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4. Drinfeld modules and local fields of positive characteristic

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The relationship between local fields and Drinfeld modules is twofold. Drinfeld modules allow explicit construction of abelian and nonabelian extensions with prescribed properties of local and global fields of positive characteristic. On the other hand, n-dimensional local fields arise in the construction of (the compactification of) moduli schemes X for Drinfeld modules, such schemes being provided with a natural stratification $X_0 \subset X_1 \subset \cdots X_i \cdots \subset X_n = X$ through smooth subvarieties X_i of dimension i.

We will survey that correspondence, but refer to the literature for detailed proofs (provided these exist so far). An important remark is in order: The contents of this article take place in characteristic $p > 0$, and are in fact locked up in the characteristic p world. No lift to characteristic zero nor even to schemes over $\mathbb{Z}/p^2$ is known!

4.1. Drinfeld modules

Let L be a field of characteristic p containing the field $\mathbb{F}_q$, and denote by $\tau = \tau_q$ raising to the qth power map $x \mapsto x^q$. If "a" denotes multiplication by $a \in L$, then $\tau a = a^q \tau$. The ring $\mathrm{End}(\mathbb{G}_{a/L})$ of endomorphisms of the additive group $\mathbb{G}_{a/L}$ equals $L\{\tau_p\} = \{\sum a_i \tau_p^i : a_i \in L\}$, the non-commutative polynomial ring in $\tau_p = (x \mapsto x^p)$ with the above commutation rule $\tau_p a = a^p \tau$. Similarly, the subring $\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a/L})$ of $\mathbb{F}_q$-endomorphisms is $L\{\tau\}$ with $\tau = \tau_p^n$ if $q = p^n$. Note that $L\{\tau\}$ is an $\mathbb{F}_q$-algebra since $\mathbb{F}_q \hookrightarrow L\{\tau\}$ is central.

Definition 1. Let $\mathcal{C}$ be a smooth geometrically connected projective curve over $\mathbb{F}_q$. Fix a closed (but not necessarily $\mathbb{F}_q$-rational) point ∞ of $\mathcal{C}$. The ring $A = \Gamma(\mathcal{C} - \{\infty\}, \mathcal{O}_{\mathcal{C}})$ is called a *Drinfeld ring*. Note that $A^* = \mathbb{F}_q^*$.

Example 1. If $\mathcal{C}$ is the projective line $\mathbb{P}^1{}_{/\mathbb{F}_q}$ and ∞ is the usual point at infinity then $A = \mathbb{F}_q[T]$.

Example 2. Suppose that $p \neq 2$, that $\mathcal{C}$ is given by an affine equation $Y^2 = f(X)$ with a separable polynomial $f(X)$ of even positive degree with leading coefficient a non-square in $\mathbb{F}_q$, and that ∞ is the point above $X = \infty$. Then $A = \mathbb{F}_q[X, Y]$ is a Drinfeld ring with $\deg_{\mathbb{F}_q}(\infty) = 2$.

Definition 2. An A-*structure* on a field L is a homomorphism of $\mathbb{F}_q$-algebras (in brief: an $\mathbb{F}_q$-ring homomorphism) $\gamma: A \to L$. Its A-*characteristic* $\mathrm{char}_A(L)$ is the maximal ideal $\ker(\gamma)$, if γ fails to be injective, and ∞ otherwise. A *Drinfeld module structure* on such a field L is given by an $\mathbb{F}_q$-ring homomorphism $\phi: A \to L\{\tau\}$ such that $\partial \circ \phi = \gamma$, where $\partial: L\{\tau\} \to L$ is the L-homomorphism sending τ to 0.

Denote $\phi(a)$ by $\phi_a \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a\,/L})$; ϕ_a induces on the additive group over L (and on each L-algebra M) a *new* structure as an A-module:

$$a * x := \phi_a(x) \quad (a \in A, x \in M). \tag{4.1.1}$$

We briefly call ϕ a Drinfeld module over L, usually omitting reference to A.

Definition 3. Let ϕ and ψ be Drinfeld modules over the A-field L. A *homomorphism* $u: \phi \to \psi$ is an element of $L\{\tau\}$ such that $u \circ \phi_a = \psi_a \circ u$ for all $a \in A$. Hence an *endomorphism* of ϕ is an element of the centralizer of $\phi(A)$ in $L\{\tau\}$, and u is an *isomorphism* if $u \in L^* \hookrightarrow L\{\tau\}$ is subject to $u \circ \phi_a = \psi_a \circ u$.

Define $\deg: a \to \mathbb{Z} \cup \{-\infty\}$ and $\deg_\tau: L\{\tau\} \to \mathbb{Z} \cup \{-\infty\}$ by $\deg(a) = \log_q |A/a|$ ($a \neq 0$; we write A/a for A/aA), $\deg(0) = -\infty$, and $\deg_\tau(f)$ = the well defined degree of f as a "polynomial" in τ. It is an easy exercise in Dedekind rings to prove the following

Proposition 1. *If ϕ is a Drinfeld module over L, there exists a non-negative integer r such that* $\deg_\tau(\phi_a) = r \deg(a)$ *for all* $a \in A$; r is called the *rank* $\mathrm{rk}(\phi)$ of ϕ.

Obviously, $\mathrm{rk}(\phi) = 0$ means that $\phi = \gamma$, i.e., the A-module structure on $\mathbb{G}_{a\,/L}$ is the tautological one.

Definition 4. Denote by $\mathcal{M}^r(1)(L)$ the set of isomorphism classes of Drinfeld modules of rank r over L.

Example 3. Let $A = \mathbb{F}_q[T]$ be as in Example 1 and let $K = \mathbb{F}_q(T)$ be its fraction field. Defining a Drinfeld module ϕ over K or an extension field L of K is equivalent to specifying $\phi_T = T + g_1\tau + \cdots + g_r\tau^r \in L\{T\}$, where $g_r \neq 0$ and $r = \mathrm{rk}(\phi)$. In the special case where $\phi_T = T + \tau$, ϕ is called the *Carlitz module*. Two such Drinfeld modules ϕ and ϕ' are isomorphic over the algebraic closure L^{alg} of L if and only if there is some $u \in L^{\mathrm{alg}\,*}$ such that $g_i' = u^{q^i - 1} g_i$ for all $i \geqslant 1$. Hence $\mathcal{M}^r(1)(L^{\mathrm{alg}})$ can be described (for $r \geqslant 1$) as an open dense subvariety of a weighted projective space of dimension $r-1$ over L^{alg}.

