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7. Parshin's higher local class field theory in characteristic p

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Parshin's theory in characteristic p is a remarkably simple and effective approach to all the main theorems of class field theory by using relatively few ingredients.

Let $F = K_n, \dots, K_0$ be an n-dimensional local field of characteristic p.

In this section we use the results and definitions of 6.1–6.5; we don't need the results of 6.6 – 6.8.

7.1

Recall that the group V_F is topologically generated by

$$1 + \theta t_n^{i_n} \dots t_1^{i_1}, \quad \theta \in \mathcal{R}^*, p \nmid (i_n, \dots, i_1)$$

(see 1.4.2). Note that

$$\begin{aligned} i_1 \dots i_n \{1 + \theta t_n^{i_n} \dots t_1^{i_1}, t_1, \dots, t_n\} &= \{1 + \theta t_n^{i_n} \dots t_1^{i_1}, t_1^{i_1}, \dots, t_n^{i_n}\} \\ = \{1 + \theta t_n^{i_n} \dots t_1^{i_1}, t_1^{i_1} \cdots t_n^{i_n}, \dots, t_n^{i_n}\} &= \{1 + \theta t_n^{i_n} \dots t_1^{i_1}, -\theta, \dots, t_n^{i_n}\} = 0, \end{aligned}$$

since $\theta^{q-1} = 1$ and V_F is $(q-1)$-divisible. We deduce that

$$K_{n+1}^{\mathrm{top}}(F) \simeq \mathbb{F}_q^*, \quad \{\theta, t_1, \dots, t_n\} \mapsto \theta, \quad \theta \in \mathcal{R}^*.$$

Recall that (cf. 6.5)

$$K_n^{\mathrm{top}}(F) \simeq \mathbb{Z} \oplus \left(\mathbb{Z}/(q-1)\right)^n \oplus V K_n^{\mathrm{top}}(F),$$

where the first group on the RHS is generated by $\{t_n, \dots, t_1\}$, and the second by $\{\theta, \dots, \widehat{t_l}, \dots\}$ (apply the tame symbol and valuation map of subsection 6.4).

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7.2. The structure of $VK_n^{\mathrm{top}}(F)$

Using the Artin–Schreier–Witt pairing (its explicit form in 6.4.3)

$$(\ ,\ ]_r\colon K_n^{\mathrm{top}}(F)/p^r \times W_r(F)/(\mathbf{F}-1)W_r(F) \to \mathbb{Z}/p^r,\ r \geqslant 1$$

and the method presented in subsection 6.4 we deduce that every element of $VK_n^{\mathrm{top}}(F)$ is uniquely representable as a convergent series

$$\sum a_{\theta,i_n,\dots,i_1}\{1+\theta t_n^{i_n}\dots t_1^{i_1}, t_1,\dots,\widehat{t_l},\dots,t_n\}, \quad a_{\theta,i_n,\dots,i_1} \in \mathbb{Z}_p,$$

where θ runs over a basis of the $\mathbb{F}_p$-space K_0, $p \nmid \gcd(i_n,\dots,i_1)$ and $l = \min\{k : p \nmid i_k\}$. We also deduce that the pairing $(\ ,\ ]_r$ is non-degenerate.

Theorem 1 (Parshin, [P2]). *Let* $J = \{j_1,\dots,j_{m-1}\}$ *run over all* $(m-1)$*-elements subsets of* $\{1,\dots,n\}$, $m \leqslant n+1$. *Let* $\mathcal{E}_J$ *be the subgroups of* V_F *generated by* $1+\theta t_n^{i_n}\dots t_1^{i_1}$, $\theta \in \mu_{q-1}$ *such that* $p \nmid \gcd(i_1,\dots,i_n)$ *and* $\min\{l : p \nmid i_l\} \notin J$. *Then the homomorphism*

$$h\colon \prod_J^{*-\text{topology}} \mathcal{E}_J \to VK_m^{\mathrm{top}}(F), \quad (\varepsilon_J) \mapsto \sum_{J=\{j_1,\dots,j_{m-1}\}} \{\varepsilon_J, t_{j_1},\dots,t_{j_{m-1}}\}$$

is a homeomorphism.

