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A. Appendix to Section 2

Masato Kurihara and Ivan Fesenko

This appendix aims to provide more details on several notions introduced in section 2, as well as to discuss some basic facts on differentials and to provide a sketch of the proof of Bloch–Kato–Gabber's theorem. The work on it was completed after sudden death of Oleg Izhboldin, the author of section 2.

A1. Definitions and properties of several basic notions (by M. Kurihara)

Before we proceed to our main topics, we collect here the definitions and properties of several basic notions.

A1.1. Differential modules.

Let A and B be commutative rings such that B is an A-algebra. We define $\Omega^1_{B/A}$ to be the B-module of regular differentials over A. By definition, this B-module $\Omega^1_{B/A}$ is a unique B-module which has the following property. For a B-module M we denote by $\mathrm{Der}_A(B,M)$ the set of all A-derivations (an A-homomorphism $\varphi\colon B \to M$ is called an A-derivation if $\varphi(xy) = x\varphi(y) + y\varphi(x)$ and $\varphi(x) = 0$ for any $x \in A$). Then, φ induces $\overline{\varphi}\colon \Omega^1_{B/A} \to M$ ($\varphi = \overline{\varphi} \circ d$ where d is the canonical derivation $d\colon B \to \Omega^1_{B/A}$), and $\varphi \mapsto \overline{\varphi}$ yields an isomorphism

$$\mathrm{Der}_A(B,M) \xrightarrow{\sim} \mathrm{Hom}_B(\Omega^1_{B/A}, M).$$

In other words, $\Omega^1_{B/A}$ is the B-module defined by the following
generators: dx for any $x \in B$
and relations:

$$d(xy) = xdy + ydx$$

$$dx = 0 \quad \text{for any } x \in A.$$

If $A = \mathbb{Z}$, we simply denote $\Omega^1_{B/\mathbb{Z}}$ by Ω^1_B.

When we consider Ω^1_A for a local ring A, the following lemma is very useful.

Lemma. *If A is a local ring, we have a surjective homomorphism*

$$A \otimes_{\mathbb{Z}} A^* \longrightarrow \Omega^1_A$$

$$a \otimes b \mapsto a d \log b = a\frac{db}{b}.$$

The kernel of this map is generated by elements of the form

$$\sum_{i=1}^{k}(a_i \otimes a_i) - \sum_{i=1}^{l}(b_i \otimes b_i)$$

for $a_i,\ b_i \in A^*$ such that $\Sigma_{i=1}^k a_i = \Sigma_{i=1}^l b_i$.

Proof. First, we show the surjectivity. It is enough to show that xdy is in the image of the above map for $x,\ y \in A$. If y is in A^*, xdy is the image of $xy \otimes y$. If y is not in A^*, y is in the maximal ideal of A, and $1 + y$ is in A^*. Since $xdy = xd(1 + y)$, xdy is the image of $x(1 + y) \otimes (1 + y)$.

Let J be the subgroup of $A \otimes A^*$ generated by the elements

$$\sum_{i=1}^{k}(a_i \otimes a_i) - \sum_{i=1}^{l}(b_i \otimes b_i)$$

for $a_i,\ b_i \in A^*$ such that $\Sigma_{i=1}^k a_i = \Sigma_{i=1}^l b_i$. Put $M = (A \otimes_{\mathbb{Z}} A^*)/J$. Since it is clear that J is in the kernel of the map in the lemma, $a \otimes b \mapsto ad\log b$ induces a surjective homomorphism $M \to \Omega^1_A$, whose injectivity we have to show.

We regard $A \otimes A^*$ as an A-module via $a(x \otimes y) = ax \otimes y$. We will show that J is a sub A-module of $A \otimes A^*$. To see this, it is enough to show

$$\sum_{i=1}^{k}(xa_i \otimes a_i) - \sum_{i=1}^{l}(xb_i \otimes b_i) \in J$$

for any $x \in A$. If $x \notin A^*$, x can be written as $x = y + z$ for some $y,\ z \in A^*$, so we may assume that $x \in A^*$. Then,

$$\begin{aligned}
&\sum_{i=1}^{k}(xa_i \otimes a_i) - \sum_{i=1}^{l}(xb_i \otimes b_i)\\
&= \sum_{i=1}^{k}(xa_i \otimes xa_i - xa_i \otimes x) - \sum_{i=1}^{l}(xb_i \otimes xb_i - xb_i \otimes x)\\
&= \sum_{i=1}^{k}(xa_i \otimes xa_i) - \sum_{i=1}^{l}(xb_i \otimes xb_i) \in J.
\end{aligned}$$

Thus, J is an A-module, and $M = (A \otimes A^*)/J$ is also an A-module.

