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14. Explicit abelian extensions of complete discrete valuation fields

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14.0

For higher class field theory Witt and Kummer extensions are very important. In fact, Parshin's construction of class field theory for higher local fields of prime characteristic [P] is based on an explicit (Artin–Schreier–Witt) pairing; see [F] for a generalization to the case of a perfect residue field. Kummer extensions in the mixed characteristic case can be described by using class field theory and Vostokov's symbol [V1], [V2]; for a perfect residue field, see [V3], [F].

An explicit description of non Kummer abelian extensions for a complete discrete valuation field K of characteristic 0 with residue field k_K of prime characteristic p is an open problem. We are interested in totally ramified extensions, and, therefore, in p-extensions (tame totally ramified abelian extensions are always Kummer and their class field theory can be described by means of the higher tame symbol defined in subsection 6.4.2).

In the case of an absolutely unramified K there is a beautiful description of all abelian totally ramified p-extensions in terms of Witt vectors over k_K by Kurihara (see section 13 and [K]). Below we give another construction of some totally ramified cyclic p-extensions for such K. The construction is complicated; however, the extensions under consideration are constructed explicitly, and eventually we obtain a certain description of the whole maximal abelian extension of K. Proofs are given in [VZ].

14.1

We recall that cyclic extensions of K of degree p can be described by means of Artin–Schreier extensions, see [FV, III.2]. Namely, for a cyclic L/K of degree p we have $L = K(x)$, $x^p - x = a$, where $v_K(a) = -1$ if L/K is totally ramified, and $v_K(a) = 0$ if L/K is unramified.

Notice that if $v_K(a_1 - a_2) \geqslant 0$, then for corresponding cyclic extensions L_1/K and L_2/K we have $L_1 K_{\mathrm{ur}} = L_2 K_{\mathrm{ur}}$. (If $v_K(a_1 - a_2) \geqslant 1$, then, moreover, $L_1 = L_2$.) We obtain immediately the following description of the maximal abelian extension of K of exponent p: $K^{\mathrm{ab},p} = K_{\mathrm{ur}}^{\mathrm{ab},p} \prod_d K_d$, where $K_d = K(x)$, $x^p - x = -p^{-1}d$, and d runs over any fixed system of representatives of k_K^* in $\mathcal{O}_K$. This is a part of a more precise statement at the end of the next subsection.

14.2

It is easy to determine whether a given cyclic extension L/K of degree p can be embedded into a cyclic extension of degree p^n, $n \geqslant 2$.

Proposition. *In the above notation, let b be the residue of pa in k_K. Then there is a cyclic extension M/K of degree p^n such that $L \subset M$ if and only if $b \in {k_K}^{p^{n-1}}$.*

The proof is based on the following theorem of Miki [M]. Let F be a field of characteristic not equal to p and let $\zeta_p \in F$. Let $L = F(\alpha)$, $\alpha^p = a \in F$. Then $a \in F^{*p} N_{F(\zeta_{p^n})/F} F(\zeta_{p^n})^*$ if and only if there is a cyclic extension M/F of degree p^n such that $L \subset M$.

Corollary. *Denote by K^{ab,p^n} (respectively $K_{\mathrm{ur}}^{\mathrm{ab},p^n}$) the maximal abelian (respectively abelian unramified) extension of K of exponent p^n. Choose $A_i \subset \mathcal{O}_K$, $1 \leqslant i \leqslant n$, in such a way that $\{\bar{d} : d \in A_i\}$ is an $\mathbb{F}_p$-basis of $k_K^{p^{i-1}}/k_K^{p^i}$ for $i \leqslant n-1$ and an $\mathbb{F}_p$-basis of $k_K^{p^{n-1}}$ for $i = n$. Let $K_{i,d}$ ($d \in A_i$) be any cyclic extension of degree p^i that contains x with $x^p - x = -p^{-1}d$. Then $K^{\mathrm{ab},p^n}/K$ is the compositum of linearly disjoint extensions $K_{i,d}/K$ ($1 \leqslant i \leqslant n$; d runs over A_i) and $K_{\mathrm{ur}}^{\mathrm{ab},p^n}/K$.*

From now on, let $p > 3$. For any $n \geqslant 1$ and any $b \in k_K^{p^{n-1}}$, we shall give a construction of a cyclic extension $K_{n,d}/K$ of degree p^n such that $x \in K_{n,d}$, $x^p - x = -p^{-1}d$, where $d \in \mathcal{O}_K$ is such that its residue $\bar{d}$ is equal to b.

14.3

Denote by G the Lubin–Tate formal group over $\mathbb{Z}_p$ such that multiplication by p in it takes the form $[p]_G(X) = pX + X^p$.

Let $\mathcal{O}$ be the ring of integers of the field E defined in (2) of Theorem 13.2, and v the valuation on E.

