sigma}$. Since h preserves the extended geodesic lamination Γ_-, $h^{-n_k}(\tau_k)$ is also geodesic and $h^{-n_k}(\rho_k)$ is a nullhomotopic loop made up of two distinct geodesic segments, a contradiction. Case 1 is entirely similar.

If σ is as in Case 3, one chooses a normal ε–neighborhood $N_\varepsilon(\sigma)$ of the closed geodesic σ so that all iterates $N_{\varepsilon,k} = (f')^k(N_\varepsilon(\sigma))$ are pairwise disjoint. The geodesic ray $\overline{\sigma}_\tau \subset L$ in Case 3 has an $\mathcal{L}_h$ projection contained entirely in

$N_\varepsilon(\sigma)$. In the argument above, view the lifts $\widetilde{N}_{\varepsilon,k}$ as slightly thickened versions of $\widetilde{\sigma}_k$, also accumulating in nonhausdorff fashion on multiple leaves of $\widetilde{\Lambda}'$. Modifications of the above argument, analogous to the treatment of Case 3 in the proof of Lemma 5.15, produce a nullhomotopic loop made up of a segment of the geodesic $\overline{\sigma}_\tau$ and a segment of the geodesic $h^{-n_k}(\tau_k)$. □

Since the leaves of $\widetilde{\Lambda}'_+$ are exactly the uniform limits of lifts of components of $(f')^k(\partial_- K')$ and all of these curves have well defined (and distinct) endpoints in $\widetilde{L}' \cup C$, we can replace each of these curves with the geodesic having the same endpoints. The strong convergence proven above implies corresponding convergence of endpoints, hence strong convergence of the geodesics with these endpoints. Thus, these geodesics are exactly the lifts of the leaves of the extended geodesic laminations Γ'_+ of the Handel–Miller theory. Similar arguments apply to the lamination Γ'_-. In standard fashion, one constructs a homeomorphism $\widetilde{g}\colon \widetilde{L}' \to \widetilde{L}'$, commuting with the covering group, fixing C pointwise, and carrying the laminations to geodesic ones. This induces the homeomorphism $g\colon L' \to L'$ of Theorem 5.8, completing the proof of that theorem.

Corollary 5.20 *Let $(M, \mathcal{F})$ be a depth one, foliated, sutured manifold. Then there is a unique closed cone $\mathcal{D} \subset H^1(M)$ which is convex, polyhedral, contains the proper foliated ray $[\mathcal{F}]$, has interior a union of foliated rays, and is such that $\partial\mathcal{D}$ contains no foliated rays.*

Proof By Theorem 3.3, Lemma 3.6 and Lemma 5.3, together with Theorem 5.4, we can suppose that the endperiodic monodromy h of a leaf L of $\mathcal{F}$ is pseudo-Anosov and consider the cone $\mathcal{D} = \mathcal{D}_h$. By Theorem 4.14, this cone is polyhedral. By Theorem 4.3, every class in $\operatorname{int}\mathcal{D}_h$ is foliated and we only need to prove that there is no foliated class in $\partial\mathcal{D}_h$. The linear inequalities $\theta_i \geq 0$ defining $\mathcal{D}_h$ are given by *integral* homology classes $\theta_i \in H_1(M;\mathbb{Z})$, these being represented by closed loops, so there is a foliated class $[\omega] \in \partial\mathcal{D}_h$ if and only if some such class corresponds to a depth one foliation. Let $\mathcal{F}'$ be a depth one foliation represented by a class $[\omega'] \in \partial\mathcal{D}_h$ and let $h'\colon L' \to L'$ be a pseudo-Anosov first return map for a noncompact leaf L' of $\mathcal{F}'$. Choose a class $[\omega''] \in \operatorname{int}\mathcal{D}_h$ on a ray through the integer lattice and as close to $[\omega']$ as desired. Thus, we can assume that the closed, nonsingular foliated form ω'' is sufficiently near ω' that the depth one foliation $\mathcal{F}''$ which it defines is also transverse to $\mathcal{L}_{h'}$. But Corollary 5.9 then implies that $\mathcal{D}_h = \mathcal{D}_{h'}$, so $[\omega'] \in \operatorname{int}\mathcal{D}_h$, a contradiction. □

6 Finiteness of the Foliation Cones.

One can construct nonmaximal foliation cones in close analogy with U. Oertel's construction of branched surfaces which carry norm minimizing representatives of elements of $H^1(M)$ [21]. While it seems to be hard to use this approach to prove convexity of the maximal cones, it does provide a proof of finiteness. Indeed, every maximal foliation cone is a union of Oertel cones and there are only finitely many of these. The proof requires a few modifications of arguments in [21] since it will not be convenient to use norm minimizing surfaces. The norm minimizing hypothesis in Oertel's argument is used to prove a crucial orientation property [21, pages 261–262] which, in our case, will follow from the depth one hypothesis. We assume familiarity with the construction of branched surfaces from surfaces in Haken normal form as carried out, for example, in [10, 20, 21].

We fix a handlebody decomposition of the sutured manifold M, requiring that ∂M be covered by a union of i–handles, $0 \le i \le 2$, $\partial_\tau M \cap \partial_\pitchfork M$ being covered by a union of 0–handles and 1–handles. All constructions of surfaces in normal form and of branched surfaces will be carried out relative to this handlebody decomposition.

We also fix normal neighborhoods $W = \partial_\tau M \times [0,2]$ and $V = \partial_\pitchfork M \times [0,2]$, arranging that the handles meeting $\partial_\tau M$ lie in $W' = \partial_\tau M \times [0,1]$ and those meeting $\partial_\pitchfork M$ lie in $V' = \partial_\pitchfork M \times [0,1]$.

Given a depth one foliation $\mathcal{F}$, we will produce a branched surface B which, in a suitable sense, "fully carries" $\mathcal{F}$. (For this, we could invoke a theorem of Gabai [14, Theorem 4.11], but we will describe a more elementary proof.)

The branched surface B will be such that every choice $\mu = \{\mu_i\}$ of strictly positive weights on the sectors B_i of B, satisfying the *branch equations* [20, page 386], defines a foliated class $[\mu] \in H^1(M)$. We call μ an *invariant measure* on B.

