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1. Higher dimensional local fields and L-functions

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1.0. Introduction

1.0.1. Recall [P1], [FP] that if X is a scheme of dimension n and

$$X_0 \subset X_1 \subset \dots X_{n-1} \subset X_n = X$$

is a flag of irreducible subschemes ($\dim(X_i) = i$), then one can define a ring

$$K_{X_0,\dots,X_{n-1}}$$

associated to the flag. In the case where everything is regularly embedded, the ring is an n-dimensional local field. Then one can form an adelic object

$$\mathbb{A}_X = \prod{}' K_{X_0,\dots,X_{n-1}}$$

where the product is taken over all the flags with respect to certain restrictions on components of adeles [P1], [Be], [Hu], [FP].

Example. Let X be an algebraic projective irreducible surface over a field k and let P be a closed point of X, $C \subset X$ be an irreducible curve such that $P \in C$.

If X and C are smooth at P, then we let $t \in \mathcal{O}_{X,P}$ be a local equation of C at P and $u \in \mathcal{O}_{X,P}$ be such that $u|_C \in \mathcal{O}_{C,P}$ is a local parameter at P. Denote by $\mathcal{C}$ the ideal defining the curve C near P. Now we can introduce a two-dimensional local field $K_{P,C}$ attached to the pair P, C by the following procedure including completions and localizations:

$$\begin{array}{ccc}
\widehat{\mathcal{O}}_{X,P} & = & k(P)[[u,t]] \supset \mathcal{C} = (t) \\
| & & \\
(\widehat{\mathcal{O}}_{X,P})_{\mathcal{C}} & = & \text{discrete valuation ring with residue field } k(P)((u)) \\
| & & \\
\widehat{\mathcal{O}}_{P,C} := \widehat{(\widehat{\mathcal{O}}_{X,P})_{\mathcal{C}}} & = & k(P)((u))[[t]] \\
| & & \\
K_{P,C} := \mathrm{Frac}\,(\widehat{\mathcal{O}}_{P,C}) & = & k(P)((u))((t))
\end{array}$$

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Note that the left hand side construction is meaningful *without* any smoothness condition.

Let K_P be the minimal subring of $K_{P,C}$ which contains $k(X)$ and $\widehat{\mathcal{O}}_{X,P}$. The ring K_P is not a field in general. Then $K \subset K_P \subset K_{P,C}$ and there is another intermediate subring $K_C = \operatorname{Frac}(\mathcal{O}_C) \subset K_{P,C}$. Note that in dimension 2 there is a duality between points P and curves C (generalizing the classical duality between points and lines in projective geometry). We can compare the structure of adelic components in dimension one and two:

$$\begin{tikzcd} K_P \arrow[dd, dash] \\ \\ K \end{tikzcd} \qquad\qquad \begin{tikzcd} & K_{P,C} \arrow[dl, dash] \arrow[dr, dash] & \\ K_P \arrow[dr, dash] & & K_C \arrow[dl, dash] \\ & K & \end{tikzcd}$$

1.0.2. In the one-dimensional case for every character $\chi\colon \mathrm{Gal}(K^{\mathrm{ab}}/K) \to \mathbb{C}^*$ we have the composite

$$\chi'\colon \mathbb{A}^* = {\prod}' K_x^* \xrightarrow{\text{reciprocity map}} \mathrm{Gal}(K^{\mathrm{ab}}/K) \xrightarrow{\chi} \mathbb{C}^*.$$

J. Tate [T] and independently K. Iwasawa introduced an analytically defined L-function

$$L(s,\chi,f) = \int_{\mathbb{A}^*} f(a)\chi'(a)|a|^s d^*a,$$

where d^* is a Haar measure on $\mathbb{A}^*$ and the function f belongs to the Bruhat–Schwartz space of functions on $\mathbb{A}$ (for the definition of this space see for instance [W1, Ch. VII]). For a special choice of f and $\chi = 1$ we get the ζ-function of the scheme X

$$\zeta_X(s) = \prod_{x\in X} (1 - N(x)^{-s})^{-1},$$

if $\dim(X) = 1$ (adding the archimedean multipliers if necessary). Here x runs through the closed points of the scheme X and $N(x) = |k(x)|$. The product converges for $\mathrm{Re}(s) > \dim X$. For $L(s,\chi,f)$ they proved the analytical continuation to the whole s-plane and the functional equation

$$L(s,\chi,f) = L(1-s,\chi^{-1},\widehat{f}),$$

using Fourier transformation ($f \mapsto \widehat{f}$) on the space $\mathbb{A}_X$ (cf. [T], [W1], [W2]).

