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Quantum invariants of periodic three-manifolds

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Abstract Let p be an odd prime and r be relatively prime to p. Let G be a finite p–group. Suppose an oriented 3–manifold $\tilde{M}$ has a free G–action with orbit space M. We consider certain Witten–Reshetikhin–Turaev $SU(2)$ invariants $w_r(M)$ in $\mathbb{Z}[\frac{1}{2r}, e^{\frac{2\pi i}{8r}}]$. We will show that $w_r(\tilde{M}) \equiv \kappa^{3 \ \mathrm{def}\ (\tilde{M}\to M)}(w_r(M))^{|G|} \pmod p$. Here $\kappa = e^{\frac{2\pi i(r-2)}{8r}}$, def denotes the signature defect, and $|G|$ is the number of elements in G. We also give a version of this result if M and $\tilde{M}$ contain framed links or colored fat graphs. We give similar formulas for non-free actions which hold for a specified finite set of values for r.

AMS Classification 57M10; 57M12

Keywords p–group action, lens space, quantum invariant, Turaev–Viro invariant, branched cover, Jones polynomial, Arf invariant

Dedicated to Rob Kirby on his sixtieth birthday

1 Introduction

Assume p is an odd prime, and that r is relatively prime to p, and $r \geq 3$. Let G be a finite p–group, with $|G|$ elements. We let ξ_a denote $e^{\frac{2\pi i}{a}}$.

Let M be an oriented closed 3–manifold with an embedded r–admissibly colored fat trivalent graph J. We include the case that J is empty. We consider the Witten–Reshetikhin–Turaev $SU(2)$ invariants $w_r(M,J) \in R_r = \mathbb{Z}[\frac{1}{2r}, \xi_t]$ [18, 13] where t is $4r$ if r is even and $8r$ if r is odd. Here $w_r(M,J)$ is the version of the WRT–invariant which is denoted $I_{-\xi_{4r}}(M,J)$ by Lickorish [9], assuming μ (also known as η in this paper) is chosen to be $\left(\xi_{2r} - \xi_{2r}^{-1}\right)/(i\sqrt{2r})$. In terms of the Kirby–Melvin [7] normalization τ_r, one has $w_r(M) = \tau_r(M)\ /\tau_r(S^1 \times S^2)$.

Let $\tilde{M}$ be an oriented closed 3–manifold with a G action. The singular set of the action is the collection of points whose isotropy subgroup is non-trivial. Let $\tilde{J}$ be an equivariant r–admissibly colored fat graph $\tilde{J}$ in $\tilde{M}$ which is disjoint from the singular set. The action is assumed to preserve the coloring and thickening of $\tilde{J}$. Burnside's theorem asserts that the center of a finite p–group is non-trivial. It follows that the quotient map $\tilde{M} \to \tilde{M}/G$ can be factored as a sequence of $\mathbb{Z}_p$ (possibly branched) covering maps. It follows that the orbit space is also a closed 3–manifold with an r–admissibly colored fat graph, the image of $\tilde{J}$. In this case, we will denote the orbit space of $\tilde{M}$ by M, and the orbit space of $\tilde{J}$ by J. We find a relationship between $w_r(M,J)$, and $w_r(\tilde{M},\tilde{J})$.

When the action is free, a signature defect of $\tilde{M} \to M$ may be defined as follows. One can arrange that some number, say n, of disjoint copies of $\tilde{M} \to M$ form the boundary of a regular G covering space of a 4–manifold W, denoted by $\tilde{W}$. Then define

$$\operatorname{def}\,(\tilde{M} \to M) = \frac{1}{n}\left(|G|\ \operatorname{Sign}(W) - \ \operatorname{Sign}(\tilde{W})\right).$$

The defect can be seen to be well-defined using Novikov additivity together with that fact the signature of an m–fold unbranched covering space of a closed manifold is m times the signature of the base manifold. This generalizes the definition of the signature defect for a finite cyclic group [5,]. As $\Omega_3(BG) = H_3(BG)$, and $H_3(BG)$ is annihilated by multiplication by $|G|$, n can be taken to be $|G|$ in the above definition. For this definition, it is not necessary that G be a p–group. We remark that $3\ \operatorname{def}\,(\tilde{M} \to M)$ is an integer. We will give a proof in section 3.

We will be working with congruences modulo the odd prime p in the ring of all algebraic integers (over $\mathbb{Z}$) after we have inverted $2r$, where r is relatively prime to p. Let κ denote $\xi_8 \xi_{4r}^{-1}$.

Theorem 1 *If G acts freely, then*

$$w_r(\tilde{M},\tilde{J}) \equiv \kappa^{3\operatorname{def}\,(\tilde{M}\to M)}(w_r(M,J))^{|G|} \pmod{p}.$$

Our equations in Theorems 1, 2 and 3 take place in R_r/pR_r. We may think of R_r as polynomials in ξ_t with coefficients in $\frac{1}{2r}\mathbb{Z}$ of degree less than $\phi(t)$. When multiplying such polynomial, one should use the t-th cyclotomic polynomial to rewrite the product as a polynomial in degree less than $\phi(t)$. Such a polynomial lies in pR_r if and only if each of its coefficients maps to zero under the map $\frac{1}{2r}\mathbb{Z} \to \mathbb{Z}_p$ given by reduction modulo p. For example we consider Theorem 1 with $r = 5$ applied to the free Z_3 action on S^3 with quotient $L(3,1)$. This action has defect $2/3$. In this case $t = 40$, and the cyclotomic polynomial tells us to reduce to polynomials of degree less than 16 via ${\xi_t}^{16} = {\xi_t}^{12} - {\xi_t}^{8} + {\xi_t}^{4} - 1$. We have:

$$w_5(L(3,1)) = \frac{3\,\xi_t + 2\,\xi_t^3 - \xi_t^5 - 4\,\xi_t^7 + 4\,\xi_t^9 + \xi_t^{11} - 2\,\xi_t^{13} - 3\,\xi_t^{15}}{10},$$

$$w_5(S^3) = \eta = \frac{\xi_t + \xi_t^3 - 2\,\xi_t^5 + 3\,\xi_t^7 + 3\,\xi_t^9 - 2\,\xi_t^{11} - 4\,\xi_t^{13} + \xi_t^{15}}{10}$$

$$w_5(S^3) - \kappa^2(w_5(L(3,1)))^3 = 3\ \frac{-\,\xi_t^5 + 2\,\xi_t^7 + 2\,\xi_t^9 - 2\,\xi_t^{11} - 2\,\xi_t^{13} + \xi_t^{15}}{20}$$

We note that $R_r/pR_r = \mathbb{Z}[\xi_t]/p\mathbb{Z}[\xi_t]$. Let f be the smallest positive integer such that $p^f \equiv 1 \pmod{t}$. $\mathbb{Z}[\xi_t]/p\mathbb{Z}[\xi_t]$ is the direct sum of $\phi(t)/f$ fields each with p^f elements [17; page 14]. It is important that the pth power map is the Frobenius automorphism of R_r/pR_r. So given $w_r(\tilde{M},\tilde{J})$ and def $(\tilde{M} \to M)$, we can always solve uniquely for $w_r(M,J)$. Theorem 1 by itself will provide no obstruction to the existence of a free G–action on a given manifold $\tilde{M}$.

