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9. Exponential maps and explicit formulas

Masato Kurihara

In this section we introduce an exponential homomorphism for the Milnor K-groups for a complete discrete valuation field of mixed characteristics.

In general, to work with the additive group is easier than with the multiplicative group, and the exponential map can be used to understand the structure of the multiplicative group by using that of the additive group. We would like to study the structure of $K_q(K)$ for a complete discrete valuation field K of mixed characteristics in order to obtain arithmetic information of K. Note that the Milnor K-groups can be viewed as a generalization of the multiplicative group. Our exponential map reduces some problems in the Milnor K-groups to those of the differential modules $\Omega^{\cdot}_{\mathcal{O}_K}$ which is relatively easier than the Milnor K-groups.

As an application, we study explicit formulas of certain type.

9.1. Notation and exponential homomorphisms

Let K be a complete discrete valuation field of mixed characteristics $(0,p)$. Let $\mathcal{O}_K$ be the ring of integers, and F be its the residue field. Denote by $\mathrm{ord}_p\colon K^* \longrightarrow \mathbb{Q}$ the additive valuation normalized by $\mathrm{ord}_p(p) = 1$. For $\eta \in \mathcal{O}_K$ we have an exponential homomorphism

$$\exp_\eta\colon \mathcal{O}_K \longrightarrow K^*, \qquad a \mapsto \exp(\eta a) = \sum_{n=0}^{\infty} (\eta a)^n/n!$$

if $\mathrm{ord}_p(\eta) > 1/(p-1)$.

For $q > 0$ let $K_q(K)$ be the qth Milnor K-group, and define $\widehat{K}_q(K)$ as the p-adic completion of $K_q(K)$, i.e.

$$\widehat{K}_q(K) = \varprojlim K_q(K) \otimes \mathbb{Z}/p^n.$$

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For a ring A, we denote as usual by Ω^1_A the module of the absolute differentials, i.e. $\Omega^1_A = \Omega^1_{A/\mathbb{Z}}$. For a field F of characteristic p and a p-base I of F, Ω^1_F is an F-vector space with basis dt $(t \in I)$. Let K be as above, and consider the p-adic completion $\widehat{\Omega}^1_{\mathcal{O}_K}$ of $\Omega^1_{\mathcal{O}_K}$

$$\widehat{\Omega}^1_{\mathcal{O}_K} = \varprojlim \Omega^1_{\mathcal{O}_K} \otimes \mathbb{Z}/p^n.$$

We take a lifting $\widetilde{I}$ of a p-base I of F, and take a prime element π of K. Then, $\widehat{\Omega}^1_{\mathcal{O}_K}$ is an $\mathcal{O}_K$-module (topologically) generated by $d\pi$ and dT $(T \in \widetilde{I})$ ([Ku1, Lemma 1.1]). If I is finite, then $\widehat{\Omega}^1_{\mathcal{O}_K}$ is generated by $d\pi$ and dT $(T \in \widetilde{I})$ in the ordinary sense. Put

$$\widehat{\Omega}^q_{\mathcal{O}_K} = \wedge^q \widehat{\Omega}^1_{\mathcal{O}_K}.$$

Theorem ([Ku3])**.** *Let $\eta \in K$ be an element such that $\mathrm{ord}_p(\eta) \geqslant 2/(p-1)$. Then for $q > 0$ there exists a homomorphism*

$$\exp^{(q)}_\eta \colon \widehat{\Omega}^q_{\mathcal{O}_K} \longrightarrow \widehat{K}_q(K)$$

such that

$$\exp^{(q)}_\eta \Big(a \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_{q-1}}{b_{q-1}}\Big) = \{\exp(\eta a), b_1, \dots, b_{q-1}\}$$

for any $a \in \mathcal{O}_K$ and any $b_1, \dots, b_{q-1} \in \mathcal{O}_K^*$.

Note that we have no assumption on F (F may be imperfect). For $b_1, \dots, b_{q-1} \in \mathcal{O}_K$ we have

$$\exp^{(q)}_\eta (a \cdot db_1 \wedge \cdots \wedge db_{q-1}) = \{\exp(\eta a b_1 \cdot \dots \cdot b_{q-1}), b_1, \dots, b_{q-1}\}.$$

9.2. Explicit formula of Sen

Let K be a finite extension of $\mathbb{Q}_p$ and assume that a primitive p^nth root ζ_{p^n} is in K. Denote by K_0 the subfield of K such that K/K_0 is totally ramified and $K_0/\mathbb{Q}_p$ is unramified. Let π be a prime element of O_K, and $g(T)$ and $h(T) \in \mathcal{O}_{K_0}[T]$ be polynomials such that $g(\pi) = \beta$ and $h(\pi) = \zeta_{p^n}$, respectively. Assume that α satisfies $\mathrm{ord}_p(\alpha) \geqslant 2/(p-1)$ and $\beta \in \mathcal{O}_K^*$. Then, Sen's formula ([S]) is

$$(\alpha, \beta) = \zeta_{p^n}^c, \qquad c = \frac{1}{p^n} \operatorname{Tr}_{K/\mathbb{Q}_p} \Big(\frac{\zeta_{p^n}}{h'(\pi)} \frac{g'(\pi)}{\beta} \log \alpha\Big)$$

where (α, β) is the Hilbert symbol defined by $(\alpha, \beta) = \gamma^{-1} \Psi_K(\alpha)(\gamma)$ where $\gamma^{p^n} = \beta$ and Ψ_K is the reciprocity map.