In order to show the bijectivity of $M \to \Omega^1_A$, we construct the inverse map $\Omega^1_A \to M$. By definition of the differential module (see the property after the definition), it is enough to check that the map

$$\varphi\colon A \longrightarrow M \qquad x \mapsto x \otimes x \quad (\text{if } x \in A^*)$$
$$x \mapsto (1+x) \otimes (1+x) \quad (\text{if } x \notin A^*)$$

is a $\mathbb{Z}$-derivation. So, it is enough to check $\varphi(xy) = x\varphi(y) + y\varphi(x)$. We will show this in the case where both x and y are in the maximal ideal of A. The remaining cases are easier, and are left to the reader. By definition, $x\varphi(y) + y\varphi(x)$ is the class of

$$\begin{aligned}
&x(1+y) \otimes (1+y) + y(1+x) \otimes (1+x)\\
&= (1+x)(1+y) \otimes (1+y) - (1+y) \otimes (1+y)\\
&\quad + (1+y)(1+x) \otimes (1+x) - (1+x) \otimes (1+x)\\
&= (1+x)(1+y) \otimes (1+x)(1+y) - (1+x) \otimes (1+x)\\
&\quad - (1+y) \otimes (1+y).
\end{aligned}$$

But the class of this element in M is the same as the class of $(1+xy) \otimes (1+xy)$. Thus, φ is a derivation. This completes the proof of the lemma. □

By this lemma, we can regard Ω^1_A as a group defined by the following generators: symbols $[a, b\}$ for $a \in A$ and $b \in A^*$
and relations:

$$[a_1 + a_2, b\} = [a_1, b\} + [a_2, b\}$$
$$[a, b_1 b_2\} = [a, b_1\} + [a_2, b_2\}$$
$$\sum_{i=1}^{k} [a_i, a_i\} = \sum_{i=1}^{l} [b_i, b_i\} \quad \text{where } a_i\text{'s and } b_i\text{'s satisfy} \quad \sum_{i=1}^{k} a_i = \sum_{i=1}^{l} b_i.$$

A1.2. n**-th differential forms.**

Let A and B be commutative rings such that B is an A-algebra. For a positive integer $n > 0$, we define $\Omega^n_{B/A}$ by

$$\Omega^n_{B/A} = \bigwedge_B \Omega^1_{B/A}.$$

Then, d naturally defines an A-homomorphism $d\colon \Omega^n_{B/A} \to \Omega^{n+1}_{B/A}$, and we have a complex

$$\dots \longrightarrow \Omega^{n-1}_{B/A} \longrightarrow \Omega^n_{B/A} \longrightarrow \Omega^{n+1}_{B/A} \longrightarrow \dots$$

which we call the *de Rham complex*.

For a commutative ring A, which we regard as a $\mathbb{Z}$-module, we simply write Ω^n_A for $\Omega^n_{A/\mathbb{Z}}$. For a local ring A, by Lemma A1.1, we have $\Omega^n_A = \bigwedge^n_A((A \otimes A^*)/J)$, where J is the group as in the proof of Lemma A1.1. Therefore we obtain

Lemma. *If A is a local ring, we have a surjective homomorphism*

$$A \otimes (A^*)^{\otimes n} \longrightarrow \Omega^n_{A/\mathbb{Z}}$$

$$a \otimes b_1 \otimes \dots \otimes b_n \mapsto a\frac{db_1}{b_1} \wedge \dots \wedge \frac{db_n}{b_n}.$$

The kernel of this map is generated by elements of the form

$$\sum_{i=1}^{k}(a_i \otimes a_i \otimes b_1 \otimes \dots \otimes b_{n-1}) - \sum_{i=1}^{l}(b_i \otimes b_i \otimes b_1 \otimes \dots \otimes b_{n-1})$$

(where $\Sigma_{i=1}^k a_i = \Sigma_{i=1}^l b_i$)
and

$$a \otimes b_1 \otimes \dots \otimes b_n \quad \textit{with } b_i = b_j \textit{ for some } i \neq j.$$

A1.3. Galois cohomology of $\mathbb{Z}/p^n(r)$ for a field of characteristic $p > 0$.

