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18. On ramification theory of monogenic extensions

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We discuss ramification theory for finite extensions L/K of a complete discrete valuation field K. This theory deals with quantities which measure wildness of ramification, such as different, the Artin (resp. Swan) characters and the Artin (resp. Swan) conductors. When the residue field extension k_L/k_K is separable there is a complete theory, e.g. [S], but in general it is not so. In the classical case (i.e. k_L/k_K separable) proofs of many results in ramification theory use the property that all finite extensions of valuation rings $\mathcal{O}_L/\mathcal{O}_K$ are monogenic which is not the case in general. Examples (e.g. [Sp]) show that the classical theorems do not hold in general. Waiting for a beautiful and general ramification theory, we consider a class of extensions L/K which has a good ramification theory. We describe this class and we will call its elements *well ramified extensions*. All classical results are generalizable for well ramified extensions, for example a generalization of the Hasse–Arf theorem proved by J. Borger. We also concentrate our attention on other ramification invariants, more appropriate and general; in particular, we consider two ramification invariants: the Kato conductor and Hyodo depth of ramification.

Here we comment on some works on general ramification theory.

The first direction aims to generalize classical ramification invariants to the general case working with (one dimensional) rational valued invariants. In his papers de Smit gives some properties about ramification jumps and considers the different and differential [Sm2]; he generalizes the Hilbert formula by using the monogenic conductor [Sm1]. We discuss works of Kato [K3-4] in subsection 18.2. In [K2] Kato describes ramification theory for two-dimensional local fields and he proves an analogue of the Hasse–Arf theorem for those Galois extensions in which the extension of the valuation rings (with respect to the discrete valuation of rank 2) is monogenic.

The second direction aims to extend ramification invariants from one dimensional to either higher dimensional or to more complicated objects which involve differential forms (as in Kato's works [K4], [K5]). By using higher local class field theory, Hyodo [H] defines generalized ramification invariants, like depth of ramification (see Theorem

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5 below). We discuss relations of his invariants with the (one dimensional) Kato conductor in subsection 18.3 below. Zhukov [Z] generalizes the classical ramification theory to the case where $|k_K : k_K^p| = p$ (see section 17 of this volume). From the viewpoint of this section the existence of Zhukov's theory is in particular due to the fact that in the case where $|k_K : k_K^p| = p$ one can reduce various assertions to the well ramified case.

18.0. Notations and definitions

In this section we recall some general definitions. We only consider complete discrete valuation fields K with residue fields k_K of characteristic $p > 0$. We also assume that $|k_K : k_K^p|$ is finite.

Definition. Let L/K be a finite Galois extension, $G = \mathrm{Gal}(L/K)$. Let $G_0 = \mathrm{Gal}(L/L \cap K_{\mathrm{ur}})$ be the inertia subgroup of G. Define functions

$$i_G, s_G : G \to \mathbb{Z}$$

by

$$i_G(\sigma) = \begin{cases} \inf_{x \in \mathcal{O}_L \setminus \{0\}} v_L(\sigma(x) - x) & \text{if } \sigma \neq 1 \\ +\infty & \text{if } \sigma = 1 \end{cases}$$

and

$$s_G(\sigma) = \begin{cases} \inf_{x \in \mathcal{O}_L \setminus \{0\}} v_L(\sigma(x)/x - 1) & \text{if } \sigma \neq 1, \sigma \in G_0 \\ +\infty & \text{if } \sigma = 1 \\ 0 & \text{if } \sigma \notin G_0. \end{cases}$$

Then $s_G(\sigma) \leqslant i_G(\sigma) \leqslant s_G(\sigma)+1$ and if k_L/k_K is separable, then $i_G(\sigma) = s_G(\sigma)+1$ for $\sigma \in G_0$. Note that the functions i_G, s_G depend not only on the group G, but on the extension L/K; we will denote i_G also by $i_{L/K}$.

Definition. The Swan function is defined as

$$\mathrm{Sw}_G(\sigma) = \begin{cases} -|k_L : k_K| s_G(\sigma), & \text{if } \sigma \in G_0 \setminus \{1\} \\ -\sum_{\tau \in G_0 \setminus \{1\}} \mathrm{Sw}_G(\tau), & \text{if } \sigma = 1 \\ 0 & \text{if } \sigma \notin G_0. \end{cases}$$

For a character χ of G its Swan conductor

$$\mathrm{sw}(\chi) = \mathrm{sw}_G(\chi) = (\mathrm{Sw}_G, \chi) = \frac{1}{|G|} \sum_{\sigma \in G} \mathrm{Sw}_G(\sigma)\chi(\sigma) \tag{1}$$

is an integer if k_L/k_K is separable (Artin's Theorem) and is not an integer in general (e.g. [Sp, Ch. I]).

18.1. Well ramified extensions

Definition. Let L/K be a finite Galois p-extension. The extension L/K is called *well ramified* if $\mathcal{O}_L = \mathcal{O}_K[\alpha]$ for some $\alpha \in L$.