Proposition. *There exist* $g_i \in \mathcal{O}$, $i \in \mathbb{Z}$, *and* $R_i \in \mathcal{O}$, $i \geqslant 0$, *satisfying the following conditions.*

(1) $g_0 \equiv 1 \mod p\mathcal{O}$, $g_i \equiv 0 \mod p\mathcal{O}$ *for* $i \neq 0$.

(2) $R_0 = T$.

(3) $v(g_i) \geqslant -i + 2 + \left[\frac{i}{p}\right] + \left[\frac{i-2}{p}\right]$ *for* $i \leqslant -1$.

(4) *Let* $g(X) = \sum_{-\infty}^{\infty} g_i X^{i(p-1)+1}$, $R(X,T) = \sum_{i=0}^{\infty} R_i X^{i(p-1)+1}$. *Then*

$$g(X) +_G [p]_G R(g(X),T) = g(X +_G R([p]_G X, T^p)).$$

Remark. We do not expect that the above conditions determine g_i and R_i uniquely. However, in [VZ] a certain canonical way to construct (g, R) by a process of the p-adic approximation is given.

Fix a system (g, R) satisfying the above conditions. Denote

$$S = \sum_{i=0}^{\infty} S_i(T) X^{i(p-1)+1} = T^{-1}X + \dots$$

the series which is inverse to R with respect to substitution in $\mathcal{O}[[X]]$.

Theorem. *Let* $d \in \mathcal{O}_K^*$. *Consider* $\beta_1, \dots, \beta_n \in K^{\mathrm{sep}}$ *such that*

$$\beta_1^p - \beta_1 = -p^{-1} \sum_{i \geqslant 0} S_i(d^{p^{n-1}})(-p)^i,$$

$$\beta_j^p - \beta_j = -p^{-1} \sum_{-\infty}^{+\infty} g_i(d^{p^{n-j}})(-p)^i \beta_{j-1}^{i(p-1)+1}, \quad j \geqslant 2.$$

Then $K_{n,d^{p^{n-1}}} = K(\beta_1, \dots, \beta_n)$ *is a cyclic extension of* K *of degree* p^n *containing a zero of the polynomial* $X^p - X + p^{-1}d^{p^{n-1}}$.

Remark. We do not know which Witt vector corresponds to $K_{n,d^{p^{n-1}}}/K$ in Kurihara's theory (cf. section 13). However, one could try to construct a parallel theory in which (the canonical character of) this extension would correspond to $(\bar{d}^{p^{n-1}}, 0, 0, \dots) \in W_n(k_K)$.

14.4

If one is interested in explicit equations for abelian extensions of K of exponent p^n for a fixed n, then it is sufficient to compute a certain p-adic approximation to g (resp. R) by polynomials in $\mathbb{Z}_{(p)}[T, T^{-1}, X, X^{-1}]$ (resp. $\mathbb{Z}_{(p)}[T, T^{-1}, X]$). Let us make this statement more precise.

In what follows we consider a fixed pair (g,R) constructed in [VZ]. Denote

$$K_{j,d^{p^{n-1}}} = K(\beta_1, \dots, \beta_j).$$

Let v be the (non-normalized) extension of the valuation of K to $K_{n,d^{p^{n-1}}}$. Then $v(\beta_j) = -p^{-1} - \dots - p^{-j}$, $j = 1, \dots, n$.

We assert that in the defining equations for $K_{n,d^{p^{n-1}}}$ the pair (g,R) can be replaced with $(\widetilde{g}, \widetilde{R})$ such that

(1) $v(\widetilde{g}_i - g_i) > n + \max_{j=1,\dots,n-1}(-j - i \cdot p^{-j} + p^{-1} + p^{-2} + \dots + p^{-j}), \quad i \in \mathbb{Z}$,

and

(2) $$v(\widetilde{R}_i - R_i) > n - i, \quad i \geqslant 0.$$

Theorem. *Assume that the pair* $(\widetilde{g}, \widetilde{R})$ *satisfies* (1) *and* (2)*. Define* $\widetilde{S}$ *as* $\widetilde{R}^{-1}$*. Let*

$$\widetilde{\beta}_1^p - \widetilde{\beta}_1 = -p^{-1} \sum_{i \geqslant 0} \widetilde{S}_i(d^{p^{n-1}})(-p)^i,$$

$$\widetilde{\beta}_j^p - \widetilde{\beta}_j = -p^{-1} \sum_{-\infty}^{+\infty} \widetilde{g}_i(d^{p^{n-j}})(-p)^i \widetilde{\beta}_{j-1}^{i(p-1)+1}, \quad j \geqslant 2.$$

Then $K(\widetilde{\beta}_1, \dots, \widetilde{\beta}_n) = K(\beta_1, \dots, \beta_n)$.

