ISSN 1464-8997 (on line) 1464-8989 (printed) **165**

Geometry & Topology Monographs
Volume 3: Invitation to higher local fields
Pages 165–195

Existence theorem for higher local fields

Kazuya Kato

0. Introduction

A field K is called an n-dimensional local field if there is a sequence of fields $k_n, \dots, k_0$ satisfying the following conditions: k_0 is a finite field, k_i is a complete discrete valuation field with residue field k_{i-1} for $i = 1, \dots, n$, and $k_n = K$.

In [9] we defined a canonical homomorphism from the nth Milnor group $K_n(K)$ (cf. [14]) of an n-dimensional local field K to the Galois group $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ of the maximal abelian extension of K and generalized the familiar results of the usual local class field theory to the case of arbitrary dimension except the "existence theorem".

An essential difficulty with the existence theorem lies in the fact that K (resp. the multiplicative group K^*) has no appropriate topology in the case where $n \geqslant 2$ (resp. $n \geqslant 3$) which would be compatible with the ring (resp. group) structure and which would take the topologies of the residue fields into account. Thus we abandon the familiar tool "topology" and define the openness of subgroups and the continuity of maps from a new point of view.

In the following main theorems the words "open" and "continuous" are not used in the topological sense. They are explained below.

Theorem 1. *Let K be an n-dimensional local field. Then the correspondence*

$$L \to N_{L/K} K_n(L)$$

is a bijection from the set of all finite abelian extensions of K to the set of all open subgroups of $K_n(K)$ of finite index.

This existence theorem is essentially contained in the following theorem which expresses certain Galois cohomology groups of K (for example the Brauer group of

K) by using the Milnor K-group of K. For a field k we define the group $H^r(k)$ ($r \geqslant 0$) as follows (cf. [9, §3.1]). In the case where $\mathrm{char}(k) = 0$ let

$$H^r(k) = \varinjlim H^r(k, \mu_m^{\otimes(r-1)})$$

(the Galois cohomology). In the case where $\mathrm{char}(k) = p > 0$ let

$$H^r(k) = \varinjlim H^r(k, \mu_m^{\otimes(r-1)}) + \varinjlim H^r_{p^i}(k).$$

Here in each case m runs over all integers invertible in k, μ_m denotes the group of all m th roots of 1 in the separable closure k^{sep} of k, and $\mu_m^{\otimes(r-1)}$ denotes its $(r-1)$ th tensor power as a $\mathbb{Z}/m$-module on which $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ acts in the natural way. In the case where $\mathrm{char}(k) = p > 0$ we denote by $H^r_{p^i}(k)$ the cokernel of

$$F - 1 : C_i^{r-1}(k) \to C_i^{r-1}(k)/\{C_i^{r-2}(k), T\}$$

where $C_i^{\cdot}$ is the group defined in [3, Ch.II,§7] (see also Milne [13, §3]). For example, $H^1(k)$ is isomorphic to the group of all continuous characters of the compact abelian group $\mathrm{Gal}(k^{\mathrm{ab}}/k)$ and $H^2(k)$ is isomorphic to the Brauer group of k.

Theorem 2. *Let K be as in Theorem 1. Then $H^r(K)$ vanishes for $r > n+1$ and is isomorphic to the group of all continuous characters of finite order of $K_{n+1-r}(K)$ in the case where $0 \leqslant r \leqslant n+1$.*

We shall explain the contents of each section.

For a category $\mathcal{C}$ the category of pro-objects $\mathrm{pro}(\mathcal{C})$ and the category of ind-objects $\mathrm{ind}(\mathcal{C})$ are defined as in Deligne [5]. Let $\mathcal{F}_0$ be the category of finite sets, and let $\mathcal{F}_1$, $\mathcal{F}_2$, ... be the categories defined by $\mathcal{F}_{n+1} = \mathrm{ind}(\mathrm{pro}(\mathcal{F}_n))$. Let $\mathcal{F}_\infty = \cup_n \mathcal{F}_n$. In section 1 we shall show that n-dimensional local fields can be viewed as ring objects of $\mathcal{F}_n$. More precisely we shall define a ring object $\underline{K}$ of $\mathcal{F}_n$ corresponding to an n-dimensional local field K such that K is identified with the ring $[e, \underline{K}]_{\mathcal{F}_\infty}$ of morphisms from the one-point set e (an object of $\mathcal{F}_0$) to $\underline{K}$, and a group object $\underline{K^*}$ such that K^* is identified with $[e, \underline{K^*}]_{\mathcal{F}_\infty}$. We call a subgroup N of $K_q(K)$ open if and only if the map

$$K^* \times \cdots \times K^* \to K_q(K)/N, \quad (x_1, \ldots, x_q) \mapsto \{x_1, \ldots, x_q\} \mod N$$

comes from a morphism $\underline{K^*} \times \cdots \times \underline{K^*} \to K_q(K)/N$ of $\mathcal{F}_\infty$ where $K_q(K)/N$ is viewed as an object of $\mathrm{ind}(\mathcal{F}_0) \subset \mathcal{F}_1$. We call a homomorphism $\varphi : K_q(K) \to \mathbb{Q}/\mathbb{Z}$ a continuous character if and only if the induced map

$$K^* \times \cdots \times K^* \to \mathbb{Q}/\mathbb{Z}, \quad (x_1, \ldots, x_q) \mapsto \varphi(\{x_1, \ldots, x_q\})$$

comes from a morphism of $\mathcal{F}_\infty$ where $\mathbb{Q}/\mathbb{Z}$ is viewed as an object of $\mathrm{ind}(\mathcal{F}_0)$. In each case such a morphism of $\mathcal{F}_\infty$ is unique if it exists (cf. Lemma 1 of section 1).

In section 2 we shall generalize the self-duality of the additive group of a one-dimensional local field in the sense of Pontryagin to arbitrary dimension.

Section 3 is a preliminary one for section 4. There we shall prove some ring-theoretic properties of $[X, \underline{K}]_{\mathcal{F}_\infty}$ for objects X of $\mathcal{F}_\infty$.

In section 4 we shall treat the norm groups of cohomological objects. For a field k denote by $\mathcal{E}(k)$ the category of all finite extensions of k in a fixed algebraic closure of k with the inclusion maps as morphisms. Let H be a functor from $\mathcal{E}(k)$ to the category Ab of all abelian groups such that $\varinjlim_{k' \in \mathcal{E}(k)} H(k') = 0$. For $w_1, \dots, w_g \in H(k)$ define the K_q-norm group $N_q(w_1, \dots, w_g)$ as the subgroup of $K_q(k)$ generated by the subgroups $N_{k'/k} K_q(k')$ where k' runs over all fields in $\mathcal{E}(k)$ such that $\{w_1, \dots, w_g\} \in \ker(H(k) \to H(k'))$ and where $N_{k'/k}$ denotes the canonical norm homomorphism of the Milnor K-groups (Bass and Tate [2, §5] and [9, §1.7]). For example, if $H = H^1$ and $\chi_1, \dots, \chi_g \in H^1(k)$ then $N_q(\chi_1, \dots, \chi_g)$ is nothing but $N_{k'/k} K_q(k')$ where k' is the finite abelian extension of k corresponding to $\cap_i \ker(\chi_i \colon \mathrm{Gal}(k^{\mathrm{ab}}/k) \to \mathbb{Q}/\mathbb{Z})$. If $H = H^2$ and $w \in H^2(k)$ then $N_1(w)$ is the image of the reduced norm map $A^* \to k^*$ where A is a central simple algebra over k corresponding to w.

As it is well known for a one-dimensional local field k the group $N_1(\chi_1, \dots, \chi_g)$ is an open subgroup of k^* of finite index for any $\chi_1, \dots, \chi_g \in H^1(k)$ and the group $N_1(w) = k^*$ for any $w \in H^2(k)$. We generalize these facts as follows.

Theorem 3. *Let K be an n-dimensional local field and let $r \geqslant 1$.*

(1) *Let $w_1, \dots, w_g \in H^r(K)$. Then the norm group $N_{n+1-r}(w_1, \dots, w_g)$ is an open subgroup of $K_{n+1-r}(K)$ of finite index.*

(2) *Let M be a discrete torsion abelian group endowed with a continuous action of* $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. *Let H be the Galois cohomology functor $H^r(\ ,M)$. Then for every $w \in H^r(K, M)$ the group $N_{n+1-r}(w)$ is an open subgroup of $K_{n+1-r}(K)$ of finite index.*

Let k be a field and let $q, r \geqslant 0$. We define a condition (N_q^r, k) as follows: for every $k' \in \mathcal{E}(k)$ and every discrete torsion abelian group M endowed with a continuous action of $\mathrm{Gal}(k'^{\mathrm{sep}}/k')$

$$N_q(w_1, \dots, w_g) = K_q(k')$$

for every $i > r$, $w_1, \dots, w_g \in H^i(k')$, $w_1, \dots, w_g \in H^i(k', M)$, and in addition $|k : k^p| \leqslant p^{q+r}$ in the case where $\mathrm{char}(k) = p > 0$.

For example, if k is a perfect field then the condition (N_0^r, k) is equivalent to $\mathrm{cd}(k) \leqslant r$ where cd denotes the cohomological dimension (Serre [16]).

Proposition 1. *Let K be a complete discrete valuation field with residue field k. Let $q \geqslant 1$ and $r \geqslant 0$. Then the two conditions (N_q^r, K) and (N_{q-1}^r, k) are equivalent.*

On the other hand by [11] the conditions (N_0^r, K) and (N_0^{r-1}, k) are equivalent for any $r \geqslant 1$. By induction on n we obtain

Corollary. *Let K be an n-dimensional local field. Then the condition (N_q^r, K) holds if and only if $q + r \geqslant n + 1$.*

We conjecture that if $q+r=q'+r'$ then the two conditions (N_q^r, k) and $(N_{q'}^{r'}, k)$ are equivalent for any field k.

Finally in section 5 we shall prove Theorem 2. Then Theorem 1 will be a corollary of Theorem 2 for $r=1$ and of [9, §3, Theorem 1] which claims that the canonical homomorphism

$$K_n(K) \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

induces an isomorphism $K_n(K)/N_{L/K}K_n(L) \xrightarrow{\sim} \mathrm{Gal}(L/K)$ for each finite abelian extension L of K.

I would like to thank Shuji Saito for helpful discussions and for the stimulation given by his research in this area (e.g. his duality theorem of Galois cohomology groups with locally compact topologies for two-dimensional local fields).

Table of contents.

Notation.

We follow the notation in the beginning of this volume. References to sections in this text mean references to sections of this work and not of the whole volume.

All fields and rings in this paper are assumed to be commutative.

Denote by Sets, Ab, Rings the categories of sets, of abelian groups and of rings respectively.

If $\mathcal{C}$ is a category and X, Y are objects of $\mathcal{C}$ then $[X,Y]_{\mathcal{C}}$ (or simply $[X,Y]$) denotes the set of morphisms $X \to Y$.

1. Definition of the continuity for higher local fields

1.1. Ring objects of a category corresponding to rings.

For a category $\mathcal{C}$ let $\mathcal{C}^\circ$ be the dual category of $\mathcal{C}$. If $\mathcal{C}$ has a final object we always denote it by e. Then, if $\theta: X \to Y$ is a morphism of $\mathcal{C}$, $[e,\theta]$ denotes the induced map $[e,X] \to [e,Y]$.

In this subsection we prove the following

Proposition 2. *Let $\mathcal{C}$ be a category with a final object e in which the product of any two objects exists. Let $\underline{R}$ be a ring object of $\mathcal{C}$ such that for a prime p the morphism $\underline{R} \to \underline{R}$, $x \mapsto px$ is the zero morphism, and via the morphism $\underline{R} \to \underline{R}$, $x \mapsto x^p$ the* *latter $\underline{R}$ is a free module of finite rank over the former $\underline{R}$. Let $R = [e, \underline{R}]$, and let A be a ring with a nilpotent ideal I such that $R = A/I$ and such that I^i/I^{i+1} is a free R-module of finite rank for any i.*

Then:

(1) *There exists a ring object $\underline{A}$ of $\mathcal{C}$ equipped with a ring isomorphism $j\colon A \xrightarrow{\sim} [e, \underline{A}]$ and with a homomorphism of ring objects $\theta\colon \underline{A} \to \underline{R}$ having the following properties:*
   (a) *$[e, \theta] \circ j\colon A \to R$ coincides with the canonical projection.*
   (b) *For any object X of $\mathcal{C}$, $[X, \underline{A}]$ is a formally etale ring over A in the sense of Grothendieck* [7, Ch. 0 §19], *and θ induces an isomorphism*

$$[X, \underline{A}]/I[X, \underline{A}] \simeq [X, \underline{R}].$$

(2) *The above triple $(\underline{A}, j, \theta)$ is unique in the following sense. If $(\underline{A}', j', \theta')$ is another triple satisfying the same condition in (1), then there exists a unique isomorphism of ring objects $\psi\colon \underline{A} \xrightarrow{\sim} \underline{A}'$ such that $[e, \psi] \circ j = j'$ and $\theta = \theta' \circ \psi$.*
(3) *The object $\underline{A}$ is isomorphic (if one forgets the ring-object structure) to the product of finitely many copies of $\underline{R}$.*
(4) *If $\mathcal{C}$ has finite inverse limits, the above assertions (1) and (2) are valid if conditions "free module of finite rank" on $\underline{R}$ and I^i/I^{i+1} are replaced by conditions "direct summand of a free module of finite rank".*

Example. Let R be a non-discrete locally compact field and A a local ring of finite length with residue field R. Then in the case where $\operatorname{char}(R) > 0$ Proposition 2 shows that there exists a canonical topology on A compatible with the ring structure such that A is homeomorphic to the product of finitely many copies of R. On the other hand, in the case where $\operatorname{char}(R) = 0$ it is impossible in general to define canonically such a topology on A. Of course, by taking a section $s\colon R \to A$ (as rings), A as a vector space over $s(R)$ has the vector space topology, but this topology depends on the choice of s in general. This reflects the fact that in the case of $\operatorname{char}(R) = 0$ the ring of R-valued continuous functions on a topological space is not in general formally smooth over R contrary to the case of $\operatorname{char}(R) > 0$.