4.2. Division points

Definition 5. For $a \in A$ and a Drinfeld module ϕ over L, write ${}_a\phi$ for the subscheme of a*-division points* of $\mathbb{G}_{a/L}$ endowed with its structure of an A-module. Thus for any L-algebra M,

$$ {}_a\phi(M) = \{x \in M : \phi_a(x) = 0\}. $$

More generally, we put ${}_{\mathfrak{a}}\phi = \bigcap_{a \in \mathfrak{a}} \phi_a$ for an arbitrary (not necessarily principal) ideal $\mathfrak{a}$ of A. It is a finite flat group scheme of degree $\mathrm{rk}(\phi) \cdot \deg(\mathfrak{a})$, whose structure is described in the next result.

Proposition 2 ([Dr], [DH, I, Thm. 3.3 and Remark 3.4])**.** *Let the Drinfeld module ϕ over L have rank $r \geqslant 1$.*

(i) *If* $\mathrm{char}_A(L) = \infty$, ${}_{\mathfrak{a}}\phi$ *is reduced for each ideal* $\mathfrak{a}$ *of* A, *and* ${}_{\mathfrak{a}}\phi(L^{\mathrm{sep}}) = {}_{\mathfrak{a}}\phi(L^{\mathrm{alg}})$ *is isomorphic with* $(A/\mathfrak{a})^r$ *as an* A*-module.*

(ii) *If* $\mathfrak{p} = \mathrm{char}_A(L)$ *is a maximal ideal, then there exists an integer* h, *the height* $\mathrm{ht}(\phi)$ *of* ϕ, *satisfying* $1 \leqslant h \leqslant r$, *and such that* ${}_{\mathfrak{a}}\phi(L^{\mathrm{alg}}) \simeq (A/\mathfrak{a})^{r-h}$ *whenever* $\mathfrak{a}$ *is a power of* $\mathfrak{p}$, *and* ${}_{\mathfrak{a}}\phi(L^{\mathrm{alg}}) \simeq (A/\mathfrak{a})^r$ *if* $(\mathfrak{a}, \mathfrak{p}) = 1$.

The absolute Galois group G_L of L acts on ${}_{\mathfrak{a}}\phi(L^{\mathrm{sep}})$ through A-linear automorphisms. Therefore, any Drinfeld module gives rise to Galois representations on its division points. These representations tend to be “as large as possible”.

The prototype of result is the following theorem, due to Carlitz and Hayes [H1].

Theorem 1. *Let* A *be the polynomial ring* $\mathbb{F}_q[T]$ *with field of fractions* K. *Let* $\rho: A \to K\{\tau\}$ *be the Carlitz module,* $\rho_T = T + \tau$. *For any non-constant monic polynomial* $a \in A$, *let* $K(a) := K({}_a\rho(K^{\mathrm{alg}}))$ *be the field extension generated by the* a*-division points.*

(i) $K(a)/K$ *is abelian with group* $(A/a)^*$. *If* σ_b *is the automorphism corresponding to the residue class of* $b \bmod a$ *and* $x \in {}_a\rho(K^{\mathrm{alg}})$ *then* $\sigma_b(x) = \rho_b(x)$.

(ii) *If* $(a) = \mathfrak{p}^t$ *is primary with some prime ideal* $\mathfrak{p}$ *then* $K(a)/K$ *is completely ramified at* $\mathfrak{p}$ *and unramified at the other finite primes.*

(iii) *If* $(a) = \prod a_i$ ($1 \leqslant i \leqslant s$) *with primary and mutually coprime* a_i, *the fields* $K(a_i)$ *are mutually linearly disjoint and* $K = \otimes_{i \leqslant i \leqslant s} K(a_i)$.

(iv) *Let* $K_+(a)$ *be the fixed field of* $\mathbb{F}_q^* \hookrightarrow (A/a)^*$. *Then* ∞ *is completely split in* $K_+(a)/K$ *and completely ramified in* $K(a)/K_+(a)$.

(v) *Let* $\mathfrak{p}$ *be a prime ideal generated by the monic polynomial* $\pi \in A$ *and coprime with* a. *Under the identification* $\mathrm{Gal}(K(a)/K) = (A/a)^*$, *the Frobenius element* $\mathrm{Frob}_{\mathfrak{p}}$ *equals the residue class of* $\pi \bmod a$.

Letting $a \to \infty$ with respect to divisibility, we obtain the field $K(\infty)$ generated over K by all the division points of ρ, with group $\mathrm{Gal}(K(\infty)/K) = \varinjlim_a (A/a)^*$, which almost agrees with the group of finite idele classes of K. It turns out that $K(\infty)$ is the maximal abelian extension of K that is tamely ramified at ∞, i.e., we get a constructive version of the class field theory of K. Hence the theorem may be seen both as a global variant of Lubin–Tate's theory and as an analogue in characteristic p of the Kronecker–Weber theorem on cyclotomic extensions of $\mathbb{Q}$.

There are vast generalizations into two directions:

(a) abelian class field theory of arbitrary global function fields $K = \mathrm{Frac}(A)$, where A is a Drinfeld ring.

(b) systems of nonabelian Galois representations derived from Drinfeld modules.

As to (a), the first problem is to find the proper analogue of the Carlitz module for an arbitrary Drinfeld ring A. As will result e.g. from Theorem 2 (see also (4.3.4)), the isomorphism classes of rank-one Drinfeld modules over the algebraic closure K^{alg} of K correspond bijectively to the (finite!) class group $\mathrm{Pic}(A)$ of A. Moreover, these Drinfeld modules $\rho^{(\mathfrak{a})}$ ($\mathfrak{a} \in \mathrm{Pic}(A)$) may be defined with coefficients in the ring $\mathcal{O}_{H_+}$ of A-integers of a certain abelian extension H_+ of K, and such that the leading coefficients of all $\rho_a^{(\mathfrak{a})}$ are units of $\mathcal{O}_{H_+}$. Using these data along with the identification of H_+ in the dictionary of class field theory yields a generalization of Theorem 1 to the case of arbitrary A. In particular, we again find an explicit construction of the class fields of K (subject to a tameness condition at ∞). However, in view of class number problems, the theory (due to D. Hayes [H2], and superbly presented in [Go2, Ch.VII]) has more of the flavour of complex multiplication theory than of classical cyclotomic theory.

