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16. Higher class field theory without using K-groups

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Let F be a complete discrete valuation field with residue field $k = k_F$ of characteristic p. In this section we discuss an alternative to higher local class field theory method which describes abelian totally ramified extensions of F without using K-groups. For n-dimensional local fields this gives a description of abelian totally ramified (with respect to the discrete valuation of rank one) extensions of F. Applications are sketched in 16.3 and 16.4.

16.1. p-class field theory

Suppose that k is perfect and $k \neq \wp(k)$ where $\wp\colon k \to k$, $\wp(a) = a^p - a$.

Let $\widetilde{F}$ be the maximal abelian unramified p-extension of F. Then due to Witt theory $\mathrm{Gal}(\widetilde{F}/F)$ is isomorphic to $\prod_\kappa \mathbb{Z}_p$ where $\kappa = \dim_{\mathbb{F}_p} k/\wp(k)$. The isomorphism is non-canonical unless k is finite where the canonical one is given by $\mathrm{Frob}_F \mapsto 1$.

Let L be a totally ramified Galois p-extension of F.

Let $\mathrm{Gal}(\widetilde{F}/F)$ act trivially on $\mathrm{Gal}(L/F)$.

Denote

$$\mathrm{Gal}(L/F)^\sim = H^1_{\mathrm{cont}}((\mathrm{Gal}(\widetilde{F}/F), \mathrm{Gal}(L/F)) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(\widetilde{F}/F), \mathrm{Gal}(L/F)).$$

Then $\mathrm{Gal}(L/F)^\sim \simeq \oplus_\kappa \mathrm{Gal}(L/F)$ non-canonically.

Put $\widetilde{L} = L\widetilde{F}$. Denote by $\varphi \in \mathrm{Gal}(\widetilde{L}/L)$ the lifting of $\varphi \in \mathrm{Gal}(\widetilde{F}/F)$.

For $\chi \in \mathrm{Gal}(L/F)^\sim$ denote

$$\Sigma_\chi = \{\alpha \in \widetilde{L} : \alpha^{\varphi\chi(\varphi)} = \alpha \quad \text{for all } \varphi \in \mathrm{Gal}(\widetilde{F}/F)\}.$$

The extension Σ_χ/F is totally ramified.

As an generalization of Neukirch's approach [N] introduce the following:

Definition. Put

$$\Upsilon_{L/F}(\chi) = N_{\Sigma_\chi/F}\pi_\chi / N_{L/F}\pi_L \mod N_{L/F}U_L$$

where π_χ is a prime element of Σ_χ and π_L is a prime element of L.

This map is well defined. Compare with 10.1.

Theorem ([F1, Th. 1.7]). *The map $\Upsilon_{L/F}$ is a homomorphism and it induces an isomorphism*

$$\mathrm{Gal}(L\cap F^{\mathrm{ab}}/F)^\sim \xrightarrow{\sim} U_F/N_{L/F}U_L \xrightarrow{\sim} U_{1,F}/N_{L/F}U_{1,L}.$$

Proof. One of the easiest ways to prove the theorem is to define and use the map which goes in the reverse direction. For details see [F1, sect. 1]. □

Problem. If π is a prime element of F, then p-class field theory implies that there is a totally ramified abelian p-extension F_π of F such that $F_\pi\widetilde{F}$ coincides with the maximal abelian p-extension of F and $\pi\in N_{F_\pi/F}F_\pi^*$. Describe F_π explicitly (like Lubin–Tate theory does in the case of finite k).

Remark. Let K be an n-dimensional local field ($K=K_n,\ \dots,,\ K_0$) with K_0 satisfying the same restrictions as k above.

For a totally ramified Galois p-extension L/K (for the definition of a totally ramified extension see 10.4) put

$$\mathrm{Gal}(L/K)^\sim = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(\widetilde{K}/K),\mathrm{Gal}(L/K))$$

where $\widetilde{K}$ is the maximal p-subextension of K_{pur}/K (for the definition of K_{pur} see (A1) of 10.1).

