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Foliation Cones; A Correction

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Abstract The proof of [1, Lemma 3.6] was incorrect. Happily, a correct proof can be given.

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The goal of [1, Section 3] was to prove that Theorem 1.1 (the main result of the paper) holds for arbitrary sutured, depth–one foliated manifolds M provided that it holds for completely reduced ones such that $\partial_\tau M$ has no toral or annular components. This was essential for the application of the Handel–Miller theory in Section 5. The proof was by induction on the cardinality r of a maximal reducing family $\{T_1, \dots, T_r\}$ for $(M, \partial_\tau M)$ [1, Definition 3.2]. This number will be called the "reducing rank".

Lemma 1 *At each step of the induction, no generality is lost by assuming that each component of $\partial_\tau M$ is neither a torus nor an annulus.*

This is essentially Lemma 3.6 of [1] and was proven by pointing out that, in either case, a small perturbation makes $\mathcal{F}$ transverse to the toral or annular component N of $\partial_\tau M$ and does not increase the reducing rank. In the case that N is a torus, this simply makes N a component of $\partial_\pitchfork M$ without changing the foliated cohomology class. There is no problem here as the resulting foliated manifold remains tautly foliated and sutured. In the case of an annulus, however, the resulting foliation is not taut and $(M, \gamma) = (M, \partial_\pitchfork M)$ does not satisfy the definition of a sutured manifold. The correct strategy for avoiding annular components of $\partial_\tau M$ depends on a strengthening of Lemma 3.5 of [1].

Recall that, in the statement of Lemma 3.5, a connected manifold M' is obtained from M by excising the interior of a normal neighborhood

$$N(T) = T \times [-1, 1]$$

of a reducing surface $T \subset M$, T being a properly imbedded torus or annulus. When T is an annulus, it is required that one component of ∂T lie on an inwardly oriented component of $\partial_\tau M$, the other on an outwardly oriented one. The strengthened version of Lemma 3.5 follows.

Lemma 2 *If M' is connected, the conclusion of* Theorem 1.1 *holds for M' if and only if it holds for M.*

The "only if" part of this lemma is the content of [1, Lemma 3.5]. The proof was omitted, but here we will give the complete proof of Lemma 2. First we use it as follows.

Proof of Lemma 1 The only problem is in the case that a component N of $\partial_\tau M$ is an annulus. In this case, N is flanked by two annular sutures (components of $\partial_\pitchfork M$) on which $\mathcal{F}$ induces the foliation by spirals. Gluing these together so as to match the foliations produces a tautly foliated, sutured manifold $(M^*, \mathcal{F}^*)$, having a toral component of $\partial_\tau M^*$ and one new properly imbedded annular reducing surface T_{r+1}. The maximal reducing family is now $\{T_1, \ldots, T_r, T_{r+1}\}$. Furthermore, $(M, \mathcal{F})$ is obtained from $(M^*, \mathcal{F}^*)$ by removing an open normal neighborhood of T_{r+1}. By Lemma 2, Theorem 1.1 will hold for M if it holds for M^*. If we perturb $\mathcal{F}^*$ to be transverse to the new toral component of $\partial_\tau M^*$, the annulus T_{r+1} is no longer a reducing surface. Any of the T_j, $1 \le j \le r$, with one boundary on N will no longer be a reducing surface, and no additional reducing surfaces will have been introduced. Hence, the maximal reducing family in the new foliated manifold is a (possibly proper) subset of $\{T_1, \ldots, T_r\}$, but the tangential boundary has one less annular component. Finitely many repetitions of this procedure removes all annular components but does not affect the induction on the reducing rank. □

The proof of Lemma 2 will use the Mayer–Vietoris sequence

$$H^1(M;\mathbb{R}) \xrightarrow{i} H^1(M';\mathbb{R}) \oplus H^1(N(T);\mathbb{R}) \xrightarrow{j} H^1(N_-;\mathbb{R}) \oplus H^1(N_+;\mathbb{R}), \qquad (*)$$

where

$$N_- = T \times [-1, -1/2]$$
$$N_+ = T \times [1/2, 1].$$

Denote by $\mathcal{O}(M)$ and $\mathcal{O}(M')$ the open sets of foliated classes in $H^1(M;\mathbb{R})$ and $H^1(M';\mathbb{R})$ respectively. Hereafter, we will omit the coefficient field $\mathbb{R}$ from the notation for homology and cohomology.