Generalization (b) is as follows. Suppose that L is a finite extension of $K = \mathrm{Frac}(A)$, where A is a general Drinfeld ring, and let the Drinfeld module ϕ over L have rank r. For each power $\mathfrak{p}^t$ of a prime $\mathfrak{p}$ of A, $G_L = \mathrm{Gal}(L^{\mathrm{sep}}/L)$ acts on ${}_{\mathfrak{p}^t}\phi \simeq (A/\mathfrak{p}^t)^r$. We thus get an action of G_L on the $\mathfrak{p}$-adic *Tate module* $T_\mathfrak{p}(\phi) \simeq (A_\mathfrak{p})^r$ of ϕ (see [DH, I sect. 4]. Here of course $A_\mathfrak{p} = \varprojlim A/\mathfrak{p}^t$ is the $\mathfrak{p}$-adic completion of A with field of fractions $K_\mathfrak{p}$. Let on the other hand $\mathrm{End}(\phi)$ be the endomorphism ring of ϕ, which also acts on $T_\mathfrak{p}(\phi)$. It is straightforward to show that (i) $\mathrm{End}(\phi)$ acts faithfully and (ii) the two actions commute. In other words, we get an inclusion

$$i\colon \mathrm{End}(\phi) \otimes_A A_\mathfrak{p} \hookrightarrow \mathrm{End}_{G_L}(T_\mathfrak{p}(\phi)) \tag{4.2.1}$$

of finitely generated free $A_\mathfrak{p}$-modules. The plain analogue of the classical Tate conjecture for abelian varieties, proved 1983 by Faltings, suggests that i is in fact bijective. This has been shown by Taguchi [Tag] and Tamagawa. Taking $\mathrm{End}(T_\mathfrak{p}(\phi)) \simeq \mathrm{Mat}(r, A_\mathfrak{p})$ and the known structure of subalgebras of matrix algebras over a field into account, this means that the subalgebra

$$K_\mathfrak{p}[G_L] \hookrightarrow \mathrm{End}(T_\mathfrak{p}(\phi) \otimes_{A_\mathfrak{p}} K_\mathfrak{p}) \simeq \mathrm{Mat}(r, K_\mathfrak{p})$$

generated by the Galois operators is as large as possible. A much stronger statement is obtained by R. Pink [P1, Thm. 0.2], who shows that the image of G_L in $\mathrm{Aut}(T_\mathfrak{p}(\phi))$ has finite index in the centralizer group of $\mathrm{End}(\phi) \otimes A_\mathfrak{p}$. Hence if e.g. ϕ has no “complex multiplications” over L^{alg} (i.e., $\mathrm{End}_{L^{\mathrm{alg}}}(\phi) = A$; this is the generic case for a Drinfeld module in characteristic ∞), then the image of G_L has finite index in $\mathrm{Aut}(T_\mathfrak{p}(\phi)) \simeq GL(r, A_\mathfrak{p})$. This is quite satisfactory, on the one hand, since we may use the Drinfeld module ϕ to construct large nonabelian Galois extensions of L with prescribed ramification properties. On the other hand, the important (and difficult) problem of estimating the index in question remains.

4.3. Weierstrass theory

Let A be a Drinfeld ring with field of fractions K, whose completion at ∞ is denoted by K_∞. We normalize the corresponding absolute value $|\ | = |\ |_\infty$ as $|a| = |A/a|$ for $0 \neq a \in A$ and let C_∞ be the completed algebraic closure of K_∞, i.e., the completion of the algebraic closure K_∞^{alg} with respect to the unique extension of $|\ |$ to K_∞^{alg}. By Krasner’s theorem, C_∞ is again algebraically closed ([BGS, p. 146], where also other facts on function theory in C_∞ may be found). It is customary to indicate the strong analogies between $A, K, K_\infty, C_\infty, \dots$ and $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \dots$, e.g. A is a discrete and cocompact subring of K_∞. But note that C_∞ fails to be locally compact since $|C_\infty : K_\infty| = \infty$.

Definition 6. A *lattice of rank* r (an r-lattice in brief) in C_∞ is a finitely generated (hence projective) discrete A-submodule Λ of C_∞ of projective rank r, where the discreteness means that Λ has finite intersection with each ball in C_∞. The *lattice function* $e_\Lambda\colon C_\infty \to C_\infty$ of Λ is defined as the product

$$e_\Lambda(z) = z \prod_{0 \neq \lambda \in \Lambda} (1 - z/\lambda). \tag{4.3.1}$$

It is entire (defined through an everywhere convergent power series), Λ-periodic and $\mathbb{F}_q$-linear. For a non-zero $a \in A$ consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & C_\infty & \xrightarrow{e_\Lambda} & C_\infty & \longrightarrow & 0 \\ & & {\scriptstyle a}\big\downarrow & & {\scriptstyle a}\big\downarrow & & {\scriptstyle \phi_a^\Lambda}\big\downarrow & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & C_\infty & \xrightarrow{e_\Lambda} & C_\infty & \longrightarrow & 0 \end{array} \tag{4.3.2}$$

with exact lines, where the left and middle arrows are multiplications by a and ϕ_a^Λ is defined through commutativity. It is easy to verify that

(i) $\phi_a^\Lambda \in C_\infty\{\tau\}$,
(ii) $\deg_\tau(\phi_a^\Lambda) = r \cdot \deg(a)$,

(iii) $a \mapsto \phi_a^\Lambda$ is a ring homomorphism $\phi^\Lambda\colon A \to C_\infty\{\tau\}$, in fact, a Drinfeld module of rank r. Moreover, all the Drinfeld modules over C_∞ are so obtained.

Theorem 2 (Drinfeld [Dr, Prop. 3.1]).