As in the cited references, one directly constructs the normal neighborhood $N(B) \subset M$, foliated by compact, oriented intervals I, B being defined as the quotient of $N(B)$ obtained by collapsing the interval fibers to points. While $N(B)$ is not exactly an interval bundle in the usual sense, the local models pictured in the cited references make clear the sense in which it can be viewed as an interval bundle over B. Hereafter, we use the term "interval bundle" in this more general sense.

A major difference from the construction in [21] is that $\partial_\tau M$ will be part of B. This implies that B can fully carry only noncompact surfaces L which accumulate on $\partial_\tau M$ exactly as does a depth one leaf (Figure 4). This requires that the corresponding measure μ take the value ∞ on the sectors of B contained in $\partial_\tau M$. Away from the tangential boundary, L will meet each interval fiber of $N(B)$ in only finitely many points.

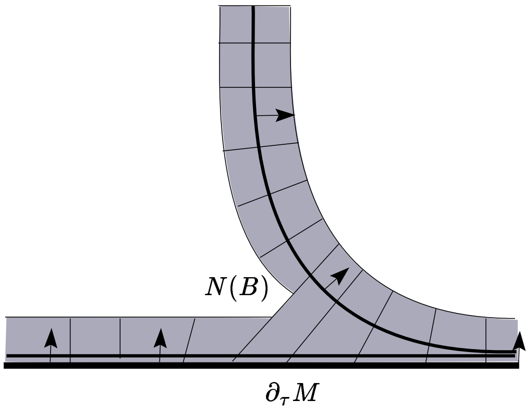


Figure 4: The normal neighborhood $N(B)$ near $\partial_\tau M$

The foliation $\mathcal{F}$ will be transverse to the fibers of the normal neighborhood $\partial_\tau M \times [0, \varepsilon] \subset W'$ for small enough $\varepsilon > 0$, so an isotopy that flows into M along the fibers of W allows us to assume that the foliation $\mathcal{F}|W$ is transverse to these fibers. Here, each noncompact leaf of $\mathcal{F}|W$ falls into finitely many disjoint, noncompact pieces, each spiraling in on one or another component of $\partial_\tau M$. It will also be convenient to choose the fibers of $V = \partial_\pitchfork M \times [0, 2]$ to be tangent to the leaves.

Choose a depth one leaf L of $\mathcal{F}$. This submanifold, being a leaf of a taut foliation, is incompressible in M. As in [4, Section 4], fix a decomposition $L = C \cup E$, where C is a compact, connected submanifold of L with boundary and, possibly, corners, E being the union of spiraling pieces. We can assume that E is exactly the intersection of L with the handles that meet $\partial_\tau M$. As in [4, pages 166–167], complete C to a properly imbedded surface $F = C \cup A$ in M, where A is made up of annuli and/or rectangles which “drop” from components of ∂C to $\partial_\tau M$. Since L is incompressible, so is F.

We would like to perform an isotopy on F, putting it into Haken normal form relative to the handlebody decomposition. As the referee pointed out, there is a problem applying standard theory here since we cannot guarantee that F is boundary incompressible. However, the very simple structure of F in the neighborhood $U = W' \cup V'$ of ∂M makes it clear that there are no boundary compressing disks contained entirely in U, so standard methods do permit an isotopy that puts A into Haken normal form relative to the handles meeting ∂M. We view this as an ambient isotopy, simultaneously moving the leaf L. Now the incompressibility of F allows us to extend this to an ambient isotopy, supported in the complement of the interiors of the handles meeting ∂M and putting all of F, hence A, into Haken normal form. The fact that $E = L \smallsetminus \operatorname{int} C$ consists of spiraling pieces near $\partial_\tau M$ now allows the entire leaf L to be isotoped to Haken normal form.

Let $N(L) \subset M$ be an imbedded normal neighborhood of L, this being an oriented J–bundle over L, where the natural orientation of the compact interval $J \subset \mathbb{R}$ agrees with the transverse orientation of L. We can assume that $N(L)$ itself is in normal form. That is, $N(L)$ meets no 3–handles and, if H is an i–handle, $1 \leq i \leq 2$, then each component of $H \cap N(L)$ is of the form $D \times J$, where D is one of finitely many *disk types* in H. Let $N^+(L)$ be the augmented J–bundle, incorporating a normal neighborhood of $\partial_\tau M$ which lies in the union of i–handles meeting $\partial_\tau M$ and is also in normal form. The orientation of the fibers respects the transverse orientations of L and $\partial_\tau M$ as indicated in Figure 4. Because L is a depth one leaf, $M_0 \smallsetminus \operatorname{int} N^+(L)$ will have a product structure $G \times I$, where G is a compact 2–manifold and I a compact interval.

In [21], the complexity of a compact, properly imbedded surface in Haken normal form is defined to be the number of disks in which the surface intersects 2–handles. Since L meets the 2–handles along $\partial_\tau M$ in infinitely many disks, we modify this definition, using only the 2–handles which do not meet the tangential boundary. We can assume that, among all normal forms in its isotopy class, L has minimal complexity. Following Oertel, we form $\overline{N}^+(L)$ by adjoining to $N^+(L)$ all products $D \times [-1,1]$, where $D \times \{\pm 1\}$ are adjacent, normally isotopic disks of $\partial N(L) \cap H$, H an i–handle with $0 \leq i \leq 2$. We can write

$$\overline{N}^+(L) = N^+(L) \cup Q_1 \cup \cdots \cup Q_r,$$

where the Q_i's are compact, connected, pairwise disjoint products. As is standard, we break $\partial\overline{N}^+(L)$ into the horizontal part $\partial_h\overline{N}^+(L)$ and the vertical part $\partial_v\overline{N}^+(L)$. Again, $\overline{N}^+(L)$ is a compact J–bundle and the complement in M_0 of its interior is a product (possibly not connected).

