ISSN 1464-8997 (on line) 1464-8989 (printed)

Geometry & Topology Monographs
Volume 3: Invitation to higher local fields
Part I, section 6, pages 61–74

6. Topological Milnor K-groups of higher local fields

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Let $F = K_n, \dots, K_0 = \mathbb{F}_q$ be an n-dimensional local field. We use the notation of section 1.

In this section we describe properties of certain quotients $K^{\mathrm{top}}(F)$ of the Milnor K-groups of F by using in particular topological considerations. This is an updated and simplified summary of relevant results in [F1–F5]. Subsection 6.1 recalls well-known results on K-groups of classical local fields. Subsection 6.2 discusses so called sequential topologies which are important for the description of subquotients of $K^{\mathrm{top}}(F)$ in terms of a simpler objects endowed with sequential topology (Theorem 1 in 6.6 and Theorem 1 in 7.2 of section 7). Subsection 6.3 introduces $K^{\mathrm{top}}(F)$, 6.4 presents very useful pairings (including Vostokov's symbol which is discussed in more detail in section 8), subsection 6.5–6.6 describe the structure of $K^{\mathrm{top}}(F)$ and 6.7 deals with the quotients $K(F)/l$; finally, 6.8 presents various properties of the norm map on K-groups. Note that subsections 6.6–6.8 are not required for understanding Parshin's class field theory in section 7.

6.0. Introduction

Let A be a commutative ring and let X be an A-module endowed with some topology. A set $\{x_i\}_{i\in I}$ of elements of X is called a set of *topological generators* of X if the sequential closure of the submodule of X generated by this set coincides with X. A set of topological generators is called a *topological basis* if for every $j \in I$ and every non-zero $a \in A$ ax_j doesn't belong to the sequential closure of the submodule generated by $\{x_i\}_{i\neq j}$.

Let I be a countable set. If $\{x_i\}$ is set of topological generators of X then every element $x \in X$ can be expressed as a convergent sum $\sum a_i x_i$ with some $a_i \in A$ (note that it is not necessarily the case that for all $a_i \in A$ the sum $\sum a_i x_i$ converges). This expression is unique if $\{x_i\}$ is a topological basis of X; then provided addition in X is sequentially continuous, we get $\sum a_i x_i + \sum b_i x_i = \sum (a_i + b_i) x_i$.

Recall that in the one-dimensional case the group of principal units $U_{1,F}$ is a multiplicative $\mathbb{Z}_p$-module with finitely many topological generators if $\operatorname{char}(F) = 0$ and infinitely many topological generators if $\operatorname{char}(F) = p$ (see for instance [FV, Ch. I §6]). This representation of $U_{1,F}$ and a certain specific choice of its generators is quite important if one wants to deduce the Shafarevich and Vostokov explicit formulas for the Hilbert symbol (see section 8).

Similarly, the group V_F of principal units of an n-dimensional local field F is topologically generated by $1 + \theta t_n^{i_n} \dots t_1^{i_1}$, $\theta \in \mu_{q-1}$ (see subsection 1.4.2). This leads to a natural suggestion to endow the Milnor K-groups of F with an appropriate topology and use the sequential convergence to simplify calculations in K-groups.

On the other hand, the reciprocity map

$$\Psi_F \colon K_n(F) \to \operatorname{Gal}(F^{\mathrm{ab}}/F)$$

is not injective in general, in particular $\ker(\Psi_F) \supset \bigcap_{l\geqslant 1} lK_n(F) \neq 0$. So the Milnor K-groups are too large from the point of view of class field theory, and one can pass to the quotient $K_n(F)/\bigcap_{l\geqslant 1} lK_n(F)$ without loosing any arithmetical information on F. The latter quotient coincides with $K_n^{\mathrm{top}}(F)$ (see subsection 6.6) which is defined in subsection 6.3 as the quotient of $K_n(F)$ by the intersection $\Lambda_n(F)$ of all neighbourhoods of 0 in $K_n(F)$ with respect to a certain topology. The existence theorem in class field theory uses the topology to characterize norm subgroups $N_{L/F}K_n(L)$ of finite abelian extensions L of F as open subgroups of finite index of $K_n(F)$ (see subsection 10.5). As a corollary of the existence theorem in 10.5 one obtains that in fact

$$\bigcap_{l\geqslant 1} lK_n(F) = \Lambda_n(F) = \ker(\Psi_F).$$

However, the class of open subgroups of finite index of $K_n(F)$ can be defined without introducing the topology on $K_n(F)$, see the paper of Kato in this volume which presents a different approach.

6.1. K-groups of one-dimensional local fields

The structure of the Milnor K-groups of a one-dimensional local field F is completely known.