1.0.3. Schemes can be classified according to their dimension

$\dim(X)$	geometric case	arithmetic case
...	...	...
2	algebraic surface $/\mathbb{F}_q$	arithmetic surface
1	algebraic curve $/\mathbb{F}_q$	arithmetic curve
0	$\mathrm{Spec}(\mathbb{F}_q)$	$\mathrm{Spec}(\mathbb{F}_1)$

where $\mathbb{F}_1$ is the "field of one element".

The analytical method works for the row of the diagram corresponding to dimension one. The problem to prove analytical continuation and functional equation for the ζ-function of arbitrary scheme X (Hasse–Weil conjecture) was formulated by A. Weil [W2] as a generalization of the previous Hasse conjecture for algebraic curves over fields of algebraic numbers, see [S1],[S2]. It was solved in the geometric situation by A. Grothendieck who employed cohomological methods [G]. Up to now there is no extension of this method to arithmetic schemes (see, however, [D]). On the other hand, a remarkable property of the Tate–Iwasawa method is that it can be simultaneously applied to the fields of algebraic numbers (arithmetic situation) and to the algebraic curves over a finite field (algebraic situation).

For a long time the author has been advocating (see, in particular, [P4], [FP]) the following:

Problem. *Extend Tate–Iwasawa's analytic method to higher dimensions.*

The higher adeles were introduced exactly for this purpose. In dimension one the adelic groups $\mathbb{A}_X$ and $\mathbb{A}_X^*$ are locally compact groups and thus we can apply the classical harmonic analysis. The starting point for that is the measure theory on locally compact local fields such as K_P for the schemes X of dimension 1. So we have the following:

Problem. *Develop a measure theory and harmonic analysis on n-dimensional local fields.*

Note that n-dimensional local fields are not locally compact topological spaces for $n > 1$ and by Weil's theorem the existence of the Haar measure on a topological group implies its locally compactness [W3, Appendix 1].

In this work several first steps in answering these problems are described.

1.1. Riemann–Hecke method

When one tries to write the ζ-function of a scheme X as a product over local fields attached to the flags of subvarieties one meets the following obstacle. For dimension greater than one the local fields are parametrized by flags and not by the closed points itself as in the Euler product. This problem is primary to any problems with the measure and integration. I think we have to return to the case of dimension one and reformulate the Tate–Iwasawa method. Actually, it means that we have to return to the Riemann–Hecke approach [He] known long before the work of Tate and Iwasawa. Of course, it was the starting point for their approach.

The main point is a reduction of the integration over ideles to integration over a single (or finitely many) local field.

Let C be a smooth irreducible complete curve defined over a field $k = \mathbb{F}_q$.

Put $K = k(C)$. For a closed point $x \in C$ denote by K_x the fraction field of the completion $\widehat{\mathcal{O}}_x$ of the local ring $\mathcal{O}_x$.

Let P be a fixed smooth k-rational point of C. Put $U = C \setminus P$, $A = \Gamma(U, \mathcal{O}_C)$. Note that A is a discrete subgroup of K_P.

A classical method to calculate ζ-function is to write it as a Dirichlet series instead of the Euler product:

$$\zeta_C(s) = \sum_{I \in \mathrm{Div}\,(\mathcal{O}_C)} |I|_C^s$$

where $\mathrm{Div}\,(\mathcal{O}_C)$ is the semigroup of effective divisors, $I = \sum_{x \in X} n_x x$, $n_x \in \mathbb{Z}$ and $n_x = 0$ for almost all $x \in C$,

$$|I|_C = \prod_{x \in X} q^{-\sum n_x |k(x):k|}.$$

Rewrite $\zeta_C(s)$ as

$$\zeta_U(s)\zeta_P(s) = \left(\sum_{I \subset U} |I|_U^s\right)\left(\sum_{\mathrm{supp}(I)=P} |I|_P^s\right).$$

