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5. Kato's higher local class field theory

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5.0. Introduction

We first recall the classical local class field theory. Let K be a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_q((X))$. The main theorem of local class field theory consists of the isomorphism theorem and existence theorem. In this section we consider the isomorphism theorem.

An outline of one of the proofs is as follows. First, for the Brauer group $\mathrm{Br}(K)$, an isomorphism

$$\mathrm{inv}\colon \mathrm{Br}(K) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$$

is established; it mainly follows from an isomorphism

$$H^1(F, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$$

where F is the residue field of K.

Secondly, we denote by $X_K = \mathrm{Hom}_{\mathrm{cont}}(G_K, \mathbb{Q}/\mathbb{Z})$ the group of continuous homomorphisms from $G_K = \mathrm{Gal}(\overline{K}/K)$ to $\mathbb{Q}/\mathbb{Z}$. We consider a pairing

$$K^* \times X_K \longrightarrow \mathbb{Q}/\mathbb{Z}$$

$$(a, \chi) \mapsto \mathrm{inv}(\chi, a)$$

where (χ, a) is the cyclic algebra associated with χ and a. This pairing induces a homomorphism

$$\Psi_K\colon K^* \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K) = \mathrm{Hom}(X_K, \mathbb{Q}/\mathbb{Z})$$

which is called the reciprocity map.

Thirdly, for a finite abelian extension L/K, we have a diagram

$$\begin{array}{ccc} L^* & \xrightarrow{\Psi_L} & \mathrm{Gal}(L^{\mathrm{ab}}/L) \\ {\scriptstyle N}\downarrow & & \downarrow \\ K^* & \xrightarrow{\Psi_K} & \mathrm{Gal}(K^{\mathrm{ab}}/K) \end{array}$$

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which is commutative by the definition of the reciprocity maps. Here, N is the norm map and the right vertical map is the canonical map. This induces a homomorphism

$$\Psi_{L/K}: K^*/NL^* \longrightarrow \mathrm{Gal}(L/K).$$

The isomorphism theorem tells us that the above map is bijective.

To show the bijectivity of $\Psi_{L/K}$, we can reduce to the case where $|L:K|$ is a prime ℓ. In this case, the bijectivity follows immediately from a famous exact sequence

$$L^* \xrightarrow{N} K^* \xrightarrow{\cup\chi} \mathrm{Br}(K) \xrightarrow{\mathrm{res}} \mathrm{Br}(L)$$

for a cyclic extension L/K (where $\cup\chi$ is the cup product with χ, and res is the restriction map).

In this section we sketch a proof of the isomorphism theorem for a higher dimensional local field as an analogue of the above argument. For the existence theorem see the paper by Kato in this volume and subsection 10.5.

5.1. Definition of $H^q(k)$

In this subsection, for any field k and $q > 0$, we recall the definition of the cohomology group $H^q(k)$ ([K2], see also subsections 2.1 and 2.2 and A1 in the appendix to section 2). If $\mathrm{char}\,(k) = 0$, we define $H^q(k)$ as a Galois cohomology group

$$H^q(k) = H^q(k, \mathbb{Q}/\mathbb{Z}(q-1))$$

where $(q-1)$ is the $(q-1)$ st Tate twist.

If $\mathrm{char}\,(k) = p > 0$, then following Illusie [I] we define

$$H^q(k, \mathbb{Z}/p^n(q-1)) = H^1(k, W_n\Omega^{q-1}_{k^{\mathrm{sep}},\log}).$$

We can explicitly describe $H^q(k, \mathbb{Z}/p^n(q-1))$ as the group isomorphic to

$$W_n(k) \otimes (k^*)^{\otimes(q-1)}/J$$

where $W_n(k)$ is the ring of Witt vectors of length n, and J is the subgroup generated by elements of the form

$w \otimes b_1 \otimes \dots \otimes b_{q-1}$ such that $b_i = b_j$ for some $i \neq j$, and
$(0, \dots, 0, a, 0, \dots, 0) \otimes a \otimes b_1 \otimes \dots \otimes b_{q-2}$, and
$(\mathbf{F} - 1)(w) \otimes b_1 \otimes \dots \otimes b_{q-1}$ ($\mathbf{F}$ is the Frobenius map on Witt vectors).