Proof. There is a sequentially continuous map $f\colon V_F \times F^{*\oplus m-1} \to \prod_J \mathcal{E}_J$ such that its composition with h coincides with the restriction of the map $\varphi\colon (F^*)^m \to K_m^{\mathrm{top}}(F)$ of 6.3 on $V_F \oplus F^{*\oplus m-1}$.

So the topology of $\prod_J^{*-\text{topology}} \mathcal{E}_J$ is $\leqslant \lambda_m$, as follows from the definition of λ_m.

Let U be an open subset in $VK_m(F)$. Then $h^{-1}(U)$ is open in the $*$-product of the topology $\prod_J \mathcal{E}_J$. Indeed, otherwise for some J there were a sequence $\alpha_J^{(i)} \notin h^{-1}(U)$ which converges to $\alpha_J \in h^{-1}(U)$. Then the sequence $\varphi(\alpha_J^{(i)}) \notin U$ converges to $\varphi(\alpha_J) \in U$ which contradicts the openness of U. $\square$

Corollary. $K_m^{\mathrm{top}}(F)$ *has no nontrivial* p*-torsion*; $\cap p^r VK_m^{\mathrm{top}}(F) = \{0\}$.

7.3

Put $\widetilde{W}(F) = \varinjlim W_r(F)/(\mathbf{F}-1)W_r(F)$ with respect to the homomorphism $\mathbf{V}\colon (a_0, \dots, a_{r-1}) \to \big(0, a_0, \dots, a_{r-1}\big)$. From the pairings (see 6.4.3)

$$K_n^{\mathrm{top}}(F)/p^r \times W_r(F)/(\mathbf{F}-1)W_r(F) \xrightarrow{(\ ,\ ]_r} \mathbb{Z}/p^r \to \frac{1}{p^r}\mathbb{Z}/\mathbb{Z}$$

one obtains a non-degenerate pairing

$$(\ ,\ ]\colon \widetilde{K}_n(F) \times \widetilde{W}(F) \to \mathbb{Q}_p/\mathbb{Z}_p$$

where $\widetilde{K}_n(F) = K_n^{\mathrm{top}}(F)/\bigcap_{r\geqslant 1} p^r K_n^{\mathrm{top}}(F)$. From 7.1 and Corollary of 7.2 we deduce

$$\bigcap_{r\geqslant 1} p^r K_n^{\mathrm{top}}(F) = \mathrm{Tors}_{p'} K_n^{\mathrm{top}}(F) = \mathrm{Tors}\, K_n^{\mathrm{top}}(F),$$

where $\mathrm{Tors}_{p'}$ is prime-to-p-torsion.

Hence

$$\widetilde{K}_n(F) = K_n^{\mathrm{top}}(F)/\,\mathrm{Tors}\, K_n^{\mathrm{top}}(F).$$

7.4. The norm map on K^{top}-groups in characteristic p

Following Parshin we present an alternative description (to that one in subsection 6.8) of the norm map on K^{top}-groups in characteristic p.

If L/F is cyclic of prime degree l, then it is more or less easy to see that

$$K_n^{\mathrm{top}}(L) = \big\langle \{L^*\} \cdot i_{F/L} K_{n-1}^{\mathrm{top}}(F) \big\rangle$$

where $i_{F/L}$ is induced by the embedding $F^* \to L^*$. For instance, if $f(L|F) = l$ then L is generated over F by a root of unity of order prime to p; if $e_i(L|F) = l$, then there is a system of local parameters $t_1, \dots, t_i', \dots, t_n$ of L such that $t_1, \dots, t_i, \dots, t_n$ is a system of local parameters of F.