There is a map $\Upsilon_{L/K}$ which induces an isomorphism [F2, Th. 3.8]

$$\mathrm{Gal}(L\cap K^{\mathrm{ab}}/K)^\sim \xrightarrow{\sim} VK_n^t(K)/N_{L/K}VK_n^t(L)$$

where $VK_n^t(K)=\{V_K\}\cdot K_{n-1}^t(K)$ and K_n^t was defined in 2.0.

16.2. General abelian local p-class field theory

Now let k be an arbitrary field of characteristic p, $\wp(k)\neq k$.

Let $\widetilde{F}$ be the maximal abelian unramified p-extension of F.

Let L be a totally ramified Galois p-extension of F. Denote

$$\mathrm{Gal}(L/F)^\sim = H^1_{\mathrm{cont}}((\mathrm{Gal}(\widetilde{F}/F),\mathrm{Gal}(L/F)) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(\widetilde{F}/F),\mathrm{Gal}(L/F)).$$

In a similar way to the previous subsection define the map

$$\Upsilon_{L/F}\colon \mathrm{Gal}(L/F)^\sim \to U_{1,F}/N_{L/F}U_{1,L}.$$

In fact it lands in $U_{1,F}\cap N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}})/N_{L/F}U_{1,L}$ and we denote this new map by the same notation.

Definition. Let $\mathbf{F}$ be complete discrete valuation field such that $\mathbf{F} \supset \widetilde{F}$, $e(\mathbf{F}|\widetilde{F}) = 1$ and $k_{\mathbf{F}} = \cup_{n\geqslant 0} k_{\widetilde{F}}^{p^{-n}}$. Put $\mathbf{L} = L\mathbf{F}$.

Denote $I(L|F) = \langle \varepsilon^{\sigma-1} : \varepsilon \in U_{1,\mathbf{L}}, \sigma \in \mathrm{Gal}(L/F)\rangle \cap U_{1,\widetilde{L}}$.

Then the sequence

$$(*) \qquad 1 \to \mathrm{Gal}(L/F)^{\mathrm{ab}} \xrightarrow{g} U_{1,\widetilde{L}}/I(L|F) \xrightarrow{N_{\widetilde{L}/\widetilde{F}}} N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}} \to 1$$

is exact where $g(\sigma) = \pi_L^{\sigma-1}$ and π_L is a prime element of L (compare with Proposition 1 of 10.4.1).

Generalizing Hazewinkel's method [H] introduce

Definition. Define a homomorphism

$$\Psi_{L/F} \colon (U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}})/N_{L/F}U_{1,L} \to \mathrm{Gal}(L \cap F^{\mathrm{ab}}/F)^{\sim}, \quad \Psi_{L/F}(\varepsilon) = \chi$$

where $\chi(\varphi) = g^{-1}(\eta^{1-\varphi})$, $\eta \in U_{1,\widetilde{L}}$ is such that $\varepsilon = N_{\widetilde{L}/\widetilde{F}}\eta$.

Properties of $\Upsilon_{L/F}, \Psi_{L/F}$.

(1) $\Psi_{L/F} \circ \Upsilon_{L/F} = \mathrm{id}$ on $\mathrm{Gal}(L \cap F^{\mathrm{ab}}/F)^{\sim}$, so $\Psi_{L/F}$ is an epimorphism.