We define $H^q(k, \mathbb{Q}_p/\mathbb{Z}_p(q-1)) = \varinjlim H^q(k, \mathbb{Z}/p^n(q-1))$, and define

$$H^q(k) = \bigoplus_{\ell} H^q(k, \mathbb{Q}_\ell/\mathbb{Z}_\ell(q-1))$$

where ℓ ranges over all prime numbers. (For $\ell \neq p$, the right hand side is the usual Galois cohomology of the $(q-1)$ st Tate twist of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$.)

Then for any k we have

$$H^1(k) = X_k \qquad (X_k \text{ is as in 5.0, the group of characters}),$$
$$H^2(k) = \mathrm{Br}(k) \qquad (\text{Brauer group}).$$

We explain the second equality in the case of $\mathrm{char}\,(k) = p > 0$. The relation between the Galois cohomology group and the Brauer group is well known, so we consider only the p-part. By our definition,

$$H^2(k, \mathbb{Z}/p^n(1)) = H^1(k, W_n\Omega^1_{k^{\mathrm{sep}},\log}).$$

From the bijectivity of the differential symbol (Bloch–Gabber–Kato's theorem in subsection A2 in the appendix to section 2), we have

$$H^2(k, \mathbb{Z}/p^n(1)) = H^1(k, (k^{\mathrm{sep}})^*/((k^{\mathrm{sep}})^*)^{p^n}).$$

From the exact sequence

$$0 \longrightarrow (k^{\mathrm{sep}})^* \xrightarrow{p^n} (k^{\mathrm{sep}})^* \longrightarrow (k^{\mathrm{sep}})^*/((k^{\mathrm{sep}})^*)^{p^n} \longrightarrow 0$$

and an isomorphism $\mathrm{Br}(k) = H^2(k, (k^{\mathrm{sep}})^*)$, $H^2(k, \mathbb{Z}/p^n(1))$ is isomorphic to the p^n-torsion points of $\mathrm{Br}(k)$. Thus, we get $H^2(k) = \mathrm{Br}(k)$.

If K is a henselian discrete valuation field with residue field F, we have a canonical map

$$i_F^K\colon H^q(F) \longrightarrow H^q(K).$$

If $\mathrm{char}\,(K) = \mathrm{char}\,(F)$, this map is defined naturally from the definition of H^q (for the Galois cohomology part, we use a natural map $\mathrm{Gal}(K^{\mathrm{sep}}/K) \longrightarrow \mathrm{Gal}(K_{\mathrm{ur}}/K) = \mathrm{Gal}(F^{\mathrm{sep}}/F)$) . If K is of mixed characteristics $(0, p)$, the prime-to-p-part is defined naturally and the p-part is defined as follows. For the class $[w \otimes b_1 \otimes \cdots \otimes b_{q-1}]$ in $H^q(F, \mathbb{Z}/p^n(q-1))$ we define $i_F^K([w \otimes b_1 \otimes \cdots \otimes b_{q-1}])$ as the class of

$$i(w) \otimes \widetilde{b_1} \otimes \cdots \otimes \widetilde{b_{q-1}}$$

in $H^1(K, \mathbb{Z}/p^n(q-1))$, where $i\colon W_n(F) \to H^1(F, \mathbb{Z}/p^n) \to H^1(K, \mathbb{Z}/p^n)$ is the composite of the map given by Artin–Schreier–Witt theory and the canonical map, and $\widetilde{b_i}$ is a lifting of b_i to K.