Proof of Proposition 2. Let X be an object of $\mathcal{C}$; put $R_X = [X, \underline{R}]$. The assumptions on $\underline{R}$ show that the homomorphism

$$R^{(p)} \otimes_R R_X \to R_X, \quad x \otimes y \mapsto xy^p$$

is bijective, where $R^{(p)} = R$ as a ring and the structure homomorphism $R \to R^{(p)}$ is $x \mapsto x^p$. Hence by [10, §1 Lemma 1] there exists a formally etale ring A_X over A with a ring isomorphism $\theta_X\colon A_X/IA_X \simeq R_X$. The property "formally etale" shows that the correspondence $X \to A_X$ is a functor $\mathcal{C}^\circ \to \mathrm{Rings}$, and that the system θ_X forms a morphism of functors. More explicitly, let n and r be sufficiently large integers, let $W_n(R)$ be the ring of p-Witt vectors over R of length n, and let $\varphi\colon W_n(R) \to A$ be the homomorphism

$$(x_0, x_1, \dots) \mapsto \sum_{i=0}^{r} p^i \widetilde{x}_i^{\,p^{r-i}}$$

where $\widetilde{x}_i$ is a representative of $x_i \in R$ in A. Then A_X is defined as the tensor product

$$W_n(R_X) \otimes_{W_n(R)} A$$

induced by φ. Since $\operatorname{Tor}_1^{W_n(R)}(W_n(R_X), R) = 0$ we have

$$\operatorname{Tor}_1^{W_n(R)}(W_n(R_X), A/I^i) = 0$$

for every i. This proves that the canonical homomorphism

$$I^i/I^{i+1} \otimes_R R_X \to I^i A_X / I^{i+1} A_X$$

is bijective for every i. Hence each functor $X \to I^i A_X/I^{i+1}A_X$ is representable by a finite product of copies of $\underline{R}$, and it follows immediately that the functor A_X is represented by the product of finitely many copies of $\underline{R}$. □

1.2. n-dimensional local fields as objects of $\mathcal{F}_n$.

Let K be an n-dimensional local field. In this subsection we define a ring object $\underline{K}$ and a group object $\underline{K^*}$ by induction on n.

Let $k_0, \dots, k_n = K$ be as in the introduction. For each i such that $\operatorname{char}(k_{i-1}) = 0$ (if such an i exists) choose a ring morphism $s_i \colon k_{i-1} \to \mathcal{O}_{k_i}$ such that the composite $k_{i-1} \to \mathcal{O}_{k_i} \to \mathcal{O}_{k_i}/\mathcal{M}_{k_i}$ is the indentity map. Assume $n \geqslant 1$ and let $\underline{k_{n-1}}$ be the ring object of $\mathcal{F}_{n-1}$ corresponding to k_{n-1} by induction on n.

If $\operatorname{char}(k_{n-1}) = p > 0$, the construction of $\underline{K}$ below will show by induction on n that the assumptions of Proposition 2 are satisfied when one takes $\mathcal{F}_{n-1}$, $\underline{k_{n-1}}$, k_{n-1} and $\mathcal{O}_K/\mathcal{M}_K^r$ ($r \geqslant 1$) as $\mathcal{C}$, $\underline{R}$, R and A. Hence we obtain a ring object $\underline{\mathcal{O}_K/\mathcal{M}_K^r}$ of $\mathcal{F}_{n-1}$. We identify $\mathcal{O}_K/\mathcal{M}_K^r$ with $[e, \underline{\mathcal{O}_K/\mathcal{M}_K^r}]$ via the isomorphism j of Proposition 2.

If $\operatorname{char}(k_{n-1}) = 0$, let $\underline{\mathcal{O}_K/\mathcal{M}_K^r}$ be the ring object of $\mathcal{F}_{n-1}$ which represents the functor

$$\mathcal{F}_{n-1}^{\circ} \to \text{Rings}, \quad X \mapsto \mathcal{O}_K/\mathcal{M}_K^r \otimes_{k_{n-1}} [X, \underline{k_{n-1}}],$$

where $\mathcal{O}_K/\mathcal{M}_K^r$ is viewed as a ring over k_{n-1} via s_{n-1}.

In each case let $\underline{\mathcal{O}_K}$ be the object "$\varprojlim$" $\underline{\mathcal{O}_K/\mathcal{M}_K^r}$ of $\operatorname{pro}(\mathcal{F}_{n-1})$. We define $\underline{K}$ as the ring object of $\mathcal{F}_n$ which corresponds to the functor

$$\operatorname{pro}(\mathcal{F}_{n-1})^{\circ} \to \text{Rings}, \quad X \mapsto K \otimes_{\mathcal{O}_K} [X, \underline{\mathcal{O}_K}].$$

Thus, $\underline{K}$ is defined canonically in the case of $\operatorname{char}(k_{n-1}) > 0$, and it depends (and doesn't depend) on the choices of s_i in the case of $\operatorname{char}(k_{n-1}) = 0$ in the following sense. Assume that another choice of sections s_i' yields $\underline{k_i}'$ and $\underline{K}'$. Then there exists an isomorphism of ring objects $\underline{K} \xrightarrow{\sim} \underline{K}'$ which induces $\underline{k_i} \xrightarrow{\sim} \underline{k_i}'$ for each i. But in general there is no isomorphism of ring objects $\psi: \underline{K} \to \underline{K}'$ such that $[e,\psi]: K \to K$ is the indentity map.

Now let $\underline{K}^*$ be the object of $\mathcal{F}_n$ which represents the functor

$$\mathcal{F}_n^\circ \to \text{Sets}, \quad X \mapsto [X, \underline{K}]^*.$$

This functor is representable because $\mathcal{F}_n$ has finite inverse limits as can be shown by induction on n.

Definition 1. We define fine (resp. cofine) objects of $\mathcal{F}_n$ by induction on n. All objects in $\mathcal{F}_0$ are called fine (resp. cofine) objects of $\mathcal{F}_0$. An object of $\mathcal{F}_n$ ($n \geqslant 1$) is called a fine (resp. cofine) object of $\mathcal{F}_n$ if and only if it is expressed as $X = \text{"}\varinjlim\text{"}\, X_\lambda$ for some objects X_λ of $\mathrm{pro}(\mathcal{F}_{n-1})$ and each X_λ is expressed as $X_\lambda = \text{"}\varprojlim\text{"}\, X_{\lambda\mu}$ for some objects $X_{\lambda\mu}$ of $\mathcal{F}_{n-1}$ satisfying the condition that all $X_{\lambda\mu}$ are fine (resp. cofine) objects of $\mathcal{F}_{n-1}$ and the maps $[e, X_\lambda] \to [e, X_{\lambda\mu}]$ are surjective for all λ, μ (resp. the maps $[e, X_\lambda] \to [e, X]$ are injective for all λ).

Recall that if $i \leqslant j$ then $\mathcal{F}_i$ is a full subcategory of $\mathcal{F}_j$. Thus each $\mathcal{F}_i$ is a full subcategory of $\mathcal{F}_\infty = \cup_i \mathcal{F}_i$.

Lemma 1.

(1) *Let K be an n-dimensional local field. Then an object of $\mathcal{F}_n$ of the form*

$$\underline{K} \times \dots \underline{K} \times \underline{K}^* \times \dots \times \underline{K}^*$$

is a fine and cofine object of $\mathcal{F}_n$. Every set S viewed as an object of $\mathrm{ind}(\mathcal{F}_0)$ *is a fine and cofine object of $\mathcal{F}_1$.*

(2) *Let X and Y be objects of $\mathcal{F}_\infty$, and assume that X is a fine object of $\mathcal{F}_n$ for some n and Y is a cofine object of $\mathcal{F}_m$ for some m. Then two morphisms $\theta, \theta': X \to Y$ coincide if $[e,\theta] = [e,\theta']$.*

As explained in 1.1 the definition of the object $\underline{K}$ depends on the sections $s_i: k_{i-1} \to \mathcal{O}_{k_i}$ chosen for each i such that $\mathrm{char}(k_{i-1}) = 0$. Still we have the following:

Lemma 2.

(1) *Let N be a subgroup of $K_q(K)$ of finite index. Then openness of N doesn't depend on the choice of sections s_i.*

(2) *Let $\varphi: K_q(K) \to \mathbb{Q}/\mathbb{Z}$ be a homomorphism of finite order. Then the continuity of χ doesn't depend on the choice of sections s_i.*

The exact meaning of Theorems 1,2,3 is now clear.

2. Additive duality

2.1. Category of locally compact objects.

If $\mathcal{C}$ is the category of finite abelian groups, let $\widetilde{\mathcal{C}}$ be the category of topological abelian groups G which possess a totally disconnected open compact subgroup H such that G/H is a torsion group. If $\mathcal{C}$ is the category of finite dimensional vector spaces over a fixed (discrete) field k, let $\widetilde{\mathcal{C}}$ be the category of locally linearly compact vector spaces over k (cf. Lefschetz [12]). In both cases the canonical self-duality of $\widetilde{\mathcal{C}}$ is well known. These two examples are special cases of the following general construction.

Definition 2. For a category $\mathcal{C}$ define a full subcategory $\widetilde{\mathcal{C}}$ of ind(pro($\mathcal{C}$)) as follows. An object X of ind(pro($\mathcal{C}$)) belongs to $\widetilde{\mathcal{C}}$ if and only if it is expressed in the form $"\varinjlim"_{j\in J}"\varprojlim"_{i\in I}X(i,j)$ for some directly ordered sets I and J viewed as small categories in the usual way and for some functor $X\colon I^\circ\times J\to\mathcal{C}$ satisfying the following conditions.

(i) If $i,i'\in I$, $i\leqslant i'$ then the morphism $X(i',j)\to X(i,j)$ is surjective for every $j\in J$. If $j,j'\in J$, $j\leqslant j'$ then the morphism $X(i,j)\to X(i,j')$ is injective for every $i\in I$.

(ii) If $i,i'\in I$, $i\leqslant i'$ and $j,j'\in J$, $j\leqslant j'$ then the square

$$\begin{array}{ccc} X(i',j) & \longrightarrow & X(i',j') \\ \downarrow & & \downarrow \\ X(i,j) & \longrightarrow & X(i,j') \end{array}$$

is cartesian and cocartesian.

It is not difficult to prove that $\widetilde{\mathcal{C}}$ is equivalent to the full subcategory of pro(ind($\mathcal{C}$)) (as well as ind(pro($\mathcal{C}$))) consisting of all objects which are expressed in the form $"\varprojlim"_{i\in I}"\varinjlim"_{j\in J}X(i,j)$ for some triple (I,J,X) satisfying the same conditions as above. In this equivalence the object
$"\varinjlim"_{j\in J}"\varprojlim"_{i\in I}X(i,j)$ corresponds to $"\varprojlim"_{i\in I}"\varinjlim"_{j\in J}X(i,j)$.

Definition 3. Let $\mathcal{A}_0$ be the category of finite abelian groups, and let $\mathcal{A}_1,\mathcal{A}_2,\dots$ be the categories defined as $\mathcal{A}_{n+1}=\widetilde{\mathcal{A}_n}$.

It is easy to check by induction on n that $\mathcal{A}_n$ is a full subcategory of the category $\mathcal{F}_n^{\mathrm{ab}}$ of all abelian group objects of $\mathcal{F}_n$ with additive morphisms.

2.2. Pontryagin duality.

The category $\mathcal{A}_0$ is equivalent to its dual via the functor

$$D_0\colon \mathcal{A}_0^\circ \xrightarrow{\sim} \mathcal{A}_0, \quad X \mapsto \mathrm{Hom}(X, \mathbb{Q}/\mathbb{Z}).$$

By induction on n we get an equivalence

$$D_n\colon \mathcal{A}_n^\circ \xrightarrow{\sim} \mathcal{A}_n, \quad \mathcal{A}_n^\circ = (\widetilde{\mathcal{A}_{n-1}})^\circ = \widetilde{\mathcal{A}_{n-1}^\circ} \xrightarrow{D_{n-1}} \widetilde{\mathcal{A}_{n-1}} = \mathcal{A}_n$$

where we use $(\widetilde{\mathcal{C}})^\circ = \widetilde{\mathcal{C}^\circ}$. As in the case of $\mathcal{F}_n$ each $\mathcal{A}_n$ is a full subcategory of $\mathcal{A}_\infty = \cup_n \mathcal{A}_n$. The functors D_n induce an equivalence

$$D\colon \mathcal{A}_\infty^\circ \xrightarrow{\sim} \mathcal{A}_\infty$$

such that $D \circ D$ coincides with the indentity functor.