For such an extension L/F define [P2]

$$N_{L/F}\colon K_n^{\mathrm{top}}(L) \to K_n^{\mathrm{top}}(F)$$

as induced by $N_{L/F}\colon L^* \to F^*$. For a separable extension L/F find a tower of subextensions

$$F = F_0 - F_1 - \cdots - F_{r-1} - F_r = L$$

such that F_i/F_{i-1} is a cyclic extension of prime degree and define

$$N_{L/F} = N_{F_1/F_0} \circ \cdots \circ N_{F_r/F_{r-1}}.$$

To prove correctness use the non-degenerate pairings of subsection 6.4 and the properties

$$(N_{L/F}\alpha, \beta]_{F,r} = (\alpha, i_{F/L}\beta]_{L,r}$$

for p-extensions;

$$t\left(N_{L/F}\alpha, \beta\right)_F = t(\alpha, i_{F/L}\beta)_L$$

for prime-to-p-extensions (t is the tame symbol of 6.4.2).

7.5. Parshin's reciprocity map

Parshin's theory [P2], [P3] deals with three partial reciprocity maps which then can be glued together.

Proposition ([P3]). *Let L/F be a cyclic p-extension. Then the sequence*

$$0 \to \widetilde{K}_n(F) \xrightarrow{i_{F/L}} \widetilde{K}_n(L) \xrightarrow{1-\sigma} \widetilde{K}_n(L) \xrightarrow{N_{L/F}} \widetilde{K}_n(F)$$

is exact and the cokernel of $N_{L/F}$ is a cyclic group of order $|L:F|$.

Proof. The sequence is dual (with respect to the pairing of 7.3) to

$$\widetilde{W}(F) \to \widetilde{W}(L) \xrightarrow{1-\sigma} \widetilde{W}(L) \xrightarrow{\mathrm{Tr}_{L/F}} \widetilde{W}(F) \to 0.$$

The norm group index is calculated by induction on degree. □

Hence the class of p-extensions of F and $\widetilde{K}_n(F)$ satisfy the classical class formation axioms. Thus, one gets a homomorphism $\widetilde{K}_n(F) \to \mathrm{Gal}(F^{\mathrm{abp}}/F)$ and

$$\Psi_F^{(p)}\colon K_n^{\mathrm{top}}(F) \to \mathrm{Gal}(F^{\mathrm{abp}}/F)$$

where F^{abp} is the maximal abelian p-extension of F. In the one-dimensional case this is Kawada–Satake's theory [KS].

The valuation map $\mathfrak{v}$ of 6.4.1 induces a homomorphism

$\Psi_F^{(\mathrm{ur})}\colon K_n^{\mathrm{top}}(F) \to \mathrm{Gal}(F_{\mathrm{ur}}/F)$,

$\{t_1, \dots, t_n\} \to$ the lifting of the Frobenius automorphism of K_0^{sep}/K_0;

and the tame symbol t of 6.4.2 together with Kummer theory induces a homomorphism

$$\Psi_F^{(p')}\colon K_n^{\mathrm{top}}(F) \to \mathrm{Gal}(F(\sqrt[q-1]{t_1}, \dots, \sqrt[q-1]{t_n})/F).$$

The three homomorphisms $\Psi_F^{(p)}$, $\Psi_F^{(\mathrm{ur})}$, $\Psi_F^{(p')}$ agree [P2], so we get the reciprocity map

$$\Psi_F\colon K_n^{\mathrm{top}}(F) \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$$

with all the usual properties.

Remark. For another rather elementary approach [F1] to class field theory of higher local fields of positive characteristic see subsection 10.2. For Kato's approach to higher class field theory see section 5 above.

References

[F1] I. Fesenko, Multidimensional local class field theory II, Algebra i Analiz (1991); English translation in St. Petersburg Math. J. 3(1992), 1103–1126.

[F2] I. Fesenko, Abelian local p-class field theory, Math. Ann. 301(1995), 561–586.

[KS] Y. Kawada and I. Satake, Class formations II, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 7(1956), 353–389.

[P1] A. N. Parshin, Class fields and algebraic K-theory, Uspekhi Mat. Nauk 30(1975), 253–254; English translation in Russian Math. Surveys.

[P2] A. N. Parshin, Local class field theory, Trudy Mat. Inst. Steklov. (1985); English translation in Proc. Steklov Inst. Math. 1985, issue 3, 157–185.

[P3] A. N. Parshin, Galois cohomology and Brauer group of local fields, Trudy Mat. Inst. Steklov. (1990); English translation in Proc. Steklov Inst. Math. 1991, issue 4, 191–201.

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