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3. The Bruhat–Tits buildings over higher dimensional local fields

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3.0. Introduction

A generalization of the Bruhat–Tits buildings for the groups $PGL(V)$ over n-dimensional local fields was introduced in [P1]. The main object of the classical Bruhat–Tits theory is a simplicial complex attached to any reductive algebraic group G defined over a field K. There are two parallel theories in the case where K has no additional structure or K is a local (or more generally, complete discrete valuation) field. They are known as the *spherical* and *euclidean* buildings correspondingly (see subsection 3.2 for a brief introduction, [BT1], [BT2] for original papers and [R], [T1] for the surveys).

In the generalized theory of buildings they correspond to local fields of dimension zero and of dimension one. The construction of the Bruhat–Tits building for the group $PGL(2)$ over two-dimensional local field was described in detail in [P2]. Later V. Ginzburg and M. Kapranov extended the theory to arbitrary reductive groups over a two-dimensional local fields [GK]. Their definition coincides with ours for $PGL(2)$ and is different for higher ranks. But it seems that they are closely related (in the case of the groups of type A_l). It remains to develop the theory for arbitrary reductive groups over local fields of dimension greater than two.

In this work we describe the structure of the higher building for the group $PGL(3)$ over a two-dimensional local field. We refer to [P1], [P2] for the motivation of these constructions.

This work contains four subsections. In 3.1 we collect facts about the Weyl group. Then in 3.2 we briefly describe the building for $PGL(2)$ over a local field of dimension not greater than two; for details see [P1], [P2]. In 3.3 we study the building for $PGL(3)$ over a local field F of dimension one and in 3.4 we describe the building over a two-dimensional local field.

We use the notations of section 1 of Part I.

If K is an n-dimensional local field, let Γ_K be the valuation group of the discrete valuation of rank n on K^*; the choice of a system of local parameters $t_1, \dots, t_n$ of K induces an isomorphism of Γ_K and the lexicographically ordered group $\mathbb{Z}^{\oplus n}$.

Let K ($K = K_2,\ K_1,\ K_0 = k$) be a two-dimensional local field. Let $O = O_K$, $M = M_K$, $\mathcal{O} = \mathcal{O}_K$, $\mathcal{M} = \mathcal{M}_K$ (see subsection 1.1 of Part I). Then $O = \mathrm{pr}^{-1}(\mathcal{O}_{K_1})$, $M = \mathrm{pr}^{-1}(\mathcal{M}_{K_1})$ where $\mathrm{pr}\colon \mathcal{O}_K \to K_1$ is the residue map. Let t_1, t_2 be a system of local parameters of K.

If $K \supset \mathcal{O}$ is the fraction field of a ring $\mathcal{O}$ we call $\mathcal{O}$-submodules $J \subset K$ fractional $\mathcal{O}$-ideals (or simply fractional ideals).

The ring O has the following properties:

(i) $O/M \simeq k, \quad K^* \simeq \langle t_1\rangle \times \langle t_2\rangle \times O^*, \quad O^* \simeq k^* \times (1+M)$;

(ii) every finitely generated fractional O-ideal is principal and equal to

$$P(i,j) = (t_1^i t_2^j) \quad \text{for some } i,j \in \mathbb{Z}$$

(for the notation $P(i,j)$ see loc.cit.);

(iii) every infinitely generated fractional O-ideal is equal to

$$P(j) = \mathcal{M}_K^j = \langle t_1^i t_2^j : i \in \mathbb{Z}\rangle \quad \text{for some } j \in \mathbb{Z}$$

(see [FP], [P2] or section 1 of Part I). The set of these ideals is totally ordered with respect to the inclusion.

3.1. The Weyl group

Let B be the image of

$$\begin{pmatrix} O & O & \dots & O \\ M & O & \dots & O \\ & & \dots & \\ M & M & \dots & O \end{pmatrix}$$

in $PGL(m,K)$. Let N be the subgroup of monomial matrices.

Definition 1. Let $T = B \bigcap N$ be the image of

$$\begin{pmatrix} O^* & \dots & 0 \\ & \ddots & \\ 0 & \dots & O^* \end{pmatrix}$$

in G.

The group

$$W = W_{K/K_1/k} = N/T$$

is called the *Weyl group*.

There is a rich structure of subgroups in G which have many common properties with the theory of BN-pairs. In particular, there are Bruhat, Cartan and Iwasawa decompositions (see [P2]).