18.1.1. Structure theorem for well ramified extensions.

Definition. We say that an extension L/K is in *case I* if k_L/k_K is separable; an extension L/K is in *case II* if $|L : K| = |k_L : k_K|$ (i.e. L/K is ferociously ramified in the terminology of 17.0) and $k_L = k_K(a)$ is purely inseparable over k_K.

Extensions in case I and case II are well ramified. An extension which is simultaneously in case I and case II is the trivial extension.

We characterize well ramified extensions by means of the function i_G in the following theorem.

Theorem 1 ([Sp, Prop. 1.5.2]). *Let L/K be a finite Galois p-extension. Then the following properties are equivalent:*

(i) *L/K is well ramified;*

(ii) *for every normal subgroup H of G the Herbrand property holds:*
for every $1 \neq \tau \in G/H$

$$i_{G/H}(\tau) = \frac{1}{e(L|L^H)} \sum_{\sigma \in \tau H} i_G(\sigma);$$

(iii) *the Hilbert formula holds:*

$$v_L(\mathcal{D}_{L/K}) = \sum_{\sigma \neq 1} i_G(\sigma) = \sum_{i \geqslant 0} (|G_i| - 1),$$

for the definition of G_i see subsection 18.2.

From the definition we immediately deduce that if M/K is a Galois subextension of a well ramified L/K then L/M is well ramified; from (ii) we conclude that M/K is well ramified.

Now we consider well ramified extensions L/K which are not in case I nor in case II.

Example. (Well ramified extension not in case I and not in case II). Let K be a complete discrete valuation field of characteristic zero. Let $\zeta_{p^2} \in K$. Consider a cyclic extension of degree p^2 defined by $L = K(x)$ where x a root of the polynomial $f(X) = X^{p^2} - (1 + u\pi)\alpha^p$, $\alpha \in U_K$, $\bar{\alpha} \notin k_K^p$, $u \in U_K$, π is a prime of K. Then $e(L|K) = p = f(L|K)^{\mathrm{ins}}$, so L/K is not in case I nor in case II. Using Theorem 1, one can show that $\mathcal{O}_L = \mathcal{O}_K[x]$ by checking the Herbrand property.

Definition. A well ramified extension which is not in case I and is not in case II is said to be in *case III*.

Note that in case III we have $e(L|K) \geqslant p, f(L|K)^{\text{ins}} \geqslant p$.

Lemma 1. *If L/K is a well ramified Galois extension, then for every ferociously ramified Galois subextension E/K such that L/E is totally ramified either $E = K$ or $E = L$.*

Proof. Suppose that there exists $K \neq E \neq L$, such that E/K is ferociously ramified and L/E is totally ramified. Let π_1 be a prime of L such that $\mathcal{O}_L = \mathcal{O}_E[\pi_1]$. Let $\alpha \in E$ be such that $\mathcal{O}_E = \mathcal{O}_K[\alpha]$. Then we have $\mathcal{O}_L = \mathcal{O}_K[\alpha, \pi_1]$. Let σ be a K-automorphism of E and denote $\widetilde{\sigma}$ a lifting of σ to $G = \mathrm{Gal}(L/K)$. It is not difficult to show that $i_G(\widetilde{\sigma}) = \min\{v_L(\widetilde{\sigma}\pi_1 - \pi_1), v_L(\sigma\alpha - \alpha)\}$. We show that $i_G(\widetilde{\sigma}) = v_L(\widetilde{\sigma}\pi_1 - \pi_1)$. Suppose we had $i_G(\widetilde{\sigma}) = v_L(\sigma\alpha - \alpha)$, then

$$(*) \qquad \frac{i_G(\widetilde{\sigma})}{e(L|E)} = v_E(\sigma\alpha - \alpha) = i_{E/K}(\sigma).$$

Furthermore, by Herbrand property we have

$$i_{E/K}(\sigma) = \frac{1}{e(L|E)} \sum_{s \in \sigma\, \mathrm{Gal}(L/E)} i_G(s) = \frac{i_G(\widetilde{\sigma})}{e(L|E)} + \frac{1}{e(L|E)} \sum_{s \neq \tilde{\sigma}} i_G(s).$$

So from $(*)$ we deduce that

$$\frac{1}{e(L|E)} \sum_{s \neq \tilde{\sigma}} i_G(s) = 0,$$

but this is not possible because $i_G(s) \geqslant 1$ for all $s \in G$. We have shown that

$$(**) \qquad i_G(s) = v_L(s\pi_1 - \pi_1) \qquad \text{for all } s \in G.$$

Now note that $\alpha \notin \mathcal{O}_K[\pi_1]$. Indeed, from $\alpha = \sum a_i \pi_1^i$, $a_i \in \mathcal{O}_K$, we deduce $\alpha \equiv a_0 \pmod{\pi_1}$ which is impossible. By $(**)$ and the Hilbert formula (cf. Theorem 1) we have

$$(***) \qquad v_L(\mathcal{D}_{L/K}) = \sum_{s \neq 1} i_G(s) = \sum_{s \neq 1} v_L(s\pi_1 - \pi_1) = v_L(f'(\pi_1)),$$

where $f(X)$ denotes the minimal polynomial of π_1 over K.