If $w_r(\tilde{M},\tilde{J}) \neq 0$ for an infinite collection of r prime to p then the values of $w_r(\tilde{M},\tilde{J})$, and $w_r(M,J)$ for this collection of r determine def $(\tilde{M} \to M)$. This is not apriori clear.

When the action is not free we have to restrict r to a few values, and we don't know an independent definition of the exponent of κ. Theorems 2 and 3 by themselves will provide no obstruction to the existence of a G–action on a given manifold $\tilde{M}$.

Theorem 2 *If r divides $\frac{p\pm1}{2}$, then for some integer δ, one has*

$$w_r(\tilde{M},\tilde{J}) \equiv \kappa^\delta (w_r(M,J))^{|G|} \pmod{p}.$$

If G is cyclic and acts semifreely and r divides $\frac{|G|\pm1}{2}$, the same conclusion holds.

We tie down the factor κ^δ in a special case.

Theorem 3 *Suppose r divides $\frac{p^s\pm1}{2}$, and $\tilde{M}$ is a p^s–fold branched cyclic cover of a knot K in a homology sphere M. Let J be a colored fat graph in M which misses K, and $\tilde{J}$ be the inverse image of J in $\tilde{M}$.*

$$w_r(\tilde{M},\tilde{J}) \equiv \mp \left(\frac{-2r}{p}\right)^s \kappa^{-3\sigma_{p^s}(K)} (w_r(M,J))^{p^s} \pmod{p},$$

Here $\sigma_{p^s}(K)$ denotes the total p^s–signature of K [8]. $\left(\frac{-2r}{p}\right)$ is a Legendre symbol.

As a corollary, we obtain the following generalization of a result of Murasugi's [11; Proposition 8]. Murasugi's hypothesis is that $\tilde{L}$ is p–periodic, ζ is a *primitive* $\frac{p\pm1}{2}$ th root of unity and $p \geq 5$. Here $V_L(\zeta)$ denotes the Jones polynomial evaluated at $t = \zeta$.

Corollary 1 *Let $\tilde{L}$ be a p^s–periodic link in S^3 with quotient link L. If $\zeta^{\frac{p^s \pm 1}{2}} = 1$ and $\zeta \neq -1$, then*

$$V_{\tilde{L}}(\zeta) \equiv V_L(\zeta^{\mp 1}) \pmod{p}.$$

L has an even number of components if and only if $\tilde{L}$ does also. In this case, we must choose the same $\sqrt{\zeta}$ when evaluating both sides of the above equation. The above equation is false, in general, for $\zeta = -1$: the trefoil is has period three with orbit knot the unknot. Making use of H. Murakami's formula [12] relating the Jones polynomial evaluated at i to the Arf invariant of a proper link we have the following corollary. First we observe: if $\tilde{L}$ is a n–periodic link in S^3 with quotient link L, and n is odd, then $\tilde{L}$ is proper if and only if L is proper.

Corollary 2 *Let $\tilde{L}$ be a n–periodic proper link in S^3 with quotient link L. Let $n = \prod p_i^{s_i}$ be the prime factorization of n. Suppose for each i, $p_i^{s_i} \equiv \pm 1 \pmod 8$ (for each i one may choose $\pm$ differently), then*

$$\mathrm{Arf}(L) \equiv \mathrm{Arf}(\tilde{L}) \pmod 2.$$

The period three action on the trefoil also shows the necessity of the condition that $p_i^{s_i} \equiv \pm 1 \pmod 8$.

In section 2 we establish versions of Theorems 1 and 2 for the related Turaev–Viro invariants by adapting an argument which Murasugi used to study the bracket polynomial of periodic links [11]. See also Traczyk's paper [15]. In fact these theorems (Theorems 4 and 5) are immediate corollaries of Theorems 1 and 2 but we prefer to give them direct proofs. The reason is that these proofs are simpler than the proofs of Theorems 1 and 2. These proofs can be used to obtain analogous results for other invariants defined by Turaev–Viro type state sums. Also Lemma 3, that we establish to prove Theorem 5, is used later in the proofs of Theorems 2 and 3.

In section 3, we relate $w_r(M, J)$ to the TQFT defined in [2]. We discuss p_1–structures. We also rephrase Theorem 1 in terms of manifolds with p_1–structure, and reduce the proof of Theorem 1 to the case $G = \mathbb{Z}_p$. We say that a regular $\mathbb{Z}_{p^s}$–cover which is a quotient of a regular $\mathbb{Z}$ cover is a simple $\mathbb{Z}_{p^s}$–cover. In section 4, we derive Theorem 1 for simple $\mathbb{Z}_{p^s}$–covers of closed manifolds. This part of the argument applies generally to quantum invariants associated to any TQFT. We also obtain a version of Theorem 1 for simple $\mathbb{Z}_{p^s}$–covers of manifolds whose boundary is a torus. In section 5, we derive Theorem 1 in the case M is a lens space and $G = \mathbb{Z}_p$. In section 6, we complete the proof of Theorem 1. One step is to show that if $\tilde{M}$ is a regular $\mathbb{Z}_p$–cover which is not a simple $\mathbb{Z}_p$–cover of M, then we may delete a simple closed curve γ in M so that the inverse image of $M-\gamma$ is a simple $\mathbb{Z}_p$–cover of $M-\gamma$. In section 7, we prove Theorem 2, Theorem 3, Corollary 1, and Corollary 2.

2 Turaev–Viro invariants

We also want to consider the associated Turaev–Viro invariants, which one may define by $\mathrm{tv}_r(M) = w_r(M)\overline{w_r(M)}$. Here conjugation is defined by the usual conjugation defined on the complex numbers. $\mathrm{tv}_r(M)$ was first defined as a state sum by Turaev and Viro [16], and later shown to be given by the above formula separately by Walker and Turaev. A very nice proof of this fact was given by Roberts [14]. We will use a state sum definition in the form used by Roberts. We pick a triangulation of M, and sum certain contributions over r–admissible colorings C of the triangulation. A coloring of a triangulation assigns to each 1–simplex a nonnegative integral color less than $r-1$. The coloring is admissible if: for each 2–simplex the colors assigned to the three edges a, b, and c satisfy $a+b+c$ is even, $a+b+c \le 2r-4$, and $|a-b| \le c \le a+b$.

$$\mathrm{tv}_r(M) = \sum_{c\in C} \prod_{v\in V} \eta^2 \prod_{e\in E} \Delta(c,e) \prod_{f\in F} \theta(c,f)^{-1} \prod_{t\in T} \mathrm{Tet}(c,t)$$