(2) Let $\mathcal{F}$ be a complete discrete valuation field such that $\mathcal{F} \supset F$, $e(\mathcal{F}|F) = 1$ and $k_{\mathcal{F}} = \cup_{n\geqslant 0} k_F^{p^{-n}}$. Put $\mathcal{L} = L\mathcal{F}$. Let

$$\lambda_{L/F} \colon (U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}})/N_{L/F}U_{1,L} \to U_{1,\mathcal{F}}/N_{\mathcal{L}/\mathcal{F}}U_{1,\mathcal{L}}$$

be induced by the embedding $F \to \mathcal{F}$. Then the diagram

$$\begin{array}{ccccc}
\mathrm{Gal}(L/F)^{\sim} & \xrightarrow{\Upsilon_{L/F}} & (U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}})/N_{L/F}U_{1,L} & \xrightarrow{\Psi_{L/F}} & \mathrm{Gal}(L \cap F^{\mathrm{ab}}/F)^{\sim} \\
\downarrow & & \lambda_{L/F}\downarrow & & \mathrm{iso}\downarrow \\
\mathrm{Gal}(\mathcal{L}/\mathcal{F})^{\sim} & \xrightarrow{\Upsilon_{\mathcal{L}/\mathcal{F}}} & U_{1,\mathcal{F}}/N_{\mathcal{L}/\mathcal{F}}U_{1,\mathcal{L}} & \xrightarrow{\Psi_{\mathcal{L}/\mathcal{F}}} & \mathrm{Gal}(\mathcal{L} \cap \mathcal{F}^{\mathrm{ab}}/\mathcal{F})^{\sim}
\end{array}$$

is commutative.

(3) Since $\Psi_{\mathcal{L}/\mathcal{F}}$ is an isomorphism (see 16.1), we deduce that $\lambda_{L/F}$ is surjective and $\ker(\Psi_{L/F}) = \ker(\lambda_{L/F})$, so

$$(U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}})/N_*(L/F) \xrightarrow{\sim} \mathrm{Gal}(L \cap F^{\mathrm{ab}}/F)^{\sim}$$

where $N_*(L/F) = U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}} \cap N_{\mathcal{L}/\mathcal{F}} U_{1,\mathcal{L}}$.

Theorem ([F3, Th. 1.9])**.** *Let L/F be a cyclic totally ramified p-extension. Then*

$$\Upsilon_{L/F} \colon \mathrm{Gal}(L/F)^{\sim} \to (U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}})/N_{L/F}U_{1,L}$$

is an isomorphism.

Proof. Since L/F is cyclic we get $I(L|F) = \{\varepsilon^{\sigma-1} : \varepsilon \in U_{1,\widetilde{L}}, \sigma \in \mathrm{Gal}(L/F)\}$, so

$$I(L|F) \cap U_{1,\widetilde{L}}^{\varphi-1} = I(L|F)^{\varphi-1}$$

for every $\varphi \in \mathrm{Gal}(\widetilde{L}/L)$.

Let $\Psi_{L/F}(\varepsilon) = 1$ for $\varepsilon = N_{\widetilde{L}/\widetilde{F}}\eta \in U_{1,F}$. Then $\eta^{\varphi-1} \in I(L|F) \cap U_{1,\widetilde{L}}^{\varphi-1}$, so $\eta \in I(L|F)L_\varphi$ where L_φ is the fixed subfield of $\widetilde{L}$ with respect to φ. Hence $\varepsilon \in N_{L_\varphi/F \cap L_\varphi} U_{1,L_\varphi}$. By induction on κ we deduce that $\varepsilon \in N_{L/F}U_{1,L}$ and $\Psi_{L/F}$ is injective. □

Remark. Miki [M] proved this theorem in a different setting which doesn't mention class field theory.

Corollary. *Let L_1/F, L_2/F be abelian totally ramified p-extensions. Assume that L_1L_2/F is totally ramified. Then*

$$N_{L_2/F}U_{1,L_2} \subset N_{L_1/F}U_{1,L_1} \iff L_2 \supset L_1.$$

Proof. Let M/F be a cyclic subextension in L_1/F. Then
$N_{\mathcal{M}/\mathcal{F}}U_{1,\mathcal{M}} \supset N_{\mathcal{L}_2/\mathcal{F}}U_{1,\mathcal{L}_2}$, so $\mathcal{M} \subset \mathcal{L}_2$ and $M \subset L_2$. Thus $L_1 \subset L_2$. □