Recall that using the Hilbert symbol and multiplicative $\mathbb{Z}_p$-basis of the group of principal units of F one obtains that

$$K_2(F) \simeq \operatorname{Tors} K_2(F) \oplus mK_2(F), \qquad \text{where } m = |\operatorname{Tors} F^*|,\ \operatorname{Tors} K_2(F) \simeq \mathbb{Z}/m$$

and $mK_2(F)$ is an uncountable uniquely divisible group (Bass, Tate, Moore, Merkur'ev; see for instance [FV, Ch. IX §4]). The groups $K_m(F)$ for $m \geqslant 3$ are uniquely divisible uncountable groups (Kahn [Kn], Sivitsky [FV, Ch. IX §4]).

6.2. Sequential topology

We need slightly different topologies from the topology of F and F^* introduced in section 1.

Definition. Let X be a topological space with topology τ. Define its *sequential saturation* λ:
a subset U of X is open with respect to λ if for every $\alpha \in U$ and a convergent (with respect to τ) sequence $X \ni \alpha_i$ to α almost all α_i belong to U. Then $\alpha_i \underset{\tau}{\to} \alpha \Leftrightarrow \alpha_i \underset{\lambda}{\to} \alpha$.

Hence the sequential saturation is the strongest topology which has the same convergent sequences and their limits as the original one. For a very elementary introduction to sequential topologies see [S].

Definition. For an n-dimensional local field F denote by λ the sequential saturation of the topology on F defined in section 1.

The topology λ is different from the old topology on F defined in section 1 for $n \geqslant 2$: for example, if $F = \mathbb{F}_p((t_1))((t_2))$ then $Y = F \setminus \{t_2^i t_1^{-j} + t_2^{-i} t_1^j : i, j \geqslant 1\}$ is open with respect to λ and is not open with respect to the topology of F defined in section 1.

Let λ_* on F^* be the sequential saturation of the topology τ on F^* defined in section 1. It is a shift invariant topology.

If $n = 1$, the restriction of λ_* on V_F coincides with the induced from λ.

The following properties of λ (λ_*) are similar to those in section 1 and/or can be proved by induction on dimension.

Properties.

(1) $\alpha_i, \beta_i \underset{\lambda}{\to} 0 \Rightarrow \alpha_i - \beta_i \underset{\lambda}{\to} 0$;

(2) $\alpha_i, \beta_i \underset{\lambda_*}{\longrightarrow} 1 \Rightarrow \alpha_i \beta_i^{-1} \underset{\lambda_*}{\longrightarrow} 1$;

(3) for every $\alpha_i \in U_F$, $\alpha_i^{p^i} \underset{\lambda_*}{\longrightarrow} 1$;

(4) multiplication is not continuous in general with respect to λ_*;

(5) every fundamental sequence with respect to λ (resp. λ_*) converges;

(6) V_F and F^{*m} are closed subgroups of F^* for every $m \geqslant 1$;

(7) The intersection of all open subgroups of finite index containing a closed subgroup H coincides with H.

Definition. For topological spaces $X_1, \dots, X_j$ define the $*$-product topology on $X_1 \times \dots \times X_j$ as the sequential saturation of the product topology.

6.3. K^{top}-groups

Definition. Let λ_m be the strongest topology on $K_m(F)$ such that subtraction in $K_m(F)$ and the natural map

$$\varphi: (F^*)^m \to K_m(F), \quad \varphi(\alpha_1, \dots, \alpha_m) = \{(\alpha_1, \dots, \alpha_m\}$$

are sequentially continuous. Then the topology λ_m coincides with its sequential saturation. Put

$$\Lambda_m(F) = \bigcap \text{ open neighbourhoods of } 0.$$

It is straightforward to see that $\Lambda_m(F)$ is a subgroup of $K_m(F)$.

Properties.

(1) $\Lambda_m(F)$ is closed: indeed $\Lambda_m(F) \ni x_i \to x$ implies that $x = x_i + y_i$ with $y_i \to 0$, so $x_i, y_i \to 0$, hence $x = x_i + y_i \to 0$, so $x \in \Lambda_m(F)$.

(2) Put $VK_m(F) = \langle \{V_F\} \cdot K_{m-1}(F) \rangle$ (V_F is defined in subsection 1.1). Since the topology with $VK_m(F)$ and its shifts as a system of fundamental neighbourhoods satisfies two conditions of the previous definition, one obtains that $\Lambda_m(F) \subset VK_m(F)$.

(3) $\lambda_1 = \lambda_*$.

Following the original approach of Parshin [P1] introduce now the following:

Definition. Set

$$K_m^{\mathrm{top}}(F) = K_m(F)/\Lambda_m(F)$$

and endow it with the quotient topology of λ_m which we denote by the same notation.

This new group $K_m^{\mathrm{top}}(F)$ is sometimes called the *topological Milnor K-group* of F.

If $\mathrm{char}(K_{n-1}) = p$ then $K_1^{\mathrm{top}} = K_1$.

If $\mathrm{char}(K_{n-1}) = 0$ then $K_1^{\mathrm{top}}(K) \neq K_1(K)$, since $1 + \mathcal{M}_{K_n}$ (which is uniquely divisible) is a subgroup of $\Lambda_1(K)$.