The Weyl group W contains the following elements of order two

$$s_i = \begin{pmatrix} 1 & \dots & 0 & & & 0 & \dots & 0 \\ & \ddots & & & & & & \\ 0 & \dots & 1 & & & & & 0 \\ 0 & \dots & & 0 & 1 & & \dots & 0 \\ 0 & & & 1 & 0 & & \dots & 0 \\ 0 & \dots & & & & 1 & \dots & 0 \\ & & & & & & \ddots & \\ 0 & \dots & 0 & & & 0 & \dots & 1 \end{pmatrix}, \quad i = 1, ..., m-1;$$

$$w_1 = \begin{pmatrix} 0 & 0 & \dots & 0 & t_1 \\ 0 & 1 & \dots & 0 & 0 \\ & & \dots & & \\ & & \dots & & \\ & & \dots & 1 & 0 \\ t_1^{-1} & 0 & \dots & 0 & 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 & 0 & \dots & 0 & t_2 \\ 0 & 1 & \dots & 0 & 0 \\ & & \dots & & \\ & & \dots & & \\ & & \dots & 1 & 0 \\ t_2^{-1} & 0 & \dots & 0 & 0 \end{pmatrix}.$$

The group W has the following properties:

(i) W is generated by the set S of its elements of order two,

(ii) there is an exact sequence

$$0 \to E \to W_{K/K_1/k} \to W_K \to 1,$$

where E is the kernel of the addition map

$$\underbrace{\Gamma_K \oplus \cdots \oplus \Gamma_K}_{m \text{ times}} \to \Gamma_K$$

and W_K is isomorphic to the symmetric group S_m;

(iii) the elements s_i, $i = 1, \dots, m-1$ define a splitting of the exact sequence and the subgroup $\langle s_1, \dots, s_{m-1} \rangle$ acts on E by permutations.

In contrast with the situation in the theory of BN-pairs the pair (W, S) is not a Coxeter group and furthermore there is no subset S of involutions in W such that (W, S) is a Coxeter group (see [P2]).

3.2. Bruhat–Tits building for $PGL(2)$ over a local field of dimension $\leqslant 2$

In this subsection we briefly recall the main constructions. For more details see [BT1], [BT2], [P1], [P2].

3.2.1. Let k be a field (which can be viewed as a 0-dimensional local field). Let V be a vector space over k of dimension two.

Definition 2. The *spherical building* of $PGL(2)$ over k is a zero-dimensional complex

$$\Delta(k) = \Delta(PGL(V), k)$$

whose vertices are lines in V.

The group $PGL(2, k)$ acts on $\Delta(k)$ transitively. The Weyl group (in this case it is of order two) acts on $\Delta(k)$ and its orbits are *apartments* of the building.

3.2.2. Let F be a complete discrete valuation field with residue field k. Let V be a vector space over F of dimension two. We say that $L \subset V$ is a lattice if L is an $\mathcal{O}_F$-module. Two submodules L and L' belong to the same class $\langle L\rangle \sim \langle L'\rangle$ if and only if $L = aL'$, with $a \in F^*$.

Definition 3. The *euclidean building* of $PGL(2)$ over F is a one-dimensional complex $\Delta(F/k)$ whose vertices are equivalence classes $\langle L\rangle$ of lattices. Two classes $\langle L\rangle$ and $\langle L'\rangle$ are connected by an edge if and only if for some choice of L, L' there is an exact sequence

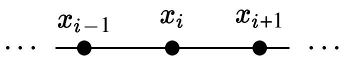
$$0 \to L' \to L \to k \to 0.$$

Denote by $\Delta_i(F/k)$ the set of i-dimensional simplices of the building $\Delta(F/k)$.

The following *link property* is important:

Let $P \in \Delta_0(F/k)$ be represented by a lattice L. Then the link of P (= the set of edges of $\Delta(F/k)$ going from P) is in one-to-one correspondence with the set of lines in the vector space $V_P = L/\mathcal{M}_F L$ (which is $\Delta(PGL(V_P), k)$).

The orbits of the Weyl group W (which is in this case an infinite group with two generators of order two) are infinite sets consisting of $x_i = \langle L_i\rangle$, $L_i = \mathcal{O}_F \oplus \mathcal{M}_F^i$.



An element w of the Weyl group acts in the following way: if $w \in E = \mathbb{Z}$ then w acts by translation of even length; if $w \notin E$ then w acts as an involution with a unique fixed point x_{i_0}: $w(x_{i+i_0}) = x_{i_0-i}$.

To formalize the connection of $\Delta(F/k)$ with $\Delta(F)$ we define a *boundary* point of $\Delta(F/k)$ as a class of half-lines such that the intersection of every two half-lines from the class is a half-line in both of them. The set of the boundary points is called the *boundary* of $\Delta(F/k)$.

There is an isomorphism between $PGL(2,F)$-sets $\Delta(F)$ and the boundary of $\Delta(F/k)$: if a half-line is represented by $L_i = \mathcal{O}_F \oplus \mathcal{M}_F^i$, $i > 0$, then the corresponding vertex of $\Delta(F)$ is the line $F \oplus (0)$ in V.

It seems reasonable to slightly change the notations to make the latter isomorphisms more transparent.

Definition 4 ([P1]). Put $\Delta_.[0](F/k) =$ the complex of classes of $\mathcal{O}_F$-submodules in V isomorphic to $F \oplus \mathcal{O}_F$ (so $\Delta_.[0](F/k)$ is isomorphic to $\Delta(F)$) and put

$$\Delta_.[1](F/k) = \Delta(F/k).$$

Define the *building of $PGL(2)$ over F* as the union

$$\Delta_.(F/k) = \Delta_.[1](F/k) \bigcup \Delta_.[0](F/k)$$

and call the subcomplex $\Delta_.[0](F/k)$ the *boundary* of the building. The discrete topology on the boundary can be extended to the whole building.