(i) *Each rank-* r *Drinfeld module* ϕ *over* C_∞ *comes via* $\Lambda \mapsto \phi^\Lambda$ *from some* r*-lattice* Λ *in* C_∞.

(ii) *Two Drinfeld modules* ϕ^Λ, $\phi^{\Lambda'}$ *are isomorphic if and only if there exists* $0 \neq c \in C_\infty$ *such that* $\Lambda' = c \cdot \Lambda$.

We may thus describe $\mathcal{M}^r(1)(C_\infty)$ (see Definition 4) as the space of r-lattices modulo similarities, i.e., as some generalized upper half-plane modulo the action of an arithmetic group. Let us make this more precise.

Definition 7. For $r \geqslant 1$ let $\mathbb{P}^{r-1}(C_\infty)$ be the C_∞-points of projective $r-1$-space and $\Omega^r := \mathbb{P}^{r-1}(C_\infty) - \bigcup H(C_\infty)$, where H runs through the K_∞-rational hyperplanes of $\mathbb{P}^{r-1}$. That is, $\underline{\omega} = (\omega_1 : \ldots : \omega_r)$ belongs to *Drinfeld's half-plane* Ω^r if and only if there is no non-trivial relation $\sum a_i\omega_i = 0$ with coefficients $a_i \in K_\infty$.

Both point sets $\mathbb{P}^{r-1}(C_\infty)$ and Ω^r carry structures of analytic spaces over C_∞ (even over K_∞), and so we can speak of holomorphic functions on Ω^r. We will not give the details (see for example [GPRV, in particular lecture 6]); suffice it to say that locally uniform limits of rational functions (e.g. Eisenstein series, see below) will be holomorphic.

Suppose for the moment that the class number $h(A) = |\mathrm{Pic}(A)|$ of A equals one, i.e., A is a principal ideal domain. Then each r-lattice Λ in C_∞ is free, $\Lambda = \sum_{1\leqslant i\leqslant r} A\omega_i$, and the discreteness of Λ is equivalent with $\underline{\omega} := (\omega_1 : \ldots : \omega_r)$ belonging to $\Omega^r \hookrightarrow \mathbb{P}^{r-1}(C_\infty)$. Further, two points $\underline{\omega}$ and $\underline{\omega}'$ describe similar lattices (and therefore isomorphic Drinfeld modules) if and only if they are conjugate under $\Gamma := GL(r, A)$, which acts on $\mathbb{P}^{r-1}(C_\infty)$ and its subspace Ω^r. Therefore, we get a canonical bijection

$$\Gamma \setminus \Omega^r \xrightarrow{\sim} \mathcal{M}^r(1)(C_\infty) \tag{4.3.3}$$

from the quotient space $\Gamma \setminus \Omega^r$ to the set of isomorphism classes $\mathcal{M}^r(1)(C_\infty)$.

In the general case of arbitrary $h(A) \in \mathbb{N}$, we let $\Gamma_i := GL(Y_i) \hookrightarrow GL(r, k)$, where $Y_i \hookrightarrow K^r$ ($1 \leqslant i \leqslant h(A)$) runs through representatives of the $h(A)$ isomorphism classes of projective A-modules of rank r. In a similar fashion (see e.g. [G1, II sect.1], [G3]), we get a bijection

$$\dot{\bigcup}_{1\leqslant i\leqslant h(A)} \Gamma_i \setminus \Omega^r \xrightarrow{\sim} \mathcal{M}^r(1)(C_\infty), \tag{4.3.4}$$

which can be made independent of choices if we use the canonical adelic description of the Y_i.

Example 4. If $r = 2$ then $\Omega = \Omega^2 = \mathbb{P}^1(C_\infty) - -\mathbb{P}^1(K_\infty) = C_\infty - K_\infty$, which rather corresponds to $\mathbb{C} - \mathbb{R} = H^+ \bigcup H^-$ (upper and lower complex half-planes) than to H^+ alone. The group $\Gamma := GL(2, A)$ acts via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z) = \frac{az+b}{cz+d}$, and thus gives rise to *Drinfeld modular forms* on Ω (see [G1]). Suppose moreover that $A = \mathbb{F}_q[T]$ as in Examples 1 and 3. We define *ad hoc* a modular form of weight k for Γ as a holomorphic function $f\colon \Omega \to C_\infty$ that satisfies

(i) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and

(ii) $f(z)$ is bounded on the subspace $\{z \in \Omega : \inf_{x \in K_\infty} |z - x| > 1\}$ of Ω.

Further, we put M_k for the C_∞-vector space of modular forms of weight k. (In the special case under consideration, (ii) is equivalent to the usual "holomorphy at cusps" condition. For more general groups Γ, e.g. congruence subgroups of $GL(2, A)$, general Drinfeld rings A, and higher ranks $r \geqslant 2$, condition (ii) is considerably more costly to state, see [G1].) Let

$$E_k(z) := \sum_{(0,0) \neq (a,b) \in A \times A} \frac{1}{(az+b)^k} \tag{4.3.5}$$

be the *Eisenstein series* of weight k. Due to the non-archimedean situation, the sum converges for $k \geqslant 1$ and yields a modular form $0 \neq E_k \in M_k$ if $k \equiv 0\ (q-1)$. Moreover, the various M_k are linearly independent and

$$M(\Gamma) := \bigoplus_{k \geqslant 0} M_k = C_\infty[E_{q-1}, E_{q^2-1}] \tag{4.3.6}$$

is a polynomial ring in the two algebraically independent Eisenstein series of weights $q-1$ and q^2-1. There is an *a priori* different method of constructing modular forms via Drinfeld modules. With each $z \in \Omega$, associate the 2-lattice $\Lambda_z := Az + A \hookrightarrow C_\infty$ and the Drinfeld module $\phi^{(z)} = \phi^{(\Lambda_z)}$. Writing $\phi_T^{(z)} = T + g(z)\tau + \Delta(z)\tau^2$, the coefficients g and Δ become functions in z, in fact, modular forms of respective weights $q-1$ and q^2-1. We have ([Go1], [G1, II 2.10])

$$g = (T^g - T)E_{q-1}, : \Delta = (T^{q^2} - T)E_{q^2-1} + (T^{q^2} - T^q)E_{q-1}^{q+1}. \tag{4.3.7}$$

The crucial fact is that $\Delta(z) \neq 0$ for $z \in \Omega$, but Δ vanishes "at infinity". Letting $j(z) := g(z)^{q+1}/\Delta(z)$ (which is a function on Ω invariant under Γ), the considerations of Example 3 show that j is a complete invariant for Drinfeld modules of rank two. Therefore, the composite map

$$j\colon \Gamma \setminus \Omega \xrightarrow{\sim} \mathcal{M}^2(1)(C_\infty) \xrightarrow{\sim} C_\infty \tag{4.3.8}$$

is bijective, in fact, biholomorphic.