6.4. Explicit pairings

Explicit pairings of the Milnor K-groups of F are quite useful if one wants to study the structure of K^{top}-groups.

The general method is as follows. Assume that there is a pairing

$$\langle\ ,\ \rangle: A \times B \to \mathbb{Z}/m$$

of two $\mathbb{Z}/m$-modules A and B. Assume that A is endowed with a topology with respect to which it has topological generators α_i where i runs over elements of a totally ordered countable set I. Assume that for every $j \in I$ there is an element $\beta_j \in B$ such that

$$\langle \alpha_j, \beta_j \rangle = 1 \mod m, \qquad \langle \alpha_i, \beta_j \rangle = 0 \mod m \quad \text{for all } i > j\,.$$

Then if a convergent sum $\sum c_i \alpha_i$ is equal to 0, assume that there is a minimal j with non-zero c_j and deduce that

$$0 = \sum c_i \langle \alpha_i, \beta_j \rangle = c_j,$$

a contradiction. Thus, $\{\alpha_i\}$ form a topological basis of A.

If, in addition, for every $\beta \in B \setminus \{0\}$ there is an $\alpha \in A$ such that $\langle \alpha, \beta \rangle \neq 0$, then the pairing $\langle\ ,\ \rangle$ is obviously non-degenerate.

Pairings listed below satisfy the assumptions above and therefore can be applied to study the structure of quotients of the Milnor K-groups of F.

6.4.1. "Valuation map".

Let $\partial\colon K_r(K_s) \to K_{r-1}(K_{s-1})$ be the border homomorphism (see for example [FV, Ch. IX §2]). Put

$$\mathfrak{v} = \mathfrak{v}_F\colon K_n(F) \xrightarrow{\partial} K_{n-1}(K_{n-1}) \xrightarrow{\partial} \dots \xrightarrow{\partial} K_0(K_0) = \mathbb{Z}, \quad \mathfrak{v}(\{t_1, \dots, t_n\}) = 1$$

for a system of local parameters $t_1, \dots, t_n$ of F. The valuation map $\mathfrak{v}$ doesn't depend on the choice of a system of local parameters.

6.4.2. Tame symbol.

Define

$$t\colon K_n(F)/(q-1) \times F^*/F^{*\,q-1} \to K_{n+1}(F)/(q-1) \to \mathbb{F}_q^* \to \mu_{q-1}, \quad q = |K_0|$$

by

$$K_{n+1}(F)/(q-1) \xrightarrow{\partial} K_n(K_{n-1})/(q-1) \xrightarrow{\partial} \dots \xrightarrow{\partial} K_1(K_0)/(q-1) = \mathbb{F}_q^* \to \mu_{q-1}.$$

Here the map $\mathbb{F}_q^* \to \mu_{q-1}$ is given by taking multiplicative representatives.

An explicit formula for this symbol (originally asked for in [P2] and suggested in [F1]) is simple: let $t_1, \dots, t_n$ be a system of local parameters of F and let $\mathbf{v} = (v_1, \dots, v_n)$ be the associated valuation of rank n (see section 1 of this volume). For elements $\alpha_1, \dots, \alpha_{n+1}$ of F^* the value $t(\alpha_1, \alpha_2, \dots, \alpha_{n+1})$ is equal to the $(q-1)$th root of unity whose residue is equal to the residue of

$$\alpha_1^{b_1} \dots \alpha_{n+1}^{b_{n+1}} (-1)^b$$

in the last residue field $\mathbb{F}_q$, where $b = \sum_{s,i<j} v_s(b_i) v_s(b_j) b_{i,j}^s$, b_j is the determinant of the matrix obtained by cutting off the jth column of the matrix $A = (v_i(\alpha_j))$ with the sign $(-1)^{j-1}$, and $b_{i,j}^s$ is the determinant of the matrix obtained by cutting off the ith and jth columns and sth row of A.

6.4.3. Artin–Schreier–Witt pairing in characteristic p.
Define, following [P2], the pairing

$$(\ , \ ]_r : K_n(F)/p^r \times W_r(F)/(\mathbf{F}-1)W_r(F) \to W_r(\mathbb{F}_p) \simeq \mathbb{Z}/p^r$$

by ($\mathbf{F}$ is the map defined in the section, Some Conventions)

$$(\alpha_0, \dots, \alpha_n, (\beta_0, \dots, \beta_r)]_r = \operatorname{Tr}_{K_0/\mathbb{F}_p} \quad (\gamma_0, \dots, \gamma_r)$$

where the i th ghost component $\gamma^{(i)}$ is given by $\operatorname{res}_{K_0}\ (\beta^{(i)}\alpha_1^{-1}d\alpha_1 \wedge \dots \wedge \alpha_n^{-1}d\alpha_n)$.
For its properties see [P2, sect. 3]. In particular,