Problem. Describe $\ker(\Psi_{L/F})$ for an arbitrary L/F. It is known [F3, 1.11] that this kernel is trivial in one of the following situations:

(1) L is the compositum of cyclic extensions M_i over F, $1 \leqslant i \leqslant m$, such that all ramification breaks of $\mathrm{Gal}(M_i/F)$ with respect to the upper numbering are not greater than every break of $\mathrm{Gal}(M_{i+1}/F)$ for all $1 \leqslant i \leqslant m-1$.

(2) $\mathrm{Gal}(L/F)$ is the product of cyclic groups of order p and a cyclic group.

No example with non-trivial kernel is known.

16.3. Norm groups

Proposition ([F3, Prop. 2.1])**.** *Let F be a complete discrete valuation field with residue field of characteristic p. Let L_1/F and L_2/F be abelian totally ramified p-extensions. Let $N_{L_1/F}L_1^* \cap N_{L_2/F}L_2^*$ contain a prime element of F. Then L_1L_2/F is totally ramified.*

Proof. If k_F is perfect, then the claim follows from p-class field theory in 16.1. If k_F is imperfect then use the fact that there is a field $\mathcal{F}$ as above which satisfies $L_1\mathcal{F} \cap L_2\mathcal{F} = (L_1 \cap L_2)\mathcal{F}$. □

Theorem (uniqueness part of the existence theorem) ([F3, Th. 2.2]). *Let $k_F \neq \wp(k_F)$. Let L_1/F, L_2/F be totally ramified abelian p-extensions. Then*

$$N_{L_2/F}L_2^* = N_{L_1/F}L_1^* \iff L_1 = L_2.$$

Proof. Use the previous proposition and corollary in 16.2. □

16.4. Norm groups more explicitly

Let F be of characteristic 0. In general if k is imperfect it is very difficult to describe $N_{L/F}U_{1,L}$. One partial case can be handled: let the absolute ramification index $e(F)$ be equal to 1 (the description below can be extended to the case of $e(F) < p-1$).

Let π be a prime element of F.

Definition.

$$\mathcal{E}_{n,\pi}\colon W_n(k_F) \to U_{1,F}/U_{1,F}^{p^n}, \quad \mathcal{E}_{n,\pi}((a_0, \dots, a_{n-1})) = \prod_{0\leqslant i\leqslant n-1} E(\widetilde{a_i}^{p^{n-i}}\pi)^{p^i}$$

where $E(X) = \exp(X + X^p/p + X^{p^2}/p^2 + \dots)$ and $\widetilde{a_i} \in \mathcal{O}_F$ is a lifting of $a_i \in k_F$ (this map is basically the same as the map ψ_n in Theorem 13.2).

The following property is easy to deduce:

Lemma. *$\mathcal{E}_{n,\pi}$ is a monomorphism. If k_F is perfect then $\mathcal{E}_{n,\pi}$ is an isomorphism.*

Theorem ([F3, Th. 3.2]). *Let $k_F \neq \wp(k_F)$ and let $e(F) = 1$. Let π be a prime element of F.*

Then cyclic totally ramified extensions L/F of degree p^n such that $\pi \in N_{L/F}L^*$ are in one-to-one correspondence with subgroups

$$\mathcal{E}_{n,\pi}\big(\mathbf{F}(w)\wp(W_n(k_F))\big)U_{1,F}^{p^n}$$

of $U_{1,F}/U_{1,F}^{p^n}$ where w runs over elements of $W_n(k_F)^*$.

Hint. Use the theorem of 16.3. If k_F is perfect, the assertion follows from p-class field theory.

Remark. The correspondence in this theorem was discovered by M. Kurihara [K, Th. 0.1], see the sequence (1) of theorem 13.2. The proof here is more elementary since it doesn't use étale vanishing cycles.

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