3.2.3. Let K be a two-dimensional local field.

Let V be a vector space over K of dimension two. We say that $L \subset V$ is a lattice if L is an O-module. Two submodules L and L' belong to the same class $\langle L\rangle \sim \langle L'\rangle$ if and only if $L = aL'$, with $a \in K^*$.

Definition 5 ([P1]). Define the vertices of the building of $PGL(2)$ over K as

$$\begin{aligned}
\Delta_0[2](K/K_1/k) &= \text{classes of } O\text{-submodules } L \subset V\colon\ L \simeq O \oplus O\\
\Delta_0[1](K/K_1/k) &= \text{classes of } O\text{-submodules } L \subset V\colon\ L \simeq O \oplus \mathcal{O}\\
\Delta_0[0](K/K_1/k) &= \text{classes of } O\text{-submodules } L \subset V\colon\ L \simeq O \oplus K.
\end{aligned}$$

Put

$$\Delta_0(K/K_1/k) = \Delta_0[2](K/K_1/k) \bigcup \Delta_0[1](K/K_1/k) \bigcup \Delta_0[0](K/K_1/k).$$

A set of $\{L_\alpha\}$, $\alpha \in I$, of O-submodules in V is called a *chain* if

(i) for every $\alpha \in I$ and for every $a \in K^*$ there exists an $\alpha' \in I$ such that $aL_\alpha = L_{\alpha'}$,

(ii) the set $\{L_\alpha, \alpha \in I\}$ is totally ordered by the inclusion.

A chain $\{L_\alpha, \alpha \in I\}$ is called a *maximal chain* if it cannot be included in a strictly larger set satisfying the same conditions (i) and (ii).

We say that $\langle L_0\rangle, \langle L_1\rangle, \ldots, \langle L_m\rangle$ belong to a *simplex* of dimension m if and only if the L_i, $i = 0, 1, \ldots, m$ belong to a maximal chain of $\mathcal{O}_F$-submodules in V. The faces and the degeneracies can be defined in a standard way (as a deletion or repetition of a vertex). See [BT2].

Let $\{L_\alpha\}$ be a maximal chain of O-submodules in the space V. There are exactly three types of maximal chains ([P2]):

(i) if the chain contains a module L isomorphic to $O \oplus O$ then all the modules of the chain are of that type and the chain is uniquely determined by its segment

$$\cdots \supset O \oplus O \supset M \oplus O \supset M \oplus M \supset \ldots .$$

(ii) if the chain contains a module L isomorphic to $O \oplus \mathcal{O}$ then the chain can be restored from the segment:

$$\cdots \supset O \oplus \mathcal{O} \supset O \oplus P(1,0) \supset O \oplus P(2,0) \supset \cdots \supset O \oplus \mathcal{M} \supset \ldots$$

(recall that $P(1,0) = M$).

(iii) if the chain contains a module L isomorphic to $\mathcal{O} \oplus \mathcal{O}$ then the chain can be restored from the segment:

$$\cdots \supset \mathcal{O} \oplus \mathcal{O} \supset P(1,0) \oplus \mathcal{O} \supset P(2,0) \oplus \mathcal{O} \supset \cdots \supset \mathcal{M} \oplus \mathcal{O} \supset \ldots .$$

3.3. Bruhat–Tits building for $PGL(3)$ over a local field F of dimension 1

Let $G = PGL(3)$.

Let F be a one-dimensional local field, $F \supset \mathcal{O}_F \supset \mathcal{M}_F$, $\mathcal{O}_F/\mathcal{M}_F \simeq k$.

Let V be a vector space over F of dimension three. Define lattices in V and their equivalence similarly to the definition of 3.2.2.

First we define the vertices of the building and then the simplices. The result will be a simplicial set $\Delta_.(G, F/k)$.

Definition 6. The *vertices* of the Bruhat–Tits building:

$$\begin{aligned}
\Delta_0[1](G,F/k) &= \{\text{classes of } \mathcal{O}_F \text{-submodules } L \subset V : L \simeq \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F\},\\
\Delta_0[0](G,F/k) &= \{\text{classes of } \mathcal{O}_F \text{-submodules } L \subset V : L \simeq \mathcal{O}_F \oplus \mathcal{O}_F \oplus F\\
&\qquad \text{or } L \simeq \mathcal{O}_F \oplus F \oplus F\},\\
\Delta_0(G,F/k) &= \Delta_0[1](G,F/k) \cup \Delta_0[0](G,F/k).
\end{aligned}$$

We say that the points of $\Delta_0[1]$ are *inner* points, the points of $\Delta_0[0]$ are *boundary* points. Sometimes we delete G and F/k from the notation if this does not lead to confusion.