Let F be a field of characteristic $p > 0$. We denote by F^{sep} the separable closure of F in an algebraic closure of F.

We consider Galois cohomology groups $H^q(F, -) := H^q(\mathrm{Gal}(F^{\mathrm{sep}}/F), -)$. For an integer $r \geqslant 0$, we define

$$H^q(F, \mathbb{Z}/p(r)) = H^{q-r}(\mathrm{Gal}(F^{\mathrm{sep}}/F), \Omega^r_{F^{\mathrm{sep}},\log})$$

where $\Omega^r_{F^{\mathrm{sep}},\log}$ is the logarithmic part of $\Omega^r_{F^{\mathrm{sep}}}$, namely the subgroup generated by $d\log a_1 \wedge \dots \wedge d\log a_r$ for all $a_i \in (F^{\mathrm{sep}})^*$.

We have an exact sequence (cf. [I, p.579])

$$0 \longrightarrow \Omega^r_{F^{\mathrm{sep}},\log} \longrightarrow \Omega^r_{F^{\mathrm{sep}}} \xrightarrow{\mathbf{F}-1} \Omega^r_{F^{\mathrm{sep}}}/d\Omega^{r-1}_{F^{\mathrm{sep}}} \longrightarrow 0$$

where $\mathbf{F}$ is the map

$$\mathbf{F}(a\frac{db_1}{b_1} \wedge \dots \wedge \frac{db_r}{b_r}) = a^p\frac{db_1}{b_1} \wedge \dots \wedge \frac{db_r}{b_r}.$$

Since $\Omega^r_{F^{\mathrm{sep}}}$ is an F-vector space, we have

$$H^n(F, \Omega^r_{F^{\mathrm{sep}}}) = 0$$

for any $n > 0$ and $r \geqslant 0$. Hence, we also have

$$H^n(F, \Omega^r_{F^{\mathrm{sep}}}/d\Omega^{r-1}_{F^{\mathrm{sep}}}) = 0$$

for $n > 0$. Taking the cohomology of the above exact sequence, we obtain

$$H^n(F, \Omega^r_{F^{\mathrm{sep}},\log}) = 0$$

for any $n \geqslant 2$. Further, we have an isomorphism

$$H^1(F, \Omega^r_{F^{\mathrm{sep}},\log}) = \mathrm{coker}(\Omega^r_F \xrightarrow{\mathbf{F}-1} \Omega^r_F / d\Omega^{r-1}_F)$$

and

$$H^0(F, \Omega^r_{F^{\mathrm{sep}},\log}) = \ker(\Omega^r_F \xrightarrow{\mathbf{F}-1} \Omega^r_F / d\Omega^{r-1}_F).$$

Lemma. *For a field F of characteristic $p > 0$ and $n > 0$, we have*

$$H^{n+1}(F, \mathbb{Z}/p(n)) = \mathrm{coker}(\Omega^n_F \xrightarrow{\mathbf{F}-1} \Omega^n_F / d\Omega^{n-1}_F)$$

and

$$H^n(F, \mathbb{Z}/p(n)) = \ker(\Omega^n_F \xrightarrow{\mathbf{F}-1} \Omega^n_F / d\Omega^{n-1}_F).$$

Furthermore, $H^n(F, \mathbb{Z}/p(n-1))$ is isomorphic to the group which has the following generators: symbols $[a, b_1, ..., b_{n-1}\}$ where $a \in F$, and $b_1, ..., b_{n-1} \in F^*$ and relations:

$$[a_1 + a_2, b_1, ..., b_{n-1}\} = [a_1, b_1, ..., b_{n-1}\} + [a_2, b_1, ..., b_{n-1}\}$$
$$[a, b_1, ..., b_i b_i', ... b_{n-1}\} = [a, b_1, ..., b_i, ... b_{n-1}\} + [a, b_1, ..., b_i', ... b_{n-1}\}$$
$$[a, a, b_2, ..., b_{n-1}\} = 0$$
$$[a^p - a, b_1, b_2, ..., b_{n-1}\} = 0$$
$$[a, b_1, ..., b_{n-1}\} = 0 \quad \text{where } b_i = b_j \text{ for some } i \neq j.$$