Here V is the set of vertices, E is the set of edges (or 1–simplexes), F is the set of faces (or 2–simplexes), and T is the set of tetrahedrons (or 3–simplexes) in the triangulation. The contributions are products of certain evaluations in the sense of Kaufman–Lins [6] of colored planar graphs. $\Delta(c,e) = \Delta_{c(e)}$, where Δ_i is the evaluation of a loop colored the color i. $\theta(c,f)$ is the evaluation of an unknotted theta curve whose edges are colored with the colors assigned by c to the edges of f. $\mathrm{Tet}(c,t)$ is the evaluation of a tetrahedron whose edges are colored with the colors assigned by c to the tetrahedron t. However we take all these evaluations in $\mathbb{C}$, taking A^2 to be $\lambda = \xi_{2r}$. Also $\eta^2 = -\frac{(\lambda-\lambda^{-1})^2}{2r}$. So $\mathrm{tv}_r(M)$ lies in $\mathbb{Z}[\frac{1}{2r},\lambda]$. This follows from the following lemmas and the formulas in [6] for these evaluations.

Lemma 1 *For j not a multiple of $2r$, $(1-\lambda^j)^{-1} \in \mathbb{Z}[\frac{1}{2r},\lambda]$.*

Proof $\prod_{s=1}^{2r-1}(x-\lambda^s) = \sum_{i=0}^{2r-1} x^i$. Letting $x=1$, $\prod_{s=1}^{2r-1}(1-\lambda^s) = 2r$. □

Lemma 2 *For $n \le r-1$, the "quantum integers" $[n] = \frac{\lambda^n-\lambda^{-n}}{\lambda-\lambda^{-1}} = \lambda^{1-n}\frac{\lambda^{2n}-1}{\lambda^2-1}$ are units in $\mathbb{Z}[\frac{1}{2r},\lambda]$.*

Since it is fixed by complex conjugation, $\mathrm{tv}_r(M) \in \mathbb{Z}[\frac{1}{2r}, \lambda + \lambda^{-1}]$.

Theorem 4 *If the action is free, then*

$$\mathrm{tv}_r(\tilde{M}) \equiv (\mathrm{tv}_r(M))^{|G|} \pmod{p}.$$

Proof A chosen triangulation T of M lifts to a triangulation $\tilde{T}$ of $\tilde{M}$. Each admissible coloring of T lifts to an admissible coloring of $\tilde{T}$. As each simplex of M is covered by $|G|$ simplexes of $\tilde{M}$, the contribution of a lifted coloring to the sum for $\mathrm{tv}_r(\tilde{M})$ is the $|G|$th power of the contribution of the original coloring to $\mathrm{tv}_r(M)$. $|G|$ acts freely on the set of colorings of $\tilde{T}$ which are not lifts of some coloring of T. Moreover the contribution of each such triangulation in a given orbit of this G action is constant. Thus the contribution of the non-equivariant colorings is a multiple of p. Making use of the equation $x^{p^s} + y^{p^s} \equiv (x+y)^{p^s} \pmod{p}$, the result follows. □

Lemma 3 *If r divides $\frac{p^s \pm 1}{2}$, $\Delta_i \equiv (\Delta_i)^{p^s} \pmod{p}$, for all i, and $\eta^2 \equiv (\eta^2)^{p^s} \pmod{p}$.*

Proof Since r divides $\frac{p^s \pm 1}{2}$, $2r$ divides $p^s \pm 1$. Thus $\lambda^{p^s \pm 1} = 1$, and $\lambda^{p^s} = \lambda^{\mp 1}$. Thus

$$\Delta_1^{p^s} = \left(-\lambda - \lambda^{-1}\right)^{p^s} \equiv -\lambda^{p^s} - \lambda^{-p^s} = -\lambda - \lambda^{-1} = \Delta_1 \pmod{p}.$$

Also $\Delta_0 = 1$. Thus $\Delta_i \equiv (\Delta_i)^{p^s} \pmod{p}$, if i is zero or one. Using the recursion formula $\Delta_{i+1} = \Delta_1 \Delta_i - \Delta_{i-1}$, $\Delta_i \equiv (\Delta_i)^{p^s} \pmod{p}$ follows by induction. Here is the inductive step:

$$\begin{aligned}(\Delta_{i+1})^{p^s} &= (\Delta_1 \Delta_i - \Delta_{i-1})^{p^s} \\ &\equiv \Delta_1^{p^s} \Delta_i^{p^s} - \Delta_{i-1}^{p^s} \equiv \Delta_1 \Delta_i - \Delta_{i-1} = \Delta_{i+1} \pmod{p}.\end{aligned}$$

It follows that $\sum_{i=0}^{r-2} \Delta_i^2 \equiv \left(\sum_{i=0}^{r-2} \Delta_i^2\right)^{p^s} \pmod{p}$. As $\eta^2 \sum_{i=0}^{r-2} \Delta_i^2 = 1$, and $\mathbb{Z}[\lambda, \frac{1}{2r}]/p(\mathbb{Z}[\lambda, \frac{1}{2r}])$ is a direct sum of fields, we have $(\eta^2)^{p^s} \equiv \eta^2 \pmod{p}$. □

Theorem 5 *If r divides $\frac{p \pm 1}{2}$, then*

$$\mathrm{tv}_r(\tilde{M}) \equiv (\mathrm{tv}_r(M))^{|G|} \pmod{p}.$$

If G is cyclic and acts semifreely and r divides $\frac{|G| \pm 1}{2}$, the same congruence holds.

Proof We pick our triangulation of the base so that the image of the fixed point set is a one dimensional subcomplex. By Lemma 3, whether a colored simplex in the base lies in the image of a simplex with a smaller orbit or not it contributes the same amount modulo p to a product associated to an equivariant coloring. Thus the proof of Theorem 4 still goes through. □

3 Quantum invariants, p_1–structures, and signature defects

Let M be a closed 3–manifold with a p_1–structure [2]. A fat colored graph in M is a trivalent graph embedded in M, with a specified 2–dimensional thickening (ie, banded in the sense of [2]) whose edges have been colored with nonnegative integers less than $r-1$. At each vertex the colors on the edges a, b, and c must satisfy the admissibility conditions: $a+b+c$ is even, $a+b+c \leq 2r-4$, and $|a-b| \leq c \leq a+b$. Let J be such a graph (possibly empty) in M. Recall the quantum invariant $\langle (M,J) \rangle_{2r} \in k_{2r}$ defined in [2]. Consider the homomorphism [10; note page 134] $\psi\colon k_{2r} \to \mathbb{C}$ which sends A to $-\xi_{4r}$, and sends κ to $\xi_8 \xi_{4r}^{-1}$. Let R_r denote the image of ψ. By abuse of notation let κ denote $\psi(\kappa)$, and η denote $\psi(\eta)$. Let $t = 4r$ if r is even and $t = 8r$ if r is odd. Then $R_r = \mathbb{Z}[\frac{1}{2r}, \xi_t]$. Further abusing notation, we let $\langle (M,J) \rangle$ denote $\psi \langle (M,J) \rangle_{2r} \in R_r$. If M is a 3–manifold without an assigned p_1–structure, we let $w_r(M,J)$ denote $\langle (M',J) \rangle$ where M' denotes M equipped with a p_1–structure with σ–invariant zero. If M already is assigned a p_1–structure, we let $w_r(M,J)$ denote $\langle (M',J) \rangle$ where M' denotes M equipped with a reassigned p_1–structure with σ–invariant zero. One has that $w_r(M,J) = \kappa^{-\sigma(M)} \langle (M,J) \rangle$. This agrees with $w_r(M,J)$ as defined in the introduction.