Now let the ideal $\mathcal{T}_{\pi_1} = \{x \in \mathcal{O}_L : x\mathcal{O}_K[\pi_1] \subset \mathcal{O}_L\}$ be the conductor of $\mathcal{O}_K[\pi_1]$ in $\mathcal{O}_L$ (cf. [S, Ch. III, §6]). We have (cf. loc.cit.)

$$\mathcal{T}_{\pi_1}\mathcal{D}_{L/K} = f'(\pi_1)\mathcal{O}_L$$

and then $(***)$ implies $\mathcal{T}_{\pi_1} = \mathcal{O}_L$, $\mathcal{O}_L = \mathcal{O}_K[\pi_1]$, which contradicts $\alpha \notin \mathcal{O}_K[\pi_1]$. □

Theorem 2 (Spriano)**.** *Let L/K be a Galois well ramified p-extension. Put $K_0 = L \cap K_{\mathrm{ur}}$. Then there is a Galois subextension T/K_0 of L/K_0 such that T/K_0 is in case I and L/T in case II.*

Proof. Induction on $|L : K_0|$.

Let M/K_0 be a Galois subextension of L/K_0 such that $|L : M| = p$. Let T/K_0 be a Galois subextension of M/K_0 such that T/K_0 is totally ramified and M/T is in case II. Applying Lemma 1 to L/T we deduce that L/M is ferociously ramified, hence in case II. □

In particular, if L/K is a Galois p-extension in case III such that $L \cap K_{\mathrm{ur}} = K$, then there is a Galois subextension T/K of L/K such that T/K is in case I, L/T in case I and $K \neq T \neq L$.

18.1.2. Modified ramification function for well ramified extensions.

In the general case one can define a filtration of ramification groups as follows. Given two integers $n, m \geqslant 0$ the (n,m)-ramification group $G_{n,m}$ of L/K is

$$G_{n,m} = \{\sigma \in G : \; v_L(\sigma(x) - x) \geqslant n + m, \text{ for all } x \in \mathcal{M}_L^m\}.$$

Put $G_n = G_{n+1,0}$ and $H_n = G_{n,1}$, so that the classical ramification groups are the G_n. It is easy to show that $H_i \geqslant G_i \geqslant H_{i+1}$ for $i \geqslant 0$.

In case I we have $G_i = H_i$ for all $i \geqslant 0$; in case II we have $G_i = H_{i+1}$ for all $i \geqslant 0$, see [Sm1]. If L/K is in case III, we leave to the reader the proof of the following equality

$$\begin{aligned} G_i &= \{\sigma \in \mathrm{Gal}(L/K) : v_L(\sigma(x) - x) \geqslant i + 1 \quad \text{for all } x \in \mathcal{O}_L\} \\ &= \{\sigma \in \mathrm{Gal}(L/K) : v_L(\sigma(x) - x) \geqslant i + 2 \quad \text{for all } x \in \mathcal{M}_L\} = H_{i+1}. \end{aligned}$$

We introduce another filtration which allows us to simultaneously deal with case I, II and III.

Definition. Let L/K be a finite Galois well ramified extension. The modified t-th *ramification group* $G[t]$ for $t \geqslant 0$ is defined by

$$G[t] = \{\sigma \in \mathrm{Gal}(L/K) \; : \; i_G(\sigma) \geqslant t\}.$$

We call an integer number m a *modified ramification jump* of L/K if $G[m] \neq G[m+1]$.

From now on we will consider only p-extensions.

Definition. For a well ramified extension L/K define the modified Hasse–Herbrand function $\mathfrak{s}_{L/K}(u)$, $u \in \mathbb{R}_{\geqslant 0}$ as

$$\mathfrak{s}_{L/K}(u) = \int_0^u \frac{|G[t]|}{e(L|K)}\, dt.$$

Put $g_i = |\mathcal{G}_i|$. If $m \leqslant u \leqslant m+1$ where m is a non-negative integer, then

$$\mathfrak{s}_{L/K}(u) = \frac{1}{e(L|K)}(g_1 + \cdots + g_m + g_{m+1}(u-m)).$$

We drop the index L/K in $\mathfrak{s}_{L/K}$ if there is no risk of confusion. One can show that the function $\mathfrak{s}$ is continuous, piecewise linear, increasing and convex. In case I, if $\varphi_{L/K}$ denotes the classical Hasse–Herbrand function as in [S, Ch. IV], then $\mathfrak{s}_{L/K}(u) = 1 + \varphi_{L/K}(u-1)$. We define a *modified upper numbering* for ramification groups by $G(\mathfrak{s}_{L/K}(u)) = G[u]$.