Lemma 3. *View $\mathbb{Q}/\mathbb{Z}$ as an object of* $\mathrm{ind}(\mathcal{A}_0) \subset \mathcal{A}_\infty \subset \mathcal{F}_\infty^{\mathrm{ab}}$. *Then:*

(1) *For every object X of $\mathcal{A}_\infty$*

$$[X, \mathbb{Q}/\mathbb{Z}]_{\mathcal{A}_\infty} \simeq [e, D(X)]_{\mathcal{F}_\infty}.$$

(2) *For all objects X, Y of $\mathcal{A}_\infty$ $[X, D(Y)]_{\mathcal{A}_\infty}$ is canonically isomorphic to the group of biadditive morphisms $X \times Y \to \mathbb{Q}/\mathbb{Z}$ in $\mathcal{F}_\infty$.*

Proof. The isomorphism of (1) is given by

$$[X, \mathbb{Q}/\mathbb{Z}]_{\mathcal{A}_\infty} \simeq [D(\mathbb{Q}/\mathbb{Z}), D(X)]_{\mathcal{A}_\infty} = [\widehat{\mathbb{Z}}, D(X)]_{\mathcal{A}_\infty} \xrightarrow{\sim} [e, D(X)]_{\mathcal{F}_\infty}$$

($\widehat{\mathbb{Z}}$ is the totally disconnected compact abelian group $\varprojlim_{n>0} \mathbb{Z}/n$ and the last arrow is the evaluation at $1 \in \widehat{\mathbb{Z}}$). The isomorphism of (2) is induced by the canonical biadditive morphism $D(Y) \times Y \to \mathbb{Q}/\mathbb{Z}$ which is defined naturally by induction on n. □

Compare the following Proposition 3 with Weil [17, Ch. II §5 Theorem 3].

Proposition 3. *Let K be an n-dimensional local field, and let V be a vector space over K of finite dimension, $V' = \mathrm{Hom}_K(V, K)$. Then*

:

(1) *The abelian group object $\underline{V}$ of $\mathcal{F}_n$ which represents the functor $X \to V \otimes_K [X, \underline{K}]$ belongs to $\mathcal{A}_n$.*

(2) *$[\underline{K}, \mathbb{Q}/\mathbb{Z}]_{\mathcal{A}_\infty}$ is one-dimensional with respect to the natural K-module structure and its non-zero element induces due to Lemma 3 (2) an isomorphism $\underline{V'} \simeq D(\underline{V})$.*

3. Properties of the ring of $\underline{K}$-valued morphisms

3.1. Multiplicative groups of certain complete rings.

Proposition 4. *Let A be a ring and let π be a non-zero element of A such that $A = \varprojlim A/\pi^n A$. Let $R = A/\pi A$ and $B = A[\pi^{-1}]$. Assume that at least one of the following two conditions is satisfied.*

(i) *R is reduced (i.e. having no nilpotent elements except zero) and there is a ring homomorphism $s: R \to A$ such that the composite $R \xrightarrow{s} A \to A/\pi A$ is the identity.*

(ii) *For a prime p the ring R is annihilated by p and via the homomorphism $R \to R$, $x \mapsto x^p$ the latter R is a finitely generated projective module over the former R.*

Then we have

$$B^* \simeq A^* \times \Gamma(\mathrm{Spec}(R), \mathbb{Z})$$

where $\Gamma(\mathrm{Spec}(R), \mathbb{Z})$ is the group of global sections of the constant sheaf $\mathbb{Z}$ on $\mathrm{Spec}(R)$ with Zariski topology. The isomorphism is given by the homomorphism of sheaves $\mathbb{Z} \to \mathcal{O}^*_{\mathrm{Spec}(R)}$, $1 \mapsto \pi$, the map

$$\Gamma(\mathrm{Spec}(R), \mathbb{Z}) \simeq \Gamma(\mathrm{Spec}(A), \mathbb{Z}) \to \Gamma(\mathrm{Spec}(B), \mathbb{Z})$$

and the inclusion map $A^* \to B^*$.

Proof. Let Aff_R be the category of affine schemes over R. In case (i) let $\mathcal{C} = \mathrm{Aff}_R$. In case (ii) let $\mathcal{C}$ be the category of all affine schemes $\mathrm{Spec}(R')$ over R such that the map

$$R^{(p)} \otimes_R R' \to R', \quad x \otimes y \mapsto xy^p$$

(cf. the proof of Proposition 2) is bijective. Then in case (ii) every finite inverse limit and finite sum exists in $\mathcal{C}$ and coincides with that taken in Aff_R. Furthermore, in this case the inclusion functor $\mathcal{C} \to \mathrm{Aff}_R$ has a right adjoint. Indeed, for any affine scheme X over R the corresponding object in $\mathcal{C}$ is $\varprojlim X_i$ where X_i is the Weil restriction of X with respect to the homomorphism $R \to R$, $x \mapsto x^{p^i}$.

Let $\underline{R}$ be the ring object of $\mathcal{C}$ which represents the functor $X \to \Gamma(X, \mathcal{O}_X)$, and let $\underline{R^*}$ be the object which represents the functor $X \to [X, \underline{R}]^*$, and $\underline{0}$ be the final object e regarded as a closed subscheme of $\underline{R}$ via the zero morphism $e \to \underline{R}$.

Lemma 4. *Let X be an object of $\mathcal{C}$ and assume that X is reduced as a scheme (this condition is always satisfied in case (ii)). Let $\theta: X \to \underline{R}$ be a morphism of $\mathcal{C}$. If $\theta^{-1}(\underline{R^*})$ is a closed subscheme of X, then X is the direct sum of $\theta^{-1}(\underline{R^*})$ and $\theta^{-1}(\underline{0})$ (where the inverse image notation are used for the fibre product).*

The group B^* is generated by elements x of A such that $\pi^n \in Ax$ for some $n \geqslant 0$. In case (i) let $\underline{A/\pi^{n+1}A}$ be the ring object of $\mathcal{C}$ which represents the functor $X \to A/\pi^{n+1}A \otimes_R [X, \underline{R}]$ where $A/\pi^{n+1}A$ is viewed as an R-ring via a fixed section s. In case (ii) we get a ring object $\underline{A/\pi^{n+1}A}$ of $\mathcal{C}$ by Proposition 2 (4).

In both cases there are morhisms $\theta_i\colon \underline{R} \to \underline{A/\pi^{n+1}A}$ ($0 \leqslant i \leqslant n$) in $\mathcal{C}$ such that the morphism

$$\underline{R} \times \cdots \times \underline{R} \to \underline{A/\pi^{n+1}A}, \quad (x_0, \ldots, x_n) \mapsto \sum_{i=0}^{n} \theta_i(x_i)\pi^i$$

is an isomorphism.

Now assume $xy = \pi^n$ for some $x, y \in A$ and take elements $x_i, y_i \in R = [e, \underline{R}]$ ($0 \leqslant i \leqslant n$) such that

$$x \mod \pi^{n+1} = \sum_{i=0}^{n} \theta_i(x_i)\pi^i, \quad y \mod \pi^{n+1} = \sum_{i=0}^{n} \theta_i(y_i)\pi^i.$$

An easy computation shows that for every $r = 0, \ldots, n$

$$\Big(\bigcap_{i=0}^{r-1} x_i^{-1}(\underline{0})\Big) \bigcap x_r^{-1}(\underline{R}^*) = \Big(\bigcap_{i=0}^{r-1} x_i^{-1}(\underline{0})\Big) \bigcap \Big(\bigcap_{i=0}^{n-r-1} y_i^{-1}(\underline{0})\Big).$$

By Lemma 4 and induction on r we deduce that $e = \mathrm{Spec}(R)$ is the direct sum of the closed open subschemes $\big(\cap_{i=0}^{r-1} x_i^{-1}(\underline{0})\big) \cap x_r^{-1}(\underline{R}^*)$ on which the restriction of x has the form $a\pi^r$ for an invertible element $a \in A$. $\square$

3.2. Properties of the ring $[X, \underline{K}]$.

Results of this subsection will be used in section 4.

Definition 4. For an object X of $\mathcal{F}_\infty$ and a set S let

$$\mathrm{lcf}(X, S) = \varinjlim_I\, [X, I]$$

where I runs over all finite subsets of S (considering each I as an object of $\mathcal{F}_0 \subset \mathcal{F}_\infty$).

Lemma 5. *Let K be an n-dimensional local field and let X be an object of $\mathcal{F}_\infty$. Then:*

(1) *The ring $[X, \underline{K}]$ is reduced.*

(2) *For every set S there is a canonical bijection*

$$\mathrm{lcf}(X, S) \xrightarrow{\sim} \Gamma(\mathrm{Spec}([X, \underline{K}]), S)$$

where S on the right hand side is regarded as a constant sheaf on $\mathrm{Spec}([X, \underline{K}])$.

Proof of (2). If I is a finite set and $\theta: X \to I$ is a morphism of $\mathcal{F}_\infty$ then X is the direct sum of the objects $\theta^{-1}(i) = X \times_I \{i\}$ in $\mathcal{F}_\infty$ ($i \in I$). Hence we get the canonical map of (2). To prove its bijectivity we may assume $S = \{0, 1\}$. Note that $\Gamma(\mathrm{Spec}(R), \{0, 1\})$ is the set of idempotents in R for any ring R. We may assume that X is an object of $\mathrm{pro}(\mathcal{F}_{n-1})$.

Let k_{n-1} be the residue field of $k_n = K$. Then

$$\Gamma(\mathrm{Spec}([X, \underline{K}]), \{0, 1\}) \simeq \Gamma(\mathrm{Spec}([X, \underline{k_{n-1}}]), \{0, 1\})$$

by (1) applied to the ring $[X, \underline{k_{n-1}}]$. □

Lemma 6. *Let K be an n-dimensional local field of characteristic $p > 0$. Let $k_0, \dots, k_n$ be as in the introduction. For each $i = 1, \dots, n$ let π_i be a lifting to K of a prime element of k_i. Then for each object X of $\mathcal{F}_\infty$ $[X, \underline{K}]^*$ is generated by the subgroups*

$$[X, \underline{K^p(\pi^{(s)})}]^*$$

where s runs over all functions $\{1, \dots, n\} \to \{0, 1, \dots, p-1\}$ and $\pi^{(s)}$ denotes $\pi_1^{s(1)} \dots \pi_n^{s(n)}$, $\underline{K^p(\pi^{(s)})}$ is the subring object of $\underline{K}$ corresponding to $K^p(\pi^{(s)})$, i.e.

$$[X, \underline{K^p(\pi^{(s)})}] = K^p(\pi^{(s)}) \otimes_{K^p} [X, \underline{K}].$$

Proof. Indeed, Proposition 4 and induction on n yield morphisms

$$\theta^{(s)}: \underline{K^*} \to \underline{K^p(\pi^{(s)})^*}$$

such that the product of all $\theta^{(s)}$ in $\underline{K^*}$ is the identity morphism $\underline{K^*} \to \underline{K^*}$. □

The following similar result is also proved by induction on n.

Lemma 7. *Let K, k_0 and $(\pi_i)_{1 \leqslant i \leqslant n}$ be as in Lemma 6. Then there exists a morphism of $\mathcal{A}_\infty$*
(cf. section 2)

$$(\theta_1, \theta_2): \underline{\Omega_K^n} \to \underline{\Omega_K^n} \times \underline{k_0}$$

such that

$$x = (1 - \mathrm{C})\theta_1(x) + \theta_2(x) d\pi_1/\pi_1 \wedge \dots \wedge d\pi_n/\pi_n$$

for every object X of $\mathcal{F}_\infty$ and for every $x \in [X, \underline{\Omega_K^n}]$ where $\underline{\Omega_K^n}$ is the object which represents the functor $X \to \Omega_K^n \otimes_K [X, \underline{K}]$ and C denotes the Cartier operator ([4], *or see 4.2 in Part I for the definition*).

Generalize the Milnor K-groups as follows.

Definition 5. For a ring R let $\Gamma_0(R) = \Gamma(\mathrm{Spec}(R), \mathbb{Z})$. The morphism of sheaves

$$\mathbb{Z} \times \mathcal{O}^*_{\mathrm{Spec}(R)} \to \mathcal{O}^*_{\mathrm{Spec}(R)}, \quad (n, x) \mapsto x^n$$

determines the $\Gamma_0(R)$-module structure on R^*. Put $\Gamma_1(R) = R^*$ and for $q \geqslant 2$ put

$$\Gamma_q(R) = \otimes^q_{\Gamma_0(R)} \Gamma_1(R)/J_q$$

where $\otimes^q_{\Gamma_0(R)}\Gamma_1(R)$ is the qth tensor power of $\Gamma_1(R)$ over $\Gamma_0(R)$ and J_q is the subgroup of the tensor power generated by elements $x_1 \otimes \cdots \otimes x_q$ which satisfy $x_i + x_j = 1$ or $x_i + x_j = 0$ for some $i \neq j$. An element $x_1 \otimes \cdots \otimes x_q \mod J_q$ will be denoted by $\{x_1, \ldots, x_q\}$.

Note that $\Gamma_q(k) = K_q(k)$ for each field k and $\Gamma_q(R_1 \times R_2) \simeq \Gamma_q(R_1) \times \Gamma_q(R_2)$ for rings R_1, R_2.

Lemma 8. *In one of the following two cases*
(i) A, R, B, π *as in Proposition 4*
(ii) *an n-dimensional local field K, an object X of $\mathcal{F}_\infty$, $A = [X, \underline{\mathcal{O}_K}]$,*
$R = [X, \underline{k_{n-1}}]$, $B = [X, \underline{K}]$,

let $U_i\Gamma_q(B)$ be the subgroup of $\Gamma_q(B)$ generated by elements $\{1+\pi^i x, y_1, \ldots, y_{q-1}\}$ such that $x \in A$, $y_j \in B^*$, $q, i \geqslant 1$.