We have defined the vertices only. For the simplices of higher dimension we have the following:

Definition 7. Let $\{L_\alpha, \alpha \in I\}$ be a set of $\mathcal{O}_F$-submodules in V. We say that $\{L_\alpha, \alpha \in I\}$ is a *chain* if

(i) for every $\alpha \in I$ and for every $a \in K^*$ there exists an $\alpha' \in I$ such that $aL_\alpha = L_{\alpha'}$,
(ii) the set $\{L_\alpha, \alpha \in I\}$ is totally ordered by the inclusion.

A chain $\{L_\alpha, \alpha \in I\}$ is called a *maximal chain* if it cannot be included in a strictly larger set satisfying the same conditions (i) and (ii).

We say that $\langle L_0\rangle, \langle L_1\rangle, \dots, \langle L_m\rangle$ belong to a *simplex* of dimension m if and only if the $L_i,\ i = 0, 1, ..., m$ belong to a maximal chain of $\mathcal{O}_F$-submodules in V. The faces and the degeneracies can be defined in a standard way (as a deletion or repetition of a vertex). See [BT2].

To describe the structure of the building we first need to determine all types of the maximal chains. Proceeding as in [P2] (for $PGL(2)$) we get the following result.

Proposition 1. *There are exactly three types of maximal chains of $\mathcal{O}_F$-submodules in the space V:*

(i) *the chain contains a module isomorphic to $\mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F$. Then all the modules from the chain are of that type and the chain has the following structure:*

$$\dots \supset \mathcal{M}_F^i L \supset \mathcal{M}_F^i L' \supset \mathcal{M}_F^i L'' \supset \mathcal{M}_F^{i+1} L \supset \mathcal{M}_F^{i+1} L' \supset \mathcal{M}_F^{i+1} L'' \supset \dots$$

where $\langle L\rangle, \langle L'\rangle, \langle L''\rangle \in \Delta_0(G, F/k)[1]$ and $L \simeq \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F$, $L' \simeq \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{M}_F,\ L'' \simeq \mathcal{O}_F \oplus \mathcal{M}_F \oplus \mathcal{M}_F$.

(ii) *the chain contains a module isomorphic to $\mathcal{O}_F \oplus \mathcal{O}_F \oplus F$. Then the chain has the following structure:*

$$\dots \supset \mathcal{M}_F^i L \supset \mathcal{M}_F^i L' \supset \mathcal{M}_F^{i+1} L \supset \dots$$

where $\langle L\rangle, \langle L'\rangle \in \Delta_0(G, F/k)[0]$ and $L \simeq \mathcal{O}_F \oplus \mathcal{O}_F \oplus F,\ L' \simeq \mathcal{M}_F \oplus \mathcal{O}_F \oplus F$.

(iii) *the chain contains a module isomorphic to $\mathcal{O}_F \oplus F \oplus F$. Then the chain has the following structure:*

$$\dots \supset \mathcal{M}_F^i L \supset \mathcal{M}_F^{i+1} L \supset \dots$$

where $\langle L\rangle \in \Delta_0(G, F/k)[0]$.

We see that the chains of the first type correspond to two-simplices, of the second type — to edges and the last type represent some vertices. It means that the simplicial set $\Delta_.$ is a disconnected union of its subsets $\Delta_.[m],\ m = 0, 1$. The dimension of the subset $\Delta_.[m]$ is equal to one for $m = 0$ and to two for $m = 1$.

Usually the buildings are defined as combinatorial complexes having a system of subcomplexes called apartments (see, for example, [R], [T1], [T2]). We show how to introduce them for the higher building.

Definition 8. Fix a basis $e_1, e_2, e_3 \in V$. The *apartment* defined by this basis is the following set

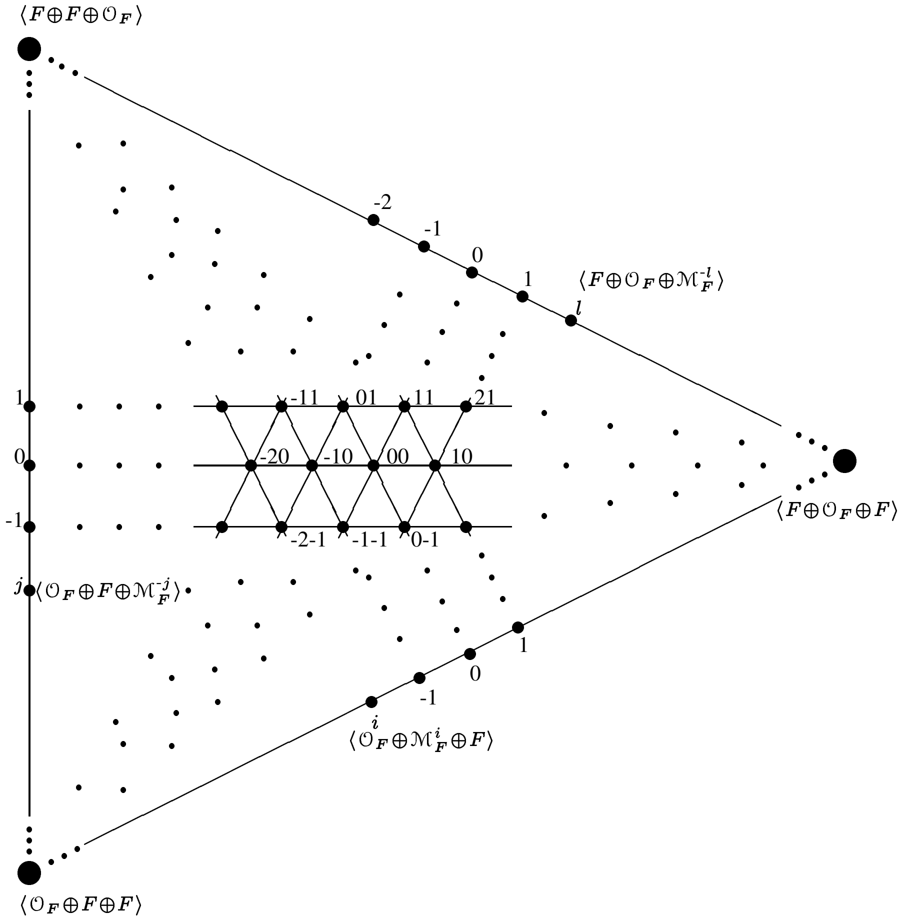
$$\Sigma_. = \Sigma_.[1] \cup \Sigma_.[0],$$