Denote $A' = A \setminus \{0\}$. For the sake of simplicity assume that $\mathrm{Pic}(U) = (0)$ and introduce A'' such that $A'' \cap k^* = (1)$ and $A' = A''k^*$. Then for every $I \subset U$ there is a unique $b \in A''$ such that $I = (b)$. We also write $|b|_* = |(b)|_*$ for $* = P, U$. Then from the product formula $|b|_C = 1$ we get $|b|_U = |b|_P^{-1}$. Hence

$$\zeta_C(s) = \left(\sum_{b \in A''} |b|_U^s\right)\left(\sum_{m \geqslant 0} q^{-ms}\right) = \left(\sum_{b \in A''} |b|_P^{-s}\right)\int_{a \in K_P^*} |a|_P^s f_+(a) d^* a$$

where in the last equality we have used local Tate's calculation, $f_+ = i^*\delta_{\widehat{\mathcal{O}}_P}$, $i\colon K_P^* \to K_P$, $\delta_{\widehat{\mathcal{O}}_P}$ is the characteristic function of the subgroup $\widehat{\mathcal{O}}_P$, $d^*(\widehat{\mathcal{O}}_P^*) = 1$. Therefore

$$\begin{aligned}\zeta_C(s) &= \sum_{b\in A''}\int_{a\in K_P^*} |ab^{-1}|_P^s f_+(a)d^*a \\ &= \sum_{b\in A''}\int_{c=ab^{-1}} |c|_P^s f_+(bc)d^*c = \int_{K_P^*} |c|_P^s F(c)d^*c,\end{aligned}$$

where $F(c) = \sum_{b\in A'} f_+(bc)$.

Thus, the calculation of $\zeta_C(s)$ is reduced to integration over the single local field K_P. Then we can proceed further using the Poisson summation formula applied to the function F.

This computation can be rewritten in a more functorial way as follows

$$\zeta_C(s) = \langle|\ |^s, f_0\rangle_G \cdot \langle|\ |^s, f_1\rangle_G = \langle|\ |^s, i^*(F)\rangle_{G\times G} = \langle|\ |^s, j_*\circ i^*(F)\rangle_G,$$

where $G = K_P^*$, $\langle f, f'\rangle_G = \int_G ff'dg$ and we introduced the functions $f_0 = \delta_{A''} =$ sum of Dirac's δ_a over all $a \in A''$ and $f_1 = \delta_{\mathcal{O}_P}$ on K_P and the function $F = f_0 \otimes f_1$ on $K_P \times K_P$. We also have the norm map $|\ |\colon G \to \mathbb{C}^*$, the convolution map $j\colon G\times G \to G$, $j(x,y) = x^{-1}y$ and the inclusion $i\colon G\times G \to K_P \times K_P$.

For the appropriate classes of functions f_0 and f_1 there are ζ-functions with a functional equation of the following kind

$$\zeta(s, f_0, f_1) = \zeta(1-s, \widehat{f}_0, \widehat{f}_1),$$

where $\widehat{f}$ is a Fourier transformation of f. We will study the corresponding spaces of functions and operations like j_* or i^* in subsection 1.3.

Remark 1. We assumed that $\mathrm{Pic}(U)$ is trivial. To handle the general case one has to consider the curve C with several points removed. Finiteness of the $\mathrm{Pic}^0(C)$ implies that we can get an open subset U with this property.

1.2. Restricted adeles for dimension 2

1.2.1. Let us discuss the situation for dimension one once more. We consider the case of the algebraic curve C as above.

One-dimensional adelic complex

$$K \oplus \prod_{x\in C}\widehat{\mathcal{O}}_x \to {\prod}'_{x\in C} K_x$$

can be included into the following commutative diagram

$$\begin{array}{ccc} K \oplus \prod_{x\in C} \widehat{\mathcal{O}}_x & \longrightarrow & \prod'_{x\in C} K_x \\ \downarrow & & \downarrow \\ K \oplus \widehat{\mathcal{O}}_P & \longrightarrow & \prod'_{x\neq P} K_x/\widehat{\mathcal{O}}_x \oplus K_P \end{array}$$

where the vertical map induces an isomorphism of cohomologies of the horizontal complexes. Next, we have a commutative diagram

$$\begin{array}{ccc} K \oplus \widehat{\mathcal{O}}_P & \longrightarrow & \prod'_{x\neq P} K_x/\widehat{\mathcal{O}}_x \oplus K_P \\ \downarrow & & \downarrow \\ K/A & \longrightarrow & \prod'_{x\neq P} K_x/\widehat{\mathcal{O}}_x \end{array}$$

where the bottom horizontal arrow is an isomorphism (the surjectivity follows from the strong approximation theorem). This shows that the complex $A \oplus \widehat{\mathcal{O}}_P \to K_P$ is quasi-isomorphic to the full adelic complex. The construction can be extended to an arbitrary locally free sheaf $\mathcal{F}$ on C and we obtain that the complex

$$W \oplus \widehat{\mathcal{F}}_P \to \widehat{\mathcal{F}}_P \otimes_{\widehat{\mathcal{O}}_P} K_P,$$

where $W = \Gamma(\mathcal{F}, C \setminus P) \subset K$, computes the cohomology of the sheaf $\mathcal{F}$.