4.4. Moduli schemes

We want to give a similar description of $\mathcal{M}^r(1)(C_\infty)$ for $r \geqslant 2$ and arbitrary A, that is, to convert (4.3.4) into an isomorphism of analytic spaces. One proceeds as follows (see [Dr], [DH], [G3]):

(a) Generalize the notion of "Drinfeld A-module over an A-field L" to "Drinfeld A-module over an A-scheme $S \to \operatorname{Spec} A$". This is quite straightforward. Intuitively, a Drinfeld module over S is a continuously varying family of Drinfeld modules over the residue fields of S.

(b) Consider the functor on A-schemes:

$$\mathcal{M}^r \colon S \longmapsto \left\{ \begin{array}{l} \text{isomorphism classes of rank-}r \\ \text{Drinfeld modules over } S \end{array} \right\}.$$

The naive initial question is to represent this functor by an S-scheme $M^r(1)$. This is impossible in view of the existence of automorphisms of Drinfeld modules even over algebraically closed A-fields.

(c) As a remedy, introduce rigidifying level structures on Drinfeld modules. Fix some ideal $0 \neq \mathfrak{n}$ of A. An $\mathfrak{n}$-*level structure* on the Drinfeld module ϕ over the A-field L whose A-characteristic doesn't divide $\mathfrak{n}$ is the choice of an isomorphism of A-modules

$$\alpha \colon (A/\mathfrak{n})^r \xrightarrow{\sim} {}_{\mathfrak{n}}\phi(L)$$

(compare Proposition 2). Appropriate modifications apply to the cases where $\operatorname{char}_A(L)$ divides $\mathfrak{n}$ and where the definition field L is replaced by an A-scheme S. Let $\mathcal{M}^r(\mathfrak{n})$ be the functor

$$\mathcal{M}^r(\mathfrak{n}) \colon S \longmapsto \left\{ \begin{array}{l} \text{isomorphism classes of rank-}r \\ \text{Drinfeld modules over } S \text{ endowed} \\ \text{with an } \mathfrak{n}\text{-level structure} \end{array} \right\}.$$

Theorem 3 (Drinfeld [Dr, Cor. to Prop. 5.4])**.** *Suppose that $\mathfrak{n}$ is divisible by at least two different prime ideals. Then $\mathcal{M}^r(\mathfrak{n})$ is representable by a smooth affine A-scheme $M^r(\mathfrak{n})$ of relative dimension $r-1$.*

In other words, the scheme $M^r(\mathfrak{n})$ carries a "tautological" Drinfeld module ϕ of rank r endowed with a level-$\mathfrak{n}$ structure such that pull-back induces for each A-scheme S a bijection

$$M^r(\mathfrak{n})(S) = \{\text{morphisms } (S, M^r(\mathfrak{n}))\} \xrightarrow{\sim} \mathcal{M}^r(\mathfrak{n})(S), \quad f \longmapsto f^*(\phi). \tag{4.4.1}$$

$M^r(\mathfrak{n})$ is called the (fine) *moduli scheme* for the moduli problem $\mathcal{M}^r(\mathfrak{n})$. Now the finite group $G(\mathfrak{n}) := GL(r, A/\mathfrak{n})$ acts on $\mathcal{M}^r(\mathfrak{n})$ by permutations of the level structures. By functoriality, it also acts on $M^r(\mathfrak{n})$. We let $M^r(1)$ be the quotient of $M^r(\mathfrak{n})$ by $G(\mathfrak{n})$ (which does not depend on the choice of $\mathfrak{n}$). It has the property that at least its L-valued points for algebraically closed A-fields L correspond bijectively and functorially to $\mathcal{M}^r(1)(L)$. It is therefore called a *coarse moduli scheme* for $\mathcal{M}^r(1)$. Combining the above with (4.3.4) yields a bijection

$$\dot{\bigcup}_{1\leqslant i\leqslant h(A)} \Gamma_i \setminus \Omega^r \xrightarrow{\sim} M^r(1)(C_\infty), \tag{4.4.2}$$

which even is an isomorphism of the underlying analytic spaces [Dr, Prop. 6.6]. The most simple special case is the one dealt with in Example 4, where $M^2(1) = \mathbb{A}^1{}_{/A}$, the affine line over A.

4.5. Compactification

It is a fundamental question to construct and study a "compactification" of the affine A-scheme $M^r(\mathfrak{n})$, relevant for example for the Langlands conjectures over K, the cohomology of arithmetic subgroups of $GL(r,A)$, or the K-theory of A and K. This means that we are seeking a proper A-scheme $\overline{M}^r(\mathfrak{n})$ with an A-embedding $M^r(\mathfrak{n}) \hookrightarrow \overline{M}^r(\mathfrak{n})$ as an open dense subscheme, and which behaves functorially with respect to the forgetful morphisms $M^r(\mathfrak{n}) \to M^r(\mathfrak{m})$ if $\mathfrak{m}$ is a divisor of $\mathfrak{n}$. For many purposes it suffices to solve the apparently easier problem of constructing similar compactifications of the generic fiber $M^r(\mathfrak{n}) \times_A K$ or even of $M^r(\mathfrak{n}) \times_A C_\infty$. Note that varieties over C_∞ may be studied by analytic means, using the GAGA principle.