(1) for $x \in K_n(F)$

$$(x, \mathbf{V}(\beta_0, \dots, \beta_{r-1})]_r = \mathbf{V}(x, (\beta_0, \dots, \beta_{r-1})]_{r-1}$$

where as usual for a field K

$$\mathbf{V}: W_{r-1}(K) \to W_r(K), \quad \mathbf{V}(\beta_0, \dots, \beta_{r-1}) = (0, \beta_0, \dots, \beta_{r-1});$$

(2) for $x \in K_n(F)$

$$(x, \mathbf{A}(\beta_0, \dots, \beta_r)]_{r-1} = \mathbf{A}(x, (\beta_0, \dots, \beta_r)]_r$$

where for a field K

$$\mathbf{A}: W_r(K) \to W_{r-1}(K), \quad \mathbf{A}(\beta_0, \dots, \beta_{r-1}, \beta_r) = (\beta_0, \dots, \beta_{r-1}).$$

(3) If $\operatorname{Tr}\theta_0 = 1$ then $\big(\{t_1, \dots, t_n\}, \theta_0\big]_1 = 1$. If i_l is prime to p then

$$\big(\{1 + \theta t_n^{i_n} \dots t_1^{i_1}, t_1, \dots, \widehat{t_l}, \dots, t_n\}, \theta_0\theta^{-1} i_l^{-1} t_1^{-i_1} \dots t_n^{-i_n}]_1 = 1.$$

6.4.4. Vostokov's symbol in characteristic 0.
Suppose that $\mu_{p^r} \leqslant F^*$ and $p > 2$. Vostokov's symbol

$$V(\ , \ )_r : K_m(F)/p^r \times K_{n+1-m}(F)/p^r \to K_{n+1}(F)/p^r \to \mu_{p^r}$$

is defined in section 8.3. For its properties see 8.3.

Each pairing defined above is sequentially continuous, so it induces the pairing of $K_m^{\mathrm{top}}(F)$.

6.5. Structure of $K^{\mathrm{top}}(F)$. I

Denote $VK_m^{\mathrm{top}}(F) = \big\langle \{V_F\} \cdot K_{m-1}^{\mathrm{top}}(F) \big\rangle$. Using the tame symbol and valuation $\mathfrak{v}$ as described in the beginning of 6.4 it is easy to deduce that

$$K_m(F) \simeq VK_m(F) \oplus \mathbb{Z}^{a(m)} \oplus (\mathbb{Z}/(q-1))^{b(m)}$$

with appropriate integer $a(m), b(m)$ (see [FV, Ch. IX, §2]); similar calculations are applicable to $K_m^{\mathrm{top}}(F)$. For example, $\mathbb{Z}^{a(m)}$ corresponds to $\oplus\langle\{t_{j_1}, \dots, t_{j_m}\}\rangle$ with $1 \leqslant j_1 < \dots < j_m \leqslant n$.

To study $VK_m(F)$ and $VK_m^{\mathrm{top}}(F)$ the following elementary equality is quite useful

$$\{1-\alpha, 1-\beta\} = \left\{\alpha, 1+\frac{\alpha\beta}{1-\alpha}\right\} + \left\{1-\beta, 1+\frac{\alpha\beta}{1-\alpha}\right\}.$$

Note that $\mathbf{v}(\alpha\beta/(1-\alpha)) = \mathbf{v}(\alpha) + \mathbf{v}(\beta)$ if $\mathbf{v}(\alpha), \mathbf{v}(\beta) > (0, \dots, 0)$.

For $\varepsilon, \eta \in V_F$ one can apply the previous formula to $\{\varepsilon, \eta\} \in K_2^{\mathrm{top}}(F)$ and using the topological convergence deduce that

$$\{\varepsilon, \eta\} = \sum \{\rho_i, t_i\}$$

with units $\rho_i = \rho_i(\varepsilon, \eta)$ sequentially continuously depending on ε, η.

Therefore $VK_m^{\mathrm{top}}(F)$ is *topologically generated* by symbols

$$\left\{1+\theta t_n^{i_n} \dots t_1^{i_1}, t_{j_1} \dots, t_{j_{m-1}}\right\}, \quad \theta \in \mu_{q-1}.$$

In particular, $K_{n+2}^{\mathrm{top}}(F) = 0$.

Lemma. $\bigcap_{l \geqslant 1} lK_m(F) \subset \Lambda_m(F)$.

Proof. First, $\bigcap lK_m(F) \subset VK_m(F)$. Let $x \in VK_m(F)$. Write

$$x = \sum \left\{\alpha_J, t_{j_1}, \dots, t_{j_{m-1}}\right\} \mod \Lambda_m(F), \quad \alpha_J \in V_F.$$

Then

$$p^r x = \sum \left\{\alpha_J^{p^r}\right\} \cdot \left\{t_{j_1}, \dots, t_{j_{m-1}}\right\} + \lambda_r, \quad \lambda_r \in \Lambda_m(F).$$

It remains to apply property (3) in 6.2. □

6.6. Structure of $K^{\mathrm{top}}(F)$. II

This subsection 6.6 and the rest of this section are not required for understanding Parshin's class field theory theory of higher local fields of characteristic p which is discussed in section 7.