Assume now that M has been assigned a p_1–structure. Let $\tilde{M}$ be a regular G covering space. Give $\tilde{M}$ the induced p_1–structure, obtained by pulling back the structure on M. The following lemma generalizes [4; 3.5]. It does not require that G be a p–group.

Lemma 4 $3 \operatorname{def}(\tilde{M} \to M) = |G|\, \sigma(M) - \sigma(\tilde{M})$. *In particular* $3 \operatorname{def}(\tilde{M} \to M)$ *is an integer.*

Proof Pick a 4–manifold W with boundary $|G|$ copies of M such that the cover extends. We may connect sum on further copies of $\mathbb{C}P(2)$ or $\overline{\mathbb{C}P(2)}$ so that the p_1–structure on M also extends. Let $\tilde{W}$ be the associated cover of W with boundary $|G|$ copies of $\tilde{M}$. We have

$$|G|\, \sigma(M) = 3 \operatorname{Sign}(W), \quad |G|\, \sigma(\tilde{M}) = 3 \operatorname{Sign}(\tilde{W}),$$
$$|G| \operatorname{def}(\tilde{M} \to M) = |G| \operatorname{Sign}(W) - \operatorname{Sign}(\tilde{W}). \qquad \square$$

Using this lemma, we rewrite Theorem 1 in an equivalent form. The conclusion is simpler. On the other hand, the hypothesis involves the notion of a p_1–structure. Since p_1–structures are sometimes a stumbling block to novices, we stated our results in the introduction without reference to p_1–structures.

Theorem 1′ *Let M have a p_1–structure, and $\tilde{M}$ be a regular G cover of M with the induced p_1–structure. Then*

$$\left\langle \left(\tilde{M}, \tilde{J}\right) \right\rangle \equiv \langle (M,J) \rangle^{|G|} \pmod{p}.$$

Note that we may define $\langle (M, \mathcal{J}) \rangle$ for $\mathcal{J}$ a linear combination over R_r of fat colored graphs in M, by extending the function $\langle (M,J) \rangle$ linearly. If $\mathcal{J} = \sum_i a_i J_i$, we define $\tilde{\mathcal{J}}$ to be $\sum_i a_i^{|G|} \tilde{J}_i$. Since the pth power map is an automorphism of R_r/pR_r, we have that if Theorem 1′ is true for a given type manifold M, then it is true for such manifolds when we replace J and $\tilde{J}$ by linear combinations over R_r of colored fat graphs: $\mathcal{J}$ and $\tilde{\mathcal{J}}$.

Finally we note that if Theorem 1′ is true for $G = \mathbb{Z}_p$, then it will follow for G a general finite p–group. In the next three sections we prove it for $G = \mathbb{Z}_p$.

4 Simple unbranched $\mathbb{Z}_{p^s}$–covers

A regular Z_{p^s}–covering space $\tilde{X}$ of X is classified by an epimorphism ϕ: $H_1(X) \to \mathbb{Z}_{p^s}$. If ϕ factors through $\mathbb{Z}$, we say $\tilde{X}$ is a simple $\mathbb{Z}_{p^s}$–cover. In this section, we prove Theorem 1′ for simple $\mathbb{Z}_{p^s}$–covers. We also obtain a version for simple $\mathbb{Z}_{p^s}$–covers of manifolds whose boundary is a torus.

If χ: $H_1(M) \to \mathbb{Z}$ is an epimorphism, let χ_{p^s}: $H_1(M) \to \mathbb{Z}_{p^s}$ denote the composition with reduction modulo p^s. Suppose $\tilde{M}$ is classified by χ_{p^s}. Consider a Seifert surface for χ ie, a closed surface in M which is Poincare dual to χ. We may and do assume that this surface is in general position with respect to the colored fat graph J. Then the intersection of J with F defines some banded colored points. This surface also acquires a p_1–structure. Thus F is an object in the cobordism category $\mathcal{C}^{p_1,c}_{2,r-1}$, [2; 4.6]. Let E be the cobordism from F to F obtained by slitting M along F. We view E as a morphism from F to F in the cobordism category $\mathcal{C}^{p_1,c}_{2,r-1}$. Then M is the mapping torus of E, and $\tilde{M}$ is the mapping torus for E^{p^s}.

We may consider the TQFT which is a functor from $\mathcal{C}^{p_1,c}_{2,r-1}$, to the category of modules over R_r obtained taking $p = 2r$ in [2] and applying the change of coefficients ψ: $k_{2r} \to R_r$.

By [2; 1.2], we have

$$\langle (M,J) \rangle = \mathrm{Trace}\,(Z(E)) \quad \text{and} \quad \left\langle \left(\tilde{M}, \tilde{J}\right) \right\rangle = \mathrm{Trace}\left(Z\left(E^{p^s}\right)\right).$$

Let $\mathcal{E}$ be the matrix for $Z(E)$ with respect to some basis for $V(F)$. $\mathcal{E}$ has entries in $R_r = \mathbb{Z}[\frac{1}{2r}, \xi_t]$. Write the entries as polynomials in ξ_t whose coefficients are quotients of integers by powers of $2r$. Let $v \in \mathbb{Z}$ such that $2rv \equiv 1 \pmod{p}$. Let $\mathcal{E}'$ denote the matrix over $\mathbb{Z}[\xi_t]$ obtained by replacing all powers of $2r$ in the denominators of entries by powers of v in the numerators of these entries. We have

$$\operatorname{Trace}(Z(E)) = \operatorname{Trace}(\mathcal{E}) \equiv \operatorname{Trace}(\mathcal{E}') \pmod{p} \text{ and,}$$
$$\operatorname{Trace}\left(Z(E)^{p^s}\right) = \operatorname{Trace}\left(\mathcal{E}^{p^s}\right) \equiv \operatorname{Trace}\left(\mathcal{E}'^{p^s}\right) \pmod{p}.$$

However all the eigenvalues of $\mathcal{E}'$ are themselves algebraic integers. The trace of $\mathcal{E}'$ is the sum of these eigenvalues counted with multiplicity. The trace of $\mathcal{E}'^{p^s}$ is the sum of p^sth powers of these eigenvalues counted with multiplicity. Therefore

$$\operatorname{Trace}\left(\mathcal{E}'^{p^s}\right) \equiv (\operatorname{Trace}(\mathcal{E}'))^{p^s} \pmod{p}.$$

Putting these equations together proves Theorem $1'$ for simple $\mathbb{Z}_{p^s}$ covers.