Theorem (Kato [K2, Th. 3]). *Let K be a henselian discrete valuation field, π be a prime element, and F be the residue field. We consider a homomorphism*

$$i = (i_F^K, i_F^K \cup \pi)\colon H^q(F) \oplus H^{q-1}(F) \longrightarrow H^q(K)$$
$$(a, b) \mapsto i_F^K(a) + i_F^K(b) \cup \pi$$

where $i_F^K(b) \cup \pi$ is the element obtained from the pairing

$$H^{q-1}(K) \times K^* \longrightarrow H^q(K)$$

which is defined by Kummer theory and the cup product, and the explicit description of $H^q(K)$ *in the case of* $\operatorname{char}(K) > 0$. *Suppose* $\operatorname{char}(F) = p$. *Then* i *is bijective in the prime-to-*p *component. In the* p*-component,* i *is injective and the image coincides with the* p*-component of the kernel of* $H^q(K) \to H^q(K_{\mathrm{ur}})$ *where* K_{ur} *is the maximal unramified extension of* K.

From this theorem and Bloch–Kato's theorem in section 4, we obtain

Corollary. *Assume that* $\operatorname{char}(F) = p > 0$, $|F : F^p| = p^{d-1}$, *and that there is an isomorphism* $H^d(F) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$.

Then, i *induces an isomorphism*

$$H^{d+1}(K) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

A typical example which satisfies the assumptions of the above corollary is a d-dimensional local field (if the last residue field is quasi-finite (not necessarily finite), the assumptions are satisfied).

5.2. Higher dimensional local fields

We assume that K is a d-dimensional local field, and F is the residue field of K, which is a $(d-1)$-dimensional local field. Then, by the corollary in the previous subsection and induction on d, there is a canonical isomorphism

$$\mathrm{inv}\colon H^{d+1}(K) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

This corresponds to the first step of the proof of the classical isomorphism theorem which we described in the introduction.

The cup product defines a pairing

$$K_d(K) \times H^1(K) \longrightarrow H^{d+1}(K) \simeq \mathbb{Q}/\mathbb{Z}.$$

This pairing induces a homomorphism

$$\Psi_K\colon K_d(K) \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K) \simeq \mathrm{Hom}(H^1(K), \mathbb{Q}/\mathbb{Z})$$

which we call the *reciprocity map*. Since the isomorphism $\mathrm{inv}\colon H^d(K) \longrightarrow \mathbb{Q}/\mathbb{Z}$ is naturally constructed, for a finite abelian extension L/K we have a commutative diagram

$$\begin{array}{ccc}
H^{d+1}(L) & \xrightarrow{\mathrm{inv}_L} & \mathbb{Q}/\mathbb{Z} \\
{\scriptstyle\mathrm{cor}}\downarrow & & \downarrow \\
H^{d+1}(K) & \xrightarrow{\mathrm{inv}_K} & \mathbb{Q}/\mathbb{Z}.
\end{array}$$

So the diagram

$$\begin{array}{ccc}
K_d(L) & \xrightarrow{\Psi_L} & \mathrm{Gal}(L^{\mathrm{ab}}/L) \\
{\scriptstyle N}\downarrow & & \downarrow \\
K_d(K) & \xrightarrow{\Psi_K} & \mathrm{Gal}(K^{\mathrm{ab}}/K)
\end{array}$$

is commutative where N is the norm map and the right vertical map is the canonical map. So, as in the classical case, we have a homomorphism

$$\Psi_{L/K}\colon K_d(K)/NK_d(L) \longrightarrow \mathrm{Gal}(L/K).$$

Isomorphism Theorem. $\Psi_{L/K}$ *is an isomorphism.*

We outline a proof. We may assume that L/K is cyclic of degree ℓ. As in the classical case in the introduction, we may study a sequence

$$K_d(L) \xrightarrow{N} K_d(K) \xrightarrow{\cup\chi} H^{d+1}(K) \xrightarrow{\mathrm{res}} H^{d+1}(L),$$

but here we describe a more elementary proof.