Then:

(1) *There is a homomorphism $\rho_0^q \colon \Gamma_q(R) \to \Gamma_q(B)/U_1\Gamma_q(B)$ such that*

$$\rho_0^q(\{x_1, \ldots, x_q\}) = \{\widetilde{x_1}, \ldots, \widetilde{x_q}\} \mod U_1\Gamma_q(B)$$

where $\widetilde{x_i} \in A$ is a representative of x_i. In case (i) (resp. (ii)) the induced map

$$\Gamma_q(R) + \Gamma_{q-1}(R) \to \Gamma_q(B)/U_1\Gamma_q(B), \quad (x, y) \mapsto \rho_0^q(x) + \{\rho_0^{q-1}(y), \pi\}$$

(*resp.*

$$\Gamma_q(R)/m + \Gamma_{q-1}(R)/m \to \Gamma_q(B)/(U_1\Gamma_q(B) + m\Gamma_q(B)),$$
$$(x, y) \mapsto \rho_0^q(x) + \{\rho_0^{q-1}(y), \pi\})$$

is bijective (resp. bijective for every non-zero integer m).

(2) *If m is an integer invertible in R then $U_1\Gamma_q(B)$ is m-divisible.*

(3) *In case (i) assume that R is additively generated by R^*. In case (ii) assume that* $\mathrm{char}\,(k_{n-1}) = p > 0$. *Then there exists a unique homomorphism*

$$\rho_i^q \colon \Omega_R^{q-1} \to U_i\Gamma_q(B)/U_{i+1}\Gamma_q(B)$$

such that

$$\rho_i^q(x\,dy_1/y_1 \wedge \cdots \wedge dy_{q-1}/y_{q-1}) = \{1 + \widetilde{x}\pi^i, \widetilde{y_1}, \ldots, \widetilde{y_{q-1}}\} \mod U_{i+1}\Gamma_q(B)$$

for every $x \in R$, $y_1, \ldots, y_{q-1} \in R^*$. The induced map

$$\Omega_R^{q-1} \oplus \Omega_R^{q-2} \to U_i\Gamma_q(B)/U_{i+1}\Gamma_q(B), \quad (x, y) \mapsto \rho_i^q(x) + \{\rho_i^{q-1}(y), \pi\}$$

is surjective. If i is invertible in R then the homomorphism ρ_i^q is surjective.

Proof. In case (i) these results follow from Proposition 4 by Bass–Tate's method [2, Proposition 4.3] for (1), Bloch's method [3, §3] for (3) and by writing down the kernel of $R \otimes R^* \to \Omega^1_R$, $x \otimes y \mapsto xdy/y$ as in [9, §1 Lemma 5].

If X is an object of $\mathrm{pro}(\mathcal{F}_{n-1})$ then case (ii) is a special case of (i) except $n = 1$ and $k_0 = \mathbb{F}_2$ where $[X, \underline{k_0}]$ is not generated by $[X, \underline{k_0}]^*$ in general. But in this exceptional case it is easy to check directly all the assertions.

For an arbitrary X we present here only the proof of (3) because the proof of (1) is rather similar.

Put $k = k_{n-1}$. For the existence of ρ_i^q it suffices to consider the cases where $X = \underline{\Omega_k^{q-1}}$ and $X = \underline{k} \times \prod^{q-1} \underline{k^*}$ ($\prod^r Y$ denotes the product of r copies of Y). Note that these objects are in $\mathrm{pro}(\mathcal{F}_{n-1})$ since $[X, \underline{\Omega_k^q}] = \Omega^q_{[X,\underline{k}]}$ for any X and q.

The uniqueness follows from the fact that $[X, \underline{\Omega_k^{q-1}}]$ is generated by elements of the form $xdc_1/c_1 \wedge \cdots \wedge dc_{q-1}/c_{q-1}$ such that $x \in [X, \underline{k}]$ and $c_1, \ldots, c_{q-1} \in k^*$.

To prove the surjectivity we may assume $X = (1+\pi^i \underline{\mathcal{O}_K}) \times \prod^{q-1} \underline{K^*}$ and it suffices to prove in this case that the typical element in $U_i\Gamma_q(B)/U_{i+1}\Gamma_q(B)$ belongs to the image of the homomorphism introduced in (3). Let $\underline{U_K}$ be the object of $\mathcal{F}_n$ which represents the functor $X \to [X, \underline{\mathcal{O}_K}]^*$. By Proposition 4 there exist morphisms $\theta_1 \colon \underline{K^*} \to \coprod_{i=0}^{p-1} \underline{U_K}\pi^i$ (the direct sum in $\mathcal{F}_n$) and $\theta_2 \colon \underline{K^*} \to \underline{K^*}$ such that $x = \theta_1(x)\theta_2(x)^p$ for each X in $\mathcal{F}_\infty$ and each $x \in [X, \underline{K^*}]$ (in the proof of (1) p is replaced by m). Since $\coprod_{i=0}^{p-1} \underline{U_K}\pi^i$ belongs to $\mathrm{pro}(\mathcal{F}_{n-1})$ and $(1+\pi^i[X, \underline{\mathcal{O}_K}])^p \subset 1+\pi^{i+1}[X, \underline{\mathcal{O}_K}]$ we are reduced to the case where X is an object of $\mathrm{pro}(\mathcal{F}_{n-1})$. □

4. Norm groups

In this section we prove Theorem 3 and Proposition 1. In subsection 4.1 we reduce these results to Proposition 6.

4.1. Reduction steps.

Definition 6. Let k be a field and let $H \colon \mathcal{E}(k) \to \mathrm{Ab}$ be a functor such that $\varinjlim_{k' \in \mathcal{E}(k)} H(k') = 0$. Let $w \in H(k)$ (cf. Introduction). For a ring R over k and $q \geqslant 1$ define the subgroup $N_q(w, R)$ (resp. $L_q(w, R)$) of $\Gamma_q(R)$ as follows.

An element x belongs to $N_q(w, R)$ (resp. $L_q(w, R)$) if and only if there exist a finite set J and element $0 \in J$,

a map $f\colon J \to J$ such that for some $n \geqslant 0$ the nth iteration f^n with respect to the composite is a constant map with value 0,
and a family $(E_j, x_j)_{j\in J}$ ($E_j \in \mathcal{E}(k)$), $x_j \in \Gamma_q(E_j \otimes_k R)$) satisfying the following conditions:

(i) $E_0 = k$ and $x_0 = x$.

(ii) $E_{f(j)} \subset E_j$ for every $j \in J$.

(iii) Let $j \in f(J)$. Then there exists a family $(y_t, z_t)_{t\in f^{-1}(j)}$ ($y_t \in (E_t \otimes_k R)^*$, $z_t \in \Gamma_{q-1}(E_j \otimes_k R)$) such that $x_t = \{y_t, z_t\}$ for all $t \in f^{-1}(j)$ and

$$x_j = \sum_{t\in f^{-1}(j)} \{N_{E_t\otimes_k R/E_j\otimes_k R}(y_t), z_t\}$$

where $N_{E_t\otimes_k R/E_j\otimes_k R}$ denotes the norm homomorphism

$$(E_t \otimes_k R)^* \to (E_j \otimes_k R)^*.$$

(iv) If $j \in J \setminus f(J)$ then w belongs to the kernel of $H(k) \to H(E_j)$ (resp. then one of the following two assertions is valid:
(a) w belongs to the kernel of $H(k) \to H(E_j)$,
(b) x_j belongs to the image of $\Gamma(\mathrm{Spec}(E_j \otimes_k R), K_q(E_j)) \to \Gamma_q(E_j \otimes_k R)$, where $K_q(E_j)$ denotes the constant sheaf on $\mathrm{Spec}(E_j \otimes_k R)$ defined by the set $K_q(E_j)$).

Remark. If the groups $\Gamma_q(E_j \otimes_k R)$ have a suitable "norm" homomorphism then x is the sum of the "norms" of x_j such that $f^{-1}(j) = \emptyset$. In particular, in the case where $R = k$ we get $N_q(w, k) \subset N_q(w)$ and $N_1(w, k) = N_1(w)$.

Definition 7. For a field k let $[\mathcal{E}(k), \mathrm{Ab}]$ be the abelian category of all functors $\mathcal{E}(k) \to \mathrm{Ab}$.

(1) For $q \geqslant 0$ let $\mathcal{N}_{q,k}$ denote the full subcategory of $[\mathcal{E}(k), \mathrm{Ab}]$ consisting of functors H such that $\varinjlim_{k'\in\mathcal{E}(k)} H(k') = 0$ and such that for every $k' \in \mathcal{E}(k)$, $w \in H(k')$ the norm group $N_q(w)$ coincides with $K_q(k')$. Here $N_q(w)$ is defined with respect to the functor $\mathcal{E}(k') \to \mathrm{Ab}$.

(2) If K is an n-dimensional local field and $q \geqslant 1$, let $\underline{\mathcal{N}}_{q,K}$ (resp. $\underline{\mathcal{L}}_{q,K}$) denote the full subcategory of $[\mathcal{E}(K), \mathrm{Ab}]$ consisting of functors H such that

$$\varinjlim_{K'\in\mathcal{E}(K)} H(K') = 0$$

and such that for every $K' \in \mathcal{E}(K)$, $w \in H(K')$ and every object X of $\mathcal{F}_\infty$ the group $N_q(w, [X, \underline{K'}])$ (resp. $L_q(w, [X, \underline{K'}])$) coincides with $\Gamma_q([X, \underline{K'}])$.

Lemma 9. *Let K be an n-dimensional local field and let H be an object of $\underline{\mathcal{L}}_{q,K}$. Then for every $w \in H(K)$ the group $N_q(w)$ is an open subgroup of $K_q(K)$ of finite index.*

Proof. Consider the case where $X = \prod^q \underline{K^*}$. We can take a system $(E_j, x_j)_{j\in J}$ as in Definition 6 such that $E_0 = K$, x_0 is the canonical element in $\Gamma_q([X, \underline{K}])$ and such that if $j \notin f(J)$ and $w \notin \ker(H(K) \to H(E_j))$ then x_j is the image of an element θ_j of $\mathrm{lcf}(X, K_q(E_j))$. Let $\theta \in \mathrm{lcf}(X, K_q(K)/N_q(w))$ be the sum of $N_{E_j/K} \circ \theta_j$ mod $N_q(w)$. Then the canonical map $[e, X] = \prod^q K^* \to K_q(K)/N_q(w)$ comes from θ. $\square$

Definition 8. Let k be a field. A collection $\{\mathcal{C}_{k'}\}_{k'\in\mathcal{E}(k)}$ of full subcategories $\mathcal{C}_{k'}$ of $[\mathcal{E}(k'), \mathrm{Ab}]$ is called *admissible* if and only if it satisfies conditions (i) – (iii) below.

(i) Let $E \in \mathcal{E}(k)$. Then every subobject, quotient object, extension and filtered inductive limit (in the category of $[\mathcal{E}(E), \mathrm{Ab}]$) of objects of $\mathcal{C}_E$ belongs to $\mathcal{C}_E$.

(ii) Let $E, E' \in \mathcal{E}(k)$ and $E \subset E'$. If H is in $\mathcal{C}_E$ then the composite functor $\mathcal{E}(E') \to \mathcal{E}(E) \xrightarrow{H} \mathrm{Ab}$ is in $\mathcal{C}_{E'}$.

(iii) Let $E \in \mathcal{E}(k)$ and H is in $[\mathcal{E}(E), \mathrm{Ab}]$. Then H is in $\mathcal{C}_E$ if conditions (a) and (b) below are satisfied for a prime p.

  (a) For some $E' \in \mathcal{E}(E)$ such that $|E' : E|$ is prime to p the composite functor $(E') \to (E) \xrightarrow{H} \mathrm{Ab}$ is in $\mathcal{C}_{E'}$.

  (b) Let q be a prime number distinct from p and let S be a direct subordered set of $\mathcal{E}(E)$. If the degree of every finite extension of the field $\varinjlim_{E'\in S} E'$ is a power of p then $\varinjlim_{E'\in S} H(E') = 0$.

Lemma 10.

(1) *For each field k and q the collection $\{\mathcal{N}_{q,k'}\}_{k'\in\mathcal{E}(k)}$ is admissible. If K is an n-dimensional local field then the collections $\{\underline{\mathcal{N}}_{q,k'}\}_{k'\in\mathcal{E}(k)}$ and $\{\underline{\mathcal{L}}_{q,k'}\}_{k'\in\mathcal{E}(k)}$ are admissible.*

(2) *Let k be a field. Assume that a collection $\{\mathcal{C}_{k'}\}_{k'\in\mathcal{E}(k)}$ is admissible. Let $r \geqslant 1$ and for every prime p there exist $E \in \mathcal{E}(k)$ such that $|E : k|$ is prime to p and such that the functor $H^r(\ ,\mathbb{Z}/p^r)\colon \mathcal{E}(E) \to \mathrm{Ab}$ is in $\mathcal{C}_E$. Then for each $k' \in \mathcal{E}(k)$, each discrete torsion abelian group M endowed with a continuous action of $\mathrm{Gal}(k'^{\mathrm{sep}}/k')$ and each $i \geqslant r$ the functor*

$$H^i(\ , M)\colon \mathcal{E}(k') \to \mathrm{Ab}$$

is in $\mathcal{C}_{k'}$.

Definition 9. For a field k, $r \geqslant 0$ and a non-zero integer m define the group $H^r_m(k)$ as follows.