There are presently three approaches towards the problem of compactification:
(a) a (sketchy) construction of the present author [G2] of a compactification $\overline{M}_\Gamma$ of M_Γ, the C_∞-variety corresponding to an arithmetic subgroup Γ of $GL(r,A)$ (see (4.3.4) and (4.4.2)). We will return to this below;
(b) an analytic compactification similar to (a), restricted to the case of a polynomial ring $A = \mathbb{F}_q[T]$, but with the advantage of presenting complete proofs, by M. M. Kapranov [K];
(c) R. Pink's idea of a modular compactification of $M^r(\mathfrak{n})$ over A through a generalization of the underlying moduli problem [P2].

Approaches (a) and (b) agree essentially in their common domain, up to notation and some other choices. Let us briefly describe how one proceeds in (a). Since there is nothing to show for $r = 1$, we suppose that $r \geqslant 2$.

We let A be any Drinfeld ring. If Γ is a subgroup of $GL(r,K)$ commensurable with $GL(r,A)$ (we call such Γ *arithmetic subgroups*), the point set $\Gamma \setminus \Omega$ is the set of C_∞-points of an affine variety M_Γ over C_∞, as results from the discussion of subsection 4.4. If Γ is the congruence subgroup $\Gamma(\mathfrak{n}) = \{\gamma \in GL(r,A)\colon \gamma \equiv 1 \bmod \mathfrak{n}\}$, then M_Γ is one of the irreducible components of $M^r(\mathfrak{n}) \times_A C_\infty$.

Definition 8. For $\underline{\omega} = (\omega_1, : \ldots : \omega_r) \in \mathbb{P}^{r-1}(C_\infty)$ put

$$r(\underline{\omega}) := \dim_K(K\omega_1 + \cdots + K\omega_r) \quad \text{and} \quad r_\infty(\underline{\omega}) := \dim_{K_\infty}(K_\infty\omega_1 + \cdots + K_\infty\omega_r).$$

Then $1 \leqslant r_\infty(\underline{\omega}) \leqslant r(\underline{\omega}) \leqslant r$ and $\Omega^r = \{\underline{\omega} \mid r_\infty(\underline{\omega}) = r\}$. More generally, for $1 \leqslant i \leqslant r$ let

$$\Omega^{r,i} := \{\underline{\omega} \colon r_\infty(\underline{\omega}) = r(\underline{\omega}) = i\}.$$

Then $\Omega^{r,i} = \dot{\bigcup} \Omega_V$, where V runs through the K-subspaces of dimension i of K^r and Ω_V is constructed from V in a similar way as is $\Omega^r = \Omega_{K^r}$ from $C_\infty^r = (K^r) \otimes C_\infty$. That is, $\Omega_V = \{\underline{\omega} \in \mathbb{P}(V \otimes C_\infty) \hookrightarrow \mathbb{P}^{r-1}(C_\infty) \colon r_\infty(\underline{\omega}) = r(\underline{\omega}) = i\}$, which has a natural structure as analytic space of dimension $\dim(V) - 1$ isomorphic with $\Omega^{\dim(V)}$. Finally, we let $\overline{\Omega}^r := \{\underline{\omega} \colon r_\infty(\underline{\omega}) = r(\underline{\omega})\} = \dot{\bigcup}_{1 \leqslant i \leqslant r} \Omega^{r,i}$.

$\overline{\Omega}^r$ along with its stratification through the $\Omega^{r,i}$ is stable under $GL(r,K)$, so this also holds for the arithmetic group Γ in question. The quotient $\Gamma \setminus \overline{\Omega}^r$ turns out to be the C_∞-points of the wanted compactification $\overline{M}_\Gamma$.

Definition 9. Let $P_i \hookrightarrow G := GL(r)$ be the maximal parabolic subgroup of matrices with first i columns being zero. Let H_i be the obvious factor group isomorphic $GL(r-i)$. Then $P_i(K)$ acts via $H_i(K)$ on K^{r-i} and thus on Ω^{r-i}. From

$$G(K)/P_i(K) \xrightarrow{\sim} \{\text{subspaces } V \text{ of dimension } r-i \text{ of } K^r\}$$

we get bijections

$$\begin{aligned} &G(K) \times_{P_i(K)} \Omega^{r-i} \xrightarrow{\sim} \Omega^{r,r-i}, \\ &(g, \omega_{i+1} : \ldots : \omega_r) \longmapsto (0 : \cdots : 0 : \omega_{i+1} : \ldots : \omega_r) g^{-1} \end{aligned} \tag{4.5.1}$$

and

$$\Gamma \setminus \Omega^{r,r-i} \xrightarrow{\sim} \dot{\bigcup}_{g \in \Gamma \setminus G(K)/P_i(K)} \Gamma(i,g) \setminus \Omega^{r-i}, \tag{4.5.2}$$

where $\Gamma(i,g) := P_i \cap g^{-1}\Gamma_g$, and the double quotient $\Gamma \setminus G(K)/P_i(K)$ is finite by elementary lattice theory. Note that the image of $\Gamma(i,g)$ in $H_i(K)$ (the group that effectively acts on Ω^{r-i}) is again an arithmetic subgroup of $H_i(K) = GL(r-i, K)$, and so the right hand side of (4.5.2) is the disjoint union of analytic spaces of the same type $\Gamma' \setminus \Omega^{r'}$.

Example 5. Let $\Gamma = \Gamma(1) = GL(r, A)$ and $i = 1$. Then $\Gamma \setminus G(K)/P_1(K)$ equals the set of isomorphism classes of projective A-modules of rank $r-1$, which in turn (through the determinant map) is in one-to-one correspondence with the class group $\mathrm{Pic}(A)$.

Let F_V be the image of Ω_V in $\Gamma \setminus \overline{\Omega}^r$. The different analytic spaces F_V, corresponding to locally closed subvarieties of $\overline{M}_\Gamma$, are glued together in such a way that F_U lies in the Zariski closure $\overline{F}_V$ of F_V if and only if U is Γ-conjugate to a K-subspace of V. Taking into account that $F_V \simeq \Gamma' \setminus \Omega^{\dim(V)} = M_{\Gamma'}(C_\infty)$ for some arithmetic subgroup Γ' of $GL(\dim(V), K)$, $\overline{F}_V$ corresponds to the compactification $\overline{M}_{\Gamma'}$ of $M_{\Gamma'}$.