First of all, using the argument in [S, Ch.5] by calculation of symbols one can obtain

$$|K_d(K) : NK_d(L)| \leqslant \ell.$$

We outline a proof of this inequality.

It is easy to see that it is sufficient to consider the case of prime ℓ. (For another calculation of the index of the norm group see subsection 6.7).

Recall that $K_d(K)$ has a filtration $U_m K_d(K)$ as in subsection 4.2. We consider $\mathrm{gr}_m K_d(K) = U_m K_d(K)/U_{m+1} K_d(K)$.

If L/K is unramified, the norm map $N\colon K_d(L) \to K_d(K)$ induces surjective homomorphisms $\mathrm{gr}_m K_d(L) \to \mathrm{gr}_m K_d(K)$ for all $m > 0$. So $U_1 K_d(K)$ is in $NK_d(L)$. If we denote by F_L and F the residue fields of L and K respectively, the norm map induces a surjective homomorphism $K_d(F_L)/\ell \to K_d(F)/\ell$ because $K_d(F)/\ell$ is isomorphic to $H^d(F, \mathbb{Z}/\ell(d))$ (cf. sections 2 and 3) and the cohomological dimension of F [K2, p.220] is d. Since $\mathrm{gr}_0 K_d(K) = K_d(F) \oplus K_{d-1}(F)$ (see subsection 4.2), the above implies that $K_d(K)/NK_d(L)$ is isomorphic to $K_{d-1}(F)/NK_{d-1}(F_L)$, which is isomorphic to $\mathrm{Gal}(F_L/F)$ by class field theory of F (we use induction on d). Therefore $|K_d(K) : NK_d(L)| = \ell$.

If L/K is totally ramified and ℓ is prime to $\mathrm{char}(F)$, by the same argument (cf. the argument in [S, Ch.5]) as above, we have $U_1 K_d(K) \subset NK_d(L)$. Let π_L be a prime element of L, and $\pi_K = N_{L/K}(\pi_L)$. Then the element $\{\alpha_1, ..., \alpha_{d-1}, \pi_K\}$ for $\alpha_i \in K^*$ is in $NK_d(L)$, so $K_d(K)/NK_d(L)$ is isomorphic to $K_d(F)/\ell$, which is isomorphic to $H^d(F, \mathbb{Z}/\ell(d))$, so the order is ℓ. Thus, in this case we also have $|K_d(K) : NK_d(L)| = \ell$.

Hence, we may assume L/K is not unramified and is of degree $\ell = p = \mathrm{char}(F)$. Note that $K_d(F)$ is p-divisible because of $\Omega_F^d = 0$ and the bijectivity of the differential symbol.

Assume that L/K is totally ramified. Let π_L be a prime element of L, and σ a generator of Gal(L/K), and put $a = \sigma(\pi_L)\pi_L^{-1} - 1$, $b = N_{L/K}(a)$, and $v_K(b-1) = i$. We study the induced maps $\mathrm{gr}_{\psi(m)} K_d(L) \to \mathrm{gr}_m K_d(K)$ from the norm map N on the subquotients by the argument in [S, Ch.5]. We have $U_{i+1}K_d(K) \subset NK_d(L)$, and can show that there is a surjective homomorphism (cf. [K1, p.669])

$$\Omega_F^{d-1} \longrightarrow K_d(K)/NK_d(L)$$

such that

$$x d\log y_1 \wedge \dots \wedge d\log y_{d-1} \mapsto \{1 + \widetilde{x}b, \widetilde{y_1}, \dots, \widetilde{y_{d-1}}\}$$

($\widetilde{x}, \widetilde{y_i}$ are liftings of x and y_i). Furthermore, from

$$N_{L/K}(1 + xa) \equiv 1 + (x^p - x)b \pmod{U_{i+1}K^*},$$

the above map induces a surjective homomorphism

$$\Omega_F^{d-1}/\big((\mathbf{F} - 1)\Omega_F^{d-1} + d\Omega_F^{d-2}\big) \longrightarrow K_d(K)/NK_d(L).$$

The source group is isomorphic to $H^d(F, \mathbb{Z}/p(d-1))$ which is of order p. So we obtain $|K_d(K) : NK_d(L)| \leqslant p$.