Proof. The first half of the lemma follows from the computation of $H^n(F, \Omega^r_{F^{\mathrm{sep}},\log})$ above and the definition of $H^q(F, \mathbb{Z}/p(r))$. Using

$$H^n(F, \mathbb{Z}/p(n-1)) = \mathrm{coker}(\Omega^{n-1}_F \xrightarrow{\mathbf{F}-1} \Omega^{n-1}_F / d\Omega^{n-2}_F)$$

and Lemma A1.2 we obtain the explicit description of $H^n(F, \mathbb{Z}/p(n-1))$. □

We sometimes use the notation $H^n_p(F)$ which is defined by

$$H^n_p(F) = H^n(F, \mathbb{Z}/p(n-1)).$$

Moreover, for any $i > 1$, we can define $\mathbb{Z}/p^i(r)$ by using the de Rham–Witt complexes instead of the de Rham complex. For a positive integer $i > 0$, following Illusie [I], define $H^q(F, \mathbb{Z}/p^i(r))$ by

$$H^q(F, \mathbb{Z}/p^i(r)) = H^{q-r}(F, W_i\Omega^r_{F^{\mathrm{sep}},\log})$$

where $W_i\Omega^r_{F^{\mathrm{sep}},\log}$ is the logarithmic part of $W_i\Omega^r_{F^{\mathrm{sep}}}$.

Though we do not give here the proof, we have the following explicit description of $H^n(F, \mathbb{Z}/p^i(n-1))$ using the same method as in the case of $i = 1$.

Lemma. *For a field F of characteristic $p > 0$ let $W_i(F)$ denote the ring of Witt vectors of length i, and let $\mathbf{F}: W_i(F) \to W_i(F)$ denote the Frobenius endomorphism. For any $n > 0$ and $i > 0$, $H^n(F, \mathbb{Z}/p^i(n-1))$ is isomorphic to the group which has the following*
generators: symbols $[a, b_1, ..., b_{n-1}\}$ where $a \in W_i(F)$, and $b_1, ..., b_{n-1} \in F^*$
and relations:

$$[a_1 + a_2, b_1, ..., b_{n-1}\} = [a_1, b_1, ..., b_{n-1}\} + [a_2, b_1, ..., b_{n-1}\}$$
$$[a, b_1, ..., b_j b_j', ...b_{n-1}\} = [a, b_1, ..., b_j, ...b_{n-1}\} + [a, b_1, ..., b_j', ...b_{n-1}\}$$
$$[(0, ..., 0, a, 0, ..., 0), a, b_2, ..., b_{n-1}\} = 0$$
$$[\mathbf{F}(a) - a, b_1, b_2, ..., b_{n-1}\} = 0$$
$$[a, b_1, ..., b_{n-1}\} = 0 \quad \textit{where } b_j = b_k \textit{ for some } j \neq k.$$

We sometimes use the notation
$$H^n_{p^i}(F) = H^n(F, \mathbb{Z}/p^i(n-1)).$$

A2. Bloch–Kato–Gabber's theorem (by I. Fesenko)

For a field k of characteristic p denote
$$\nu_n = \nu_n(k) = H^n(k, \mathbb{Z}/p(n)) = \ker(\wp: \Omega^n_k \to \Omega^n_k / d\Omega^{n-1}_k),$$
$$\wp = \mathbf{F} - 1: \left(a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}\right) \mapsto (a^p - a)\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n} + d\Omega^{n-1}_k.$$

Clearly, the image of the differential symbol
$$d_k: K_n(k)/p \to \Omega^n_k, \qquad \{a_1, \ldots, a_n\} \mapsto \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}$$
is inside $\nu_n(k)$. We shall sketch the proof of Bloch–Kato–Gabber's theorem which states that d_k is an isomorphism between $K_n(k)/p$ and $\nu_n(k)$.