Following Oertel [20, 21], we cut $\overline{N}^+(L)$ along $\partial N(L) \cap Q_j$, $1 \leq j \leq r$, obtaining a compact interval bundle $\widehat{N}^+(L)$ which generally has many components. There is a canonical immersion $\iota \colon \widehat{N}^+(L) \to M$ which is an imbedding on each component of $\widehat{N}^+(L)$ and has image $\overline{N}^+(L)$. We can identify the components of $\widehat{N}^+(L)$ with their images under ι, noting that one component of

$\widehat{N}^+(L)$ is identified with $N^+(L)$. One must be concerned with the possibility that some component(s) $Q = Q_j$ may have opposed orientations along the two components of $\partial_h Q$, as indicated in Figure 5.

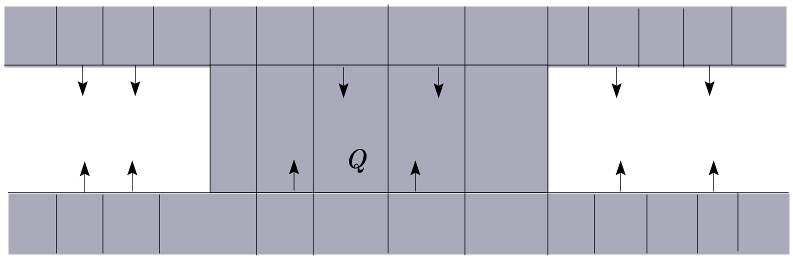


Figure 5: Opposed orientations on $\partial_h Q$

Lemma 6.1 *There is a continuous choice of orientation on the interval fibers of* $\overline{N}^+(L)$ *which agrees with the orientation on the fibers of* $N^+(L)$.

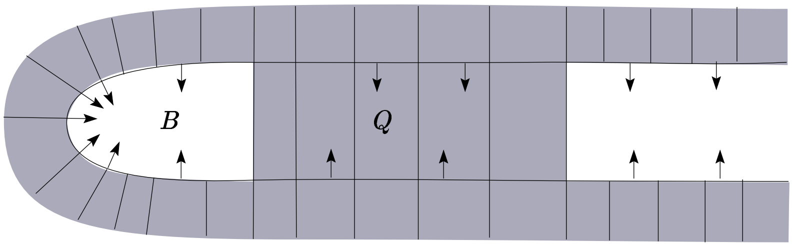


Figure 6

Proof We assume that some component $Q = Q_j$ of $\widehat{N}^+(L)$ has opposed orientations on $\partial_h Q$, as indicated in Figure 5, and deduce a contradiction. Write $Q = R \times [0,1]$ and suppose that $C \subset R$ is any closed loop (respectively, properly imbedded arc). Then $C \times [0,1]$ can be interpreted as a proper homotopy of $C \times \{0\}$ to $C \times \{1\}$ in the complement of the depth one leaf L in M_0. Because of the opposed orientations on $\partial_h Q$, the homotopy meets that leaf at $C \times \{0,1\}$ *on the same side of* L. Since $M_0 \smallsetminus L$ is a product, this homotopy can be compressed to a homotopy in L, keeping $C \times \{0,1\}$ pointwise fixed. Remark that $R \times \{0\}$ and $R \times \{1\}$ are disjoint subsurfaces of L and C is arbitrary, so the well understood structure of orientable surfaces implies that these subsurfaces are annuli (respectively, disks) and are separated by an annulus (respectively, a disk). One concludes that the picture is as in Figure 6, where B and Q are both solid tori (respectively, 3–cells). An isotopy of $N(L)$ across $B \cup Q$, as indicated in Figure 7, then produces a normal form in the isotopy class of L with strictly smaller complexity. This contradicts the fact that L already has minimal complexity. $\square$

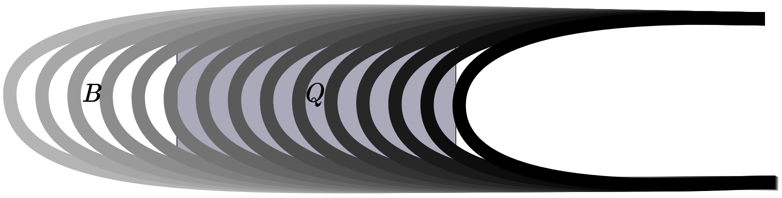


Figure 7: A complexity reducing isotopy

Corollary 6.2 *There is a smooth, oriented, one dimensional foliation $\mathcal{L}$ of M which is transverse to $\partial_\tau M$, tangent to $\partial_\pitchfork M$ and in $\overline{N}^+(L)$ coincides with the interval fibration.*

Indeed, $P = M_0 \smallsetminus \operatorname{int} \overline{N}^+(L)$ is a product, so the corollary follows directly from Lemma 6.1. Note that $\mathcal{L}|P$ is the product fibration of P by compact intervals.

The branched surface B is the quotient of $\overline{N}^+(L)$ by the interval fibers. We write $\overline{N}^+(L) = N(B)$ and view B as imbedded in $N(B)$ transverse to the fibers in the standard way. By a small isotopy of $N(B)$ we can assume that the branch locus meets itself transversely and without triple points.

Remark In [21], certain "trivial" components of $\widehat{N}^+(L)$ are deleted, thereby eliminating disks of contact and insuring that B is Reebless as well as incompressible. For our purposes, there seems to be no compelling reason to incorporate this step.

If an imbedded surface $F \subset N(B)$ is transverse to the fibers, meets every fiber and meets the fibers away from $\partial_\tau M$ only finitely often, F is said to be *fully carried* by B. Because $\partial_\tau M$ can be viewed as part of B with branch locus the junctures, any surface fully carried by B will be noncompact with ends spiraling in on $\partial_\tau M$ over these junctures. Such a surface defines an invariant measure μ on B which has value ∞ on sectors of B in $\partial_\tau M$ and positive integer values on other sectors. Conversely, an invariant measure with these properties corresponds to such an imbedded surface. More generally, we allow the finite values of μ to be positive real numbers, always requiring that the branch equations be satisfied.