If m is a modified ramification jumps, then the number $\mathfrak{s}_{L/K}(m)$ is called a *modified upper ramification jump* of L/K.

For well ramified extensions we can show the Herbrand theorem as follows.

Lemma 2. *For $u \geqslant 0$ we have* $\mathfrak{s}_{L/K}(u) = \dfrac{1}{e(L|K)} \sum_{\sigma \in G} \inf(i_G(\sigma), u).$

The proof goes exactly as in [S, Lemme 3, Ch.IV, §3].

Lemma 3. *Let H be a normal subgroup of G and $\tau \in G/H$ and let $j(\tau)$ be the upper bound of the integers $i_G(\sigma)$ where σ runs over all automorphisms of G which are congruent to τ modulo H. Then we have*

$$i_{L^H/K}(\tau) = \mathfrak{s}_{L/L^H}(j(\tau)).$$

For the proof see Lemme 4 loc.cit. (note that Theorem 1 is fundamental in the proof). In order to show Herbrand theorem, we have to show the multiplicativity in the tower of extensions of the function $\mathfrak{s}_{L/K}$.

Lemma 4. *With the above notation, we have* $\mathfrak{s}_{L/K} = \mathfrak{s}_{L^H/K} \circ \mathfrak{s}_{L/L^H}$.

For the proof see Prop. 15 loc.cit.

Corollary. *If L/K is well ramified and H is a normal subgroup of $G = \mathrm{Gal}(L/K)$, then the Herbrand theorem holds:*

$$(G/H)(u) = G(u)H/H \quad \textit{for all } u \geqslant 0.$$

It is known that the upper ramification jumps (with respect the classical function φ) of an abelian extension in case I are integers. This is the Hasse–Arf theorem. Clearly the same result holds with respect the function $\mathfrak{s}$. In fact, if m is a classical ramification jump and $\varphi_{L/K}(m)$ is the upper ramification jump, then the modified ramification jump is $m+1$ and the modified upper ramification jumps is $\mathfrak{s}_{L/K}(m+1) = 1 + \varphi_{L/K}(m)$ which is an integer. In case II it is obvious that the modified upper ramification jumps are integers. For well ramified extensions we have the following theorem, for the proof see the end of 18.2.

Theorem 3 (Borger)**.** *The modified upper ramification jumps of abelian well ramified extensions are integers.*

18.2. The Kato conductor

We have already remarked that the Swan conductor $\mathrm{sw}(\chi)$ for a character χ of the Galois group G_K is not an integer in general. In [K3] Kato defined a modified Swan conductor in case I, II for any character χ of G_K; and [K4] contains a definition of an integer valued conductor (which we will call the Kato conductor) for characters of degree 1 in the general case i.e. not only in cases I and II.

We recall its definition. The map $K^* \to H^1(K, \mathbb{Z}/n(1))$ (cf. the definition of $H^q(K)$ in subsection 5.1) induces a pairing

$$\{\ ,\ \}: H^q(K) \times K_r(K) \to H^{q+r}(K),$$

which we briefly explain only for K of characteristic zero, in characteristic $p > 0$ see [K4, (1.3)]. For $a \in K^*$ and a fixed $n \geqslant 0$, let $\{a\} \in H^1(K, \mathbb{Z}/n(1))$ be the image under the connecting homomorphism $K^* \to H^1(K, \mathbb{Z}/n(1))$ induced by the exact sequence of G_K-modules

$$1 \longrightarrow \mathbb{Z}/n(1) \longrightarrow K_s^* \xrightarrow{n} K_s^* \longrightarrow 1.$$

For $a_1, ..., a_r \in K^*$ the symbol $\{a_1, ..., a_r\} \in H^r(K, \mathbb{Z}/n(r))$ is the cup product $\{a_1\} \cup \{a_2\} \cup \dots \cup \{a_r\}$. For $\chi \in H^q(K)$ and $a_1, \dots, a_r \in K^*$ $\{\chi, a_1, ..., a_r\} \in H_n^{q+r}(K)$ is the cup product $\{\chi\} \cup \{a_1\} \cup \dots \cup \{a_r\}$. Passing to the limit we have the element $\{\chi, a_1, ..., a_r\} \in H^{q+r}(K)$.

Definition. Following Kato, we define an increasing filtration $\{\mathrm{fil}_m H^q(K)\}_{m \geqslant 0}$ of $H^q(K)$ by

$$\mathrm{fil}_m H^q(K) = \{\chi \in H^q(K) : \{\chi|_M, U_{m+1,M}\} = 0 \quad \text{for every } M\}$$

where M runs through all complete discrete valuation fields satisfying $\mathcal{O}_K \subset \mathcal{O}_M$, $\mathcal{M}_M = \mathcal{M}_K \mathcal{O}_M$; here $\chi|_M$ denotes the image of $\chi \in H^q(K)$ in $H^q(M)$.