A2.1. Surjectivity of the differential symbol $d_k: K_n(k)/p \to \nu_n(k)$.

It seems impossible to suggest a shorter proof than original Kato's proof in [K, §1]. We can argue by induction on n; the case of $n = 1$ is obvious, so assume $n > 1$.

Definitions–Properties.

(1) Let $\{b_i\}_{i \in I}$ be a p-base of k (I is an ordered set). Let S be the set of all strictly increasing maps
$$s: \{1, \ldots, n\} \to I.$$
For two maps $s, t: \{1, \ldots, n\} \to I$ write $s < t$ if $s(i) \leqslant t(i)$ for all i and $s(i) \neq t(i)$ for some i.

(2) Denote $d\log a := a^{-1}da$. Put

$$\omega_s = d\log b_{s(1)} \wedge \cdots \wedge d\log b_{s(n)}.$$

Then $\{\omega_s : s \in S\}$ is a basis of Ω_k^n over k.

(3) For a map $\theta\colon I \to \{0, 1, \dots, p-1\}$ such that $\theta(i) = 0$ for almost all i set

$$b_\theta = \prod b_i^{\theta(i)}.$$

Then $\{b_\theta \omega_s\}$ is a basis of Ω_k^n over k^p.

(4) Denote by $\Omega_k^n(\theta)$ the k^p-vector space generated by $b_\theta\omega_s, s \in S$. Then $\Omega_k^n(0) \cap d\Omega_k^{n-1} = 0$. For an extension l of k, such that $k \supset l^p$, denote by $\Omega_{l/k}^n$ the module of relative differentials. Let $\{b_i\}_{i\in I}$ be a p-base of l over k. Define $\Omega_{l/k}^n(\theta)$ for a map $\theta\colon I \to \{0, 1, \dots, p-1\}$ similarly to the previous definition. The cohomology group of the complex

$$\Omega_{l/k}^{n-1}(\theta) \to \Omega_{l/k}^n(\theta) \to \Omega_{l/k}^{n+1}(\theta)$$

is zero if $\theta \neq 0$ and is $\Omega_{l/k}^n(0)$ if $\theta = 0$.

We shall use *Cartier's theorem* (which can be more or less easily proved by induction on $|l : k|$): the sequence

$$0 \to l^*/k^* \to \Omega_{l/k}^1 \to \Omega_{l/k}^1/dl$$

is exact, where the second map is defined as $b \mod k^* \to d\log b$ and the third map is the map $a\,d\log b \mapsto (a^p - a)d\log b + dl$.

Proposition. *Let $\Omega_k^n(<s)$ be the k-subspace of Ω_k^n generated by all ω_t for $s > t \in S$. Let $k^{p-1} = k$ and let a be a non-zero element of k. Let I be finite. Suppose that*

$$(a^p - a)\omega_s \in \Omega_k^n(<s) + d\Omega_k^{n-1}.$$

Then there are $v \in \Omega_k^n(<s)$ and

$$x_i \in k^p(\{b_j : j \leqslant s(i)\}) \quad \textit{for} \quad 1 \leqslant i \leqslant n$$

such that

$$a\omega_s = v + d\log x_1 \wedge \cdots \wedge d\log x_n.$$

Proof of the surjectivity of the differential symbol. First, suppose that $k^{p-1} = k$ and I is finite. Let $S = \{s_1, \dots, s_m\}$ with $s_1 > \cdots > s_m$. Let $s_0\colon \{1, \dots, n\} \to I$ be a map such that $s_0 > s_1$. Denote by A the subgroup of Ω_k^n generated by $d\log x_1 \wedge \cdots \wedge d\log x_n$. Then $A \subset \nu_n$. By induction on $0 \leqslant j \leqslant m$ using the proposition it is straightforward to show that $\nu_n \subset A + \Omega_k^n(<s_j)$, and hence $\nu_n = A$.