Now assume that L/K is ferociously ramified, i.e. F_L/F is purely inseparable of degree p. We can use an argument similar to the previous one. Let h be an element of $\mathcal{O}_L$ such that $F_L = F(\overline{h})$ ($\overline{h} = h \mod \mathcal{M}_L$). Let σ be a generator of Gal(L/K), and put $a = \sigma(h)h^{-1} - 1$, and $b = N_{L/K}(a)$. Then we have a surjective homomorphism (cf. [K1, p.669])

$$\Omega_F^{d-1}/\big((\mathbf{F} - 1)\Omega_F^{d-1} + d\Omega_F^{d-2}\big) \longrightarrow K_d(K)/NK_d(L)$$

such that

$$x d\log y_1 \wedge \dots \wedge d\log y_{d-2} \wedge d\log N_{F_L/F}(\overline{h}) \mapsto \{1 + \widetilde{x}b, \widetilde{y_1}, \dots, \widetilde{y_{d-2}}, \pi\}$$

(π is a prime element of K). So we get $|K_d(K) : NK_d(L)| \leqslant p$.

So in order to obtain the bijectivity of $\Psi_{L/K}$, we have only to check the surjectivity. We consider the most interesting case $\mathrm{char}(K) = 0$, $\mathrm{char}(F) = p > 0$, and $\ell = p$. To show the surjectivity of $\Psi_{L/K}$, we have to show that there is an element $x \in K_d(K)$ such that $\chi \cup x \neq 0$ in $H^{d+1}(K)$ where χ is a character corresponding to L/K. We may assume a primitive p-th root of unity is in K. Suppose that L is given by an equation $X^p = a$ for some $a \in K \setminus K^p$. By Bloch–Kato's theorem (bijectivity of the cohomological symbols in section 4), we identify the kernel of multiplication by p on $H^{d+1}(K)$ with $H^{d+1}(K, \mathbb{Z}/p(d))$, and with $K_{d+1}(K)/p$. Then our aim is to show that there is an element $x \in K_d(K)$ such that $\{x, a\} \neq 0$ in $k_{d+1}(K) = K_{d+1}(K)/p$.

(Remark. The pairing $K_1(K)/p \times K_d(K)/p \to K_{d+1}(K)/p$ coincides up to a sign with Vostokov's symbol defined in subsection 8.3 and the latter is non-degenerate which provides an alternative proof).

We use the notation of section 4. By the Proposition in subsection 4.2, we have

$$K_{d+1}(K)/p = k_{d+1}(K) = U_{e'}k_{d+1}(K)$$

where $e' = v_K(p)p/(p-1)$. Furthermore, by the same proposition there is an isomorphism

$$H^d(F, \mathbb{Z}/p(d-1)) = \Omega_F^{d-1}/\big((\mathbf{F}-1)\Omega_F^{d-1} + d\Omega_F^{d-2}\big) \longrightarrow k_{d+1}(K)$$

such that

$$x d\log y_1 \wedge \ldots \wedge d\log y_{d-1} \mapsto \{1+\widetilde{x}b, \widetilde{y_1}, \ldots, \widetilde{y_{d-1}}, \pi\}$$

where π is a uniformizer, and b is a certain element of K such that $v_K(b) = e'$. Note that $H^d(F, \mathbb{Z}/p(d-1))$ is of order p.