Then one can show $H^q(K) = \cup_{m \geqslant 0} \mathrm{fil}_m H^q(K)$ [K4, Lemma (2.2)] which allows us to give the following definition.

Definition. For $\chi \in H^q(K)$ the *Kato conductor* of χ is the integer $\mathrm{ksw}(\chi)$ defined by

$$\mathrm{ksw}(\chi) = \min\{m \geqslant 0 : \chi \in \mathrm{fil}_m H^q(K)\}.$$

This integer $\mathrm{ksw}(\chi)$ is a generalization of the classical Swan conductor as stated in the following proposition.

Proposition 1. *Let $\chi \in H^1(K)$ and let L/K be the corresponding finite cyclic extension and suppose that L/K is in case I or II. Then*
(a) $\mathrm{ksw}(\chi) = \mathrm{sw}(\chi)$ *(see formula* (1) *).*
(b) Let t be the maximal modified ramification jump. Then

$$\mathrm{ksw}(\chi) = \begin{cases} \mathfrak{s}_{L/K}(t) - 1 & \textit{case I} \\ \mathfrak{s}_{L/K}(t) & \textit{case II.} \end{cases}$$

Proof. (a) See [K4, Prop. (6.8)]. (b) This is a computation left to the reader. □

We compute the Kato conductor in case III.

Theorem 4 (Spriano)**.** *If L/K is a cyclic extension in case III and if χ is the corresponding element of $H^1(K)$, then* $\mathrm{ksw}(\chi) = \mathrm{sw}(\chi) - 1$. *If t is the maximal modified ramification jump of L/K, then* $\mathrm{ksw}(\chi) = \mathfrak{s}_{L/K}(t) - 1$.

Before the proof we explain how to compute the Kato conductor $\mathrm{ksw}(\chi)$ where $\chi \in H^1(K)$. Consider the pairing $H^1(K) \times K^* \to H^2(K)$, ($q = 1 = r$). It coincides with the symbol $(\cdot,\cdot)$ defined in [S, Ch. XIV]. In particular, if $\chi \in H^1(K)$ and $a \in K^*$, then $\{\chi, a\} = 0$ if and only if the element a is a norm of the extension L/K corresponding to χ. So we have to compute the minimal integer m such that $U_{m+1,M}$ is in the norm of the cyclic extension corresponding to $\chi|_M$ when M runs through all complete discrete valuation fields satisfying $\mathcal{M}_M = \mathcal{M}_K \mathcal{O}_M$. The minimal integer n such that $U_{n+1,K}$ is contained in the norm of L/K is not, in general, the Kato conductor (for instance if the residue field of K is algebraically closed)

Here is a characterization of the Kato conductor which helps to compute it and does not involve extensions M/K, cf. [K4, Prop. (6.5)].

Proposition 2. *Let K be a complete discrete valuation field. Suppose that $|k_K : k_K^p| = p^c < \infty$, and $H_p^{c+1}(k_K) \neq 0$. Then for $\chi \in H^q(K)$ and $n \geqslant 0$*

$$\chi \in \mathrm{fil}_n H^q(K) \iff \{\chi, U_{n+1} K^M_{c+2-q}(K)\} = 0 \quad \textit{in } H^{c+2}(K),$$

for the definition of $U_{n+1} K^M_{c+2-q}(K)$ see subsection 4.2.

In the following we will only consider characters χ such that the corresponding cyclic extensions L/K are p-extension, because $\mathrm{ksw}(\chi) = 0$ for tame characters χ, cf. [K4, Prop. (6.1)]. We can compute the Kato conductor in the following manner.

Corollary. *Let K be as in Proposition 2. Let $\chi \in H^1(K)$ and assume that the corresponding cyclic extension L/K is a p-extension. Then the minimal integer n such that*

$$U_{n+1,K} \subset N_{L/K} L^*$$

is the Kato conductor of χ.

Proof. By the hypothesis (i.e. $U_{n+1,K} \subset N_{L/K}L^*$) we have $\mathrm{ksw}(\chi) \geqslant n$. Now $U_{n+1,K} \subset N_{L/K}L^*$, implies that $U_{n+1}K_{c+1}(K)$ is contained in the norm group $N_{L/K}K_{c+1}(L)$. By [K1, II, Cor. at p. 659] we have that $\{\chi, U_{n+1}K_{c+1}(K)\} = 0$ in $H^{c+2}(K)$ and so by Proposition 2 $\mathrm{ksw}(\chi) \leqslant n$. □

Beginning of the proof of Theorem 4. Let L/K be an extension in case III and let $\chi \in H^1(K)$ be the corresponding character. We can assume that $H_p^{c+1}(k_K) \neq 0$, otherwise we consider the extension $k = \cup_{i \geqslant 0} k_K(T^{p^{-i}})$ of the residue field k_K, preserving a p-base, for which $H_p^{c+1}(k) \neq 0$ (see [K3, Lemma (3-9)]).