This fact is essential for the analytical approach to the ζ-function of the curve C. To understand how to generalize it to higher dimensions we have to recall another applications of this diagram, in particular, the so called Krichever correspondence from the theory of integrable systems.

Let z be a local parameter at P, so $\widehat{\mathcal{O}}_P = k[[z]]$. The Krichever correspondence assigns points of infinite dimensional Grassmanians to (C, P, z) and a torsion free coherent sheaf of $\mathcal{O}_C$-modules on C. In particular, there is an injective map from classes of triples (C, P, z) to $A \subset k((z))$. In [P5] it was generalized to the case of algebraic surfaces using the higher adelic language.

1.2.2. Let X be a projective irreducible algebraic surface over a field k, $C \subset X$ be an irreducible projective curve, and $P \in C$ be a smooth point on both C and X.

In dimension two we start with the adelic complex

$$\mathbb{A}_0 \oplus \mathbb{A}_1 \oplus \mathbb{A}_2 \to \mathbb{A}_{01} \oplus \mathbb{A}_{02} \oplus \mathbb{A}_{12} \to A_{012},$$

where

$$A_0 = K = k(X), \qquad A_1 = \prod_{C\subset X} \widehat{\mathcal{O}}_C, \qquad A_2 = \prod_{x\in X} \widehat{\mathcal{O}}_x,$$

$$A_{01} = \prod\nolimits'_{C\subset X} K_C, A_{02} = \prod\nolimits'_{x\in X} K_x, A_{12} = \prod\nolimits'_{x\in C} \widehat{\mathcal{O}}_{x,C}, A_{012} = \mathbb{A}_X = \prod\nolimits' K_{x,C}.$$

In fact one can pass to another complex whose cohomologies are the same as of the adelic complex and which is a generalization of the construction for dimension one. We have to make the following assumptions: $P \in C$ is a smooth point on both C and X, and the surface $X \setminus C$ is affine. The desired complex is

$$A \oplus A_C \oplus \widehat{\mathcal{O}}_P \to B_C \oplus B_P \oplus \widehat{\mathcal{O}}_{P,C} \to K_{P,C}$$

where the rings B_x, B_C, A_C and A have the following meaning. Let $x \in C$. Let

$B_x = \bigcap\limits_{D \neq C} (K_x \cap \widehat{\mathcal{O}}_{x,D})$ where the intersection is taken inside K_x;

$B_C = K_C \cap (\bigcap\limits_{x \neq P} B_x)$ where the intersection is taken inside $K_{x,C}$;

$A_C = B_C \cap \widehat{\mathcal{O}}_C$, $A = K \cap (\bigcap_{x \in X \setminus C} \widehat{\mathcal{O}}_x)$.

This can be easily extended to the case of an arbitrary torsion free coherent sheaf $\mathcal{F}$ on X.

1.2.3. Returning back to the question about the ζ-function of the surface X over $k = \mathbb{F}_q$ we suggest to write it as the product of three Dirichlet series

$$\zeta_X(s) = \zeta_{X \setminus C}(s)\zeta_{C \setminus P}(s)\zeta_P(s) = \Big(\sum_{I \subset X \setminus C} |I|_X^s\Big)\Big(\sum_{I \subset C \setminus P} |I|_X^s\Big)\Big(\sum_{I \subset \mathrm{Spec}(\widehat{\mathcal{O}}_{P,C})} |I|_X^s\Big).$$

Again we can assume that the surface $U = X \setminus C$ has the most trivial possible structure. Namely, $\mathrm{Pic}(U) = (0)$ and $\mathrm{Ch}(U) = (0)$. Then every rank 2 vector bundle on U is trivial. In the general case one can remove finitely many curves C from X to pass to the surface U satisfying these properties (the same idea was used in the construction of the higher Bruhat–Tits buildings attached to an algebraic surface [P3, sect. 3]).

Therefore any zero-ideal I with support in $X \setminus C$, $C \setminus P$ or P can be defined by functions from the rings A, A_C and $\mathcal{O}_P$, respectively. The fundamental difference between the case of dimension one and the case of surfaces is that zero-cycles I and ideals of finite colength on X are not in one-to-one correspondence.

Remark 2. In [P2], [FP] we show that the functional equation for the L-function on an algebraic surface over a finite field can be rewritten using the K_2-adeles. Then it has the same shape as the functional equation for algebraic curves written in terms of $\mathbb{A}^*$-adeles (as in [W1]).