This shows that for any uniformizer π of K, and for any lifting $t_1, \ldots, t_{d-1}$ of a p-base of F, there is an element $x \in \mathcal{O}_K$ such that

$$\{1+\pi^{e'}x, t_1, \ldots, t_{d-1}, \pi\} \neq 0$$

in $k_{d+1}(K)$.

If the class of a is not in $U_1k_1(K)$, we may assume a is a uniformizer or a is a part of a lifting of a p-base of F. So it is easy to see by the above property that there exists an x such that $\{a, x\} \neq 0$. If the class of a is in $U_{e'}k_1(K)$, it is also easily seen from the description of $U_{e'}k_{d+1}(K)$ that there exists an x such that $\{a, x\} \neq 0$.

Suppose $a \in U_ik_1(K) \setminus U_{i+1}k_1(K)$ such that $0 < i < e'$. We write $a = 1 + \pi^i a'$ for a prime element π and $a' \in \mathcal{O}_K^*$. First, we assume that p does not divide i. We use a formula (which holds in $K_2(K)$)

$$\{1-\alpha, 1-\beta\} = \{1-\alpha\beta, -\alpha\} + \{1-\alpha\beta, 1-\beta\} - \{1-\alpha\beta, 1-\alpha\}$$

for $\alpha \neq 0, 1$, and $\beta \neq 1, \alpha^{-1}$. From this formula we have in $k_2(K)$

$$\{1+\pi^i a', 1+\pi^{e'-i}b\} = \{1+\pi^{e'}a'b, \pi^i a'\}$$

for $b \in \mathcal{O}_K$. So for a lifting $t_1, \ldots, t_{d-1}$ of a p-base of F we have

$$\begin{aligned}\{1+\pi^i a', 1+\pi^{e'-i}b, t_1, \ldots, t_{d-1}\} &= \{1+\pi^{e'}a'b, \pi^i, t_1, \ldots, t_{d-1}\} \\ &= i\{1+\pi^{e'}a'b, \pi, t_1, \ldots, t_{d-1}\}\end{aligned}$$

in $k_{d+1}(K)$ (here we used $\{1+\pi^{e'}x, u_1, \ldots, u_d\} = 0$ for any units u_i in $k_{d+1}(K)$ which follows from $\Omega_F^d = 0$ and the calculation of the subquotients $\mathrm{gr}_m k_{d+1}(K)$ in subsection 4.2). So we can take $b \in \mathcal{O}_K$ such that the above symbol is non-zero in $k_{d+1}(K)$. This completes the proof in the case where i is prime to p.

Next, we assume p divides i. We also use the above formula, and calculate

$$\begin{aligned}\{1+\pi^i a', 1+(1+b\pi)\pi^{e'-i-1}, \pi\} &= \{1+\pi^{e'-1}a'(1+b\pi), 1+b\pi, \pi\}\\ &= \{1+\pi^{e'}a'b(1+b\pi), a'(1+b\pi), \pi\}.\end{aligned}$$

Since we may think of a' as a part of a lifting of a p-base of F, we can take some $x = \{1+(1+b\pi)\pi^{e'-i-1}, \pi, t_1, \ldots, t_{d-2}\}$ such that $\{a, x\} \neq 0$ in $k_{d+1}(K)$.

If ℓ is prime to $\mathrm{char}(F)$, for the extension L/K obtained by an equation $X^\ell = a$, we can find x such that $\{a, x\} \neq 0$ in $K_{d+1}(K)/\ell$ in the same way as above, using $K_{d+1}(K)/\ell = \mathrm{gr}_0 K_{d+1}(K)/\ell = K_d(F)/\ell$. In the case where $\mathrm{char}(K) = p > 0$ we can use Artin–Schreier theory instead of Kummer theory, and therefore we can argue in a similar way to the previous method. This completes the proof of the isomorphism theorem.

Thus, the isomorphism theorem can be proved by computing symbols, once we know Bloch–Kato's theorem. See also a proof in [K1].

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