An invariant measure μ well defines a homomorphism

$$[\mu]\colon\ \pi_1(M) \to \mathbb{R}.$$

Indeed, given a loop σ in M, there is a free homotopy of σ to a loop σ' which is transverse to B and meets B only in the ineriors of sectors B_i. If $x \in \sigma' \cap B_i$, we assign the value $\pm\mu_i$ to x according as the orientation of σ' at x does or does not agree with the transverse orientation of B there. The sum of these signed weights is an invariant of the free homotopy class of σ because of the branch equations. Thus, $[\mu] \in H^1(M)$.

Lemma 6.3 *Each cohomology class $[\mu]$, determined as above by a strictly positive invariant measure μ on B, is a foliated class. As μ varies over all such measures, these classes form a convex cone $\mathcal{D}_B$ in $H^1(M)$.*

Proof Let $\mathcal{Z}$ be the core lamination of the foliation $\mathcal{L}$ of Corollary 6.2 and let Φ_t parametrize $\mathcal{L}$ as a flow (stationary exactly along $\partial_\tau M$). One can use this parametrization to define the "length" of any compact subarc of a leaf of $\mathcal{L}$, remarking that there is a positive upper bound m to the lengths of the fibers of $N(B)$ that do not meet $\partial_\tau M$ and to the lengths of the fibers of P. Let $\overline{\Gamma} = \lim_{k\to\infty} \Gamma_k^*/\tau_k$ be a homology direction of $\mathcal{Z}$, where Γ_k^*/τ_k are *long, almost closed orbits* as defined in the proof of Lemma 4.5. If $a > 0$ is the minimum value taken by μ, one checks that the values of $[\mu]$ on the long, almost closed orbits cluster at a value no smaller than $a/2m$. By Lemma 4.7, it follows that $[\mu]$ is strictly positive on the cone $\mathbf{C}_{\mathcal{Z}}$ of structure cycles. In particular, none of these structure cycles bound, so $[\mu] \in \operatorname{int}\mathcal{D}_{\mathcal{Z}}$ is a foliated class (Theorem 4.3). The fact these foliated classes form a convex subcone $\mathcal{D}_B \subset \mathcal{D}_{\mathcal{Z}}$ is elementary. □

The branched surfaces B obtained by our procedure are finite in number, up to small isotopy. Indeed, each i–handle determines finitely many disk types, up to transverse isotopy, and contributes at most one disk of each type to the construction of B. Since $\mathcal{D}_B$ is a convex cone of foliated classes, it is contained in a unique maximal foliation cone as in Theorem 5.20. This guarantees that there are only finitely many of the latter.

Theorem 6.4 *There are only finitely many maximal foliation cones.*

The proof of Theorem 1.1 is complete. The assertion that the proper foliated ray $[\mathcal{F}]$ determines $\mathcal{F}$ up to isotopy was proven in [4].

Remark The "Oertel cone" $\mathcal{D}_B$ may not have full dimension, but at least some of the branched surfaces produce cones of full dimension. Indeed, the nonempty interior of each foliation cone is a finite union of Oertel cones.

7 Computing examples

If M reduces to a sutured manifold that is completely disk decomposable [13], the foliation cones are easy to compute. Furthermore, in many examples it is easy to compute the Thurston norm for the 3-manifold M using the foliation cones. In these examples, each foliation cone is the union of cones over top dimensional faces of the Thurston ball and the set

$$\mathfrak{C} = \{C = (-C_i) \cap C_j \mid C_i, C_j \text{ are foliation cones}\}$$

is the set of cones over the top dimensional faces of the Thurston ball. If M^* denotes the sutured manifold obtained by reversing the orientation of $\partial_\tau M$ (but not of M) the cones

$$\{-C_i \mid C_i \text{ is a foliation cone for } M\}$$

become the foliation cones for M^*. The lattice points in the cones $(-C_i) \cap C_j$ correspond to foliations of both M and M^*. Thus, in the examples mentioned above, the lattice points in the interior of each cone $C \in \mathfrak{C}$ correspond to foliations that can be spun both ways at $\partial_\tau M$. In general it is very hard to compute the Thurston norm and these results do not hold.

Remarks We routinely make the identification

$$H_2(M, \partial M) = H^1(M).$$

When representing foliated classes by disks of a disk decomposition, it is more natural to view the foliation cones in $H_2(M, \partial M)$. When defining these cones by inequalities $\Gamma_\iota \geq 0$, where Γ_ι is (the homology class of) a loop in $\mathcal{Z}$ of minimal period ι, it is more natural to view the cones in $H^1(M)$.

We compute the Thurston norm in a sutured manifold by doubling along the sutures, computing the norm of the doubled class in the doubled manifold, and dividing by 2. Up to a factor 2, this is the same as Scharlemann's definition of the Thurston norm for a sutured manifold [26, Definition 7.4] (take $\beta = \emptyset$ in Scharlemann's definition).

In the following examples if $\{R_1, \dots, R_n\}$ is a Markov partition for $\mathcal{Z} \cap L$, we will let $a_{ij} = 1$ iff $h(R_i) \cap R_j \neq \emptyset$ (zero otherwise), and $A = [a_{ij}]$. Then

$$\Sigma_A = \{(\dots i_k\, i_{k+1} \dots) \mid a_{i_k i_{k+1}} = 1, \forall\, k \in \mathbb{Z}\}$$

are the allowable sequences (see Section 4). If $\iota \in \Sigma_A$ is periodic, the transverse loop Γ_ι is well defined as a homology class.

If the sutured manifold M is completely disk decomposable, using disks

$$D_1, \dots, D_n \subset M,$$

we will let α_i be the curves in M defined, up to homology, by the condition that the intersection products with the disks are

$$\alpha_i \cdot D_j = \delta_{ij}, \quad 1 \leq i, j \leq n.$$

We will use the notation $\mathbf{e}_i = [D_i] \in H_2(M, \partial M)$, $0 \leq i \leq n$. (In many of the examples there will be a disk D_0 so that $\mathbf{e}_0 + \mathbf{e}_1 + \dots + \mathbf{e}_n = 0$). Then $H_2(M, \partial M) = \mathbb{R}^n$ is generated by $\mathbf{e}_1, \dots, \mathbf{e}_n$. The minimal period loops Γ_ι, thought of as classes in $H_1(M)$, will be expressed in terms of the basis $\{\alpha_1, \dots, \alpha_n\}$.