where

$$\Sigma_0[1] = \{\langle L\rangle : L = a_1e_1 \oplus a_2e_2 \oplus a_3e_3, \text{ where } a_1, a_2, a_3 \text{ are } \mathcal{O}_F\text{-submodules in } F \text{ isomorphic to } \mathcal{O}_F\}$$

$$\Sigma_0[0] = \{\langle L\rangle : L = a_1e_1 \oplus a_2e_2 \oplus a_3e_3, \text{ where } a_1, a_2, a_3 \text{ are } \mathcal{O}_F\text{-submodules in } F \text{ isomorphic either to } \mathcal{O}_F \text{ or to } F \text{ and at least one } a_i \text{ is isomorphic to } F\}.$$

$\Sigma_.[m]$ is the minimal subcomplex having $\Sigma_0[m]$ as vertices.



It can be shown that the building $\Delta_.(G, F/k)$ is glued from the apartments, namely

$$\Delta_.(G, F/k) = \bigsqcup_{\text{all bases of } V} \Sigma_. \,/\, \text{an equivalence relation}$$

(see [T2]).

We can make this description more transparent by drawing all that in the picture above where the dots of different kinds belong to the different parts of the building. In contrast with the case of the group $PGL(2)$ it is not easy to draw the whole building and we restrict ourselves to an apartment.

Here the inner vertices are represented by the lattices

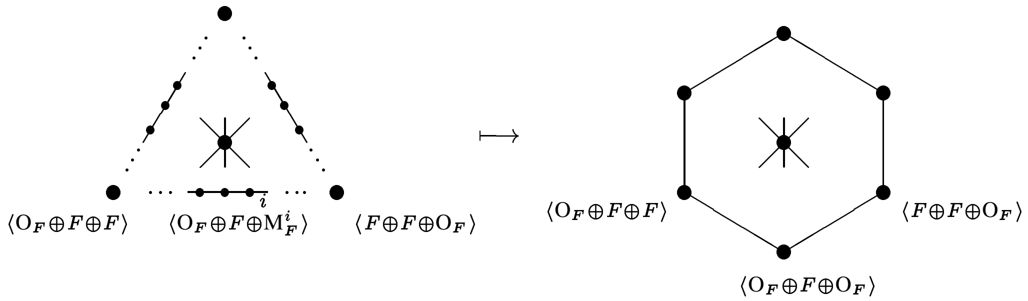
$$ij = \langle \mathcal{O}_F \oplus \mathcal{M}_F^i \oplus \mathcal{M}_F^j \rangle, \quad i, j \in \mathbb{Z}.$$

The definition of the boundary gives a topology on $\Delta_0(G, F/k)$ which is discrete on both subsets $\Delta_0[1]$ and $\Delta_0[0]$. The convergence of the inner points to the boundary points is given by the following rules:

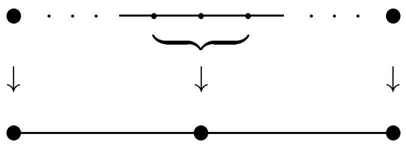
$$\langle \mathcal{O}_F \oplus \mathcal{M}_F^i \oplus \mathcal{M}_F^j \rangle \xrightarrow{j \to -\infty} \langle \mathcal{O}_F \oplus \mathcal{M}_F^i \oplus F \rangle,$$
$$\langle \mathcal{O}_F \oplus \mathcal{M}_F^i \oplus \mathcal{M}_F^j \rangle \xrightarrow{j \to \infty} \langle F \oplus F \oplus \mathcal{O}_F \rangle,$$

because $\langle \mathcal{O}_F \oplus \mathcal{M}_F^i \oplus \mathcal{M}_F^j \rangle = \langle \mathcal{M}_F^{-j} \oplus \mathcal{M}_F^{-j+i} \oplus \mathcal{O}_F \rangle$. The convergence in the other two directions can be defined along the same line (and it is shown on the picture). It is easy to extend it to the higher simplices.