We now wish to obtain a version of Theorem $1'$ for manifolds whose boundary is a torus. Let N be a compact oriented 3–manifold with p_1–structure with boundary $S^1 \times S^1$. $S^1 \times S^1$ acquires a p_1–structure as the boundary. Let J be a colored fat graph in N which is disjoint from the boundary. Then (N, J) defines an element of $V(S^1 \times S^1)$ under the above TQFT. We denote this element by $[N, J]$.

If $\chi\colon H_1(N) \to \mathbb{Z}$, let $\chi_{p^s}\colon H_1(N) \to \mathbb{Z}$ denote the composition with reduction modulo $p^s\colon \mathbb{Z} \to \mathbb{Z}_{p^s}$. Suppose χ_{p^s} restricted to the boundary is an epimorphism. Suppose $\tilde{N}$ is a regular Z_{p^s} covering space given by χ_{p^s}. $\tilde{N}$ has an induced p_1–structure. Suppose that we have identified the boundary with $S^1 \times S^1$ so that χ restricted to the boundary is $\pi_{1*}\colon H_1(S^1 \times S^1) \to H_1(S^1)$, followed by the standard isomorphism. Here π_1 denotes projection on the first factor. We can always identify the boundary in this way.

Let $\tilde{N}$ be the regular Z_{p^s} covering space given by χ_{p^s}, and $\tilde{J}$ the colored fat graph in $\tilde{N}$ given by the inverse image of J. The boundary of $\tilde{N}$ is naturally identified with $\tilde{S^1} \times S^1$. Equip $S^1 \times D^2$ with a p_1–structure extending the p_1–structure that $S^1 \times S^1$ acquires as the boundary of N. Equip $(\tilde{S^1}) \times D^2$ with a p_1–structure extending the p_1–structure $(\tilde{S^1}) \times S^1$ acquires as the boundary of $\tilde{N}$. This is the same p_1–structure it gets as the cover of $S^1 \times S^1$. We have $[\tilde{N}, \tilde{J}] \in V(\tilde{S^1} \times S^1)$. Also $\partial(S^1 \times D^2) = \partial N$, and $\partial(\tilde{S^1} \times D^2) = \partial \tilde{N}$. For $0 \le i \le r-2$, let $e_i = [S_i] \in V(S^1 \times S^1)$, where S_i is $S^1 \times D^2$ with the core with the standard thickening colored i. Similarly let $\tilde{e}_i = [\tilde{S}_i] \in V(\tilde{S^1} \times S^1)$, where $\tilde{S}_i$ is $\tilde{S}^1 \times D^2$ with the core with the standard thickening colored i.

Proposition 1 *If* $[N,J]=\sum_{i=0}^{r-2} a_i e_i$, *and* $[\tilde{N},\tilde{J}]=\sum_{i=0}^{r-2} \tilde{a}_i \tilde{e}_i$, *then* $\tilde{a}_j \equiv a_j^{p^s} \pmod{p}$, *for all* j, *such that* $0 \le j \le r-2$.

Proof Note $(\tilde{N},\tilde{J}) \cup_{\tilde{S^1}\times S^1} -\tilde{S}_j$ is a simple cover of $(N,J)\cup_{S^1\times S^1} -S_j$. Moreover $\left\langle (\tilde{N},\tilde{J}) \cup_{\tilde{S^1}\times S^1} -\tilde{S}_j \right\rangle = \tilde{a}_j$ and $\langle (N,J)\cup_{S^1\times S^1} -S_j\rangle = a_j$. By Theorem $1'$, $\tilde{a}_j \equiv a_j^{p^s} \pmod{p}$. □

Proposition 2 *If* $\mathcal{J}$ *is a linear combination over* R_r *of colored fat graphs in* $S^1\times D^2$ *then* $[S^1\times D^2,\mathcal{J}]\in V(S^1\times S^1)$ *determines* $[\tilde{S}^1\times D^2,\tilde{\mathcal{J}}]\in V(\tilde{S}^1\times S^1)$ *modulo* p.

Proof Just take $N=S^1\times D^2$, and sum over the terms in $\mathcal{J}$. □

See the paragraph following the statement of Theorem $1'$ for the definition of $\tilde{\mathcal{J}}$.

5 $\mathbb{Z}_p$–covers of lens spaces

$L(m,q)$ can be described as $-m/q$ surgery to an unknot in S^3. A meridian of this unknot becomes a curve in $L(m,q)$ which we refer to as a meridian of $L(m,q)$. Below, we verify directly that Theorem 1 (and thus Theorem $1'$) holds when M is a lens space, J a meridian colored c, and $G=\mathbb{Z}_p$. By general position any fat graph J in a lens space can be isotoped into a tubular neighborhood of any meridian. Without changing the invariant of the lens space with J or the cover of the lens space with $\tilde{J}$ one can replace J by a linear combination of this meridian with various colorings and $\tilde{J}$ by the same linear combination of the inverse image of this meridian with the same colorings in the covering space. This follows from Proposition 2. Thus it will follow that Theorem 1 (and thus Theorem $1'$) will hold if M is a lens space, and J is any fat colored graph in M, and $G=\mathbb{Z}_p$. This is a step in the proof of Theorem $1'$ for $G=\mathbb{Z}_p$.

Consider the p–fold cyclic cover $L(m,q)\to L(mp,q)$. We assume m, q are greater than zero. q must be relatively prime to m and p.

Lemma 5

$$\operatorname{def}\ (L(m,q)\to L(mp,q)) = \frac{1}{m}\left(\operatorname{def}\ \left(S^3\to L(mp,q)\right) - \operatorname{def}\ \left(S^3\to L(m,q)\right)\right).$$

Proof Suppose $Z \to X$ is a regular $\mathbb{Z}_{mp}$ covering of 4–manifolds with boundary, and on the boundary we have mp copies of the regular $\mathbb{Z}_{mp}$ covering $S^3 \to L(mp,q)$. Let Y denote Z modulo the action of $\mathbb{Z}_m \subset \mathbb{Z}_{mp}$. Then $Z \to Y$ is a regular $\mathbb{Z}_m$ covering, and on the boundary we have mp copies of the regular $\mathbb{Z}_m$ covering $S^3 \to L(m,q)$. Moreover $Y \to X$ is a regular $\mathbb{Z}_p$ covering and on the boundary we have mp copies of the regular $\mathbb{Z}_p$ covering $L(m,q) \to L(mp,q)$. We have:

$$mp\,(\mathrm{def}\ (L(m,q) \to L(mp,q)) = p\ \mathrm{Sign}(X) - \mathrm{Sign}(Y),$$

$$mp\,\left(\mathrm{def}\ (S^3 \to L(m,q)\right) = m\ \mathrm{Sign}(Y) - \mathrm{Sign}(Z) \quad \text{and}$$

$$mp\ \mathrm{def}\ (S^3 \to L(mp,q) = mp\ \mathrm{Sign}(X) - \mathrm{Sign}(Z).$$

The result follows. □

Suppose $\mathcal{H}$ is a normal subgroup of a finite group $\mathcal{G}$ which acts freely on M. By the above argument, one can show more generally that:

$$\mathrm{def}\ (M \to M/\mathcal{G}) = \mathrm{def}\ (M \to M/\mathcal{H}) + |\mathcal{H}|\ \mathrm{def}\ (M/\mathcal{H} \to M/\mathcal{G})\,.$$

According to Hirzebruch [5] (with different conventions)

$$3\ \mathrm{def}\ \left(S^3 \to L(m,q)\right) = 12m\ s(q,m) \in \mathbb{Z}.$$

See also [8; 3.3], whose conventions we follow.