In our examples, the sutured manifolds will be of the form $M_S(\kappa)$ as in the introduction, where κ is a knot or link. They will generally be denoted simply by M unless some confusion is possible. They will be pictured "from the inside". That is, $M_S(\kappa)$ will be the complement of the interior of the handlebody that is drawn. Our convention will be that S_+ is the part of $\partial_\tau M$ oriented outwards from M (away from the viewer), S_- the part oriented inwards. A decomposing disk D_i with positive sign (often suppressed) has boundary oriented counterclockwise from the viewer's perspective, the boundary of $-D_i$ being oriented clockwise.

Finally, the following observation will often be used.

Proposition 7.1 *Suppose that*

$$(M, \gamma) \overset{D}{\rightsquigarrow} (M', \gamma')$$

is a disk decomposition by a nonseparating disk that meets the sutures twice. Then the regluing map

$$p\colon\, M' \to M$$

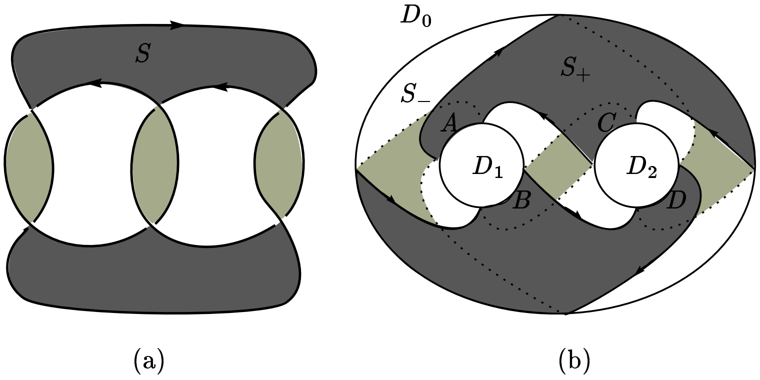


Figure 8: (a) A Seifert surface for $(2,2,2)$ (b) The sutured manifold M obtained from $(2,2,2)$

induces a surjection

$$p^*\colon\ H^1(M)\to H^1(M')$$

having one-dimensional kernel generated by the Poincaré dual of D. Furthermore, the foliation cones in $H^1(M)$ are exactly the preimages of the foliation cones in $H^1(M')$.

Indeed, an elementary Mayer–Vietoris argument, expressing M as the union of M' and $D\times I$, proves the assertion. This is similar to the procedure in Section 3, but easier and more natural for disk decomposable examples.

Example 1 Cutting the complement of the three component link $(2,2,2)$ apart along the Seifert surface given in Figure 8(a), gives the sutured manifold M, everything outside the solid two-holed torus in Figure 8(b). We let D_1 and D_2 be the two disks indicated in the figure and D_0 the outside disk.

If one carries out the disk decomposition of M using the disks $+D_1,+D_2$ to obtain a foliation $\mathcal{F}$ of M, one can piece together a core K of a leaf L of $\mathcal{F}$ as displayed in Figure 9, the entire leaf L being displayed in Figure 10(a) with the core shaded. (It should be remarked that this choice of K is not entirely in accord with the conventions of Sections 4 and 5, where the components of $L\setminus K$ are all unbounded. The current choice, however, is adequate to cover the invariant set Z and is convenient for illustrating the pseudo-Anosov dynamics.) In Figure 10(a), the endperiodic map h moves points near the ends in the directions indicated by the arrows, but also involves a lateral exchange

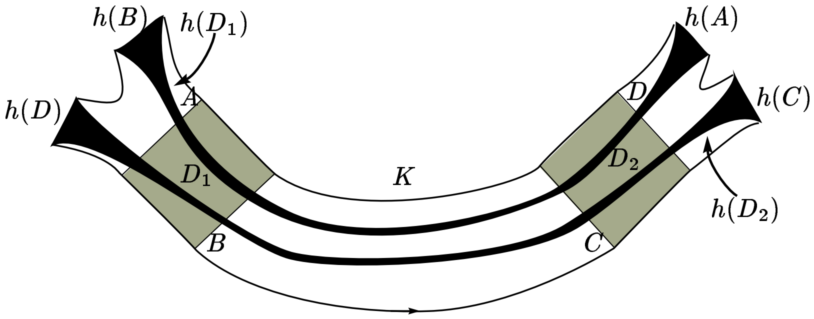


Figure 9: The core K of the leaf L of Example 1

as indicated by the labels. With some thought, this can all be deduced from Figure 8. In Figure 9, the disks D_1 and D_2 serve as the rectangles of a Markov partition. The intersections $h(D_i) \cap D_j$ are also displayed, showing that the incidence matrix is

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

That is, in this case every sequence is an allowable sequence. The loops corresponding to sequences of minimal period are

$$\Gamma_{(\ldots 111\ldots)} = \alpha_1$$
$$\Gamma_{(\ldots 222\ldots)} = \alpha_2,$$

so the foliation cone to which $\mathcal{F}$ belongs is defined by the inequalities

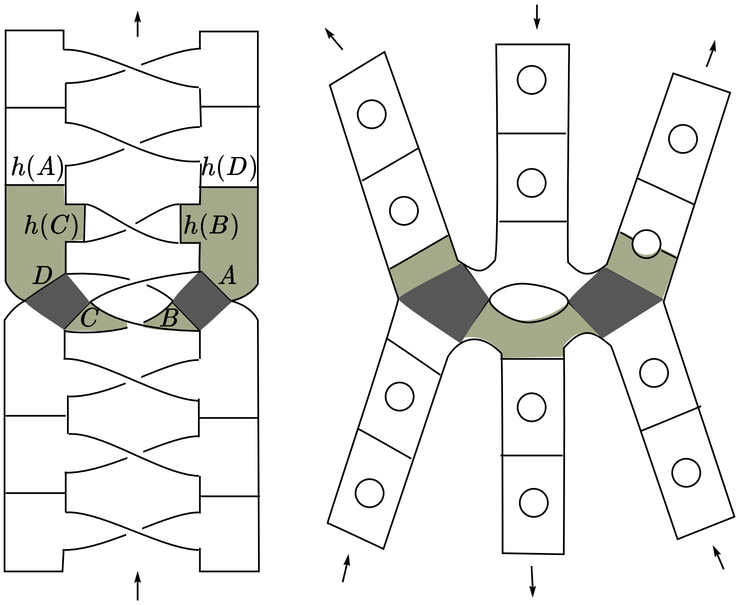
$$\{\alpha_1 \geq 0\} = \text{the right half plane}$$
$$\{\alpha_2 \geq 0\} = \text{the upper half plane.}$$

A base for this cone is the segment $[\mathbf{e}_1, \mathbf{e}_2]$ (see Figure 11).