If $\mathrm{char}(k) = 0$ let

$$H^r_m(k) = H^r(k, \mu_m^{\otimes(r-1)}).$$

If $\mathrm{char}(k) = p > 0$ and $m = m'p^i$ where m' is prime to p and $i \geqslant 0$ let

$$H^r_m(k) = H^r_{m'}(k, \mu_{m'}^{\otimes(r-1)}) \oplus \mathrm{coker}(F - 1\colon C_i^{r-1}(k) \to C_i^{r-1}(k)/\{C_i^{r-2}(k), T\})$$

(where $C_i^{\cdot}$ is the group defined in [3, Ch.II,§7], $C_i^r = 0$ for $r < 0$).

By the above results it suffices for the proof of Theorem 3 to prove the following Proposition 5 in the case where m is a prime number.

Proposition 5. *Let K be an n-dimensional local field. Let $q, r \geqslant 1$ and let m be a non-zero integer. Then the functor $H_m^r \colon \mathcal{E}(K) \to \mathrm{Ab}$ is in $\mathcal{L}_{q,K}$ if $q + r = n + 1$ and in $\mathcal{N}_{q,K}$ if $q + r > n + 1$.*

Now we begin the proofs of Proposition 1 and Proposition 5.

Definition 10. Let K be a complete discrete valuation field, $r \geqslant 0$ and m be a non-zero integer.

(1) Let $H^r_{m,\mathrm{ur}}$ and $H^r_m / H^r_{m,\mathrm{ur}}$ be the functors $\mathcal{E}(K) \to \mathrm{Ab}$:

$$H^r_{m,\mathrm{ur}}(K') = \ker(H^r_m(K') \to H^r_m(K'_{\mathrm{ur}})),$$
$$(H^r_m / H^r_{m,\mathrm{ur}})(K') = H^r_m(K')/H^r_{m,\mathrm{ur}}(K')$$

where K'_{ur} is the maximal unramified extension of K'.

(2) Let I^r_m (resp. J^r_m) be the functor $\mathcal{E}(K) \to \mathrm{Ab}$ such that $I^r_m(K') = H^r_m(k')$ (resp. $J^r_m(K') = H^r_m(k')$) where k' is the residue field of K' and such that the homomorphism $I^r_m(K') \to I^r_m(K'')$ (resp. $J^r_m(K') \to J^r_m(K'')$) for $K' \subset K''$ is $j_{k''/k'}$ (resp. $e(K''|K')j_{k''/k'}$) where k'' is the residue field of K'', $j_{k''/k'}$ is the canonical homomorphism induced by the inclusion $k' \subset k''$ and $e(K''|K')$ is the index of ramification of K''/K'.

Lemma 11. *Let K and m be as in Definition 10.*

(1) *For $r \geqslant 1$ there exists an exact sequence of functors*

$$0 \to I^r_m \to H^r_{m,\mathrm{ur}} \to J^{r-1}_m \to 0.$$

(2) *J^r_m is in $\mathcal{N}_{1,K}$ for every $r \geqslant 0$.*

(3) *Let $q, r \geqslant 1$. Then I^r_m is in $\mathcal{N}_{q,K}$ if and only if $H^r_m \colon \mathcal{E}(k) \to \mathrm{Ab}$ is in $\mathcal{N}_{q-1,k}$ where k is the residue field of K.*

Proof. The assertion (1) follows from [11]. The assertion (3) follows from the facts that $1 + \mathcal{M}_K \subset N_{L/K}(L^*)$ for every unramified extension L of K and that there exists a canonical split exact sequence

$$0 \to K_q(k) \to K_q(K)/U_1 K_q(K) \to K_{q-1}(k) \to 0. \qquad \square$$

The following proposition will be proved in 4.4.

Proposition 6. *Let K be a complete discrete valuation field with residue field k. Let $q, r \geqslant 1$ and m be a non-zero integer. Assume that $|k : k^p| \leqslant p^{q+r-2}$ if $\mathrm{char}(k) = p > 0$. Then:*

(1) $H^r_m/H^r_{m,\mathrm{ur}}$ *is in* $\mathcal{N}_{q,K}$.
(2) *If* K *is an* n*-dimensional local field with* $n \geqslant 1$ *then* $H^r_m/H^r_{m,\mathrm{ur}}$ *is in* $\underline{\mathcal{N}}_{q,K}$.

Proposition 1 follows from this proposition by Lemma 10 and Lemma 11 (note that if $\mathrm{char}(k) = p > 0$ and $i \geqslant 0$ then $H^r_{p^i}(k)$ is isomorphic to $\ker(p^i\colon H^r(k) \to H^r(k))$ as it follows from [11]).

Lemma 12. *Let* K *be an* n*-dimensional local field and let* X *be an object of* $\mathcal{F}_\infty$. *Consider the following cases.*
(i) $q > n+1$ *and* m *is a non-zero integer.*
(ii) $q = n+1$, $\mathrm{char}(K) = p > 0$ *and* m *is a power of* p.
(iii) $q = n+1$ *and* m *is a non-zero integer.*

Let $x \in \Gamma_q([X, \underline{K}]$. *Then in cases (i) and (ii)* (*resp. in case (iii)*) *there exist a triple* $(J, 0, f)$ *and a family* $(E_j, x_j)_{j\in J}$ *which satisfy all the conditions in Definition 6 with* $k = K$ *except condition (iv), and which satisfy the following condition:*
(iv)' *If* $j \in J \setminus f(J)$ *then* $x_j \in m\Gamma_q([X, \underline{E_j}])$
(*resp.* x_j *belongs to* $m\Gamma_q([X, \underline{E_j}])$
or to the image of $\mathrm{lcf}(X, K_q(E_j)) \to \Gamma_q([X, \underline{E_j}])$).

Corollary. *Let* K *be an* n*-dimensional local field. Then* $mK_{n+1}(K)$ *is an open subgroup of finite index of* $K_{n+1}(K)$ *for every non-zero integer* m.

This corollary follows from case (iii) above by the argument in the proof of Lemma 9.

Proof of Lemma 12. We may assume that m is a prime number.

First we consider case (ii). By Lemma 6 we may assume that there are elements $b_1, \dots, b_{n+1} \in [X, \underline{K}]^*$ and $c_1, \dots, c_{n+1} \in K^*$ such that $x = \{b_1, \dots, b_{n+1}\}$ and $b_i \in [X, \underline{K^p(c_i)}]^*$ for each i. We may assume that $|K^p(c_1, \dots, c_r) : K^p| = p^r$ and $c_{r+1} \in K^p(c_1, \dots, c_r)$ for some $r \leqslant n$. Let $J = \{0, 1, \dots, r\}$, and define $f\colon J \to J$ by $f(j) = j-1$ for $j \geqslant 1$ and $f(0) = 0$. Put $E_j = K(c_1^{1/p}, \dots, c_j^{1/p})$ and $x_j = \{b_1^{1/p}, \dots, b_j^{1/p}, b_{j+1}, \dots, b_{n+1}\}$. Then
$x_r = p\{b_1^{1/p}, \dots, b_{r+1}^{1/p}, b_{r+2}, \dots, b_{n+1}\}$ in $\Gamma_{n+1}([X, \underline{E_r}])$.

Next we consider cases (i) and (iii). If K is a finite field then the assertion for (i) follows from Lemma 13 below and the assertion for (iii) is trivial. Assume $n \geqslant 1$ and let k be the residue field of K. By induction on n Lemma 8 (1) (2) and case (ii) of Lemma 12 show that we may assume $x \in U_1\Gamma_q([X, \underline{K}])$, $\mathrm{char}(K) = 0$ and $m = \mathrm{char}(k) = p > 0$. Furthermore we may assume that K contains a primitive pth root ζ of 1. Let $e_K = v_K(p)$ and let π be a prime element of K. Then

$$U_i\Gamma_q([X, \underline{\mathcal{O}_K}]) \subset pU_1\Gamma_q([X, \underline{\mathcal{O}_K}]), \quad \text{if } i > pe_K/(p-1).$$

From this and Lemma 8 (3) (and a computation of the map $x \mapsto x^p$ on $U_1\Gamma_q([X, \underline{\mathcal{O}_K}])$) it follows that $U_1\Gamma_q([X, \underline{K}])$ is p-divisible if $q > n+1$ and that there is a surjective homomorphism

$$[X, \underline{\Omega_k^{n-1}}]/(1-\mathrm{C})[X, \underline{\Omega_k^{n-1}}] \to U_1\Gamma_{n+1}([X,\underline{K}])/pU_1\Gamma_{n+1}([X,\underline{K}]),$$
$$x dy_1/y_1 \wedge \dots \wedge dy_{n-1}/y_{n-1} \mapsto \{1+\widetilde{x}(\zeta-1)^p, \widetilde{y_1}, \dots, \widetilde{y_{n_1}}, \pi\}$$

where C is the Cartier operator. By Lemma 7

$$[X, \underline{\Omega_k^{n-1}}]/(1-\mathrm{C})[X, \underline{\Omega_k^{n-1}}] = \mathrm{lcf}(X, \Omega_k^{n-1}/(1-\mathrm{C})\Omega_k^{n-1}). \qquad \square$$

Lemma 13. *Let K be a finite field and let X be an object of $\mathcal{F}_\infty$. Then*

(1) $\Gamma_q[X,\underline{K}] = 0$ *for* $q \geqslant 2$.

(2) *For every finite extension L of K the norm homomorphism $[X,\underline{L}]^* \to [X,\underline{K}]^*$ is surjective.*

Proof. Follows from Lemma 5 (2). $\square$

Proof of Proposition 5 assuming Proposition 6. If K is a finite field, the assertion of Proposition 5 follows from Lemma 13.

Let $n \geqslant 1$. Let k be the residue field of K. Let I_m^r and J_m^r be as in Definition 10. Assume $q+r = n+1$ (resp. $q+r > n+1$). Using Lemma 8 (1) and the fact that

$$U_1\Gamma_q([X,\underline{K}]) \subset N_{L/K}\Gamma_q([X,\underline{L}])$$

for every unramified extension L/K we can deduce that I_m^r is in $\underline{\mathcal{L}}_{q,K}$ (resp. $\underline{\mathcal{N}}_{q,K}$) from the induction hypothesis $H_m^r\colon \mathcal{E}(k) \to \mathrm{Ab}$ is in $\underline{\mathcal{L}}_{q-1,k}$ (resp. $\underline{\mathcal{N}}_{q-1,k}$). We can deduce J_m^{r-1} is in $\underline{\mathcal{L}}_{q,K}$ (resp. $\underline{\mathcal{N}}_{q,K}$) from the hypothesis $H_m^{r-1}\colon \mathcal{E}(k) \to \mathrm{Ab}$ is in $\underline{\mathcal{L}}_{q,k}$ (resp. $\underline{\mathcal{N}}_{q,k}$). Thus $H^r_{m,\mathrm{ur}}$ is in $\underline{\mathcal{L}}_{q,K}$ (resp. $\underline{\mathcal{N}}_{q,K}$). $\square$

4.2. Proof of Proposition 6.

Let k be a field and let m be a non-zero integer. Then $\oplus_{r\geqslant 0}H_m^r(k)$ (cf. Definition 9) has a natural right $\oplus_{q\geqslant 0}K_q(k)$-module structure (if m is invertible in k this structure is defined by the cohomological symbol $h^q_{m,k}\colon K_q(k)/m \to H^q(k, \mu_m^{\otimes q})$ and the cup-product, cf. [9, §3.1]). We denote the product in this structure by $\{w, a\}$ ($a \in \oplus_{q\geqslant 0}K_q(k)$, $w \in \oplus_{r\geqslant 0}H_m^r(k)$).

Definition 11. Let K be a complete discrete valuation field with residue field k such that $\mathrm{char}(k) = p > 0$. Let $r \geqslant 1$. We call an element w of $H_p^r(K)$ *standard* if and only if w is in one of the following forms (i) or (ii).

(i) $w = \{\chi, a_1, \dots, a_{r-1}\}$ where χ is an element of $H_p^1(K)$ corresponding to a totally ramified cyclic extension of K of degree p, and $a_1, \dots, a_{r-1}$ are elements of $\mathcal{O}_K^*$ such that

$$|k^p(\overline{a_1}, \dots, \overline{a_{r-1}}) : k^p| = p^{r-1}$$

($\overline{a_i}$ denotes the residue of a_i).

(ii) $w = \{\chi, a_1, \dots, a_{r-2}, \pi\}$ where χ is an element of $H_p^1(K)$ corresponding to a cyclic extension of K of degree p whose residue field is an inseparable extension of k of degree p, π is a prime element of K and $a_1, \dots, a_{r-2}$ are elements of $\mathcal{O}_K^*$ such that $|k^p(\overline{a_1}, \dots, \overline{a_{r-2}}) : k^p| = p^{r-2}$.

Lemma 14. *Let K and k be as in Definition 11. Assume that $|k : k^p| = p^{r-1}$. Then for every element $w \in H_p^r(K) \setminus H_{p,\mathrm{ur}}^r(K)$ there exists a finite extension L of K such that $|L : K|$ is prime to p and such that the image of w in $H_p^r(L)$ is standard.*

Proof. If $\operatorname{char}(K) = p$ the proof goes just as in the proof of [8, §4 Lemma 5] where the case of $r = 2$ was treated.