To treat the general case put $c(k) = \operatorname{coker}(k_n(k) \to \nu_n(k))$. Since every field is the direct limit of finitely generated fields and the functor c commutes with direct limits, it is sufficient to show that $c(k) = 0$ for a finitely generated field k. In particular, we may assume that k has a finite p-base. For a finite extension k' of k there is a commutative diagram

$$\begin{array}{ccc} k_n(k') & \longrightarrow & \nu_n(k') \\ {\scriptstyle N_{k'/k}}\big\downarrow & & {\scriptstyle \mathrm{Tr}_{k'/k}}\big\downarrow \\ k_n(k) & \longrightarrow & \nu_n(k). \end{array}$$

Hence the composite $c(k) \to c(k') \xrightarrow{\mathrm{Tr}_{k'/k}} c(k)$ is multiplication by $|k' : k|$. Therefore, if $|k' : k|$ is prime to p then $c(k) \to c(k')$ is injective.

Now pass from k to a field l which is the compositum of all l_i where $l_{i+1} = l_i(\sqrt[p-1]{l_{i-1}})$, $l_0 = k$. Then $l = l^{p-1}$. Since l/k is separable, l has a finite p-base and by the first paragraph of this proof $c(l) = 0$. The degree of every finite subextension in l/k is prime to p, and by the second paragraph of this proof we conclude $c(k) = 0$, as required. $\square$

Proof of Proposition. First we prove the following lemma which will help us later for fields satisfying $k^{p-1} = k$ to choose a specific p-base of k.

Lemma. *Let l be a purely inseparable extension of k of degree p and let $k^{p-1} = k$. Let $f\colon l \to k$ be a k-linear map. Then there is a non-zero $c \in l$ such that $f(c^i) = 0$ for all $1 \leqslant i \leqslant p-1$.*

Proof of Lemma. The l-space of k-linear maps from l to k is one-dimensional, hence $f = ag$ for some $a \in l$, where $g\colon l = k(b) \to \Omega^1_{l/k}/dl \xrightarrow{\sim} k,\ x \mapsto xd\log b \bmod dl$ for every $x \in l$. Let $\alpha = gd\log b$ generate the one-dimensional space $\Omega^1_{l/k}/dl$ over k. Then there is $h \in k$ such that $g^p d\log b - h\alpha \in dl$. Let $z \in k$ be such that $z^{p-1} = h$. Then $((g/z)^p - g/z)d\log b \in dl$ and by Cartier's theorem we deduce that there is $w \in l$ such that $(g/z)d\log b = d\log w$. Hence $\alpha = zd\log w$ and $\Omega^1_{l/k} = dl \cup kd\log l$.

If $f(1) = ad\log b \neq 0$, then $f(1) = gd\log c$ with $g \in k, c \in l^*$ and hence $f(c^i) = 0$ for all $1 \leqslant i \leqslant p-1$. $\square$

Now for $s\colon \{1, \dots, n\} \to I$ as in the statement of the Proposition denote

$$k_0 = k^p(\{b_i : i < s(1)\}), \quad k_1 = k^p(\{b_i : i \leqslant s(1)\}), \quad k_2 = k^p(\{b_i : i \leqslant s(n)\}).$$

Let $|k_2 : k_1| = p^r$.

Let $a = \sum_\theta x_\theta^p b_\theta$. Assume that $a \notin k_2$. Then let θ, j be such that $j > s(n)$ is the maximal index for which $\theta(j) \neq 0$ and $x_\theta \neq 0$.

$\Omega^n_k(\theta)$-projection of $(a^p - a)\omega_s$ is equal to $-x_\theta^p b_\theta \omega_s \in \Omega^n_k(<s)(\theta) + d\Omega^{n-1}_k(\theta)$. Log differentiating, we get

$$-x_\theta^p\Big(\sum_i \theta(i) d\log b_i\Big) b_\theta \wedge \omega_s \in d\Omega^n_k(<s)(\theta)$$

which contradicts $-x_\theta^p\theta(j)b_\theta d\log b_j\wedge\omega_s\notin d\Omega_k^n(<s)(\theta)$. Thus, $a\in k_2$.