The details of the gluing procedure are quite technical and complicated and cannot be presented here (see [G2] and [K] for some special cases). Suffice it to say that for each boundary component F_V of codimension one, a vertical coordinate t_V may be specified such that F_V is locally given by $t_V = 0$. The result (we refrain from stating a "theorem" since proofs of the assertions below in full strength and generality are published neither in [G2] nor in [K]) will be a normal projective C_∞-variety $\overline{M}_\Gamma$ provided with an open dense embedding $i: M_\Gamma \hookrightarrow \overline{M}_\Gamma$ with the following properties:

- $\overline{M}_\Gamma(C_\infty) = \Gamma \setminus \overline{\Omega}^r$, and the inclusion $\Gamma \setminus \Omega^r \hookrightarrow \Gamma \setminus \overline{\Omega}^r$ corresponds to i;
- $\overline{M}_\Gamma$ is defined over the same finite abelian extension of K as is M_Γ;
- for $\Gamma' \hookrightarrow \Gamma$, the natural map $M_{\Gamma'} \to M_\Gamma$ extends to $\overline{M}_{\Gamma'} \to \overline{M}_\Gamma$;
- the F_V correspond to locally closed subvarieties, and $\overline{F}_V = \cup F_U$, where U runs through the K-subspaces of V contained up to the action of Γ in V;
- $\overline{M}_\Gamma$ is "virtually non-singular", i.e., Γ contains a subgroup Γ' of finite index such that $\overline{M}_{\Gamma'}$ is non-singular; in that case, the boundary components of codimension one present normal crossings.

Now suppose that $\overline{M}_\Gamma$ is non-singular and that $x \in \overline{M}_\Gamma(C_\infty) = \bigcup_{1\leqslant i\leqslant r} \Omega^{r,i}$ belongs to $\Omega^{r,1}$. Then we can find a sequence $\{x\} = X_0 \subset \cdots X_i \cdots \subset X_{r-1} = \overline{M}_\Gamma$ of smooth subvarieties $X_i = \overline{F}_{V_i}$ of dimension i. Any holomorphic function around x (or more generally, any modular form for Γ) may thus be expanded as a series in t_V with coefficients in the function field of $\overline{F}_{V_{r-1}}$, etc. Hence $\overline{M}_\Gamma$ (or rather its completion at the X_i) may be described through $(r-1)$-dimensional local fields with residue field C_∞. The expansion of some standard modular forms can be explicitly calculated, see [G1, VI] for the case of $r = 2$. In the last section we shall present at least the vanishing orders of some of these forms.

Example 6. Let A be the polynomial ring $\mathbb{F}_q[T]$ and $\Gamma = GL(r, A)$. As results from Example 3, (4.3.3) and (4.4.2),

$$M_\Gamma(C_\infty) = M^r(1)(C_\infty) = \{(g_1, \dots, g_r) \in C_\infty^r : g_r \neq 0\}/C_\infty^*,$$

where C_∞^* acts diagonally through $c(g_1, \dots, g_r) = (\dots, c^{q^i-1} g_i, \dots)$, which is the open subspace of weighted projective space $\mathbb{P}^{r-1}(q-1, \dots, q^r-1)$ with non-vanishing last coordinate. The construction yields

$$\overline{M}_\Gamma(C_\infty) = \mathbb{P}^{r-1}(q-1, \dots, q^r-1)(C_\infty) = \dot{\bigcup}_{1\leqslant i\leqslant r} M^i(1)(C_\infty).$$

Its singularities are rather mild and may be removed upon replacing Γ by a congruence subgroup.

4.6. Vanishing orders of modular forms

In this final section we state some results about the vanishing orders of certain modular forms along the boundary divisors of $\overline{M}_\Gamma$, in the case where Γ is either $\Gamma(1) = GL(r,A)$ or a full congruence subgroup $\Gamma(\mathfrak{n})$ of $\Gamma(1)$. These are relevant for the determination of K- and Chow groups, and for standard conjectures about the arithmetic interpretation of partial zeta values.

In what follows, we suppose that $r \geqslant 2$, and put $z_i := \frac{\omega_i}{\omega_r}$ $(1 \leqslant i \leqslant r)$ for the coordinates $(\omega_1 : \ldots : \omega_r)$ of $\underline{\omega} \in \Omega^r$. Quite generally, $\underline{a} = (a_1, \ldots, a_r)$ denotes a vector with r components.

Definition 10. The *Eisenstein series* E_k of weight k on Ω^r is defined as

$$E_k(\underline{\omega}) := \sum_{0 \neq \underline{a} \in A^r} \frac{1}{(a_1 z_1 + \cdots + a_r z_r)^k}.$$

Similarly, we define for $\underline{u} \in \mathfrak{n}^{-1} \times \cdots \times \mathfrak{n}^{-1} \subset K^r$

$$E_{k,\underline{u}}(\underline{\omega}) = \sum_{\substack{0 \neq \underline{a} \in K^r \\ \underline{a} \equiv \underline{u} \bmod A^r}} \frac{1}{(a_1 z_1 + \cdots + a_r z_r)^k}.$$

These are modular forms for $\Gamma(1)$ and $\Gamma(\mathfrak{n})$, respectively, that is, they are holomorphic, satisfy the obvious transformation values under $\Gamma(1)$ (resp. $\Gamma(\mathfrak{n})$), and extend to sections of a line bundle on $\overline{M}_\Gamma$. As in Example 4, there is a second type of modular forms coming directly from Drinfeld modules.