So by the above Corollary we have to compute the minimal integer n such that $U_{n+1,K} \subset N_{L/K}L^*$.

Let T/K be the totally ramified extension defined by Lemma 1 (here T/K is uniquely determined because the extension L/K is cyclic). Denote by $U_{v,L}$ for $v \in \mathbb{R}, v \geqslant 0$ the group $U_{n,L}$ where n is the smallest integer $\geqslant v$.

If t is the maximal modified ramification jump of L/K, then

$$U_{\mathfrak{s}_{L/T}(t)+1,T} \subset N_{L/T}L^* \tag{1}$$

because L/T is in case II and its Kato conductor is $\mathfrak{s}_{L/T}(t)$ by Proposition 1 (b). Now consider the totally ramified extension T/K. By [S, Ch. V, Cor. 3 §6] we have

$$N_{T/K}(U_{s,T}) = U_{\mathfrak{s}_{T/K}(s+1)-1,K} \quad \text{if} \quad \mathrm{Gal}(T/K)_s = \{1\}. \tag{2}$$

Let $t' = i_{T/K}(\tau)$ be the maximal modified ramification jump of T/K. Let r be the maximum of $i_{L/K}(\sigma)$ where σ runs over all representatives of the coset $\tau\,\mathrm{Gal}(L/T)$. By Lemma 3 $t' = \mathfrak{s}_{L/T}(r)$. Note that $r < t$ (we explain it in the next paragraph), so

$$t' = \mathfrak{s}_{L/T}(r) < \mathfrak{s}_{L/T}(t). \tag{3}$$

To show that $r < t$ it suffices to show that for a generator ρ of $\mathrm{Gal}(L/K)$

$$i_{L/K}(\rho^{p^m}) > i_{L/K}(\rho^{p^{m-1}})$$

for $|T:K| \leqslant p^m \leqslant |L:K|$. Write $\mathcal{O}_L = \mathcal{O}_K(a)$ then

$$\rho^{p^m}(a) - a = \rho^{p^{m-1}}(b) - b, \quad b = \sum_{i=0}^{p-1} \rho^{p^{m-1}i}(a).$$

Then $b = pa + \pi^i f(a)$ where π is a prime element of L, $f(X) \in \mathcal{O}_K[X]$ and $i = i_{L/K}(\rho^{p^{m-1}})$. Hence $i_{L/K}(\rho^{p^m}) = v_L(\rho^{p^m}(a) - a) \geqslant \min(i + v_L(p), 2i)$, so $i_{L/K}(\rho^{p^m}) > i_{L/K}(\rho^{p^{m-1}})$, as required.

Now we use the fact that the number $\mathfrak{s}_{L/K}(t)$ is an integer (by Borger's Theorem). We shall show that $U_{\mathfrak{s}_{L/K}(t),K} \subset N_{L/K}L^*$.

By (3) we have $\mathrm{Gal}(T/K)_{\mathfrak{s}_{L/T}(t)} = \{1\}$ and so we can apply (2). By (1) we have $U_{\mathfrak{s}_{L/T}(t)+1,T} \subset N_{L/T}L^*$, and by applying the norm map $N_{T/K}$ we have (by (2))

$$N_{T/K}(U_{\mathfrak{s}_{L/T}(t)+1,T}) = U_{\mathfrak{s}_{T/K}(\mathfrak{s}_{L/T}(t)+2)-1,K} \subset N_{L/K}L^*.$$

Thus it suffices to show that the smallest integer $\geqslant \mathfrak{s}_{T/K}(\mathfrak{s}_{L/T}(t)+2)-1$ is $\mathfrak{s}_{L/K}(t)$. Indeed we have

$$\mathfrak{s}_{T/K}(\mathfrak{s}_{L/T}(t)+2)-1 = \mathfrak{s}_{T/K}(\mathfrak{s}_{L/T}(t)) + \frac{2}{|T:K|} - 1 = \mathfrak{s}_{L/K}(t) - 1 + \frac{2}{p^e}$$

where we have used Lemma 4. By Borger's theorem $\mathfrak{s}_{L/K}(t)$ is an integer and thus we have shown that $\mathrm{ksw}(\chi) \leqslant \mathfrak{s}_{L/K}(t) - 1$.

Now we need a lemma which is a key ingredient to deduce Borger's theorem.