Claim 1 *The Poincaré dual* $[\alpha_T]$ *of* $[T] \in H^2(M, \partial M)$ *spans* $\ker i = \ker \varphi$, *where* $\varphi : H^1(M) \to H^1(M')$ *is the restriction map.*

Proof Examining the degree 0 terms of the Mayer–Vietoris sequence reveals that i has 1–dimensional kernel. The Poincaré dual of $[T]$ is represented by a closed 1–form α_T, chosen to have compact support in $T \times (-1/2, 1/2)$. Since T does not separate M, $0 \neq [\alpha_T] \in H^1(M;\mathbb{R})$. Since T does separate $N(T)$, $(0,0) = ([\alpha_T|M'], [\alpha_T|N(T)])$ and the assertion for $\ker i$ follows. If

$$p : H^1(M';) \oplus H^1(N(T);) \to H^1(M';)$$

is projection onto the first summand, we write $\varphi = p \circ i$ and prove that p is one–to–one on $\operatorname{im} i$. By exactness of $(*)$, $\operatorname{im} i = \ker j$, so an element of $\operatorname{im} i$ annihilated by p must be of the form $(0, [\eta])$, where

$$0 = j(0, [\eta]) = (-[\eta|N_+], -[\eta|N_-]).$$

Since the inclusions $N_\pm \hookrightarrow N(T)$ are homotopy equivalences, $[\eta] = 0$. □

Claim 2 *For* $\varphi : H^1(M) \to H^1(M')$ *as above,*

$$\varphi(\mathcal{O}(M)) = \mathcal{O}(M')$$
$$\varphi^{-1}(\mathcal{O}(M')) = \mathcal{O}(M).$$

Proof If $[\omega] \in \mathcal{O}(M)$, we assume that ω is a foliated form and let $\mathcal{F}$ be the corresponding foliation. By a theorem of Roussarie and Thurston [4, 5, 2], an isotopy moves $T \times \{\pm 1\}$ to a position everywhere transverse to $\mathcal{F}$ (fixing $\partial T \times \{\pm 1\}$ in the case that T is an annulus). Thus, $[\omega|M'] = \varphi[\omega]$ is a foliated class.

Now let $[\omega'] \in \mathcal{O}(M')$, where ω' is a foliated form. Let $\mathcal{F}'$ be the foliation defined by ω'. Since $T_+ = T \times \{1\}$ and $T_- = T \times \{-1\}$ are homologous in M', we see that the forms $\omega'|T_\pm$ induce cohomologous forms on T. If T is an annulus, the foliations $\mathcal{F}|T_+$ and $\mathcal{F}|T_-$ are either both product foliations or both foliations by spirals. In either case, $\mathcal{F}'$ extends across $N \times [-1,1]$ to provide a taut foliation of M with foliated class $[\omega]$ such that $\varphi[\omega] = [\omega']$. In the case that T is a torus, the theorem of Laudenbach and Blank [3] implies that, after an isotopy in M' supported near $T_\pm$, $\mathcal{F}'$ again extends across $N \times [-1,1]$

and provides the foliated class $[\omega]$ such that $\varphi[\omega] = [\omega']$. Together with the previous paragraph, this proves that

$$\varphi(\mathcal{O}(M)) = \mathcal{O}(M')$$
$$\mathcal{O}(M) \subseteq \varphi^{-1}(\mathcal{O}(M')).$$