The next theorem relates the structure of $VK_m^{\mathrm{top}}(F)$ with the structure of a simpler object.

Theorem 1 ([F5, Th. 4.6]). *Let* $\mathrm{char}(K_{n-1}) = p$. *The homomorphism*

$$g\colon \prod_J V_F \to VK_m(F), \quad (\beta_J) \mapsto \sum_{J=\{j_1,\dots,j_{m-1}\}} \left\{\beta_J, t_{j_1}, \dots, t_{j_{m-1}}\right\}$$

induces a homeomorphism between $\prod V_F/g^{-1}(\Lambda_m(F))$ *endowed with the quotient of the* $*$*-topology and* $VK_m^{\mathrm{top}}(F)$*;* $g^{-1}(\Lambda_m(F))$ *is a closed subgroup.*

Since every closed subgroup of V_F is the intersection of some open subgroups of finite index in V_F (property (7) of 6.2), we obtain the following:

Corollary. $\Lambda_m(F) = \bigcap$ *open subgroups of finite index in* $K_m(F)$.

Remarks. 1. If F is of characteristic p, there is a complete description of the structure of $K_m^{\mathrm{top}}(F)$ in the language of topological generators and relations due to Parshin (see subsection 7.2).

2. If $\mathrm{char}(K_{n-1}) = 0$, then the border homomorphism in Milnor K-theory (see for instance [FV, Ch. IX §2]) induces the homomorphism

$$VK_m(F) \to VK_m(K_{n-1}) \oplus VK_{m-1}(K_{n-1}).$$

Its kernel is equal to the subgroup of $VK_m(F)$ generated by symbols $\{u, \dots\}$ with u in the group $1 + \mathcal{M}_F$ which is uniquely divisible. So

$$VK_m^{\mathrm{top}}(F) \simeq VK_m^{\mathrm{top}}(K_{n-1}) \oplus VK_{m-1}^{\mathrm{top}}(K_{n-1})$$

and one can apply Theorem 1 to describe $VK_m^{\mathrm{top}}(F)$.

Proof of Theorem 1. Recall that every symbol $\{\alpha_1, \dots, \alpha_m\}$ in $K_m^{\mathrm{top}}(F)$ can be written as a convergent sum of symbols $\{\beta_J, t_{j_1}, \dots, t_{j_{m-1}}\}$ with β_J sequentially continuously depending on α_i (subsection 6.5). Hence there is a sequentially continuous map $f: V_F \times F^{*\oplus m-1} \to \prod_J V_F$ such that its composition with g coincides with the restriction of the map $\varphi: (F^*)^m \to K_m^{\mathrm{top}}(F)$ on $V_F \oplus F^{*\oplus m-1}$.

So the quotient of the $*$-topology of $\prod_J V_F$ is $\leqslant \lambda_m$, as follows from the definition of λ_m. Indeed, the sum of two convergent sequences x_i, y_i in $\prod_J V_F/g^{-1}(\Lambda_m(F))$ converges to the sum of their limits.

Let U be an open subset in $VK_m(F)$. Then $g^{-1}(U)$ is open in the $*$-product of the topology $\prod_J V_F$. Indeed, otherwise for some J there were a sequence $\alpha_J^{(i)} \notin g^{-1}(U)$ which converges to $\alpha_J \in g^{-1}(U)$. Then the properties of the map φ of 6.3 imply that the sequence $\varphi(\alpha_J^{(i)}) \notin U$ converges to $\varphi(\alpha_J) \in U$ which contradicts the openness of U. $\square$

Theorem 2 ([F5, Th. 4.5])**.** *If* $\mathrm{char}(F) = p$ *then* $\Lambda_m(F)$ *is equal to* $\bigcap_{l \geqslant 1} lK_m(F)$ *and is a divisible group.*

Proof. Bloch–Kato–Gabber's theorem (see subsection A2 in the appendix to section 2) shows that the differential symbol

$$d: K_m(F)/p \longrightarrow \Omega_F^m, \qquad \{\alpha_1, \dots, \alpha_m\} \longmapsto \frac{d\alpha_1}{\alpha_1} \wedge \dots \wedge \frac{d\alpha_m}{\alpha_m}$$

is injective. The topology of Ω_F^m induced from F (as a finite dimensional vector space) is Hausdorff, and d is continuous, so $\Lambda_m(F) \subset pK_m(F)$.

Since $VK_m(F)/\Lambda_m(F) \simeq \prod \mathcal{E}_J$ doesn't have p-torsion by Theorem 1 in subsection 7.2, $\Lambda_m(F) = p\Lambda_m(F)$. □

Theorem 3 ([F5, Th. 4.7])**.** *If* $\mathrm{char}(F) = 0$ *then* $\Lambda_m(F)$ *is equal to* $\bigcap_{l \geqslant 1} lK_m(F)$ *and is a divisible group. If a primitive* l*th root* ζ_l *belongs to* F*, then* ${}_lK_m^{\mathrm{top}}(F) = \{\zeta_l\} \cdot K_{m-1}^{\mathrm{top}}(F)$*.*

Proof. To show that $p^rVK_m(F) \supset \Lambda_m(F)$ it suffices to check that $p^rVK_m(F)$ is the intersection of open neighbourhoods of $p^rVK_m(F)$.