Thus, there is the structure of a simplicial topological space on the apartment and then we define it on the whole building using the gluing procedure. This topology is stronger than the topology usually introduced to connect the inner part and the boundary together. The connection with standard "compactification" of the building is given by the following map:



This map is bijective on the inner simplices and on a part of the boundary can be described as



We note that the complex is not a CW-complex but only a closure finite complex. This "compactification" was used by G. Mustafin [M].

We have two kinds of connections with the buildings for other fields and groups. First, for the local field F there are two local fields of dimension 0, namely F and k. Then for every $P \in \Delta_0[1](PGL(V), F/k)$ the Link(P) is equal to $\Delta_.(PGL(V_P), k)$ where $V_P = L/\mathcal{M}_F L$ if $P = \langle L \rangle$ and the Link(P) is the boundary of the Star(P). Since the apartments for the $PGL(3,k)$ are hexagons, we can also observe this property on the picture. The analogous relation with the building of $PGL(3,K)$ is more complicated. It is shown on the picture above.

The other relations work if we change the group G but not the field. We see that three different lines go out from every inner point in the apartment. They represent the apartments of the group $PGL(2, F/k)$. They correspond to different embeddings of the $PGL(2)$ into $PGL(3)$.

Also we can describe the action of the Weyl group W on an apartment. If we fix a basis, the extension

$$0 \to \Gamma_F \oplus \Gamma_F \to W \to S_3 \to 1$$

splits. The elements from $S_3 \subset W$ act either as rotations around the point 00 or as reflections. The elements of $\mathbb{Z} \oplus \mathbb{Z} \subset W$ can be represented as triples of integers (according to property (ii) in the previous subsection). Then they correspond to translations of the lattice of inner points along the three directions going from the point 00.

If we fix an embedding $PGL(2) \subset PGL(3)$ then the apartments and the Weyl groups are connected as follows:

$$\Sigma_.(PGL(2)) \subset \Sigma_.(PGL(3)),$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & W' & \longrightarrow & S_2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & W & \longrightarrow & S_3 & \longrightarrow & 1 \end{array}$$

where W' is a Weyl group of the group $PGL(2)$ over the field F/k.

3.4. Bruhat–Tits building for $PGL(3)$ over a local field of dimension 2

Let K be a two-dimensional local field. Denote by V a vector space over K of dimension three. Define lattices in V and their equivalence in a similar way to 3.2.3. We shall consider the following *types* of lattices:

$\Delta_0[2]$	222	$\langle O \oplus O \oplus O \rangle$
$\Delta_0[1]$	221	$\langle O \oplus O \oplus \mathcal{O} \rangle$
	211	$\langle O \oplus \mathcal{O} \oplus \mathcal{O} \rangle$
$\Delta_0[0]$	220	$\langle O \oplus O \oplus K \rangle$
	200	$\langle O \oplus K \oplus K \rangle$

To define the buildings we repeat the procedure from the previous subsection.

Definition 10. The *vertices* of the Bruhat–Tits building are the elements of the following set:

$$\Delta_0(G, K/K_1/k) = \Delta_0[2] \cup \Delta_0[1] \cup \Delta_0[0].$$

To define the simplices of higher dimension we can repeat word by word Definitions 7 and 8 of the previous subsection replacing the ring $\mathcal{O}_F$ by the ring O (note that we work only with the types of lattices listed above). We call the subset $\Delta[1]$ the *inner boundary* of the building and the subset $\Delta[0]$ the *external boundary*. The points in $\Delta[2]$ are the *inner* points.

To describe the structure of the building we first need to determine all types of the maximal chains. Proceeding as in [P2] for $PGL(2)$ we get the following result.

Proposition 2. *Let $\{L_\alpha\}$ be a maximal chain of O-submodules in the space V. There are exactly five types of maximal chains:*

(i) *If the chain contains a module L isomorphic to $O \oplus O \oplus O$ then all the modules of the chain are of that type and the chain is uniquely determined by its segment*

$$\cdots \supset O \oplus O \oplus O \supset M \oplus O \oplus O \supset M \oplus M \oplus O \supset M \oplus M \oplus M \supset \dots$$

(ii) *If the chain contains a module L isomorphic to $O \oplus O \oplus \mathcal{O}$ then the chain can be restored from the segment:*

$$\cdots \supset \text{``}\,\mathcal{O} \oplus \underbrace{\mathcal{O} \oplus \mathcal{O}\,\text{''} \supset \cdots \supset O \oplus O \oplus \mathcal{O} \supset M \oplus O \oplus \mathcal{O} \supset M \oplus M \oplus \mathcal{O} \supset \cdots \supset \mathcal{M} \oplus \mathcal{M}}_{\text{quotient } \simeq K_1 \oplus K_1} \oplus \mathcal{O}$$

$$= \mathcal{M} \oplus \underbrace{\mathcal{M} \oplus \mathcal{O} \supset \cdots \supset \mathcal{M} \oplus \mathcal{M} \oplus O \supset \mathcal{M} \oplus \mathcal{M} \oplus M \supset \cdots \supset \mathcal{M}}_{\text{quotient } \simeq K_1} \oplus \mathcal{M} \oplus \mathcal{M} \supset \dots$$

Here the modules isomorphic to $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$ do not belong to this chain and are inserted as in the proof of Proposition 1 of [P2].