There is a simple symmetry of M cyclically permuting $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_0$. This symmetry takes the foliation cone with base $[\mathbf{e}_1, \mathbf{e}_2]$ to the foliation cone with base $[\mathbf{e}_2, \mathbf{e}_0]$ and then to the foliation cone with base $[\mathbf{e}_0, \mathbf{e}_1]$. This gives Figure 11, where the unit ball for the Thurston norm on M is also drawn with dashed line segments.

Example 2 Carrying out the disk decomposition of M (as in Figure 8) using the disks $+D_1, -D_2$ to obtain a foliation $\mathcal{F}$ of M, one can piece together the core of a leaf L of $\mathcal{F}$ as displayed in Figure 12 (the entire leaf L is displayed in Figure 10(b) with the core shaded). Then

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$



(a) (b)

Figure 10: (a) The leaf L of Example 1 (b) The leaf L of Example 2

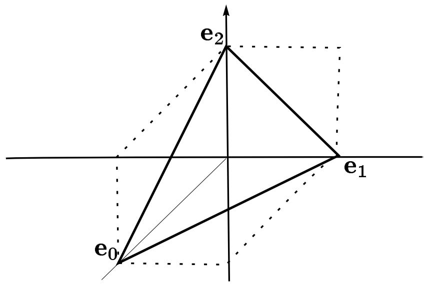


Figure 11: Foliation cones for $(2,2,2)$

That is, in this case, 1 can follow 2 and 2 can follow 1 or 2, but 1 cannot follow 1. The loops of minimal period are

$$\Gamma_{(\dots 1212\dots)} = \alpha_1 - \alpha_2$$
$$\Gamma_{(\dots 2222\dots)} = -\alpha_2$$

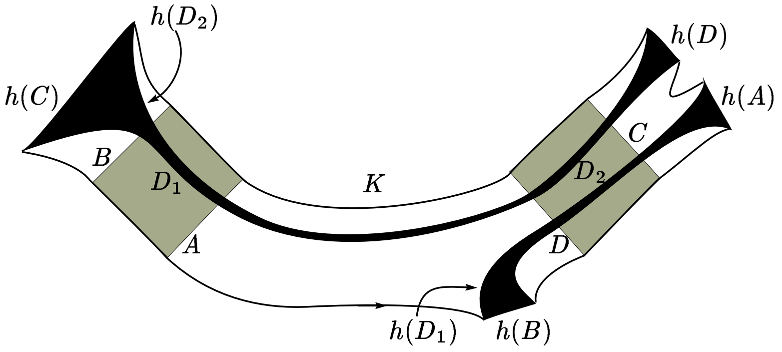


Figure 12: The core of the leaf L in Example 2

The foliation cone to which $\mathcal{F}$ belongs is defined by the inequalities

$$\begin{aligned}\{\alpha_1 \geq \alpha_2\} &= \text{the half plane below the line } \alpha_1 = \alpha_2\\ \{\alpha_2 \leq 0\} &= \text{the lower half plane.}\end{aligned}$$

A base for this cone is the segment $[\mathbf{e}_0, \mathbf{e}_1]$ (see Figure 11).

Example 3 If one decomposes the sutured manifold M of Figure 8 using the disk $-D_0$, one obtains a product sutured torus and thus a depth one foliation $\mathcal{F}$ of M. A leaf L of this foliation and the disk D_0 is given in Figure 13. If t is translation to the left by one unit and d is a Dehn twist in the dotted curve, then $h = d^2 \circ t$ is the pseudo-Anosov endperiodic monodromy of the foliation $\mathcal{F}$. The foliation $\mathcal{F}$, the leaf L and the endperiodic monodromy are the same as in Example 1 (see Figure 10(a)). The disk D_0 does contain $Z = \mathcal{Z} \cap L$ but is not a Markov partition for Z. However the two components of $h(D_0) \cap D_0$ will be a Markov partition for Z and can be used to compute the foliation cone with base $[\mathbf{e}_1, \mathbf{e}_2]$ much as above.

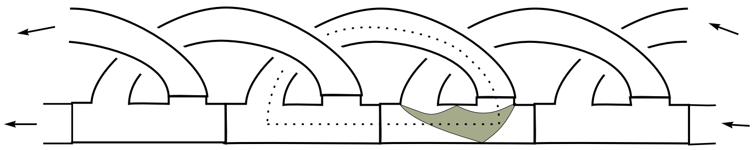


Figure 13: The leaf L of Example 3 showing the disk D_0 (shaded)

Remark There are 13 knots of ten or fewer crossings that have unique Seifert surface S and whose complements, cut apart along the Seifert surface, reduce via disks meeting the sutures twice to the sutured manifold M in Figure 8(b). They are:

$$8_{15}, 9_{25}, 9_{39}, 9_{41}, 9_{49}, 10_{58}, 10_{135}, 10_{144}, 10_{163}, 10_{165}, 10_{49}, 10_{66}, 10_{80}$$