We can assume that μ_p is contained in F applying the standard argument by using $(p, |F(\mu_p) : F|) = 1$ and l-divisibility of $VK_m(F)$ for l prime to p.

If $r = 1$ then one can use Bloch–Kato's description of

$$U_iK_m(F) + pK_m(F)/U_{i+1}K_m(F) + pK_m(F)$$

in terms of products of quotients of $\Omega^j_{K_{n-1}}$ (section 4). $\Omega^j_{K_{n-1}}$ and its quotients are finite-dimensional vector spaces over K_{n-1}/K^p_{n-1}, so the intersection of all neighborhoods of zero there with respect to the induced from K_{n-1} topology is trivial. Therefore the injectivity of d implies $\Lambda_m(F) \subset pK_m(F)$.

Thus, the intersection of open subgroups in $VK_m(F)$ containing $pVK_m(F)$ is equal to $pVK_m(F)$.

Induction Step.

For a field F consider the pairing

$$(\ ,\ )_r \colon K_m(F)/p^r \times H^{n+1-m}(F, \mu_{p^r}^{\otimes n-m}) \to H^{n+1}(F, \mu_{p^r}^{\otimes n})$$

given by the cup product and the map $F^* \to H^1(F, \mu_{p^r})$. If $\mu_{p^r} \subset F$, then Bloch–Kato's theorem shows that $(\ ,\ )_r$ can be identified (up to sign) with Vostokov's pairing $V_r(\ ,\ )$.

For $\chi \in H^{n+1-m}(F, \mu_{p^r}^{\otimes n-m})$ put

$$A_\chi = \{\alpha \in K_m(F) : (\alpha, \chi)_r = 0\}.$$

One can show [F5, Lemma 4.7] that A_χ is an open subgroup of $K_m(F)$.

Let α belong to the intersection of all open subgroups of $VK_m(F)$ which contain $p^rVK_m(F)$. Then $\alpha \in A_\chi$ for every $\chi \in H^{n+1-m}(F, \mu_{p^r}^{\otimes n-m})$.

Set $L = F(\mu_{p^r})$ and $p^s = |L : F|$. From the induction hypothesis we deduce that $\alpha \in p^sVK_m(F)$ and hence $\alpha = N_{L/F}\beta$ for some $\beta \in VK_m(L)$. Then

$$0 = (\alpha, \chi)_{r,F} = (N_{L/F}\beta, \chi)_{r,F} = (\beta, i_{F/L}\chi)_{r,L}$$

where $i_{F/L}$ is the natural map. Keeping in mind the identification between Vostokov's pairing V_r and $(\ ,\ )_r$ for the field L we see that β is annihilated by $i_{F/L}K_{n+1-m}(F)$

with respect to Vostokov's pairing. Using explicit calculations with Vostokov's pairing one can directly deduce that

$$\beta \in (\sigma - 1)K_m(L) + p^{r-s} i_{F/L} K_m(F) + p^r K_m(L),$$

and therefore $\alpha \in p^r K_m(F)$, as required.

Thus $p^r K_m(F) = \bigcap$ open neighbourhoods of $p^r V K_m(F)$.

To prove the second statement we can assume that l is a prime. If $l \neq p$, then since $K_m^{\mathrm{top}}(F)$ is the direct sum of several cyclic groups and $VK_m^{\mathrm{top}}(F)$ and since l-torsion of $K_m^{\mathrm{top}}(F)$ is p-divisible and $\cap_r p^r V K_m^{\mathrm{top}}(F) = \{0\}$, we deduce the result.

Consider the most difficult case of $l = p$. Use the exact sequence

$$0 \to \mu_{p^s}^{\otimes n} \to \mu_{p^{s+1}}^{\otimes n} \to \mu_p^{\otimes n} \to 0$$

and the following commutative diagram (see also subsection 4.3.2)

$$\begin{array}{ccccc}
\mu_p \otimes K_{m-1}(F)/p & \longrightarrow & K_m(F)/p^s & \xrightarrow{p} & K_m(F)/p^{s+1} \\
\downarrow & & \downarrow & & \downarrow \\
H^{m-1}(F, \mu_p^{\otimes m}) & \longrightarrow & H^m(F, \mu_{p^s}^{\otimes m}) & \longrightarrow & H^m(F, \mu_{p^{s+1}}^{\otimes m}).
\end{array}$$

We deduce that $px \in \Lambda_m(F)$ implies $px \in \bigcap p^r K_m(F)$, so $x = \{\zeta_p\} \cdot a_{r-1} + p^{r-1} b_{r-1}$ for some $a_i \in K_{m-1}^{\mathrm{top}}(F)$ and $b_i \in K_m^{\mathrm{top}}(F)$.