1.3. Types for dimension 1

We again discuss the case of dimension one. If D is a divisor on the curve C then the Riemann–Roch theorem says

$$l(D) - l(K_C - D) = \deg(D) + \chi(\mathcal{O}_C),$$

where as usual $l(D) = \dim \Gamma(C, \mathcal{O}_X(D))$ and K_C is the canonical divisor. If $\mathbb{A} = \mathbb{A}_C$ and $\mathbb{A}_1 = \mathbb{A}(D)$ then

$$H^1(C, \mathcal{O}_X(D)) = \mathbb{A}/(\mathbb{A}(D) + K), \qquad H^0(C, \mathcal{O}_X(D)) = \mathbb{A}(D) \cap K$$

where $K = \mathbb{F}_q(C)$. We have the following topological properties of the groups:

$\mathbb{A}$	locally compact group,
$\mathbb{A}(D)$	compact group,
K	discrete group,
$\mathbb{A}(D) \cap K$	finite group,
$\mathbb{A}(D) + K$	group of finite index of $\mathbb{A}$.

The group $\mathbb{A}$ is dual to itself. Fix a rational differential form $\omega \in \Omega^1_K$, $\omega \neq 0$ and an additive character ψ of $\mathbb{F}_q$. The following bilinear form

$$\langle (f_x), (g_x) \rangle = \sum_x \operatorname{res}_x(f_x g_x \omega), \ (f_x), (g_x) \in \mathbb{A}$$

is non-degenerate and defines an auto-duality of $\mathbb{A}$.

If we fix a Haar measure dx on $\mathbb{A}$ then we also have the Fourier transform

$$f(x) \mapsto \widehat{f}(x) = \int_{\mathbb{A}} \psi(\langle x, y \rangle) f(y) dy$$

for functions on $\mathbb{A}$ and for distributions F defined by the Parseval equality

$$(\widehat{F}, \widehat{\phi}) = (F, \phi).$$

One can attach some functions and/or distributions to the subgroups introduced above

δ_D = the characteristic function of $\mathbb{A}(D)$

δ_{H^1} = the characteristic function of $\mathbb{A}(D) + K$

$\delta_K = \sum_{\gamma \in K} \delta_\gamma$ where δ_γ is the delta-function at the point γ

$\delta_{H^0} = \sum_{\gamma \in \mathbb{A}(D) \cap K} \delta_\gamma.$

There are two fundamental rules for the Fourier transform of these functions

$$\widehat{\delta}_D = \operatorname{vol}(\mathbb{A}(D)) \delta_{\mathbb{A}(D)^\perp},$$

where

$$\mathbb{A}(D)^{\perp} = \mathbb{A}((\omega) - D),$$

and

$$\widehat{\delta}_{\Gamma} = \mathrm{vol}(\mathbb{A}/\Gamma)^{-1}\delta_{\Gamma^{\perp}}$$

for any discrete co-compact group Γ. In particular, we can apply that to $\Gamma = K = \Gamma^{\perp}$. We have

$$(\delta_K, \delta_D) = \#(K \cap \mathbb{A}(D)) = q^{l(D)},$$
$$(\widehat{\delta}_K, \widehat{\delta}_D) = \mathrm{vol}(\mathbb{A}(D))\mathrm{vol}(\mathbb{A}/K)^{-1}(\delta_K, \delta_{K_C - D}) = q^{\deg D} q^{\chi(\mathcal{O}_C)} q^{l(K_C - D)}$$

and the Parseval equality gives us the Riemann–Roch theorem.

The functions in these computations can be classified according to their types. There are four types of functions which were introduced by F. Bruhat in 1961 [Br].

Let V be a finite dimensional vector space over the adelic ring $\mathbb{A}$ (or over an one-dimensional local field K with finite residue field $\mathbb{F}_q$). We put

$$\begin{aligned}
\mathcal{D} &= \{\text{locally constant functions with compact support}\},\\
\mathcal{E} &= \{\text{locally constant functions}\},\\
\mathcal{D}' &= \{\text{dual to } \mathcal{D} \ = \ \text{all distributions}\},\\
\mathcal{E}' &= \{\text{dual to } \mathcal{E} \ = \ \text{distributions with compact support}\}.
\end{aligned}$$

Every V has a filtration $P \supset Q \supset R$ by compact open subgroups such that all quotients P/Q are finite dimensional vector spaces over $\mathbb{F}_q$.