Definition 11. For $\underline{\omega} \in \Omega^r$ write $\Lambda_{\underline{\omega}} = Az_1 + \cdots + Az_r$ and $e_{\underline{\omega}}$, $\phi^{\underline{\omega}}$ for the lattice function and Drinfeld module associated with $\Lambda_{\underline{\omega}}$, respectively. If $a \in A$ has degree $d = \deg(a)$,

$$\phi_a^{\underline{\omega}} = a + \sum_{1 \leqslant i \leqslant r \cdot d} \ell_i(a, \underline{\omega}) \tau^i.$$

The $\ell_i(a, \underline{\omega})$ are modular forms of weight $q^i - 1$ for Γ. This holds in particular for

$$\Delta_a(\underline{\omega}) := \ell_{rd}(a, \underline{\omega}),$$

which has weight $q^{rd} - 1$ and vanishes nowhere on Ω^r. The functions g and Δ in Example 4 merely constitute a very special instance of this construction. We further let, for $\underline{u} \in (\mathfrak{n}^{-1})^r$,

$$e_{\underline{u}}(\underline{\omega}) := e_{\underline{\omega}}(u_1 z_1 + \cdots + u_r z_r),$$

the $\mathfrak{n}$-division point of type $\underline{u}$ of $\phi^{\underline{\omega}}$. If $\underline{u} \notin A^r$, $e_{\underline{u}}(\underline{\omega})$ vanishes nowhere on Ω^r, and it can be shown that in this case,

$$e_{\underline{u}}^{-1} = E_{1,\underline{u}}. \tag{4.6.1}$$

We are interested in the behavior around the boundary of $\overline{M}_\Gamma$ of these forms. Let us first describe the set $\{\overline{F}_V\}$ of boundary divisors, i.e., of irreducible components, all of codimension one, of $\overline{M}_\Gamma - M_\Gamma$. For $\Gamma = \Gamma(1) = GL(r, A)$, there is a natural bijection

$$\{\overline{F}_V\} \xrightarrow{\sim} \mathrm{Pic}(A) \tag{4.6.2}$$

described in detail in [G1, VI 5.1]. It is induced from $V \mapsto$ inverse of $\Lambda^{r-1}(V \cap A^r)$. (Recall that V is a K-subspace of dimension $r-1$ of K^r, thus $V \cap A^r$ a projective module of rank $r-1$, whose $(r-1)$-th exterior power $\Lambda^{r-1}(V \cap A^r)$ determines an element of $\mathrm{Pic}(A)$.) We denote the component corresponding to the class $(\mathfrak{a})$ of an ideal $\mathfrak{a}$ by $\overline{F}_{(\mathfrak{a})}$. Similarly, the boundary divisors of $\overline{M}_\Gamma$ for $\Gamma = \Gamma(\mathfrak{n})$ could be described via generalized class groups. We simply use (4.5.1) and (4.5.2), which now give

$$\{\overline{F}_V\} \xrightarrow{\sim} \Gamma(\mathfrak{n}) \setminus GL(r, K)/P_1(K). \tag{4.6.3}$$

We denote the class of $\nu \in GL(r, K)$ by $[\nu]$. For the description of the behavior of our modular forms along the $\overline{F}_V$, we need the partial zeta functions of A and K. For more about these, see [W] and [G1, III].

Definition 12. We let

$$\zeta_K(s) = \sum |\mathfrak{a}|^{-s} = \frac{P(q^{-s})}{(1-q^{-s})(1-q^{1-s})}$$

be the zeta function of K with numerator polynomial $P(X) \in \mathbb{Z}[X]$. Here the sum is taken over the positive divisors $\mathfrak{a}$ of K (i.e., of the curve $\mathcal{C}$ with function field K). Extending the sum only over divisors with support in $\mathrm{Spec}(A)$, we get

$$\zeta_A(s) = \sum_{0 \neq \mathfrak{a} \subset A \text{ ideal}} |\mathfrak{a}|^{-s} = \zeta_K(s)(1 - q^{-d_\infty s}),$$

where $d_\infty = \deg_{\mathbb{F}_q}(\infty)$. For a class $\mathfrak{c} \in \mathrm{Pic}(A)$ we put

$$\zeta_{\mathfrak{c}}(s) = \sum_{\mathfrak{a} \in \mathfrak{c}} |\mathfrak{a}|^{-s}.$$

If finally $\mathfrak{n} \subset K$ is a fractional A-ideal and $t \in K$, we define

$$\zeta_{t \bmod \mathfrak{n}}(s) = \sum_{\substack{a \in K \\ a \equiv t \bmod \mathfrak{n}}} |a|^{-s}.$$

Among the obvious distribution relations [G1, III sect.1] between these, we only mention

$$\zeta_{(\mathfrak{n}^{-1})}(s) = \frac{|\mathfrak{n}|^s}{q-1} \zeta_{0 \bmod \mathfrak{n}}(s). \tag{4.6.4}$$

We are now in a position to state the following theorems, which may be proved following the method of [G1, VI].

Theorem 4. *Let $a \in A$ be non-constant and $\mathfrak{c}$ a class in* $\mathrm{Pic}(A)$. *The modular form Δ_a for $GL(r,A)$ has vanishing order*

$$\mathrm{ord}_{\mathfrak{c}}(\Delta_a) = -(|a|^r - 1)\zeta_{\mathfrak{c}}(1-r)$$

at the boundary component $\overline{F}_{\mathfrak{c}}$ corresponding to $\mathfrak{c}$.

Theorem 5. *Fix an ideal $\mathfrak{n}$ of A and $\underline{u} \in K^r - A^r$ such that $\underline{u} \cdot \mathfrak{n} \subset A^r$, and let $e_{\underline{u}}^{-1} = E_{1,\underline{u}}$ be the modular form for $\Gamma(\mathfrak{n})$ determined by these data. The vanishing order $\mathrm{ord}_{[\nu]}$ of $E_{1,\underline{u}}(\underline{\omega})$ at the component corresponding to $\nu \in GL(r,K)$ (see (4.6.2)) is given as follows: let $\pi_1 \colon K^r \to K$ be the projection to the first coordinate and let $\mathfrak{a}$ be the fractional ideal $\pi_1(A^r \cdot \nu)$. Write further $\underline{u} \cdot \nu = (v_1, \dots, v_r)$. Then*

$$\mathrm{ord}_{[\nu]} E_{1,\underline{u}}(\omega) = \frac{|\mathfrak{n}|^{r-1}}{|\mathfrak{a}|^{r-1}}(\zeta_{v_1 \bmod \mathfrak{a}}(1-r) - \zeta_{0 \bmod \mathfrak{a}}(1-r)).$$

Note that the two theorems do not depend on the full strength of properties of $\overline{M}_\Gamma$ as stated without proofs in the last section, but only on the *normality* of $\overline{M}_\Gamma$, which is *proved* in $[K]$ for $A = \mathbb{F}_q[T]$, and whose generalization to arbitrary Drinfeld rings is straightforward (even though technical).

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