Let $m(1)<\dots<m(r-n)$ be integers such that the union of m's and s's is equal to $[s(1),s(n)]\cap\mathbb{Z}$. Apply the Lemma to the linear map

$$f\colon k_1\to\Omega^r_{k_2/k_0}/d\Omega^{r-1}_{k_2/k_0}\stackrel{\sim}{\to}k_0,\quad b\mapsto ba\omega_s\wedge d\log b_{m(1)}\wedge\dots\wedge d\log b_{m(r-n)}.$$

Then there is a non-zero $c\in k_1$ such that

$$c^ia\omega_s\wedge d\log b_{m(1)}\wedge\dots\wedge d\log b_{m(r-n)}\in d\Omega^{r-1}_{k_2/k_0}\qquad\text{for } 1\leqslant i\leqslant p-1\,.$$

Hence $\Omega^r_{k_2/k_0}(0)$-projection of $c^ia\omega_s\wedge d\log b_{m(1)}\wedge\dots\wedge d\log b_{m(r-n)}$ for $1\leqslant i\leqslant p-1$ is zero.

If $c\in k_0$ then $\Omega^r_{k_2/k_0}(0)$-projection of $a\omega_s\wedge d\log b_{m(1)}\wedge\dots\wedge d\log b_{m(r-n)}$ is zero. Due to the definition of k_0 we get

$$\beta=(a^p-a)\omega_s\wedge d\log b_{m(1)}\wedge\dots\wedge d\log b_{m(r-n)}\in d\Omega^{r-1}_{k_2/k_0}.$$

Then $\Omega^r_{k_2/k_0}(0)$-projection of β is zero, and so is $\Omega^r_{k_2/k_0}(0)$-projection of

$$a^p\omega_s\wedge d\log b_{m(1)}\wedge\dots\wedge d\log b_{m(r-n)},$$

a contradiction. Thus, $c\notin k_0$.

From $dk_0\subset\sum_{i<s(1)}k^pdb_i$ we deduce $dk_0\wedge\Omega_k^{n-1}\subset\Omega_k^n(<s)$. Since $k_0(c)=k_0(b_{s(1)})$, there are $a_i\in k_0$ such that $b_{s(1)}=\sum_{i=0}^{p-1}a_ic^i$. Then

$$a d\log b_{s(1)}\wedge\dots\wedge d\log b_{s(n)}\equiv a'd\log b_{s(2)}\dots\wedge d\log b_{s(n)}\wedge d\log c\mod\Omega_k^n(<s).$$

Define $s'\colon\{1,\dots,n-1\}\to I$ by $s'(j)=s(j+1)$. Then

$$a\omega_s=v_1+a'\omega_{s'}\wedge d\log c\qquad\text{with } v_1\in\Omega_k^n(<s)$$

and $c^ia'\omega_{s'}\wedge d\log c\wedge d\log b_{m(1)}\wedge\dots\wedge d\log b_{m(r-n)}\in d\Omega^{r-1}_{k_2/k_0}$. The set

$$I'=\{c\}\cup\{b_i: s(1)<i\leqslant s(n)\}$$

is a p-base of k_2/k_0. Since c^ia' for $1\leqslant i\leqslant p-1$ have zero $k_2(0)$-projection with respect to I', there are $a_0'\in k_0$, $a_1'\in\oplus_{\theta\ne0}k_1b'_\theta$ with $b'_\theta=\prod_{s(1)<i\leqslant s(n)}b_i^{\theta(i)}$ such that $a'=a_0'+a_1'$.

The image of $a\omega_s\wedge d\log b_{m(1)}\wedge\dots\wedge d\log b_{m(r-n)}$ with respect to the Artin–Schreier map belongs to $\Omega^r_{k_2/k_0}$ and so is

$$(a'^p-a')d\log c\wedge\omega_{s'}\wedge d\log b_{m(1)}\wedge\dots\wedge d\log b_{m(r-n)}$$

which is the image of

$$a'd\log c\wedge\omega_{s'}\wedge d\log b_{m(1)}\wedge\dots\wedge d\log b_{m(r-n)}.$$

Then a'^p-a_0', as $k_0(0)$-projection of a'^p-a', is zero. So $a'-a'^p=a_1'$.

Note that $d(a_1'\omega_{s'})\wedge d\log c\in d\Omega^n_{k/k_0}(<s)=d\Omega^{n-1}_{k/k_0}(<s)\wedge d\log c$.

Hence $d(a_1'\omega_{s'}) \in d\Omega^{n-1}_{k/k_0}(<s) + d\log c \wedge d\Omega^{n-2}_{k/k_0}$. Therefore $d(a_1'\omega_{s'}) \in d\Omega^{n-1}_{k/k_1}(<s)$ and $a_1'\omega_{s'} = \alpha + \beta$ with $\alpha \in \Omega^{n-1}_{k/k_1}(<s)$, $\beta \in \ker(d\colon \Omega^{n-1}_{k/k_1} \to \Omega^n_{k/k_1})$.