If $\operatorname{char}(K) = 0$ we may assume that K contains a primitive pth root of 1. Then the cohomological symbol $h_{p,K}^r\colon K_r(K)/p \to H_p^r(K)$ is surjective and

$$\operatorname{coker}(h_{p,K}^r\colon U_1K_r(K) \to H_p^r(K)) \simeq \nu_{r-1}(k)$$

by [11] and $|k : k^p| = p^{r-1}$.

Here we are making the following:

Definition 12. Let K be a complete discrete valuation field. Then $U_iK_q(K)$ for $i, q \geqslant 1$ denotes $U_i\Gamma_q(K)$ of Lemma 8 case (i) (take $A = \mathcal{O}_K$ and $B = K$).

Definition 13. Let k be a field of characteristic $p > 0$. As in Milne [13] denote by $\nu_r(k)$ the kernel of the homomorphism

$$\Omega_k^r \to \Omega_k^r/d(\Omega_k^{r-1}), \quad x\,dy_1/y_1 \wedge \dots \wedge dy_r/y_r \mapsto (x^p - x)dy_1/y_1 \wedge \dots \wedge dy_r/y_r.$$

By [11, Lemma 2] for every element α of $\nu_{r-1}(k)$ there is a finite extension k' of k such that
$|k' : k|$ is prime to p and the image of α in $\nu_{r-1}(k')$ is the sum of elements of type

$$dx_1/x_1 \wedge \dots \wedge dx_r/x_r.$$

Hence we can follow the method of the proof of [8, §4 Lemma 5 or §2 Proposition 2]. □

Proof of Proposition 6. If m is invertible in k then $H_m^r = H_{m,\mathrm{ur}}^r$. Hence we may assume that $\operatorname{char}(k) = p > 0$ and $m = p^i$, $i \geqslant 1$. Since $\ker(p\colon H_{p^i}^r/H_{p^i,\mathrm{ur}}^r \to H_{p^i}^r/H_{p^i,\mathrm{ur}}^r)$ is isomorphic to $H_p^r/H_{p,\mathrm{ur}}^r$ by [11], we may assume $m = p$.

The proof of part (1) is rather similar to the proof of part (2). So we present here only the proof of part (2), but the method is directly applicable to the proof of (1).

The proof is divided in several steps. In the following K always denotes an n-dimensional local field with $n \geqslant 1$ and with residue field k such that
$\operatorname{char}(k) = p > 0$, except in Lemma 21. X denotes an object of $\mathcal{F}_\infty$.

Step 1. In this step w denotes a standard element of $H_p^r(K)$ and $\overline{w}$ is its image in $(H_p^r/H_{p,\mathrm{ur}}^r)(K)$. We shall prove here that $U_1\Gamma_q([X,\underline{K}]) \subset N(\overline{w},[X,\underline{K}])$. We fix a presentation of w as in (i) or (ii) of Definition 11. Let L be a cyclic extension of K corresponding to χ. In case (i) (resp. (ii)) let h be a prime element of L (resp. an element of $\mathcal{O}_L$ such that the residue class $\overline{h}$ is not contained in k). Let G be the subgroup of K^* generated by $a_1,\dots,a_{r-1}$ (resp. by $a_1,\dots,a_{r-2},\pi$), by $1+\mathcal{M}_K$ and $N_{L/K}(h)$. Let l be the subfield of k generated by the residue classes of $a_1,\dots,a_{r-1}$ (resp. $a_1,\dots,a_{r-2},N_{L/K}(h)$).

Let $i \geqslant 1$. Let $G_{i,q}$ be the subgroup of $U_i\Gamma_q([X,\underline{K}])$ generated by $\{U_i\Gamma_{q-1}([X,\underline{K}]),G\}$ and $U_{i+1}\Gamma_q([X,\underline{K}])$. Under these notation we have the following Lemma 15, 16 ,17.

Lemma 15.

(1) $G_{i,q} \subset N_q(w,[X,\underline{K}]) + U_{i+1}\Gamma_q([X,\underline{K}])$.

(2) *The homomorphism ρ_i^q of Lemma 8 (3) induces the surjections*

$$[X,\underline{\Omega_k^{q-1}}] \to [X,\underline{\Omega_{k/l}^{q-1}}] \xrightarrow{\overline{\rho_i^q}} U_1\Gamma_q([X,\underline{K}])/G_{i,q}.$$

(3) *If ρ_i^q is defined using a prime element π which belongs to G then the above homomorphism $\overline{\rho_i^q}$ annihilates the image of the exterior derivation $d\colon [X,\underline{\Omega_{k/l}^{q-2}}] \to [X,\underline{\Omega_{k/l}^{q-1}}]$.*

Lemma 16. *Let a be an element of K^* such that $v_K(a)=i$ and $a = a_1^{s(1)}\dots a_{r-1}^{s(r-1)}N_{L/K}(h)^{s(r)}$ (resp. $a = a_1^{s(1)}\dots a_{r-2}^{s(r-2)}\pi^{s(r-1)}N_{L/K}(h)^{s(r)}$) where s is a map $\{0,\dots,r\}\to\mathbb{Z}$ such that $p \nmid s(j)$ for some $j \neq r$.*

Then $1 - x^pa \in N_1(w,[X,\underline{K}])$ for each $x \in [X,\underline{\mathcal{O}_K}]$.

Proof. It follows from the fact that $w \in \{H_p^{r-1}(K),a\}$ and $1-x^pa$ is the norm of $1-xa^{1/p} \in [X,\underline{K(a^{1/p})}]^*$ ($\underline{K(a^{1/p})}$ denotes the ring object which represents the functor $X \to K(a^{1/p})\otimes_K [X,\underline{K}]$). □

Lemma 17. *Let σ be a generator of* $\mathrm{Gal}(L/K)$ *and let $a = h^{-1}\sigma(h)-1$, $b = N_{L/K}(a)$, $t = v_K(b)$. Let $f=1$ in case (i) and let $f=p$ in case (ii). Let $N\colon [X,\underline{L}]^* \to [X,\underline{K}]^*$ be the norm homomorphism. Then:*

(1) *If $f|i$ and $1 \leqslant i < t$ then for every $x \in \mathcal{M}_L^{i/f}[X,\underline{\mathcal{O}_L}]$*

$$N(1+x) \equiv 1 + N(x) \mod \mathcal{M}_K^{i+1}[X,\underline{\mathcal{O}_K}].$$

(2) *For every $x \in [X,\underline{\mathcal{O}_K}]$*

$$N(1+xa) \equiv 1 + (x^p - x)b \mod \mathcal{M}_K^{t+1}[X,\underline{\mathcal{O}_K}].$$

In case (ii) for every integer r prime to p and every $x \in [X, \underline{\mathcal{O}_K}]$

$$N(1 + xh^r a) \equiv 1 + x^p N(h)^r b \mod \mathcal{M}_K^{t+1}[X, \underline{\mathcal{O}_K}].$$

(3)

$$1 + \mathcal{M}_K^{t+1}[X, \underline{\mathcal{O}_K}] \subset N(1 + \mathcal{M}_L^{t/f+1}[X, \underline{\mathcal{O}_L}]).$$

Proof. Follows from the computation of the norm homomorphism $L^* \to K^*$ in Serre [15, Ch. V §3] and [8, §1]. □

From these lemmas we have

(1) If $0 < i < t$ then

$$U_i\Gamma_q([X, \underline{K}]) \subset N_q(w, [X, \underline{K}]) + U_{i+1}\Gamma_q([X, \underline{K}]).$$

(2) $U_{t+1}\Gamma_q([X, \underline{K}]) \subset N_q(w, [X, \underline{K}])$.

(3) In case (ii) let $a_{r-1} = N_{L/K}(h)$. then in both cases (i) and (ii) the homomorphism

$$\begin{aligned} &[X, \underline{\Omega_k^{q+r-2}}] \to U_1\Gamma_q([X, \underline{K}])/N_q(w, [X, \underline{K}]), \\ &xd\overline{a_1}/\overline{a_1} \wedge \cdots \wedge d\overline{a_{r-1}}/\overline{a_{r-1}} \wedge dy_1/y_1 \wedge \cdots \wedge dy_{q-1}/y_{q-1} \\ &\qquad\qquad \mapsto \{1 + \widetilde{x}b, \widetilde{y_1}, \ldots, \widetilde{y_{q-1}}\}, \end{aligned}$$

($x \in [X, \underline{k}], y_i \in [X, \underline{k^*}]$) annihilates $(1 - \mathrm{C})[X, \underline{\Omega_k^{q+r-2}}]$.

Lemma 7 and (1), (2), (3) imply that $U_1\Gamma_q([X, \underline{K}])$ is contained in the sum of $N_q(w, [X, \underline{K}])$ and the image of $\operatorname{lcf}(X, U_{t+1}K_q(K))$.

Lemma 18. *For each $u \in \mathcal{O}_K$ there exists an element ψ of $H^1_{p,\mathrm{ur}}(K)$ such that $(1 + ub)N_{L/K}(h)^{-1}$ is contained in the norm group $N_{L'/K}L'^*$ where L' is the cyclic extension of K corresponding to $\chi + \psi$ (χ corresponds to L/K).*

Proof. Follows from [9, §3.3 Lemma 15] (can be proved using the formula

$$N_{L_{\mathrm{ur}}/K_{\mathrm{ur}}}(1 + xa) \equiv 1 + (x^p - x)b \mod b\mathcal{M}_{K_{\mathrm{ur}}}$$

for $x \in \mathcal{O}_{K_{\mathrm{ur}}}$. □

Lemma 18 shows that $1 + ub$ is contained in the subgroup generated by $N_{L/K}L^*$ and $N_{L'/K}L'^*$, $\chi_L = 0$, $\chi_{L'} \in H^1_{p,\mathrm{ur}}(L')$.

Step 2. Next we prove that

$$U_1\Gamma_q([X, \underline{K}]) \subset N(\overline{w}, [X, \underline{K}])$$

for every $w \in H^r_p(K)$ where $\overline{w}$ is the image of w in $(H^r_p/H^r_{p,\mathrm{ur}})(K)$.

Lemma 19. *Let $q, r \geqslant 1$ and let $w \in H_p^r(K)$. Then there exists $i \geqslant 1$ such that $p^i\Gamma_q([X, \underline{K'}])$ and $U_{e(K'|K)i}\Gamma_q([X, \underline{K'}])$ are contained in $N_q(w_{K'}, [X, \underline{K'}])$ for every $K' \in \mathcal{E}(K)$ where $w_{K'}$ denotes the image of w in $H_p^r(K')$ and $e(K'|K)$ denotes the ramification index of K'/K.*

Lemma 20. *Let $i \geqslant 1$ and $x \in U_1\Gamma_q([X, \underline{K}])$; (resp. $x = \{u_1, \dots, u_q\}$ with $u_i \in [X, \underline{\mathcal{O}_K^*}]$; resp. $x \in \Gamma_q([X, \underline{K}])$).*

Then there exists a triple $(J, 0, f)$ and a family $(E_j, x_j)_{j \in J}$ which satisfy all the conditions of Definition 6 except (iv) and satisfy condition (iv)' below.

(iv)' *If $j \notin f(J)$ then x_j satisfy one of the following three properties:*

(a) $x_j \in p^i\Gamma_q([X, \underline{E_j}])$.

(b) $x_j \in U_{e(E_j|K)i}\overline{\Gamma_q}([X, \underline{E_j}])$; *(resp. (b) $x_j \in U_1\Gamma_q([X, \underline{E_j}])$).*

(c) *Let $\overline{E_j}$ be the residue field of E_j. There are elements $c_1, \dots, c_{q-1}$ of $\mathcal{O}_{E_j}^*$ such that*
$x_j \in \{U_1\Gamma_1([X, \underline{E_j}]), c_1, \dots, c_{q-1}\}$ and $|\overline{E_j}^p(c_1, \dots, c_{q-1}) : \overline{E_j}^p| = p^{q-1}$; (resp. (c) There are elements $b_1, \dots, b_q$ of $[X, \underline{\mathcal{O}_{E_j}^*}]$ and $c_1, \dots, c_q$ of $\mathcal{O}_{E_j}^*$ such that $x_j = \{b_1, \dots, b_q\}$ and such that for each m the residue class $\overline{b_m} \in [X, \underline{\overline{E_j}}]$ belongs to $[X, \underline{\overline{E_j}}]^p[\overline{c_m}]$ and $|\overline{E_j}^p(c_1, \dots, c_q) : \overline{E_j}^p| = p^q$); (resp. (c) There are elements $b_1, \dots, b_{q-1}$ of $[X, \underline{\mathcal{O}_{E_j}^*}]$ and $c_1, \dots, c_{q-1}$ of $\mathcal{O}_{E_j}^*$ such that $x_j \in \{[X, \underline{E_j}]^*, b_1, \dots, b_{q-1}\}$ and such that for each m the residue class $\overline{b_m} \in [X, \underline{\overline{E_j}}]$ belongs to $[X, \underline{\overline{E_j}}]^p[\overline{c_m}]$ and $|\overline{E_j}^p(c_1, \dots, c_{q-1}) : \overline{E_j}^p| = p^{q-1}$).

Using Lemma 19 and 20 it suffices for the purpose of this step to consider the following elements
$\{u, c_1, \dots, c_{q-1}\} \in U_1\Gamma_q([X, \underline{K}])$ such that $u \in U_1\Gamma_1([X, \underline{K}])$, $c_1, \dots, c_{q-1} \in \mathcal{O}_K^*$ and $|k^p(\overline{c_1}, \dots, \overline{c_{q-1}} : k^p| = p^{q-1}$.