If V, V' are the vector spaces over $\mathbb{F}_q$ of finite dimension then for every homomorphism $i\colon V \to V'$ there are two maps

$$\mathcal{F}(V) \xrightarrow{i_*} \mathcal{F}(V'), \qquad \mathcal{F}(V') \xrightarrow{i^*} \mathcal{F}(V),$$

of the spaces $\mathcal{F}(V)$ of all functions on V (or V') with values in $\mathbb{C}$. Here i^* is the standard inverse image and i_* is defined by

$$i_* f(v') = \begin{cases} 0, & \text{if } v' \notin \mathrm{im}(i) \\ \sum_{v \mapsto v'} f(v), & \text{otherwise.} \end{cases}$$

The maps i_* and i^* are dual to each other.

We apply these constructions to give a more functorial definition of the Bruhat spaces. For any triple P, Q, R as above we have an epimorphism $i\colon P/R \to P/Q$ with the corresponding map for functions $\mathcal{F}(P/Q) \xrightarrow{i^*} \mathcal{F}(P/R)$ and a monomorphism $j\colon Q/R \to P/R$ with the map for functions $\mathcal{F}(Q/R) \xrightarrow{j_*} \mathcal{F}(P/R)$.

Now the Bruhat spaces can be defined as follows

$$\begin{aligned}
\mathcal{D} &= \varinjlim_{j_*} \varinjlim_{i^*} \mathcal{F}(P/Q),\\
\mathcal{E} &= \varprojlim_{j^*} \varinjlim_{i^*} \mathcal{F}(P/Q),\\
\mathcal{D}' &= \varprojlim_{j^*} \varprojlim_{i_*} \mathcal{F}(P/Q),\\
\mathcal{E}' &= \varinjlim_{j_*} \varprojlim_{i_*} \mathcal{F}(P/Q).
\end{aligned}$$

The spaces don't depend on the choice of the chain of subspaces P, Q, R. Clearly we have

$$\begin{aligned}
&\delta_D \in \mathcal{D}(\mathbb{A}),\\
&\delta_K \in \mathcal{D}'(\mathbb{A}),\\
&\delta_{H^0} \in \mathcal{E}'(\mathbb{A}),\\
&\delta_{H^1} \in \mathcal{E}(\mathbb{A}).
\end{aligned}$$

We have the same relations for the functions $\delta_{\mathcal{O}_P}$ and $\delta_{A''}$ on the group K_P considered in section 1.

The Fourier transform preserves the spaces $\mathcal{D}$ and $\mathcal{D}'$ but interchanges the spaces $\mathcal{E}$ and $\mathcal{E}'$. Recalling the origin of the subgroups from the adelic complex we can say that, in dimension one the types of the functions have the following structure

$$\begin{tikzcd}
& \mathcal{E} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{D} \arrow[dr, dash] & & \mathcal{D}' \arrow[dl, dash] \\
& \mathcal{E}' &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbf{01} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbf{1} \arrow[dr, dash] & & \mathbf{0} \arrow[dl, dash] \\
& \emptyset &
\end{tikzcd}$$

corresponding to the full simplicial division of an edge. The Fourier transform is a reflection of the diagram with respect to the middle horizontal axis.

The main properties of the Fourier transform we need in the proof of the Riemann-Roch theorem (and of the functional equation of the ζ-function) can be summarized as the commutativity of the following cube diagram

$$\begin{tikzcd}
& \mathcal{D}\otimes\mathcal{D}' \arrow[rr, "j_*"] \arrow[dl, leftrightarrow] \arrow[dd, "i^*" near start] & & \mathcal{E} \arrow[dl, leftrightarrow] \arrow[dd, "\alpha^*"] \\
\mathcal{D}\otimes\mathcal{D}' \arrow[rr, "i^*" near start] \arrow[dd, "j_*"'] & & \mathcal{E}' \arrow[dd, "\beta_*" near end] & \\
& \mathcal{E}' \arrow[rr, "\beta_*" near start] \arrow[dl, leftrightarrow] & & \mathcal{F}(\mathbb{F}_1) \arrow[dl, dash] \\
\mathcal{E} \arrow[rr, "\alpha^*"] & & \mathcal{F}(\mathbb{F}_1) &
\end{tikzcd}$$

coming from the exact sequence

$$\mathbb{A} \xrightarrow{i} \mathbb{A} \oplus \mathbb{A} \xrightarrow{j} \mathbb{A},$$

with $i(a) = (a, a)$, $j(a, b) = a - b$, and the maps

$$\mathbb{F}_1 \xrightarrow{\alpha} \mathbb{A} \xrightarrow{\beta} \mathbb{F}_1$$

with $\alpha(0) = 0$, $\beta(a) = 0$. Here $\mathbb{F}_1$ is the field of one element, $\mathcal{F}(\mathbb{F}_1) = \mathbb{C}$ and the arrows with heads on both ends are the Fourier transforms.