Since $k(0)$-projection of a_1' is zero, $\Omega^{n-1}_{k/k_1}(0)$-projection of $a_1'\omega_{s'}$ is zero. Then we deduce that $\beta(0) = \sum_{x_t\in k_1, t<s'} x_t^p\omega_t$, so $a_1'\omega_{s'} = \alpha + \beta(0) + (\beta - \beta(0))$. Then $\beta - \beta(0) \in \ker(d\colon \Omega^{n-1}_{k/k_1} \to \Omega^n_{k/k_1})$, so $\beta - \beta(0) \in d\Omega^{n-2}_{k/k_1}$. Hence $(a' - a'^p)\omega_{s'} = a_1'\omega_{s'}$ belongs to $\Omega^{n-1}_{k/k_1}(<s') + d\Omega^{n-2}_k$. By induction on n, there are $v' \in \Omega^{n-1}_k(<s')$, $x_i \in k^p\{b_j : j \leqslant s(i)\}$ such that $a'\omega_{s'} = v' + d\log x_2 \wedge \dots \wedge d\log x_n$. Thus, $a\omega_s = v_1 \pm d\log c \wedge v' \pm d\log c \wedge d\log x_2 \wedge \dots \wedge d\log x_n$. □

A2.2. Injectivity of the differential symbol.

We can assume that k is a finitely generated field over $\mathbb{F}_p$. Then there is a finitely generated algebra over $\mathbb{F}_p$ with a local ring being a discrete valuation ring $\mathcal{O}$ such that $\mathcal{O}/\mathcal{M}$ is isomorphic to k and the field of fractions E of $\mathcal{O}$ is purely transcendental over $\mathbb{F}_p$.

Using standard results on $K_n(l(t))$ and $\Omega^n_{l(t)}$ one can show that the injectivity of d_l implies the injectivity of $d_{l(t)}$. Since $d_{\mathbb{F}_p}$ is injective, so is d_E.

Define $k_n(\mathcal{O}) = \ker(k_n(E) \to k_n(k))$. Then $k_n(\mathcal{O})$ is generated by symbols and there is a homomorphism

$$k_n(\mathcal{O}) \to k_n(k), \quad \{a_1, \dots, a_n\} \to \{\overline{a_1}, \dots, \overline{a_n}\},$$

where $\bar{a}$ is the residue of a. Let $k_n(\mathcal{O}, \mathcal{M})$ be its kernel.

Define $\nu_n(\mathcal{O}) = \ker(\Omega^n_{\mathcal{O}} \to \Omega^n_{\mathcal{O}}/d\Omega^{n-1}_{\mathcal{O}})$, $\nu_n(\mathcal{O}, \mathcal{M}) = \ker(\nu_n(\mathcal{O}) \to \nu_n(k))$. There is a homomorphism $k_n(\mathcal{O}) \to \nu_n(\mathcal{O})$ such that

$$\{a_1, \dots, a_n\} \mapsto d\log a_1 \wedge \dots \wedge d\log a_n.$$

So there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & k_n(\mathcal{O},\mathcal{M}) \arrow[r] \arrow[d, "\varphi"'] & k_n(\mathcal{O}) \arrow[r] \arrow[d] & k_n(k) \arrow[r] \arrow[d, "d_k"'] & 0 \\
0 \arrow[r] & \nu_n(\mathcal{O},\mathcal{M}) \arrow[r] & \nu_n(\mathcal{O}) \arrow[r] & \nu_n(k) & .
\end{tikzcd}$$

Similarly to A2.1 one can show that φ is surjective [BK, Prop. 2.4]. Thus, d_k is injective. □

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Department of Mathematics Tokyo Metropolitan University
Minami-Osawa 1-1, Hachioji, Tokyo 192-03, Japan
E-mail: m-kuri@comp.metro-u.ac.jp

Department of Mathematics University of Nottingham
Nottingham NG7 2RD England
E-mail: ibf@maths.nott.ac.uk