(with notation as in Rolfsen [23]). If κ is one of the first ten of these knots and M is the sutured manifold of Figure 8(b), then there is a disk decomposition of $M_S(\kappa)$ to M using two disks, each of which meet the sutures twice. By Proposition 7.1, the foliation cones for $M_S(\kappa)$ are obtained from the foliation cones in Figure 11 by crossing with $\mathbb{R}^2$. If κ is one of the last three of these knots, then there is a disk decomposition of $M_S(\kappa)$ to M using four disks, each of which meet the sutures twice, and the foliation cones for $M_S(\kappa)$ are obtained from the foliation cones in Figure 11 by crossing with $\mathbb{R}^4$. Finally, an additional knot 10_{53} has two disjoint, nonisotopic Seifert surfaces S_1 and S_2. If we cut the knot complement apart along $S_1 \cup S_2$, we get two disjoint sutured manifolds X and Y. The component X is the sutured manifold $X(2;1,2)$ defined in [5, page 385] and $Y = M_S(8_{15})$. By results in [5], the foliation cones in $H^1(X) = \mathbb{R}^4$ consist of two half spaces, while those in $H^1(Y) = \mathbb{R}^4$ were determined above. Using Mayer–Vietoris as in [5, section 4], one shows that

$$\mathbb{R}^4 = H^1(M_{S_i}(10_{53})) \hookrightarrow H^1(X) \oplus H^1(Y), \quad i = 1, 2,$$

is a diagonal inclusion sending each foliated class to the direct sum of foliated classes. One then assembles the cone structure from that for X and Y.

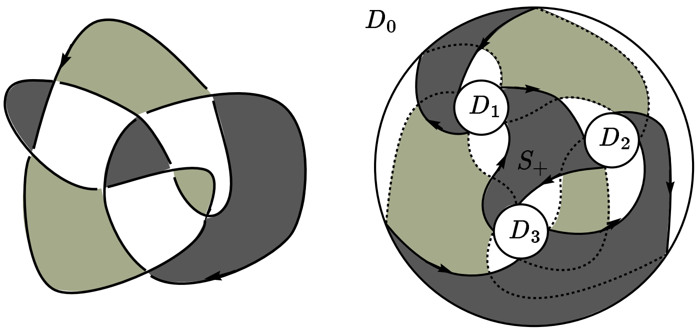
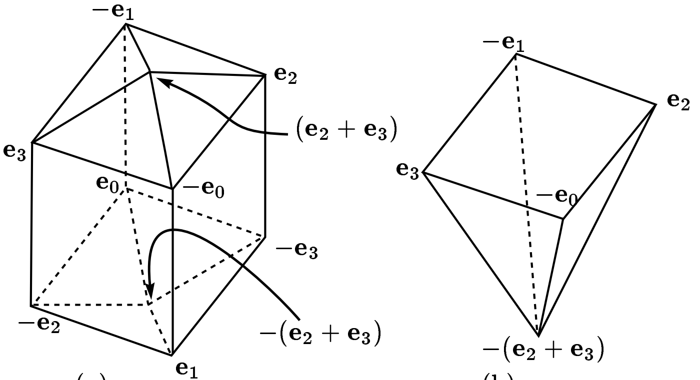


Figure 14: The 2–component link of Example 4 and the sutured manifold obtained from this link

Example 4 Consider the 2–component link with Seifert surface in Figure 14 and the sutured manifold M obtained from the link complement by cutting apart along the Seifert surface. We let D_1, D_2, D_3 be the three disks indicated in the figure, D_0 the outside disk. Then $H_2(M, \partial M) = \mathbb{R}^3$. The Thurston ball for M and bases for the foliation cones are given in Figure 15. One computes the Thurston ball by noting that the given vertices all have norm one and, in this case, the cones over the top dimensional faces of the Thurston ball are all of the form $-C_i \cap C_j$ where C_i and C_j are foliation cones.



(a) (b)

Figure 15: (a) The Thurston ball and (b) the foliation cones for Example 4

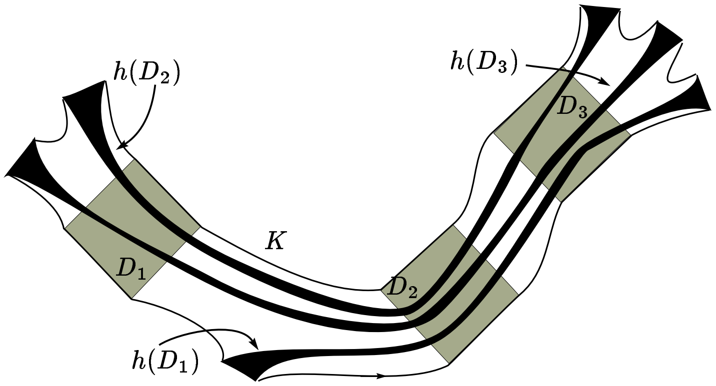


Figure 16: The core K of the leaf in Example 4

We compute the foliation cone with square base in Figure 15(b). If one carries out the disk decomposition of M using the disks $-D_1, +D_2, +D_3$ to obtain a foliation $\mathcal{F}$ of M, one can piece together the core of a depth one leaf L of $\mathcal{F}$ as displayed in Figure 16. Then

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

That is, in this case, 1 can follow 2 and 3, while 2 and 3 can each follow 1, 2 and 3. The minimal period loops are

$$\begin{aligned} \Gamma_{(\ldots 1212\ldots)} &= \alpha_2 - \alpha_1 \\ \Gamma_{(\ldots 2222\ldots)} &= \alpha_2 \\ \Gamma_{(\ldots 1313\ldots)} &= \alpha_3 - \alpha_1 \\ \Gamma_{(\ldots 3333\ldots)} &= \alpha_3. \end{aligned}$$

The foliation cone with square base is defined by the inequalities

$$\begin{aligned} \alpha_2 &\geq \alpha_1 \\ \alpha_2 &\geq 0 \\ \alpha_3 &\geq \alpha_1 \\ \alpha_3 &\geq 0. \end{aligned}$$

If $\kappa = 9_{38}$ or 10_{97} and M is the sutured manifold in Figure 14, then one can do a disk decomposition of $M_S(\kappa)$ to M using a disk that meets the sutures twice. By Proposition 7.1, the foliation cones for $M_S(\kappa)$ are obtained from the foliation cones in Figure 15(b) by crossing with $\mathbb{R}$.