Define $\psi \colon K_{m-1}^{\mathrm{top}}(F) \to K_m^{\mathrm{top}}(F)$ as $\psi(\alpha) = \{\zeta_p\} \cdot \alpha$; it is a continuous map. Put $D_r = \psi^{-1}(p^r K_m^{\mathrm{top}}(F))$. The group $D = \cap D_r$ is the kernel of ψ. One can show [F5, proof of Th. 4.7] that $\{a_r\}$ is a Cauchy sequence in the space $K_{m-1}^{\mathrm{top}}(F)/D$ which is complete. Hence there is $y \in \bigcap \left(a_{r-1} + D_{r-1}\right)$. Thus, $x = \{\zeta_p\} \cdot y$ in $K_m^{\mathrm{top}}(F)$.

Divisibility follows. □

Remarks. 1. Compare with Theorem 8 in 2.5.

2. For more properties of $K_m^{\mathrm{top}}(F)$ see [F5].

3. Zhukov [Z, §7–10] gave a description of $K_n^{\mathrm{top}}(F)$ in terms of topological generators and relations for some fields F of characteristic zero with small $v_F(p)$.

6.7. The group $K_m(F)/l$

6.7.1. If a prime number l is distinct from p, then, since V_F is l-divisible, we deduce from 6.5 that

$$K_m(F)/l \simeq K_m^{\mathrm{top}}(F)/l \simeq (\mathbb{Z}/l)^{a(m)} \oplus (\mathbb{Z}/d)^{b(m)}$$

where $d = \gcd(q-1, l)$.

6.7.2. The case of $l = p$ is more interesting and difficult. We use the method described at the beginning of 6.4.

If $\operatorname{char}(F) = p$ then the Artin–Schreier pairing of 6.4.3 for $r = 1$ helps one to show that $K_n^{\mathrm{top}}(F)/p$ has the following topological $\mathbb{Z}/p$-basis:

$$\left\{1 + \theta t_n^{i_n} \dots t_1^{i_1}, t_n, \dots, \widehat{t_l}, \dots, t_1\right\}$$

where $p \nmid \gcd(i_1, \dots, i_n)$, $0 < (i_1, \dots, i_n)$, $l = \min\{k : p \nmid i_k\}$ and θ runs over all elements of a fixed basis of K_0 over $\mathbb{F}_p$.

If $\operatorname{char}(F) = 0$, $\zeta_p \in F^*$, then using Vostokov's symbol (6.4.4 and 8.3) one obtains that $K_n^{\mathrm{top}}(F)/p$ has the following topological $\mathbb{Z}_p$-basis consisting of elements of two types:

$$\omega_*(j) = \left\{1 + \theta_* t_n^{pe_n/(p-1)} \dots t_1^{pe_1/(p-1)}, t_n, \dots, \widehat{t_j}, \dots, t_1\right\}$$

where $1 \leqslant j \leqslant n$, $(e_1, \dots, e_n) = \mathbf{v}_F(p)$ and $\theta_* \in \mu_{q-1}$ is such that
$1 + \theta_* t_n^{pe_n/(p-1)} \dots t_1^{pe_1/(p-1)}$ doesn't belong to F^{*p}
and

$$\left\{1 + \theta t_n^{i_n} \dots t_1^{i_1}, t_n, \dots, \widehat{t_l}, \dots, t_1\right\}$$

where $p \nmid \gcd(i_1, \dots, i_n)$, $0 < (i_1, \dots, i_n) < p(e_1, \dots, e_n)/(p-1)$,
$l = \min\{k : p \nmid i_k\}$, where θ runs over all elements of a fixed basis of K_0 over $\mathbb{F}_p$.

If $\zeta_p \notin F^*$, then pass to the field $F(\zeta_p)$ and then go back, using the fact that the degree of $F(\zeta_p)/F$ is relatively prime to p. One deduces that $K_n^{\mathrm{top}}(F)/p$ has the following topological $\mathbb{Z}_p$-basis:

$$\left\{1 + \theta t_n^{i_n} \dots t_1^{i_1}, t_n, \dots, \widehat{t_l}, \dots, t_1\right\}$$

where $p \nmid \gcd(i_1, \dots, i_n)$, $0 < (i_1, \dots, i_n) < p(e_1, \dots, e_n)/(p-1)$,
$l = \min\{k : p \nmid i_k\}$, where θ runs over all elements of a fixed basis of K_0 over $\mathbb{F}_p$.

6.8. The norm map on K^{top}-groups

Definition. Define the norm map on $K_n^{\mathrm{top}}(F)$ as induced by $N_{L/F}\colon K_n(L) \to K_n(F)$.

Alternatively in characteristic p one can define the norm map as in 7.4.