Thus

$$3\ \mathrm{def}\ (L(m,q) \to L(mp,q)) = \frac{1}{m}\,(12mp\ s(q,mp) - 12m\ s(q,m)) \in \mathbb{Z}.$$

We will need reciprocity for generalized Gauss sums in the form due to Siegel [1; Formula 2.8]. Here is a slightly less general form which suffices for our purposes:

$$\sum_{k=0}^{\gamma-1} \xi_{2\gamma}^{\alpha k^2+\beta k} = \xi_8\ \xi_{8\alpha\gamma}^{-\beta^2} \sqrt{\frac{\gamma}{\alpha}}\ \sum_{k=0}^{\alpha-1} \xi_{2\alpha}^{-(\gamma k^2+\beta k)}$$

where $\alpha,\beta,\gamma \in \mathbb{Z}$ and $\alpha,\ \gamma > 0$ and $\alpha\gamma+\beta$ is even.

We use this reciprocity to rewrite the sum

$$\begin{aligned}\sum_{n=1}^{m} \xi_m^{(qr)n^2+(ql\pm1)n} &= \sum_{n=1}^{m} \xi_{2m}^{(2qr)n^2+2(ql\pm1)n} \\ &= \xi_8\ \xi_{16mqr}^{-4(ql\pm1)^2} \sqrt{\frac{m}{2qr}} \sum_{n=1}^{2qr} \xi_{4qr}^{-mn^2-2(ql\pm1)n}.\end{aligned}$$

We substitute this into a formula from [3; section 2].

$$w_r(L(m,q),\mu_c) = \frac{i(-1)^{c+1}}{\sqrt{2rm}} \sum_{\pm} \pm \xi_{4rmq}^{mb-mq\Phi(U)+q^2l^2+1\pm 2ql} \sum_{n=1}^{m} \xi_m^{(qr)n^2+(ql\pm1)n},$$

where $U = \left(\begin{smallmatrix} q & b \\ m & d \end{smallmatrix}\right)$, and $l = c+1$. We remark that the derivation given in [3] is valid for $m \geq 1$. Also using

$$mb - mq\Phi(U) + q^2l^2 + 1 = q^2(l^2-1) + 12mq\ s(q,m)$$

again from [3], we obtain:

$$w_r(L(m,q),\mu_c) = \frac{\xi_8^3(-1)^{c+1}}{2r\sqrt{q}} \xi_{4rm}^{12m\ s(q,m)} \xi_{4rmq}^{-q^2-1} \sum_{\pm} \pm \sum_{n=1}^{2qr} \xi_{4qr}^{-mn^2-2(ql\pm1)n}.$$

Similarly

$$w_r(L(mp,q),\mu_c) = \frac{\xi_8^3(-1)^{c+1}}{2r\sqrt{q}} \xi_{4rmp}^{12mp\ s(q,mp)} \xi_{4rmpq}^{-q^2-1} \sum_{\pm} \pm \sum_{n=1}^{2qr} \xi_{4qr}^{-mpn^2-2(ql\pm1)n}.$$

So we now work with congruences modulo p in the ring of algebraic integers after we have inverted $2r$ and q. We need to show that

$$w_r(L(m,q),\mu_c) \equiv \kappa^{3\ \mathrm{def}\ (L(m,q)\to L(mp,q))} \left(w_r(L(mp,q),\mu_c)\right)^p \pmod{p}. \quad (5.1)$$

We have that:

$$\begin{aligned} \kappa^{3\ \mathrm{def}\ (L(m,q)\to L(mp,q))} &= \left(\xi_8\xi_{4r}^{-1}\right)^{\frac{1}{m}(12mp\ s(q,mp)-12m\ s(q,m))} \\ &= \xi_8^{\frac{1}{m}(12mp\ s(q,mp)-12m\ s(q,m))} \xi_{4rm}^{12m\ s(q,m)-12mp\ s(q,mp)}, \end{aligned}$$

$$\left(\xi_{4rmpq}^{-q^2-1}\right)^p = \xi_{4rmq}^{-q^2-1}$$

$$(-1)^p = -1 \text{ and,}$$

$$(2r)^p \equiv 2r \pmod{p}.$$

Note that $\xi_{4qr}^{-mn^2-2(ql\pm1)n}$ only depends on n modulo $2qr$.

Thus

$$\begin{aligned} \left(\sum_{n=1}^{2qr} \xi_{4qr}^{-mpn^2-2(ql\pm1)n}\right)^p &\equiv \sum_{n=1}^{2qr} \xi_{4qr}^{-m(pn)^2-2(ql\pm1)pn} \\ &\equiv \sum_{n=1}^{2qr} \xi_{4qr}^{-mn^2-2(ql\pm1)n} \pmod{p}. \end{aligned}$$

Here we have made use of the fact that as n ranges over all the congruence classes modulo $2qr$ so does pn. Let $\binom{q}{p}$ denote the Legendre–Jacobi symbol.

$$(\sqrt{q})^p = q^{\frac{p-1}{2}}(\sqrt{q}) \equiv \left(\frac{q}{p}\right)\sqrt{q} \pmod p$$

Equation (5.1) will follow when we show:

$$\left(\frac{q}{p}\right)\xi_8^3 = \xi_8^{\frac{1}{m}(12mp\ s(q,mp)-12m\ s(q,m))}\xi_8^{3p},$$

or

$$\xi_8^{3-3p+\frac{1}{m}(12m\ s(q,m)-12mp\ s(q,mp))} = \left(\frac{q}{p}\right). \tag{5.2}$$

We will make use of a congruence of Dedekind's [14; page 160 (73.8)]: for k positive and odd:

$$12k\ s(q,k) \equiv k+1-2\left(\frac{q}{k}\right) \pmod 8.$$

We first consider the case that m is odd. We have:

$$\begin{aligned}12m\ s(q,m)-12mp\ s(q,mp) &\equiv m(1-p)+2\left(\left(\frac{q}{mp}\right)-\left(\frac{q}{m}\right)\right)\\ &= m(1-p)+2\left(\frac{q}{m}\right)\left(\left(\frac{q}{p}\right)-1\right) \pmod 8.\end{aligned}$$