For each $i = 1, \dots, q-1$ and each $s \geqslant 0$ take a p^s th root $c_{i,s}$ of $-c_i$ satisfying $c_{i,s+1}^p = c_{i,s}$. Note that $N_{k(c_{i,s+1})/k(c_{i,s})}(-c_{i,s+1}) = -c_{i,s}$. For each $m \geqslant 0$ write m in the form $(q-1)s + r$ ($s \geqslant 0$, $0 \leqslant r < q-1$). Let E_m be the finite extension of K of degree p^m generated by $c_{i,s+1}$ ($1 \leqslant i \leqslant r$) and $c_{i,s}$ ($r+1 \leqslant i \leqslant q-1$) and let

$$x_m = \{u, -c_{1,s+1}, \dots, -c_{r,s+1}, -c_{r+1,s}, \dots, -c_{q-1,s}\} \in \Gamma_q([X, \underline{E_m}]).$$

Then $E_\infty = \varinjlim E_m$ is a henselian discrete valuation field with residue field $\overline{E_\infty}$ satisfying $|\overline{E_\infty} : \overline{E_\infty}^p| \leqslant p^{r-1}$. Hence by Lemma 14 and Lemma 21 below there exists $m < \infty$ such that for some finite extension E'_m of E_m of degree prime to p the image of w in $H_p^r(E'_m)$ is standard. Let $J = \{0, 1, \dots, m, m'\}$, $f(j) = j-1$ for $1 \leqslant j \leqslant m$, $f(0) = 0$, $f(m') = m$, $E_{m'} = E'_m$ and

$$x_{m'} = \{u^{1/|E'_m : E_m|}, c_1, \dots, c_{q-1}\}.$$

Then from Step 1 we deduce $\{u, c_1, \dots, c_{q-1}\} \in N_q(\overline{w}, [X, \underline{K}])$.

Lemma 21. *Let K be a henselian discrete valuation field, and let $\widehat{K}$ be its completion. Then $H^r_m(K) \simeq H^r_m(\widehat{K})$ for every r and m.*

Proof. If m is invertible in K this follows from the isomorphism $\mathrm{Gal}(\widehat{K}^{\mathrm{sep}}/\widehat{K}) \simeq \mathrm{Gal}(K^{\mathrm{sep}}/K)$ (cf. [1, Lemma 2.2.1]). Assume $\mathrm{char}(K) = p > 0$ and $m = p^i$ ($i \geqslant 1$). For a field k of characteristic $p > 0$ the group $H^r_{p^i}(k)$ is isomorphic to $(H^1_{p^i}(k) \otimes k^* \otimes \cdots \otimes k^*)/J$ where J is the subgroup of the tensor product generated by elements of the form (cf. [9, §2.2 Corollary 4 to Proposition 2])

(i) $\chi \otimes a_1 \otimes \cdots \otimes a_{r-1}$ such that $a_i = a_j$ for some $i \neq j$,

(ii) $\chi \otimes a_1 \otimes \cdots \otimes a_{r-1}$ such that $a_i \in N_{k_\chi/k} k_\chi^*$ for some i where k_χ is the extension of k corresponding to χ.

By the above isomorphism of the Galois groups $H^1_{p^i}(K) \simeq H^1_{p^i}(\widehat{K})$. Furthermore if L is a cyclic extension of K then $1 + \mathcal{M}^n_K \subset N_{L/K} L^*$ and $1 + \mathcal{M}^n_{\widehat{K}} \subset N_{L\widehat{K}/\widehat{K}} (L\widehat{K})^*$ for sufficiently large n. Since $K^*/(1 + \mathcal{M}^n_K) \simeq \widehat{K}^*/(1 + \mathcal{M}^n_{\widehat{K}})$, the lemma follows. □

Step 3. In this step we prove that the subgroup of $\Gamma_q([X, \underline{K}])$ generated by $U_1\Gamma_q([X, \underline{K}])$ and elements of the form $\{u_1, \dots, u_q\}$ ($u_i \in [X, \underline{\mathcal{O}^*_K}]$) is contained in $N_q(\overline{w}, [X, \underline{K}])$. By Lemma 20 it suffices to consider elements $\{b_1, \dots, b_q\}$ such that $b_i \in [X, \underline{\mathcal{O}^*_K}]$ and such that there are elements $c_j \in \mathcal{O}^*_K$ satisfying

$$|k^p(\overline{c_1}, \dots, \overline{c_q}) : k^p| = p^q$$

and $\overline{b_i} \in [X, \underline{k}]^p[\overline{c_i}]$ for each i. Define fields E_m as in Step 2 replacing $q - 1$ by q. Then $E_\infty = \varinjlim E_m$ is a henselian discrete valuation field with residue field $\overline{E_\infty}$ satisfying $|\overline{E_\infty} : \overline{E_\infty}^p| \leqslant p^{r-2}$. Hence $H^r_p(\widehat{E_\infty}) = H^r_{p,\mathrm{ur}}(\widehat{E_\infty})$. By Lemma 21 there exists $m < \infty$ such that $w_{E_m} \in H^r_{p,\mathrm{ur}}(E_m)$.

Step 4. Let w be a standard element. Then there exists a prime element π of K such that $\pi \in N_1(w, [X, \underline{K}]) = \Gamma_q([X, \underline{K}])$.

Step 5. Let w be any element of $H^r_p(K)$. To show that $\Gamma_q([X, \underline{K}]) = N_q(\overline{w}, [X, \underline{K}])$ it suffices using Lemma 20 to consider elements of $\Gamma_q([X, \underline{K}])$ of the form $\{x, b_1, \dots, b_{q-1}\}$ ($x \in [X, \underline{K}]^*$, $b_i \in [X, \underline{\mathcal{O}^*_K}]$) such that there are elements $c_1, \dots, c_{q-1} \in \mathcal{O}^*_K$ satisfying $|k^p(\overline{c_1}, \dots, \overline{c_{q-1}}) : k^p| = p^{q-1}$ and $\overline{b_i} \in [X, \underline{k}]^p[\overline{c_i}]$ for each i. The fields E_m are defined again as in Step 2, and we are reduced to the case where w is standard. □

5. Proof of Theorem 2

Let K be an n-dimensional local field. By [9, §3 Proposition 1] $H^r(K) = 0$ for $r > n+1$ and there exists a canonical isomorphism $H^{n+1}(K) \simeq \mathbb{Q}/\mathbb{Z}$.
For $0 \leqslant r \leqslant n+1$ the canonical pairing

$$\{\ ,\ \}\colon H^r(K) \times K_{n+1-r}(K) \to H^{n+1}(K)$$

(see subsection 4.2) induces a homomorphism

$$\Phi_K^r\colon H^r(K) \to \mathrm{Hom}(K_{n+1-r}(K), \mathbb{Q}/\mathbb{Z}).$$

if $w \in H^r(K)$ with $r \geqslant 1$ (resp. $r = 0$) then $\Phi_K^r(w)$ annihilates the norm group $N_{n+1-r}(w)$ (resp. $\Phi_K^r(w)$ annihilates $mK_{n+1}(K)$ where m is the order of w). Since $N_{n+1-r}(w)$ (resp. $mK_{n+1}(K)$) is open in $K_{n+1-r}(K)$ by Theorem 3 (resp. Corollary to Lemma 12), $\Phi_K^r(w)$ is a continuous character of $K_{n+1-r}(K)$ of finite order.

5.1. Continuous characters of prime order.

In this subsection we shall prove that for every prime p the map Φ_K^r ($0 \leqslant r \leqslant n+1$) induces a bijection between $H_p^r(K)$ (cf. Definition 10) and the group of all continuous characters of order p of $K_{n+1-r}(K)$. We may assume that $n \geqslant 1$ and $1 \leqslant r \leqslant n$. Let k be the residue field of K. In the case where $\mathrm{char}(k) \neq p$ the above assertion follows by induction on n from the isomorphisms

$$H_p^r(k) \oplus H_p^{r-1}(k) \simeq H_p^r(K), \quad K_q(k)/p \oplus K_q(k)/p \simeq K_q(K)/p.$$

Now we consider the case of $\mathrm{char}(k) = p$.

Definition 14. Let K be a complete discrete valuation field with residue field k of characteristic $p > 0$. For $r \geqslant 1$ and $i \geqslant 0$ we define the subgroup $T_i H_p^r(K)$ of $H_p^r(K)$ as follows.

(1) If $\mathrm{char}(K) = p$ then let $\delta_K^r\colon \Omega_K^{r-1} = C_1^{r-1}(K) \to H_p^r(K)$ be the canonical projection. Then $T_i H_p^r(K)$ is the subgroup of $H_p^r(K)$ generated by elements of the form

$$\delta_K^r(x\,dy_1/y_1 \wedge \dots \wedge dy_{r-1}/y_{r-1}), \quad x \in K, y_1, \dots, y_{r-1} \in K^*, v_K(x) \geqslant -i.$$

(2) If $\mathrm{char}(K) = 0$ then let ζ be a primitive pth root of 1, and let $L = K(\zeta)$.
Let $j = (pe_K/(p-1) - i)e(L|K)$ where $e_K = v_K(p)$ and $e(L|K)$ is the ramification index of L/K. If $j \geqslant 1$ let $U_j H_p^r(L)$ be the image of $U_j K_r(L)$ (cf. Definition 12) under the cohomological symbol $K_r(L)/p \to H_p^r(L)$. If $j \leqslant 0$, let $U_j H_p^r(L) = H_p^r(L)$. Then $T_i H_p^r(K)$ is the inverse image of $U_j H_p^r(L)$ under the canonical injection $H_p^r(K) \to H_p^r(L)$.

Remark. $T_iH_p^1(K)$ coincides with the subgroup consisting of elements which corresponds to cyclic extensions of K of degree p with ramification number $\leqslant i$ (the ramification number is defined as t of Lemma 17).

Let K be as in Definition 14, and assume that $|k : k^p| < \infty$. Fix $q, r \geqslant 1$ such that $|k : k^p| = p^{q+r-2}$. Let $T_i = T_iH_p^r(K)$, for $i \geqslant 0$; let U_i be the image of $U_iK_q(K)$ in $K_q(K)/p$ for $i \geqslant 1$, and let $U_0 = K_q(K)/p$. Let $e = v_K(p)$ ($= \infty$ if $\mathrm{char}(K) = p$). Fix a prime element π of K. Via the homomorphism

$$(x, y) \mapsto \rho_i^q(x) + \{\rho_i^{q-1}(y), \pi\}$$

of Lemma 8 whose kernel is known by [11], we identify U_i/U_{i+1} with the following groups:

(1) $K_q(k)/p \oplus K_{q-1}(k)/p$ if $i = 0$.

(2) Ω_k^{q-1} if $0 < i < pe/(p-1)$ and i is prime to p.

(3) $\Omega_k^{q-1}/\Omega_{k,d=0}^{q-1} \oplus \Omega_k^{q-2}/\Omega_{k,d=0}^{q-2}$ if $0 < i < pe/(p-1)$ and $p|i$.

(4) $\Omega_k^{q-1}/D_{a,k}^{q-1} \oplus \Omega_k^{q-2}/D_{a,k}^{q-2}$ if $\mathrm{char}(K) = 0$, $pe/(p-1)$ is an integer and $i = pe/(p-1)$.

(5) 0 if $i > pe/(p-1)$.

Here in (3) $\Omega_{k,d=0}^q$ ($q \geqslant 0$) denotes the kernel of the exterior derivation $d\colon \Omega_k^q \to \Omega_k^{q+1}$. In (4) a denotes the residue class of $p\pi^{-e}$ where $e = v_K(p)$ and $D_{a,k}$ denotes the subgroup of Ω_k^q generated by $d(\Omega_k^{q-1})$ and elements of the form

$$(x^p + ax)dy_1/y_1 \wedge \cdots \wedge dy_q/y_q.$$

Note that $H_p^{r+1}(K) \simeq H_p^{q+r-1}(k)$ by [11]. Let $\delta = \delta_k^{q+r-1}\colon \Omega_k^{q+r-2} \to H_p^{q+r-1}(k)$ (Definition 14).