6.8.1. Put $u_{i_1,\dots,i_n} = U_{i_1,\dots,i_n}/U_{i_1+1,\dots,i_n}$.

Proposition ([F2, Prop. 4.1] and [F3, Prop. 3.1])**.** *Let L/F be a cyclic extension of prime degree l such that the extension of the last finite residue fields is trivial. Then there is s and a local parameter $t_{s,L}$ of L such that $L = F(t_{s,L})$. Let $t_1, \dots, t_n$ be a system of local parameters of F, then $t_1, \dots, t_{s,L}, \dots, t_n$ is a system of local parameters of L.*

Let $l = p$. *For a generator* σ *of* $\mathrm{Gal}(L/F)$ *let*

$$\frac{\sigma t_{s,L}}{t_{s,L}} = 1 + \theta_0 t_n^{r_n} \cdots t_{s,L}^{r_s} \cdots t_1^{r_1} + \cdots$$

Then

(1) *if* $(i_1, \dots, i_n) < (r_1, \dots, r_n)$ *then*

$$N_{L/F}\colon u_{i_1,\dots,i_n,L} \to u_{pi_1,\dots,i_s,\dots,pi_n,F}$$

sends $\theta \in K_0$ *to* θ^p;

(2) *if* $(i_1, \dots, i_n) = (r_1, \dots, r_n)$ *then*

$$N_{L/F}\colon u_{i_1,\dots,i_n,L} \to u_{pi_1,\dots,i_s,\dots,pi_n,F}$$

sends $\theta \in K_0$ *to* $\theta^p - \theta\theta_0^{p-1}$;

(3) *if* $(j_1, \dots, j_n) > 0$ *then*

$$N_{L/F}\colon u_{j_1+r_1,\dots,pj_s+r_s,\dots,j_n+r_n,L} \to u_{j_1+pr_1,\dots,j_s+r_s,\dots,j_n+pr_n,F}$$

sends $\theta \in K_0$ *to* $-\theta\theta_0^{p-1}$.

Proof. Similar to the one-dimensional case [FV, Ch. III §1]. □

6.8.2. If L/F is cyclic of prime degree l then

$$K_n^{\mathrm{top}}(L) = \left\langle \{L^*\} \cdot i_{F/L} K_{n-1}^{\mathrm{top}}(F) \right\rangle$$

where $i_{F/L}$ is induced by the embedding $F^* \to L^*$. For instance (we use the notations of section 1), if $f(L|F) = l$ then L is generated over F by a root of unity of order prime to p; if $e_i(L|F) = l$, then use the previous proposition.

Corollary 1. *Let* L/F *be a cyclic extension of prime degree* l. *Then*

$$|K_n^{\mathrm{top}}(F) : N_{L/F} K_n^{\mathrm{top}}(L)| = l.$$

If L/F is as in the preceding proposition, then the element

$$\left\{1 + \theta_* t_n^{pr_n} \cdots t_{s,F}^{r_s} \cdots t_1^{pr_1}, t_1, \dots, \widehat{t_s}, \dots, t_n\right\},$$

where the residue of θ_* in K_0 doesn't belong to the image of the map

$$\mathcal{O}_F \xrightarrow{\theta \mapsto \theta^p - \theta\theta_0^{p-1}} \mathcal{O}_F \to K_0,$$

is a generator of $K_n^{\mathrm{top}}(F)/N_{L/F} K_n^{\mathrm{top}}(L)$.

If $f(L|F) = 1$ and $l \neq p$, then

$$\left\{\theta_*, t_1, \dots, \widehat{t_s}, \dots, t_n\right\}$$

where $\theta_* \in \mu_{q-1} \setminus \mu_{q-1}^l$ is a generator of $K_n^{\mathrm{top}}(F)/N_{L/F} K_n^{\mathrm{top}}(L)$.

If $f(L|F) = l$, then

$$\{t_1, \dots, t_n\}$$

is a generator of $K_n^{\mathrm{top}}(F)/N_{L/F}K_n^{\mathrm{top}}(L)$.

Corollary 2. *$N_{L/F}$ (closed subgroup) is closed and $N_{L/F}^{-1}$ (open subgroup) is open.*

Proof. Sufficient to show for an extension of prime degree; then use the previous proposition and Theorem 1 of 6.6. □

6.8.3.

Theorem 4 ([F2, §4], [F3, §3])**.** *Let L/F be a cyclic extension of prime degree l with a generator σ then the sequence*

$$K_n^{\mathrm{top}}(F)/l \oplus K_n^{\mathrm{top}}(L)/l \xrightarrow{i_{F/L} \oplus (1-\sigma)} K_n^{\mathrm{top}}(L)/l \xrightarrow{N_{L/F}} K_n^{\mathrm{top}}(F)/l$$

is exact.

Proof. Use the explicit description of K_n^{top}/l in 6.7. □

This theorem together with the description of the torsion of $K_n^{\mathrm{top}}(F)$ in 6.6 imply:

Corollary. *Let L/F be cyclic with a generator σ then the sequence*

$$K_n^{\mathrm{top}}(L) \xrightarrow{1-\sigma} K_n^{\mathrm{top}}(L) \xrightarrow{N_{L/F}} K_n^{\mathrm{top}}(F)$$

is exact.

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