(iii) *All the modules $L_\alpha \simeq O \oplus \mathcal{O} \oplus \mathcal{O}$. Then the chain contains a piece*

$$\cdots \supset \text{``}\mathcal{O} \oplus \mathcal{O} \underbrace{\oplus\, \mathcal{O}\text{''} \supset \cdots \supset O \oplus \mathcal{O} \oplus \mathcal{O} \supset M \oplus \mathcal{O} \oplus \mathcal{O} \supset \cdots \supset \mathcal{M} \oplus \mathcal{O} \oplus}_{\text{quotient } \simeq K_1} \mathcal{O}$$

$$= \mathcal{M} \oplus \mathcal{O} \underbrace{\oplus\, \mathcal{O} \supset \cdots \supset \mathcal{M} \oplus O \oplus \mathcal{O} \supset \cdots \supset \mathcal{M} \oplus \mathcal{M} \oplus}_{\text{quotient } \simeq K_1} \mathcal{O}$$

$$= \mathcal{M} \oplus \mathcal{M} \underbrace{\oplus\, \mathcal{O} \supset \cdots \supset \mathcal{M} \oplus \mathcal{M} \oplus O \supset \cdots \supset \mathcal{M} \oplus \mathcal{M} \oplus}_{\text{quotient } \simeq K_1} \mathcal{M} \supset \dots$$

and can also be restored from it. Here the modules isomorphic to $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$ do not belong to this chain and are inserted as in the proof of Proposition 1 of [P2].

(iv) *If there is an $L_\alpha \simeq O \oplus O \oplus K$ then one can restore the chain from*

$$\cdots \supset O \oplus O \oplus K \supset M \oplus O \oplus K \supset M \oplus M \oplus K \supset \dots$$

(v) *If there is an* $L_\alpha \simeq O \oplus K \oplus K$ *then the chain can be written down as*

$$\dots \supset M^i \oplus K \oplus K \supset M^{i+1} \oplus K \oplus K \supset \dots$$

We see that the chains of the first three types correspond to two-simplices, of the fourth type — to edges of the external boundary and the last type represents a vertex of the external boundary. As above we can glue the building from apartments. To introduce them we can again repeat the corresponding definition for the building over a local field of dimension one (see Definition 4 of the previous subsection). Then the apartment $\Sigma_.$ is a union

$$\Sigma_. = \Sigma_.[2] \cup \Sigma_.[1] \cup \Sigma_.[0]$$

where the pieces $\Sigma_.[i]$ contain the lattices of the types from $\Delta_.[i]$.

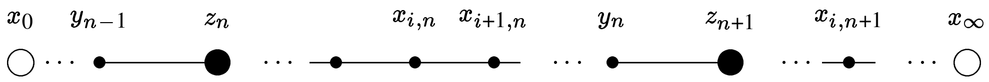
The combinatorial structure of the apartment can be seen from two pictures at the end of the subsection. There we removed the external boundary $\Sigma_.[0]$ which is simplicially isomorphic to the external boundary of an apartment of the building $\Delta_.(PGL(3), K/K_1/k)$. The dots in the first picture show a convergence of the vertices inside the apartment. As a result the building is a simplicial topological space.

We can also describe the relations of the building with buildings of the same group G over the complete discrete valuation fields K and K_1. In the first case there is a projection map

$$\pi\colon \Delta_.(G, K/K_1/k) \to \Delta_.(G, K/K_1).$$

Under this map the big triangles containing the simplices of type (i) are contracted into points, the triangles containing the simplices of type (ii) go to edges and the simplices of type (iii) are mapped isomorphically to simplices in the target space. The external boundary don't change.

The lines

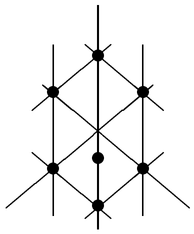


can easily be visualized inside the apartment. Only the big white dots corresponding to the external boundary are missing. We have three types of lines going from the inner points under the angle $2\pi/3$. They correspond to different embeddings of $PGL(2)$ into $PGL(3)$.

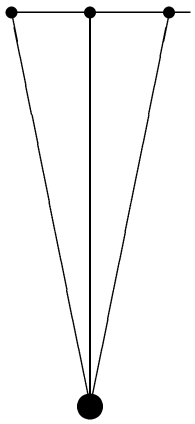
Using the lines we can understand the action of the Weyl group W on an apartment. The subgroup S_3 acts in the same way as in 3.2. The free subgroup E (see 3.1) has six types of translations along these three directions. Along each line we have two opportunities which were introduced for $PGL(2)$.