If m is odd, the equation (5.2) becomes:

$$(3-3p)m+m(1-p)+2\left(\frac{q}{m}\right)\left(\left(\frac{q}{p}\right)-1\right) \equiv \begin{cases} 0, & \text{for } \binom{q}{p}=1 \\ 4, & \text{for } \binom{q}{p}=-1 \end{cases} \pmod 8,$$

which is easily checked. For the rest of this section, we consider the case that m is even. In this case, q must be odd. First we use Dedekind reciprocity [14; page 148 (69.6)] to rewrite $\frac{1}{m}(12m\ s(q,m)-12mp\ s(q,mp))$ as:

$$\begin{aligned}&\frac{1}{mq}(-12mq\ s(m,q)+m^2+q^2+1-3mq+12mqp\ s(mp,q)\\ &\qquad\qquad -m^2p^2-q^2-1+3mpq)\end{aligned}$$

$$\text{or}\quad \frac{1}{q}\left(-12q\ s(m,q)+m-3q+12qp\ s(mp,q)-mp^2+3pq\right).$$

As q is odd, $q^2=1 \pmod 8$. So modulo eight the above expression is

$$q\left(-12q\ s(m,q)+m-3q+12qp\ s(mp,q)-mp^2+3pq\right).$$

As p is odd, $p^2 = 1 \pmod 8$, and $m - mp^2 = 0 \pmod 8$. The expression, modulo eight, becomes

$$-12q^2\ s(m,q) - 3q^2 + 12q^2 p\ s(mp,q) + 3pq^2.$$

So the exponent of ξ_8 in Equation (5.2) modulo eight is:

$$q\left(-12q\ s(m,q) + 12qp\ s(mp,q)\right). \tag{5.3}$$

Again using Dedekind's congruence, we have:

$$12q\ s(m,q) \equiv q + 1 - 2q\binom{m}{q} \pmod 8.$$

Similarly

$$12q\ s(mp,q) \equiv q + 1 - 2q\binom{mp}{q} \pmod 8.$$

The expression (5.3) modulo eight, becomes

$$q(p-1)(q+1) + 2\binom{m}{q}\left(1 - p\binom{p}{q}\right)$$

Thus we only need to see

$$\xi_8^{q(p-1)(q+1)+2\binom{m}{q}(1-p\binom{p}{q})} = \binom{q}{p}.$$

Using quadratic reciprocity for Jacobi–Legendre symbols, this becomes

$$(-1)^{\frac{p-1}{2}\frac{q+1}{2}+\binom{m}{q}\frac{\left(1-p\binom{p}{q}\right)}{2}} = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}\binom{p}{q},$$

or

$$(-1)^{\frac{p-1}{2}+\frac{\left(1-p\binom{p}{q}\right)}{2}} = \binom{p}{q},$$

which is easily checked.

6 Unbranched $\mathbb{Z}_p$–covers

Lemma 6 *If $\phi \in H^1(M, \mathbb{Z}_p)$ is not the reduction of an integral class, then there is a simple closed curve γ in M such that the restriction of ϕ to $M - \gamma$ is the reduction of an integral class.*

Proof We consider the Bockstein homomorphism β associated to the short exact sequence of coefficients: $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_p \to 0$. Let γ be a simple closed curve which represents the element which is Poincare dual to $\beta(\phi) \in H^2(M, \mathbb{Z})$. The restriction of $\beta(\phi)$ to $H^2(M - \gamma, \mathbb{Z})$ is zero. This is easily seen using the geometric description of Poincare duality which makes use of dual cell decompositions.

Naturality of the long exact Bockstein sequence completes the proof. $\square$

It follows that any non-simple unbranched cyclic covering of a closed 3–manifold can be decomposed as the union of two simple coverings one of which is a covering of a solid torus. Let N denote M with a tubular neighborhood of γ deleted. We may replace N in M by a solid torus with a linear combination of colored cores obtaining a new manifold (M', J'), such that $\langle (M', J') \rangle = \langle (M, J) \rangle$. By Proposition 2, $\left\langle \left(\tilde{M}', \tilde{J}' \right) \right\rangle = \left\langle \left(\tilde{M}, \tilde{J} \right) \right\rangle$. M' is a union of two solid tori, and so is a lens space or $S^1 \times S^2$. Note that any cover of $S^1 \times S^2$ is simple. As Theorem $1'$ when $G = \mathbb{Z}_p$ has already been established for M', we have now established Theorem $1'$ when $G = \mathbb{Z}_p$.

7 Branched $\mathbb{Z}_{p^s}$–covers

We need to refine the last statement of Lemma 3:

Lemma 7 *If r divides $\frac{p^s \pm 1}{2}$, $\eta^{p^s} \equiv \mp \left(\frac{-2r}{p}\right)^s \eta \pmod{p}$.*

Proof We have that $\eta = -\frac{(\lambda - \lambda^{-1})}{\sqrt{2r}} i$, $\lambda^{p^s} \equiv \lambda^{\mp 1} \pmod{p}$, $i^{p^s} = \left(\frac{-1}{p}\right)^s i$, $(\sqrt{2r})^{p^s} \equiv \left(\frac{2r}{p}\right)^s \sqrt{2r} \pmod{p}$. $\square$

We need the following which follows from Proposition 1, Lemma 3, and Lemma 7. In the notation developed at the end of section 4 , let $\omega = \eta \sum_{i=o}^{r-2} \Delta_i S_i$.

Proposition 3 *If r divides $\frac{p^s \pm 1}{2}$, $\tilde{\omega} \equiv \mp \left(\frac{-2r}{p}\right)^s \eta \sum_{i=o}^{r-2} \Delta_i \tilde{S}_i \pmod{p}$.*

7.1 Proof of Theorem 2 for G cyclic

Suppose M is a 3–manifold, L is a link in this 3–manifold and there exists a homomorphism ϕ: $H_1(M-L) \to \mathbb{Z}_{p^s}$ which sends each meridian of L to a unit of $\mathbb{Z}_{p^s}$. Then we may form a branched cover $\tilde{M}$ of M branched along L. Every semi-free $\mathbb{Z}_{p^s}$ action on an oriented manifold arises in this way. Then we may pick some parallel curve to each component of L whose homology class maps to zero. Perform integral surgery to M along L with framing given by these parallel curves, to form P. Then we may complete the regular unbranched cover of $M-L$ given by ϕ to a regular unbranched cover $\tilde{P}$ of P. If we then do surgery to P along an original meridian (with the framing given by a parallel meridian) of each component of L, we recover M. Similarly if we do surgery to $\tilde{P}$ along the inverse images of these meridians of L then we recover $\tilde{M}$. Note that the inverse image of each meridian of L is a single component in $\tilde{M}$ and $\tilde{P}$. We give M a p_1–structure with σ invariant zero. Then P receives a p_1–structure as the result of p_1–surgery on M [2; page 925]. $\tilde{P}$ receives p_1–structure as the cover of P. $\tilde{M}$ receives a p_1–structure as the result of p_1–surgery on $\tilde{P}$.