The existence of our exponential homomorphism introduced in the previous subsection helps to provide a new proof of this formula by reducing it to Artin–Hasse's formula for (α, ζ_{p^n}). In fact, put $k = \mathbb{Q}_p(\zeta_{p^n})$, and let η be an element of k such that $\mathrm{ord}_p(\eta) = 2/(p-1)$. Then, the commutative diagram

$$\begin{array}{ccc} \widehat{\Omega}^1_{\mathcal{O}_K} & \xrightarrow{\exp_\eta} & \widehat{K}_2(K) \\ {\scriptstyle\mathrm{Tr}}\downarrow & & {\scriptstyle N}\downarrow \\ \widehat{\Omega}^1_{\mathcal{O}_k} & \xrightarrow{\exp_\eta} & \widehat{K}_2(k) \end{array}$$

($N\colon \widehat{K}_2(K) \longrightarrow \widehat{K}_2(k)$ is the norm map of the Milnor K-groups, and $\mathrm{Tr}\colon \widehat{\Omega}^1_{\mathcal{O}_K} \longrightarrow \widehat{\Omega}^1_{\mathcal{O}_k}$ is the trace map of differential modules) reduces the calculation of the Hilbert symbol of elements in K to that of the Hilbert symbol of elements in k (namely reduces the problem to Iwasawa's formula [I]).

Further, since any element of $\widehat{\Omega}^1_{\mathcal{O}_k}$ can be written in the form $a d\zeta_{p^n}/\zeta_{p^n}$, we can reduce the problem to the calculation of (α, ζ_{p^n}).

In the same way, we can construct a formula of Sen's type for a higher dimensional local field (see [Ku3]), using a commutative diagram

$$\begin{array}{ccc} \widehat{\Omega}^q_{\mathcal{O}_K\{\{T\}\}} & \xrightarrow{\exp_\eta} & \widehat{K}_{q+1}(K\{\{T\}\}) \\ {\scriptstyle\mathrm{res}}\downarrow & & {\scriptstyle\mathrm{res}}\downarrow \\ \widehat{\Omega}^{q-1}_{\mathcal{O}_K} & \xrightarrow{\exp_\eta} & \widehat{K}_q(K) \end{array}$$

where the right arrow is the residue homomorphism $\{\alpha, T\} \mapsto \alpha$ in [Ka], and the left arrow is the residue homomorphism $\omega dT/T \mapsto \omega$. The field $K\{\{T\}\}$ is defined in Example 3 of subsection 1.1 and $\mathcal{O}_K\{\{T\}\} = \mathcal{O}_{K\{\{T\}\}}$.

9.3. Some open problems

Problem 1. Determine the kernel of $\exp_\eta^{(q)}$ completely. Especially, in the case of a d-dimensional local field K, the knowledge of the kernel of $\exp_\eta^{(d)}$ will give a lot of information on the arithmetic of K by class field theory. Generally, one can show that

$$p d\widehat{\Omega}^{q-2}_{\mathcal{O}_K} \subset \ker(\exp_p^{(q)}\colon \widehat{\Omega}^{q-1}_{\mathcal{O}_K} \longrightarrow \widehat{K}_q(K)).$$

For example, if K is absolutely unramified (namely, p is a prime element of K) and $p > 2$, then $p d\widehat{\Omega}^{q-2}_{\mathcal{O}_K}$ coincides with the kernel of $\exp_p^{(q)}$ ([Ku2]). But in general, this is not true. For example, if $K = \mathbb{Q}_p\{\{T\}\}(\sqrt[p]{pT})$ and $p > 2$, we can show that the kernel of $\exp_p^{(2)}$ is generated by $p d\mathcal{O}_K$ and the elements of the form $\log(1 - x^p) dx/x$ for any $x \in \mathcal{M}_K$ where $\mathcal{M}_K$ is the maximal ideal of $\mathcal{O}_K$.

Problem 2. Can one generalize our exponential map to some (formal) groups? For example, let G be a p-divisible group over K with $|K:\mathbb{Q}_p|<\infty$. Assume that the $[p^n]$-torsion points $\ker[p^n]$ of $G(K^{\mathrm{alg}})$ are in $G(K)$. We define the Hilbert symbol $K^*\times G(K)\longrightarrow \ker[p^n]$ by $(\alpha,\beta)=\Psi_K(\alpha)(\gamma)-_G\gamma$ where $[p^n]\gamma=\beta$. Benois obtained an explicit formula ([B]) for this Hilbert symbol, which is a generalization of Sen's formula. Can one define a map $\exp_G\colon \Omega^1_{\mathcal{O}_K}\otimes \mathrm{Lie}(G)\longrightarrow K^*\times G(K)/\sim$ (some quotient of $K^*\times G(K)$) by which we can interpret Benois's formula? We also remark that Fukaya recently obtained some generalization ([F]) of Benois's formula for a higher dimensional local field.

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Department of Mathematics Tokyo Metropolitan University
Minami-Osawa 1-1, Hachioji, Tokyo 192-03, Japan
E-mail: m-kuri@comp.metro-u.ac.jp