Lemma 5. *Let L/K be a Galois extension in case III. If $k_L = k_K(a^{1/f})$ then $a \in k_K \setminus k_K^p$ where $f = |L:T| = f(L/K)^{\mathrm{ins}}$. Let α be a lifting of a in K and let $M = K(\beta)$ where $\beta^f = \alpha$.*

If $\sigma \in \mathrm{Gal}(L/K)$ and $\sigma' \in \mathrm{Gal}(LM/M)$ is such that $\sigma'|_L = \sigma$ then

$$i_{LM/M}(\sigma') = e(LM|L)\, i_{L/K}(\sigma).$$

Proof. (After J. Borger). Note that the extension M/K is in case II and LM/M is in case I, in particular it is totally ramified. Let $x \in \mathcal{O}_L$ such that $\mathcal{O}_L = \mathcal{O}_K[x]$. One can check that $x^f - \alpha \in \mathcal{M}_L \setminus \mathcal{M}_L^2$. Let $g(X)$ be the minimal polynomial of β over K. Then $g(X+x)$ is an Eisenstein polynomial over L (because $g(X+x) \equiv X^f + x^f - \alpha \equiv X^f \mod \mathcal{M}_L$) and $\beta - x$ is a root of $g(X+x)$. So $\beta - x$ is a prime of LM and we have

$$i_{LM/M}(\sigma') = v_{LM}(\sigma'(\beta - x) - (\beta - x)) = v_{LM}(\sigma'(x) - x) = e(LM|L)\, i_{L/K}(\sigma).$$

□

Proof of Theorem 3 and Theorem 4. Now we deduce simultaneously the formula for the Kato conductor in case III and Borger's theorem. We compute the classical Artin conductor $A(\chi|_M)$. By the preceding lemma we have

(2)

$$\begin{aligned} A(\chi|_M) &= \frac{1}{e(LM|M)} \sum_{\sigma' \in \mathrm{Gal}(LM/M)} \chi|_M(\sigma')\, i_{LM/M}(\sigma') \\ &= \frac{e(LM|L)}{e(LM|M)} \sum_{\sigma' \in \mathrm{Gal}(LM/M)} \chi|_M(\sigma')\, i_{L/K}(\sigma) = \frac{1}{e(L|K)} \sum_{\sigma \in G} \chi(\sigma)\, i_{L/K}(\sigma). \end{aligned}$$

Since $A(\chi|_M)$ is an integer by Artin's theorem we deduce that the latter expression is an integer. Now by the well known arguments one deduces the Hasse–Arf property for L/K.

The above argument also shows that the Swan conductor (=Kato conductor) of LM/M is equal to $A(\chi|_M)-1$, which shows that $\mathrm{ksw}(\chi) \geqslant A(\chi|_M)-1 = \mathfrak{s}_{L/K}(t)-1$, so $\mathrm{ksw}(\chi) = \mathfrak{s}_{L/K}(t) - 1$ and Theorem 4 follows. □

18.3. More ramification invariants

18.3.1. Hyodo's depth of ramification. This ramification invariant was introduced by Hyodo in [H]. We are interested in its link with the Kato conductor.

Let K be an m-dimensional local field, $m \geqslant 1$. Let $t_1, \dots, t_m$ be a system of local parameters of K and let $\mathbf{v}$ be the corresponding valuation.

Definition. Let L/K be a finite extension. The *depth of ramification* of L/K is

$$d_K(L/K) = \inf\{\mathbf{v}(\mathrm{Tr}_{L/K}(y)/y) : y \in L^*\} \in \mathbb{Q}^m.$$

The right hand side expression exists; and, in particular, if $m = 1$ then $d_K(L/K) = v_K(\mathcal{D}_{L/K}) - (1 - v_K(\pi_L))$, see [H]. The main result about the depth is stated in the following theorem (see [H, Th. (1-5)]).

Theorem 5 (Hyodo)**.** *Let L be a finite Galois extension of an m-dimensional local field K. For $l \geqslant 1$ define*

$$\mathbf{j}(l) = \mathbf{j}_{L/K}(l) = \begin{cases} \max\{\mathbf{i} : 1 \leqslant \mathbf{i} \in \mathbb{Z}^m, \quad |\Psi_{L/K}(U_{\mathbf{i}}K_m^{\mathrm{top}}(K))| \geqslant p^l\} & \textit{if it exists} \\ 0 & \textit{otherwise} \end{cases}$$

where $\Psi_{L/K}$ is the reciprocity map; the definition of $U_{\mathbf{i}}K_m^{\mathrm{top}}(K)$ is given in 17.0. Then

$$(3) \qquad (p-1)\sum_{l \geqslant 1} \mathbf{j}(l)/p^l \leqslant d_K(L/K) \leqslant (1-p^{-1})\sum_{l\geqslant 1}\mathbf{j}(l).$$

Furthermore, these inequalities are the best possible (*cf.* [H, Prop. (3-4) and Ex. (3-5)]).

For $\mathbf{i} \in \mathbb{Z}^m$, let $G^{\mathbf{i}}$ be the image of $U_{\mathbf{i}}K_m^{\mathrm{top}}(K)$ in $\mathrm{Gal}(L/K)$ under the reciprocity map $\Psi_{L/K}$. The numbers $\mathbf{j}(l)$ are called jumping number (by Hyodo) and in the classical case, i.e. $m = 1$, they coincide with the upper ramification jumps of L/K.