Now let J denote colored fat graph in M disjoint from L. Now let $\mathcal{J}^+$ denote the linear combination of colored fat graphs in P given by J together the result of replacing the meridians of L by ω. As usual in this subject, the union of linear combinations is taken to be the linear combination obtained by expanding multilinearly. Then $\langle(M,J)\rangle = \langle(P,\mathcal{J}^+)\rangle$, by [2]. By Theorem 1, $\left\langle\left(\tilde{P},\tilde{\mathcal{J}^+}\right)\right\rangle \equiv \langle(P,\mathcal{J}^+)\rangle^{p^s} \pmod p$. Using Proposition 3 and by [2], we have $\left\langle\left(\tilde{M},\tilde{J}\right)\right\rangle = \left(\mp\left(\frac{-2r}{p}\right)^s\right)^{\beta_0(L)}\left\langle\left(\tilde{P},\tilde{\mathcal{J}^+}\right)\right\rangle$. As some power of κ^3 is minus one, and changing the p_1–structure on $\tilde{M}$, has the effect of multiplying $\langle(M,J)\rangle$ by a power of κ, this yields Theorem 2 for semifree actions of cyclic groups. □

7.2 Proof of Theorem 2 for general p–groups.

Now we assume r divides $\frac{p\pm1}{2}$. Thus we have the congruence for every $\mathbb{Z}_p$ action by 7.1. However we can write the projection from $\tilde{M}$ to M as a sequence of quotients of $\mathbb{Z}_p$ actions. □

7.3 Proof of Theorem 3

In the argument of 7.1, if M is a homology sphere, and L is a knot K, then the longitude of K maps to zero under ϕ, and P is obtained by zero framed surgery along L. $\tilde{M}$ is a rational homology sphere, and $\tilde{P}$ is obtained by zero framed surgery along the lift of K. The trace of both surgeries have signature zero. If we give M a p_1–structure with σ invariant zero, then P also has a p_1–structure with σ invariant zero. Also $\tilde{M}$ and $\tilde{P}$ have p_1–structures with the same σ invariant. $\tilde{P}$ has a p_1–structure with σ invariant 3 def $(\tilde{P} \to P)$ by Lemma 4, but this is $-3\sigma_{p^s}(K)$. Thus $(w_r(M,J))^{p^s} = \langle (M,J)\rangle^{p^s} \equiv \mp\left(\frac{-2r}{p}\right)^s \left\langle \left(\tilde{M},\tilde{J}\right)\right\rangle = \mp(\left(\frac{-2r}{p}\right))^s \kappa^{3\sigma_{p^s}(K)} w_r(\tilde{M},\tilde{J})$, modulo p. This proves Theorem 3. □

7.4 Proof of Corollary 1

We obtain Corollary 1 from Theorem 3 by taking M to be S^3, K to be the unknot and J to be L colored one with the framing given by a Seifert surface for J. Using [6; page 7], one has that

$$w_r(M,J) = \eta\ \langle L\rangle_{A=-\xi_{4r}} = \eta\ V_L\left(\xi_r^{-1}\right)\left(-\xi_{2r} - \xi_{2r}^{-1}\right).$$

In evaluating the Jones polynomial at ξ_r^{-1}, we choose $\sqrt{\xi_r^{-1}}$ to be ξ_{2r}^{-1} (for the time being.) Since the induced framing of $\tilde{L}$ is also given by the Seifert surface which is the inverse image of the Seifert surface for L,

$$w_r\left(\tilde{M},\tilde{J}\right) = \eta\ \left\langle \tilde{L}\right\rangle_{A=-\xi_{4r}} = \eta\ V_{\tilde{L}}\left(\xi_r^{-1}\right)\left(-\xi_{2r} - \xi_{2r}^{-1}\right).$$

By Theorem 3, Lemma 3 and Lemma 7, $V_{\tilde{L}}(\xi_r^{-1}) \equiv \left(V_L(\xi_r^{-1})\right)^{p^s} \pmod p$. This means that the difference is p times an algebraic integer. $V_L(t)$ is a Laurent polynomial in $\sqrt{t}$ with integer coefficients. We have that

$$\left(\sqrt{\xi_r^{-1}}\right)^{p^s} = \xi_{2r}^{-p^s} = \xi_{2r}^{\pm 1} = \sqrt{\xi_r^{\pm 1}}$$

Thus:

$$\left(V_L(\xi_r^{-1})\right)^{p^s} \equiv V_L(\xi_r^{\pm 1}) \pmod p.$$

and so

$$V_{\tilde{L}}(\xi_r^{-1}) \equiv V_L(\xi_r^{\pm 1}) \pmod p.$$

As all primitive $2r$th roots of unity are conjugate over $\mathbb{Z}$,

$$V_{\tilde{L}}(\zeta^{-1}) \equiv V_L(\zeta^{\pm 1}) \pmod p$$

holds if ζ is any primitive rth root such that r divides $\frac{p^s \pm 1}{2}$, and $r > 2$. We must choose the same $\sqrt{\zeta}$ when evaluating both sides.

For any link $\mathcal{L}$, let $\#(L)$ denote the number of components of $\mathcal{L}$. One has that $V_{\mathcal{L}}(1) = (-2)^{\#(L)-1}$. One has that $\#(\tilde{L}) \equiv \#(L) \pmod{p-1}$. So $V_{\tilde{L}}(1) \equiv V_L(1) \pmod p$. Thus the stated congruence holds if ζ is any $\frac{p^s \pm 1}{2}$-th root of unity, and $\zeta \neq -1$. □

7.5 Proof of Corollary 2

It suffices to prove the result for $n = p^s$ where $p^s \equiv \pm 1 \pmod 8$. We apply Corollary 1 with $\zeta = i$ and choose $\sqrt{i} = e^{\frac{10\pi i}{8}}$. By H. Murakami [12], for any proper link $V_{\mathcal{L}}(i) = (-1)^{\mathrm{Arf}(\mathcal{L})}(\sqrt{2})^{\#(L)-1}$, with the above choice of $\sqrt{i}$. Since 2 is a square modulo p, $\sqrt{2}^{p-1} = 1$. Thus $(\sqrt{2})^{\#(L)-1} \equiv (\sqrt{2})^{\#(\tilde{L})-1} \pmod p$. Thus we conclude $(-1)^{\mathrm{Arf}(L)} = (-1)^{\mathrm{Arf}(\tilde{L})} \pmod p$. Therefore $\mathrm{Arf}(L) \equiv \mathrm{Arf}(\tilde{L}) \pmod 2$. □

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