Proof. It is easy to check by induction on j that $\widetilde{\beta}_j \in K_{j,d^{p^{n-1}}}$ and $v(\widetilde{\beta}_j - \beta_j) > n - j$, $j = 1, \dots, n$.

Remark. For a fixed n, one may take $\widetilde{R}_i = 0$ for $i \geqslant n$, $\widetilde{g}_i = 0$ for all sufficiently small or sufficiently large i.

14.5

If we consider non-strict inequalities in (1) and (2), then we obtain an extension $\widetilde{K}_{n,d^{p^{n-1}}}$ such that $\widetilde{K}_{n,d^{p^{n-1}}} K_{\mathrm{ur}} = K_{n,d^{p^{n-1}}} K_{\mathrm{ur}}$. In particular, let $n = 2$. Calculation of (R,g) in [VZ] shows that

$$g_i \overset{p^2\mathcal{O}}{\equiv} \begin{cases} 0, & i < -1 \\ p \cdot \frac{T^{1-p}-1}{2}, & i = -1 \\ 1 + p \cdot \frac{T^{1-p}-1}{2}(1 - T^p), & i = 0 \end{cases}$$

Therefore, one may take $\widetilde{g}_i = 0$ for $i < -1$ or $i > 0$, $\widetilde{g}_{-1} = p \cdot \frac{T^{1-p}-1}{2}$, $\widetilde{g}_0 = 1 + p \cdot \frac{T^{1-p}-1}{2}(1 - T^p)$. Further, one may take $\widetilde{R} = TX$. Thus, we obtain the following

Theorem. *For any* $d \in \mathcal{O}_K^*$, *let* $\widetilde{K}_{1,d} = K(y)$, *where* $y^p - y = -p^{-1}d$. *Next, let* $\widetilde{K}_{2,d^p} = K(y_1, y_2)$, *where*

$$y_1^p - y_1 = -p^{-1}d^p,$$
$$y_2^p - y_2 = -p^{-1}y_1 + p^{-1} \cdot \frac{d^{1-p}-1}{2} y_1^{-p+2} - \frac{d^{1-p}-1}{2}(1-d^p)y_1.$$

Then

1. *All* $\widetilde{K}_{1,d}/K$ *are cyclic of degree* p, *and all* $\widetilde{K}_{2,d^p}/K$ *are cyclic of degree* p^2.
2. $K^{\mathrm{ab},p^2}/K$ *is the compositum of linearly disjoint extensions described below:*
   (a) $\widetilde{K}_{1,d}/K$, *where* d *runs over a system of representatives of an* $\mathbb{F}_p$*-basis of* k_K/k_K^p;
   (b) $\widetilde{K}_{2,d^p}/K$, *where* d *runs over a system of representatives of an* $\mathbb{F}_p$*-basis of* k_K;
   (c) $K_{\mathrm{ur}}^{\mathrm{ab},p^2}/K$.

14.6

One of the goals of developing explicit constructions for abelian extensions would be to write down explicit formulas for class field theory. We are very far from this goal in the case of non Kummer extensions of an absolutely unramified higher local field. However, the K-group involved in the reciprocity map can be computed for such fields in a totally explicit way.

Let K be an absolutely unramified n-dimensional local field with any perfect residue field. Then [Z, §11] gives an explicit description of

$$U(1)K_n^{\mathrm{top}}K = \langle \{\alpha, \beta_1, \ldots, \beta_{n-1}\} : \alpha, \beta_i \in K^*, v(\alpha - 1) > 0 \rangle.$$

Notice that the structure of $K_n^{\mathrm{top}}K/U(1)K_n^{\mathrm{top}}K$, i.e., the quotient group responsible for tamely ramified extensions, is well known. We cite here a result in the simplest possible case $K = \mathbb{Q}_p\{\{t\}\}$.

Theorem. *Let* $K = \mathbb{Q}_p\{\{t\}\}$.

1. *For every* $\alpha \in U_1K_2^{\mathrm{top}}(K)$ *there are* $n_j \in \mathbb{Z}_p$, $j \in \mathbb{Z} \setminus \{0\}$ *which are uniquely determined modulo* $p^{v_{\mathbb{Q}_p}(j)+1}$ *and there is* $n_0 \in \mathbb{Z}_p$ *which is uniquely determined such that*
$$\alpha = \sum_j n_j \{1 - pt^j, t\}.$$
2. *For any* $j \neq 0$ *we have*
$$p^{v_{\mathbb{Q}_p}(j)+1}\{1 - pt^j, t\} = 0.$$

Proof. Use explicit class field theory of section 10 and the above mentioned theorem of Miki.

Question. How does $\{1 - pt^j, t\}$ act on $K_{n,d^{p^{n-1}}}$?

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