For local fields (i.e. 1-dimensional local fields) one can show that the first inequality in (3) is actually an equality. Hyodo stated ([H, p.292]) *"It seems that we can define nice ramification groups only when the first equality of (3) holds."*

For example, if L/K is of degree p, then the inequalities in (3) are actually equalities and in this case we actually have a nice ramification theory. For an abelian extension L/K [H, Prop. (3-4)] shows that the first equality of (3) holds if at most one diagonal component of $E(L/K)$ (for the definition see subsection 1.2) is divisible by p.

Extensions in case I or II verify the hypothesis of Hyodo's proposition, but it is not so in case III. We shall show below that the first equality does not hold in case III.

18.3.2. The Kato conductor and depth of ramification.

Consider an m-dimensional local field K, $m \geqslant 1$. Proposition 2 of 18.2 shows (if the first residue field is of characteristic $p > 0$) that for $\chi \in H^1(K)$, $\chi \in \mathrm{fil}_n H^1(K)$ if and only if the induced homomorphism $K_m(K) \to \mathbb{Q}/\mathbb{Z}$ annihilates $U_{n+1}K_m(K)$ (cf. also in [K4, Remark (6.6)]). This also means that the Kato conductor of the extension L/K corresponding to χ is the m-th component of the last ramification jump $\mathbf{j}(1)$ (recall that $\mathbf{j}(1) = \max\{\mathbf{i} : 1 \leqslant \mathbf{i} \in \mathbb{Z}^m, \quad |G^{\mathbf{i}}| \geqslant p\}$).

Example. Let L/K as in Example of 18.1.1 and assume that K is a 2-dimensional local field with the first residue field of characteristic $p > 0$ and let $\chi \in H^1(K)$ be the corresponding character. Let $\mathbf{j}(l)_i$ denotes the i-th component of $\mathbf{j}(l)$. Then by Theorem 3 and by the above discussion we have

$$\mathrm{ksw}(\chi) = \mathbf{j}(1)_2 = \mathfrak{s}_{L/K}(pe/(p-1)) - 1 = \frac{(2p-1)e}{p-1} - 1.$$

If T/K is the subextension of degree p, we have

$$d_K(T/K)_2 = p^{-1}(p-1)\mathbf{j}(2)_2 \implies \mathbf{j}(2)_2 = \frac{pe}{p-1} - 1.$$

The depth of ramification is easily computed:

$$d_K(L/K)_2 = d_K(T/K)_2 + d_K(L/T)_2 = \frac{(p-1)}{p}\left(\frac{2pe}{p-1} - 1\right).$$

The left hand side of (3) is $(p-1)(\mathbf{j}(1)/p + \mathbf{j}(2)/p^2)$, so for the second component we have

$$(p-1)\left(\frac{\mathbf{j}(1)_2}{p} + \frac{\mathbf{j}(2)_2}{p^2}\right) = 2e - \frac{(p^2-1)}{p^2} \neq d_K(L/K)_2.$$

Thus, the first equality in (3) does not hold for the extension L/K.

If K is a complete discrete valuation (of rank one) field, then in the well ramified case straightforward calculations show that

$$e(L|K)d_K(L/K) = \begin{cases} \sum_{\sigma\neq 1} s_G(\sigma) & \text{case I,II} \\ \sum_{\sigma\neq 1} s_G(\sigma) - e(L|K) + 1 & \text{case III} \end{cases}$$

Let $\chi \in H^1(K)$ and assume that the corresponding extension L/K is well ramified. Let t denote the last ramification jump of L/K; then from the previous formula and Theorem 4 we have

$$e(L|K)\,\mathrm{ksw}(\chi) = \begin{cases} d_L(L/K) + t & \text{case I,II} \\ d_L(L/K) + t - 1 & \text{case III} \end{cases}$$

In the general case, we can indicate the following relation between the Kato conductor and Hyodo's depth of ramification.

Theorem 6 (Spriano)**.** *Let $\chi \in H^1(K, \mathbb{Z}/p^n)$ and let L/K be the corresponding cyclic extension. Then*

$$\mathrm{ksw}(\chi) \leqslant d_K(L/K) + \frac{t}{e(L|K)}$$

where t is the maximal modified ramification jump.

Proof. In [Sp, Prop. 3.7.3] we show that

$$(*) \qquad \mathrm{ksw}(\chi) \leqslant \left[\frac{1}{e(L|K)}\left(\sum_{\sigma\in G} \mathrm{Sw}_G(\sigma)\chi(\sigma) - M_{L/K}\right)\right],$$

where $[x]$ indicates the integer part of $x \in \mathbb{Q}$ and the integer $M_{L/K}$ is defined by

$$(**) \qquad d_L(L/K) + M_{L/K} = \sum_{\sigma\neq 1} s_G(\sigma).$$

Thus, the inequality in the statement follows from $(*)$ and $(**)$. $\square$

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