In particular, the commutativity of the diagram implies the Parseval equality used above:

$$\begin{aligned}\langle \widehat{F}, \widehat{G} \rangle &= \beta_* \circ i^*(\widehat{F} \otimes \widehat{G}) \\ &= \beta_* \circ i^*(\widehat{F \otimes G}) = \beta_* j_* \widehat{(F \otimes G)} \\ &= \alpha^* \circ j_*(F \otimes G) = \beta_* \circ i^*(F \otimes G) \\ &= \langle F, G \rangle.\end{aligned}$$

Remark 3. These constructions can be extended to the function spaces on the groups $G(\mathbb{A})$ or $G(K)$ for a local field K and a group scheme G.

1.4. Types for dimension 2

In order to understand the types of functions in the case of dimension 2 we have to look at the adelic complex of an algebraic surface. We will use physical notations and denote a space by the discrete index which corresponds to it. Thus the adelic complex can be written as

$$\emptyset \to \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2} \to \mathbf{01} \oplus \mathbf{02} \oplus \mathbf{12} \to \mathbf{012},$$

where $\emptyset$ stands for the augmentation map corresponding to the inclusion of H^0. Just as in the case of dimension one we have a duality of $\mathbb{A} = \mathbb{A}_{012} = \mathbf{012}$ with itself defined by a bilinear form

$$\langle (f_{x,C}), (g_{x,C}) \rangle = \sum_{x,C} \operatorname{res}_{x,C}(f_{x,C}\, g_{x,C}\, \omega), \qquad (f_{x,C}), (g_{x,C}) \in \mathbb{A}$$

which is also non-degenerate and defines the autoduality of $\mathbb{A}$.

It can be shown that

$$\mathbb{A}_0 = \mathbb{A}_{01} \cap \mathbb{A}_{02}, \quad \mathbb{A}_{01}^{\perp} = \mathbb{A}_{01}, \quad \mathbb{A}_{02}^{\perp} = \mathbb{A}_{02}, \quad \mathbb{A}_0^{\perp} = \mathbb{A}_{01} \oplus \mathbb{A}_{02},$$

and so on. The proofs depend on the following residue relations for a rational differential form $\omega \in \Omega^2_{k(X)}$

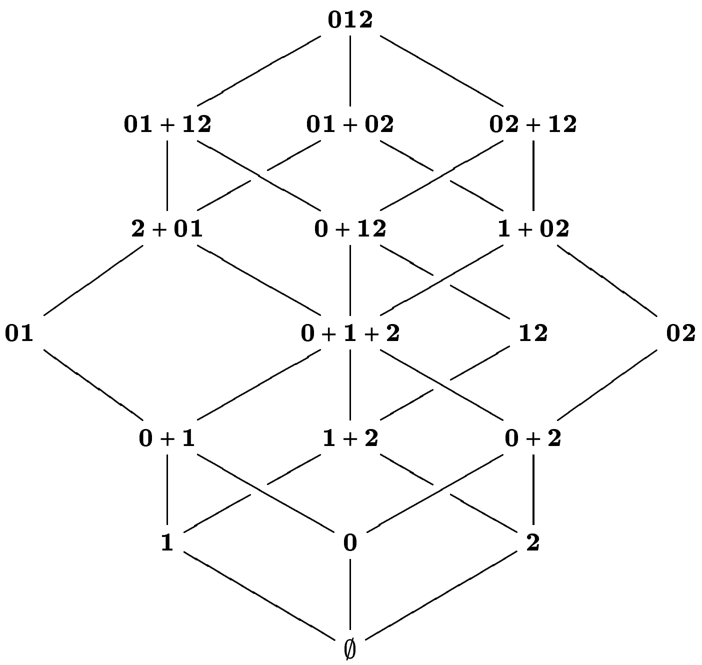
$$\text{for all } x \in X \qquad \sum_{C \ni x} \operatorname{res}_{x,C}(\omega) = 0,$$

$$\text{for all } C \subset X \qquad \sum_{x \in C} \operatorname{res}_{x,C}(\omega) = 0.$$

We see that the subgroups appearing in the adelic complex are not closed under the duality. It means that the set of types in dimension two will be greater then the set of types coming from the components of the adelic complex. Namely, we have:

Theorem 1 ([P4])**.** *Fix a divisor D on an algebraic surface X and let $\mathbb{A}_{12} = \mathbb{A}(D)$. Consider the lattice $\mathfrak{L}$ of the commensurability classes of subspaces in $\mathbb{A}_X$ generated by subspaces $\mathbb{A}_{01}, \mathbb{A}_{02}, \mathbb{A}_{12}$.*

The lattice $\mathfrak{L}$ is isomorphic to a free distributive lattice in three generators and has the structure shown in the diagram.