Lemma 22. *In the canonical pairing*

$$H_p^r(K) \times K_q(K)/p \to H_p^{q+r}(K) \simeq H_p^{q+r-1}(k)$$

T_i annihilates U_{i+1} for each $i \geqslant 0$. Furthermore,

(1) $T_0 = H_{p,\mathrm{ur}}^r(k) \simeq H_p^r(k) \oplus H_p^{r-1}(k)$, *and the induced pairing*

$$T_0 \times U_0/U_1 \to H_p^{q+r-1}(k)$$

is identified with the direct sum of the canonical pairings

$$H_p^r(k) \times K_{q-1}(k)/p \to H_p^{q+r-1}(k), \quad H_p^{r-1}(k) \times K_q(k)/p \to H_p^{q+r-1}(k).$$

(2) *If $0 < i < pe/(p-1)$ and i is prime to p then there exists an isomorphism*

$$T_i/T_{i-1} \simeq \Omega_k^{r-1}$$

such that the induced pairing $T_i/T_{i-1} \times U_i/U_{i+1} \to H_p^{q+r-1}(k)$ is identified with

$$\Omega_k^{r-1} \times \Omega_k^{q-1} \to H_p^{q+r-1}(k), \quad (w, v) \mapsto \delta(w \wedge v).$$

(3) *If* $0 < i < pe/(p-1)$ *and* $p|i$ *then there exists an isomorphism*

$$T_i/T_{i-1} \simeq \Omega_k^{r-1}/\Omega_{k,d=0}^{r-1} \oplus \Omega_k^{r-2}/\Omega_{k,d=0}^{r-2}$$

such that the induced pairing is identified with

$$(w_1 \oplus w_2, v_1 \oplus v_2) \mapsto \delta(dw_1 \wedge v_2 + dw_2 \wedge v_1).$$

(4) *If* $\mathrm{char}(K) = 0$ *and* $pe/(p-1)$ *is not an integer, then* $H_p^r(K) = T_i$ *for the maximal integer* i *smaller than* $pe/(p-1)$. *Assume that* $\mathrm{char}(K) = 0$ *and* $pe/(p-1)$ *is an integer. Let* a *be the residue element of* $p\pi^{-e}$ *and let for* $s \geqslant 0$

$$\nu_s(a, F) = \ker(\Omega_{k,d=0}^s \to \Omega_k^s, \quad w \mapsto \mathrm{C}(w) + aw)$$

(C *denotes the Cartier operator*). *Then there exists an isomorphism*

$$T_{pe/(p-1)}/T_{pe/(p-1)-1} \simeq \nu_r(a,k) \oplus \nu_{r-1}(a,k)$$

such that the induced pairing is identified with

$$(w_1 \oplus w_2, v_1 \oplus v_2) \mapsto \delta(w_1 \wedge v_2 + w_2 \wedge v_1).$$

Proof. If $\mathrm{char}(K) = p$ the lemma follows from a computation in the differential modules Ω_K^s ($s = r-1, q+r-1$). In the case where $\mathrm{char}(K) = 0$ let ζ be a primitive pth root of 1 and let $L = K(\zeta)$. Then the cohomological symbol $K_r(L)/p \to H_p^r(L)$ is surjective and the structure of $H_p^r(L)$ is explicitly given in [11]. Since

$$H_p^r(K) \simeq \{x \in H_p^r(L) : \sigma(x) = x \quad \text{for all } \sigma \in \mathrm{Gal}(L/K)\},$$

the structure of $H_p^r(K)$ is deduced from that of $H_p^r(L)$ and the description of the pairing

$$H_p^r(K) \times K_q(K)/p \to H_p^{q+r}(K)$$

follows from a computation of the pairing

$$K_r(L)/p \times K_q(L)/p \to K_{q+r}(L)/p. \qquad \square$$

Lemma 23. *Let* K *be an* n*-dimensional local field such that* $\mathrm{char}(K) = p > 0$. *Then the canonical map* $\delta_K^n \colon \Omega_K^n \to H_p^{n+1}(K) \simeq \mathbb{Z}/p$ (*cf. Definition 14*) *comes from a morphism* $\underline{\Omega_K^n} \to \mathbb{Z}/p$ *of* $\mathcal{A}_\infty$.

Proof. Indeed it comes from the composite morphism of $\mathcal{F}_\infty$

$$\underline{\Omega_K^n} \xrightarrow{\theta_2} k_0 \xrightarrow{\mathrm{Tr}_{k_0/\mathbb{F}_p}} \mathbb{F}_p$$

defined by Lemma 7. $\square$

Now let K be an n-dimensional local field ($n \geqslant 1$) with residue field k such that $\mathrm{char}(k) = p > 0$. Let $1 \leqslant r \leqslant n$, $q = n+1-r$, and let T_i and U_i ($i \geqslant 0$) be as in Lemma 22.

The injectivity of the map induced by Φ_K^r

$$H_p^r(K) \to \mathrm{Hom}(K_{n+1-r}(K)/p, \mathbb{Z}/p)$$

follows by induction on n from the injectivity of $T_i/T_{i-1} \to \mathrm{Hom}(U_i/U_{i+1}, \mathbb{Z}/p)$, $i \geqslant 1$. Note that this injectivity for all prime p implies the injectivity of Φ_K^r.

Now let $\varphi\colon K_{n+1-r}(K) \to \mathbb{Z}/p$ be a continuous character of order p. We prove that there is an element w of $H_p^r(K)$ such that $\varphi = \Phi_K^r(w)$.

The continuity of φ implies that there exists $i \geqslant 1$ such that

$$\varphi(\{x_1, \dots, x_{n+1-r}\}) = 0 \quad \text{for all } x_1, \dots, x_{n+1-r} \in 1 + \mathcal{M}_K^i .$$

Using Graham's method [6, Lemma 6] we deduce that $\varphi(U_i) = 0$ for some $i \geqslant 1$. We prove the following assertion (A_i) ($i \geqslant 0$) by downward induction on i.

(A_i) The restriction of φ to U_i coincides with the restriction of $\Phi_K^r(w)$ for some $w \in H_p^r(K)$.

Indeed, by induction on i there exists $w \in H_p^r(K)$ such that the continuous character $\varphi' = \varphi - \Phi_K^r(w)$ annihilates U_{i+1}.

In the case where $i \geqslant 1$ the continuity of φ' implies that the map

$$\Omega_k^{n-r} \oplus \Omega_k^{n-r-1} \xrightarrow{\text{Lemma 8}} U_i/U_{i+1} \xrightarrow{\varphi'} \mathbb{Z}/p$$

comes from a morphism of $\mathcal{F}_\infty$. By additive duality of Proposition 3 and Lemma 23 applied to k the above map is expressed in the form

$$(v_1, v_2) \mapsto \delta_k^n(w_1 \wedge v_2 + w_2 \wedge v_1)$$

for some $w_1 \in \Omega_k^n, w_2 \in \Omega_k^{r-1}$. By the following argument the restriction of φ' to U_i/U_{i+1} is induced by an element of T_i/T_{i-1}. For example, assume $\mathrm{char}(K) = 0$ and $i = pe/(p-1)$ (the other cases are treated similarly and more easily). Since φ' annihilates $d(\Omega_k^{n-r-1}) \oplus d(\Omega_k^{n-r-2})$ and δ_k^n annihilates $d(\Omega_k^{n-2})$ we get

$$\delta_k^n(dw_1 \wedge v_2) = \pm\delta_k^n(w_2 \wedge dv_2) = 0 \quad \text{for all } v_2 .$$

Therefore $dw_1 = 0$. For every $x \in F$, $y_1, \dots, y_{n-r-1} \in F^*$ we have

$$\delta_k^n\big((\mathrm{C}(w_1)+aw_1)\wedge x\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_{n-r-1}}{y_{n-r-1}}\big) = \delta_k^n\big(w_1\wedge(x^p+ax)\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_{n-r-1}}{y_{n-r-1}}\big) = 0$$

(where a is as in Lemma 22 (4)). Hence $w_1 \in \nu_r(a,k)$ and similarly $w_2 \in \nu_{r-1}(a,k)$.

In the case where $i = 0$ Lemma 22 (1) and induction on n imply that there is an element $w \in T_0$ such that $\varphi' = \Phi_K^r(w)$.

5.2. Continuous characters of higher orders.

In treatment of continuous characters of higher order the following proposition will play a key role.

Proposition 7. *Let K be an n-dimensional local field. Let p be a prime number distinct from the characteristic of K. Assume that K contains a primitive pth root ζ of 1. Let $r \geqslant 0$ and $w \in H^r(K)$. Then the following two conditions are equivalent.*
(1) $w = pw'$ *for some* $w' \in H^r(K)$.
(2) $\{w, \zeta\} = 0$ *in* $H^{r+1}(K)$.

Proof. We may assume that $0 \leqslant r \leqslant n$. Let $\delta_r\colon H^r(K) \to H^{r+1}(K, \mathbb{Z}/p)$ be the connecting homomorphism induced by the exact sequence of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-modules

$$0 \to \mathbb{Z}/p \to \varinjlim_i \mu_{p^i}^{\otimes(r-1)} \xrightarrow{p} \varinjlim_i \mu_{p^i}^{\otimes(r-1)} \to 0.$$

Condition (1) is clearly equivalent to $\delta_r(w) = 0$.

First we prove the proposition in the case where $r = n$. Since the kernel of

$$\delta_n\colon H^n(K) \to H^{n+1}(K, \mathbb{Z}/p) \simeq \mathbb{Z}/p$$

is contained in the kernel of the homomorphism $\{\ ,\zeta\}\colon H^n(K) \to H^{n+1}(K)$ it suffices to prove that the latter homomorphism is not a zero map. Let i be the maximal natural number such that K contains a primitive p^ith root of 1. Since the image χ of a primitive p^ith root of 1 under the composite map

$$K^*/K^{*p} \simeq H^1(K, \mu_p) \simeq H^1(K, \mathbb{Z}/p) \to H^1(K)$$

is not zero, the injectivity of Φ_K^1 shows that there is an element a of $K_n(K)$ such that $\{\chi, a\} \neq 0$. Let w be the image of a under the composite map induced by the cohomological symbol

$$K_n(K)/p^i \to H^n(K, \mu_{p^i}^{\otimes n}) \simeq H^n(K, \mu_{p^i}^{\otimes(n-1)}) \to H^n(K).$$

Then $\{\chi, a\} = \pm\{w, \zeta\}$.

Next we consider the general case of $0 \leqslant r \leqslant n$. Let w be an element of $H^r(K)$ such that $\{w, \zeta\} = 0$. Since the proposition holds for $r = n$ we get $\{\delta_r(w), a\} = \delta_n(\{w, a\}) = 0$ for all $a \in K_{n-r}(K)$. The injectivity of Φ_K^{r+1} implies $\delta_r(w) = 0$. □

Remark. We conjecture that condition (1) is equivalent to condition (2) for every field K.

This conjecture is true if $\oplus_{r\geqslant 1} H^r(K)$ is generated by $H^1(K)$ as a $\oplus_{q\geqslant 0} K_q(K)$-module.

Completion of the proof of Theorem 2. Let φ be a non-zero continuous character of $K_{n+1-r}(K)$ of finite order, and let p be a prime divisor of the order of φ. By induction on the order there exists an element w of $H^r(K)$ such that $p\varphi = \Phi_K^r(w)$.

If $\operatorname{char}(K) = p$ then $H^r(K)$ is p-divisible. If $\operatorname{char}(K) \neq p$, let $L = K(\zeta)$ where ζ is a primitive pth root of 1 and let w_L be the image of w in $H^r(L)$. Then $\Phi^r_L(w_L)\colon K_{n+1-r}(L) \to \mathbb{Q}/\mathbb{Z}$ coincides with the composite

$$K_{n+1-r}(L) \xrightarrow{N_{L/K}} K_{n+1-r}(K) \xrightarrow{p\varphi} \mathbb{Q}/\mathbb{Z}$$

and hence $\{w_L, \zeta, a\} = 0$ in $H^{n+1}(L)$ for all $a \in K_{n-r}(L)$. The injectivity of Φ^{r+1}_L and Proposition 7 imply that $w_L \in pH^r(L)$. Since $|L : K|$ is prime to p, w belongs to $pH^r(K)$.

Thus there is an element w' of $H^r(K)$ such that $w = pw'$. Then $\varphi - \Phi^r_K(w')$ is a continuous character annihilated by p. $\square$

References

[1] M. Artin, Dimension cohomologique; premiers résultats, Théorie des topos et cohomologie etale des schémas, Tome 3, Exposé X, Lecture Notes in Math. 305, Springer, Berlin 1973, 43–63

[2] H. Bass, J. Tate, The Milnor ring of a global field, Algebraic K-theory II, Lecture Notes in Math. 342, Springer, Berlin 1972, 349–446.

[3] S. Bloch, Algebraic K-theory and crystalline cohomology, Publ. Math. IHES 47, 1977, 187–268.

[4] P. Cartier, Une nouvelle opération sur les formes différentielles, C. R. Acad. Sc. Paris 244, 1957, 426–428.

[5] P. Deligne, Cohomologie à support propre et construction du functeur $f^!$, in R. Hartshorne, Residue and duality, Lecture Notes in Math. 20, Springer, Berlin 1966, 404–421.

[6] J. Graham, Continuous symbols on fields of formal power series, Algebraic K-theory II, Lecture Notes in Math. 342, Springer, Berlin 1972, 474–486.

[7] A. Grothendieck, Elements de Géométrie Algébrique IV, Première Partie, Publ. Math. IHES 20, 1964.

[8] K. Kato, A generalization of local class field theory by using K-groups I, J. Fac. Sci. Univ. Tokyo Sec. IA 26 No.2, 1979, 303–376.

[9] K. Kato, A generalization of local class field theory by using K-groups II, J. Fac. Sci. Univ. Tokyo Sec. IA 27 No.3, 1980, 603–683.

[10] K. Kato, A generalization of local class field theory by using K-groups III, J. Fac. Sci. Univ. Tokyo Sec. IA 29 No.1, 1982, 31–42.

[11] K. Kato, Galois cohomology of complete discrete valuation fields, Algebraic K-theory, Part II (Oberwolfach, 1980), Lecture Notes in Math. 967, 1982, 215–238.

[12] S. Lefschetz, Algebraic Topology, Amer. Math. Soc. Colloq. Publ., 1942.

[13] J. S. Milne, Duality in flat cohomology of a surface, Ann. Sc. Ec. Norm. Sup. 4ème série 9, 1976, 171–202.

[14] J. Milnor, Algebraic K-theory and quadratic forms, Invent. Math. 9, 1970, 318–344.

[15] J.-P. Serre, Corps Locaux, Hermann, Paris 1962.

[16] J.-P. Serre, Cohomologie Galoisienne, Lecture Notes in Math. 5, Springer, Berlin 1965.

[17] A. Weil, Basic Number Theory, Springer, Berlin 1967.

Department of Mathematics University of Tokyo
3-8-1 Komaba Meguro-Ku Tokyo 153-8914 Japan