Namely, if $w \in \Gamma_K \simeq \mathbb{Z} \oplus \mathbb{Z} \subset W$ then $w = (0,1)$ acts as a shift of the whole structure to the right: $w(x_{i,n}) = x_{i,n+2}$, $w(y_n) = y_{n+2}$, $w(z_n) = z_{n+2}, w(x_0) = x_0$, $w(x_\infty) = x_\infty$.

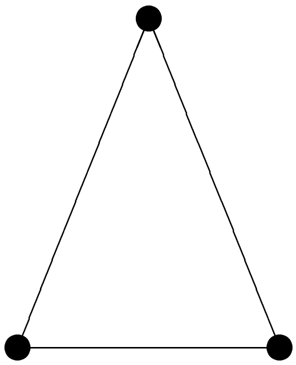
The element $w = (1,0)$ acts as a shift on the points $x_{i,n}$ but leaves fixed the points in the inner boundary $w(x_{i,n}) = x_{i+2,n}$, $w(y_n) = y_n$, $w(z_n) = z_n$, $w(x_0) = x_0$, $w(x_\infty) = x_\infty$, (see [P2, Theorem 5, v]).



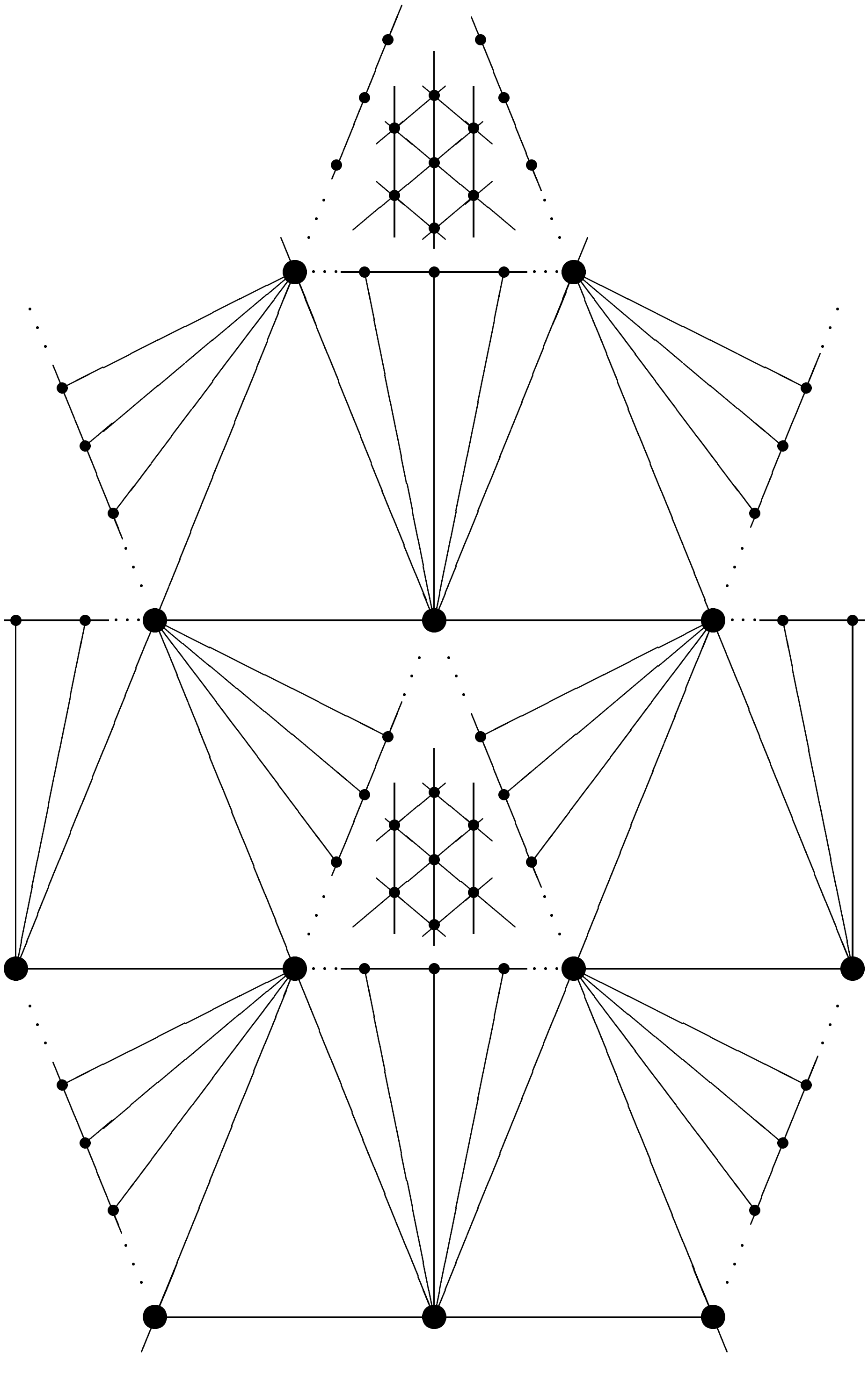
simplices of type (i)



simplices of type (ii)



simplices of type (iii)



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