Finally we must show that, if $\varphi[\omega] = [\omega'] \in \mathcal{O}(M')$, then $[\omega] \in \mathcal{O}(M)$. Write $i[\omega] = ([\omega'], [\eta])$ and choose the representative form ω' to be foliated. After an isotopy, we can also assume that $T \times \{1\}$ is transverse to ω'. Exactness of the sequence $(*)$ implies that ω' and η restrict to cohomologous forms on $N_\pm$. Replacing η with a cohomologous form $\eta + df$, we can assume without loss of generality that $\eta|N_+ = \omega'|N_+$. By an isotopy within $N(T)$, we compress $N(T)$ into a neighborhood of N_+ where η is nonsingular, keeping N_+ itself pointwise fixed and carrying $T \times \{-1\}$ to a position transverse to η. Reversing this isotopy, we see that no generality is lost in assuming that η is a foliated form on $N(T)$. As above, there is an isotopy of η, supported in a neighborhood of N_- in $N(T)$, to a form η' agreeing with ω' on $N_\pm$. Then ω' and η' assemble to a foliated form $\widetilde{\omega}$ on M and

$$i([\widetilde{\omega}]) = ([\omega'], [\eta']) = i([\omega]).$$

By Claim 1, $[\omega] = [\widetilde{\omega}] + c[\alpha_T]$, where $c \in \mathbb{R}$ and the class $[\alpha_T]$ is Poincaré dual to $[T] = [T \times \{-1\}]$. Since $\widetilde{\omega}$ is transverse to $T \times \{-1\}$, we can choose α_T to be compactly supported near $T \times \{-1\}$ and to vanish identically on a vector field v (on $M \smallsetminus \partial_\tau M$) such that $\widetilde{\omega}(v) > 0$ everywhere. That is, the closed form $\widehat{\omega} = \widetilde{\omega} + c\alpha_T$ is nonsingular. Also, α_T is bounded, implying that $\widehat{\omega}$ also blows up nicely at $\partial_\tau M$ and $[\omega] = [\widehat{\omega}]$ is a foliated class. □

Let $\ell_T \subset H^1(M)$ be the one–dimensional subspace spanned by the class $[\alpha_T]$.

Claim 3 *If U is a connected component of $\mathcal{O}(M)$, then $\ell_T \subset \overline{U}$.*

Proof Indeed, let ω be a foliated form representing an element of U. As in the previous proof, α_T can be chosen to vanish identically on a vector field v such that $\omega(v) > 0$ everywhere. Thus, for each $c[\alpha_t] \in \ell_T$, the line segment

$$t[\omega] + (1-t)c[\alpha_T], \quad 0 \le t \le 1,$$

connects $c[\alpha_T]$ to $[\omega]$ and lies in U for $t > 0$. □

Proof of Lemma 2 First we eliminate trivial cases. By Claim 2, $0 \in \mathcal{O}(M)$ if and only if $0 \in \mathcal{O}(M')$. By [1, Proposition 3.7], 0 is a foliated class if and only if the manifold is a product $S \times I$ of a compact surface S and a compact interval I. In turn, this is the case if and only if the entire cohomology space is the unique foliation cone and satisfies Theorem 1.1 of [1] trivially. Thus, we assume that neither M nor M' is a product. Claim 2 also allows us to assume that neither $\mathcal{O}(M)$ nor $\mathcal{O}(M')$ are empty.

Since $\mathcal{O}(M')$ is open in the vector space $H^1(M')$, Claim 2 implies that the linear map

$$\varphi : H^1(M) \to H^1(M')$$

is surjective. If Theorem 1.1 holds on M', we can use this surjection to pull the cone structure back to $H^1(M)$ and use Claim 2 to verify that Theorem 1.1 holds on M. For the converse, suppose the theorem holds on M. If $\mathcal{C} \subset H^1(M)$ is a foliation cone, the fact that it is neither empty nor the entire vector space implies that it is defined by a finite set of nontrivial linear inequalities $\theta_i \geq 0$, $1 \leq i \leq q$. By Claim 3, $\ell_T \subset \mathcal{C}$, hence $\theta_i|\ell_T \geq 0$ and this implies that $\theta_i|\ell_T \equiv 0$. By Claim 1, the linear functionals θ_i pass to nontrivial linear functionals $\widetilde{\theta}_i$ on $H^1(M')$. The convex, polyhedral cone $\mathcal{C}'$ defined by the linear inequalities $\widetilde{\theta}_i \geq 0$ is precisely the image of $\mathcal{C}$ under φ and has $\mathcal{C}$ as its entire pre–image. By Claim 2, Theorem 1.1 follows easily for M'. $\square$

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