Example 5 If $\kappa = 10_{55}$ or 10_{63} and M is the sutured manifold consisting of the complement of the pretzel link $(2,4,2)$ cut apart along the Seifert surface (as in Figure 17), then one can do a disk decomposition of $M_S(\kappa)$ using two disks, each disk meeting the sutures twice, obtaining M. By Proposition 7.1, the foliation cones for $M_S(\kappa)$ are obtained from the foliation cones in Figure 17 by crossing with $\mathbb{R}^2$. Figure 17 also gives the Thurston ball (dashed) for M. As before if one carries out a disk decomposition of M with the disks $+D_0, -D_1$, it is easy to trace out a suitable core of the leaf L. Figure 18 gives this core as well as the rectangles R_i used in the Markov partition. The disk D_1 is represented as hexagonal since it meets the sutures 6 times. The rectangular disk $D_0 = R_1$ meets the sutures 4 times. Notice that D_1 is divided into three rectangles R_2, R_3, and R_4 of the partition. These look like pentagons, but the barycenter of D_1 is not to be viewed as a vertex of these rectangles. The incidence matrix for this scheme is

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The minimal period loops are

$$\begin{aligned} \Gamma_{(\dots 234234 \dots)} &= \alpha_2 - \alpha_1 \\ \Gamma_{(\dots 131313 \dots)} &= (-\alpha_2) + (\alpha_2 - \alpha_1) = -\alpha_1. \end{aligned}$$

The foliation cone is defined by the inequalities

$$\begin{aligned} \alpha_2 &\geq \alpha_1 \\ \alpha_1 &\leq 0 \end{aligned}$$

and has base $[\mathbf{e}_0, \mathbf{e}_2]$.

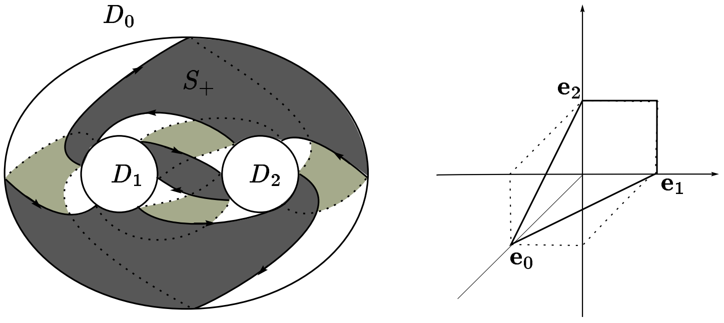


Figure 17: The sutured manifold, foliation cones, and Thurston ball for $(2, 4, 2)$

Remark In Example 5 one is tempted to take $\{R'_1 = D_0, R'_2 = D_1\}$ as "Markov partition". The hexagon $h(R'_2)$ stretches and passes properly through both R'_2 and R'_1, while $h(R'_1)$ meets only R'_2. In fact, the resulting 2×2 incidence matrix does determine the correct cone. In general, however, one must stick to the standard definition of "Markov partition", requiring that the elements be rectangles with one pair of opposite edges in the stable set and one pair of opposite edges in the unstable set. This is illustrated by the sutured manifold in Figure 19. We omit the details, but it happens in this case that, carrying out the disk decomposition by $-D_1$ and $-D_2$ and using $\{D_1, D_2\}$ as "Markov partition", one does not get the entire cone with base $[-\mathbf{e}_1, (3\mathbf{e}_1 - \mathbf{e}_2)/4]$. In

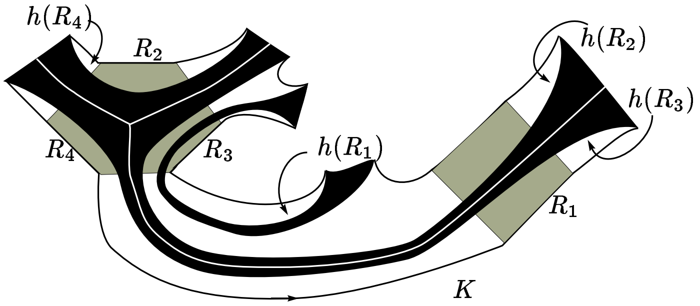


Figure 18: The core K of the leaf in Example 5

this case, D_2 is a decagon. Figure 19 also gives the bases for the foliation cones and the Thurston ball (dashed) for this example.

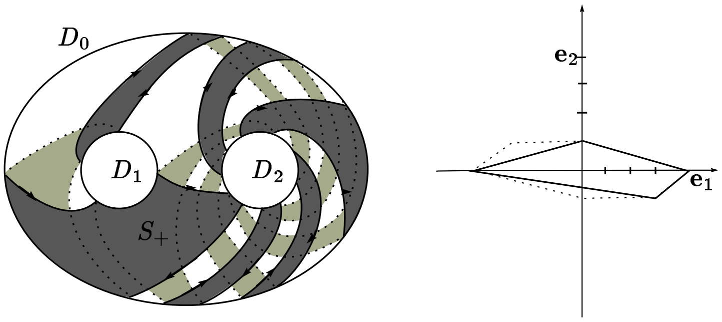


Figure 19: Example in which $\{D_1, D_2\}$ does not work as Markov partition

Remark Of the 249 knots of ten or fewer crossings, 117 are fibered and 111 belong to the class $\mathcal{B}$ of knots described in [5]. The preceding examples indicate how to compute the foliation cones of 18 of the remaining 21 knots. The three remaining knots, $9_{35}, 10_{101}, 10_{120}$, of ≤ 10 crossings each have a unique Seifert surface. Let M be the knot complement cut apart along this Seifert surface. The knot 9_{35} is the pretzel knot $(3,3,3)$, M is completely reduced and $H_2(M, \partial M) = \mathbb{R}^2$. For the knot 10_{101}, decomposition by a disk meeting the sutures twice replaces M with a completely reduced, sutured manifold M' with $H_2(M', \partial M') = \mathbb{R}^3$. For the knot 10_{120}, M is completely reduced and $H_2(M, \partial M) = \mathbb{R}^4$. In all cases the foliation cones are easily computed as above.

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