Remark 4. Two subspaces V, V' are called commensurable if $(V + V')/V \cap V'$ is of finite dimension. In the one-dimensional case *all* the subspaces of the adelic complex are commensurable (even the subspaces corresponding to different divisors). In this case we get a free distributive lattice in two generators (for the theory of lattices see [Bi]).

Just as in the case of curves we can attach to every node some space of functions (or distributions) on $\mathbb{A}$. We describe here a particular case of the construction, namely, the space $\mathcal{F}_{02}$ corresponding to the node $\mathbf{02}$. Also we will consider not the full adelic group but a single two-dimensional local field $K = \mathbb{F}_q((u))((t))$.

In order to define the space we use the filtration in K by the powers $\mathcal{M}^n$ of the maximal ideal $\mathcal{M} = \mathbb{F}_q((u))[[t]]t$ of K as a discrete valuation (of rank 1) field. Then we try to use the same procedure as for the local field of dimension 1 (see above).

If $P \supset Q \supset R$ are the elements of the filtration then we need to define the maps

$$\mathcal{D}(P/R) \xrightarrow{i_*} \mathcal{D}(P/Q), \qquad \mathcal{D}(P/R) \xrightarrow{j^*} \mathcal{D}(Q/R)$$

corresponding to an epimorphism $i: P/R \to P/Q$ and a monomorphism $j: Q/R \to P/R$. The map j^* is a restriction of the locally constant functions with compact support and it is well defined. To define the direct image i_* one needs to integrate along the fibers of the projection i. To do that we have to choose a Haar measure on the fibers for *all* P, Q, R in a consistent way. In other words, we need a system of Haar measures on all quotients P/Q and by transitivity of the Haar measures in exact sequences it is enough to do that on all quotients $\mathcal{M}^n/\mathcal{M}^{n+1}$.

Since $\mathcal{O}_K/\mathcal{M} = \mathbb{F}_q((u)) = K_1$ we can first choose a Haar measure on the residue field K_1. It will depend on the choice of a fractional ideal $\mathcal{M}^i_{K_1}$ normalizing the Haar measure. Next, we have to extend the measure on all $\mathcal{M}^n/\mathcal{M}^{n+1}$. Again, it is enough to choose a second local parameter t which gives an isomorphism

$$t^n: \mathcal{O}_K/\mathcal{M} \to \mathcal{M}^n/\mathcal{M}^{n+1}.$$

Having made these choices we can put as above

$$\mathcal{F}_{02} = \varprojlim_{j^*} \varprojlim_{i_*} \mathcal{D}(P/Q)$$

where the space $\mathcal{D}$ was introduced in the previous section.

We see that contrary to the one-dimensional case the space $\mathcal{F}_{02}$ is not intrinsically defined. But the choice of all additional data can be easily controlled.

Theorem 2 ([P4])**.** *The set of the spaces $\mathcal{F}_{02}$ is canonically a principal homogeneous space over the valuation group Γ_K of the field K.*

Recall that Γ_K is non-canonically isomorphic to the lexicographically ordered group $\mathbb{Z} \oplus \mathbb{Z}$.

One can extend this procedure to other nodes of the diagram of types. In particular, for $\mathbf{012}$ we get the space which does not depend on the choice of the Haar measures.

The standard subgroup of the type $\mathbf{02}$ is $B_P = \mathbb{F}_p[[u]]((t))$ and it is clear that

$$\delta_{B_P} \in \mathcal{F}_{02}.$$

The functions δ_{B_C} and $\delta_{\widehat{\mathcal{O}}_{P,C}}$ have the types $\mathbf{01}$, $\mathbf{12}$ respectively.

Remark 5. Note that the whole structure of all subspaces in $\mathbb{A}$ or K corresponding to different divisors or coherent sheaves is more complicated. The spaces $\mathbb{A}(D)$ of type $\mathbf{12}$ are no more commensurable. To describe the whole lattice one has to introduce several equivalence relations (commensurability up to compact subspace, a locally compact subspace and so on).

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