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Bracelets and the Goussarov Filtration of the Space of Knots

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Abstract Following Goussarov's paper "Interdependent Modifications of Links and Invariants of Finite Degree" [3] we describe an alternative finite type theory of knots. While (as shown by Goussarov) the alternative theory turns out to be equivalent to the standard one, it nevertheless has its own share of intrinsic beauty.

AMS Classification 57M27

Keywords Bracelets, interdependent modifications, Goussarov, Vassiliev

In Memory of Mikhail Nikolaevitch Goussarov

1 Introduction

There is a well known notion of Vassiliev finite type invariants of knots (see e.g. [1]). A knot invariant I is called "Vassiliev of type n" if, like a polynomial of degree n, its higher than nth iterated differences ("derivatives") vanish. That is, one picks a knot K and (say) some number $m > n$ of crossings and then looks at the alternating sum of the values of I evaluated on the 2^m knots obtained from K by flipping the crossings in some subset of the m chosen crossings (with signs determined by the parity of the number of crossings flipped). If this sum vanishes for all K and all choices of $m > n$ crossings, then I is of Vassiliev type n.

A different way of saying this is to say that we look at K and at some number m of possible simple modifications to K (of the form $\times \to \times$ or $\times \to \times$) which can (but don't need to) be performed simultaneously. We then look at iterated differences of values of I evaluated on K with just some of these modification applied, and if this vanishes whenever $m > n$, then I is of Vassiliev type n.

But why restrict to just "simple modification"? Goussarov's novel idea in his paper 'Interdependent Modifications of Links and Invariants of Finite Degree" [3]

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was to allow arbitrary modifications to K. That is, we pick some number m of intervals along K and allow them to make completely arbitrary detours, provided none of the original paths and none of the re-routed paths ever intersect. We can then form the same sort of alternating sum of values of a knot invariant I, and make a similar definition of "Goussarov type n", if this alternating sum vanishes whenever the number of detours m is bigger than n. (We will repeat this definition in more precise terms in Section 3).

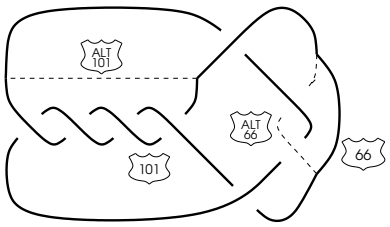
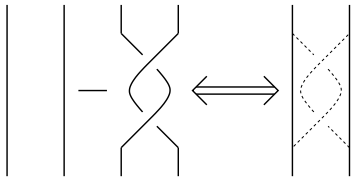


Figure 1: 6_1 and two detours

An example appears in Figure 1; if we travel the main road, it is the knot 6_1. If we choose route ALT 66 over route 66, the knot becomes the more complicated 8_3. If we choose route ALT 101 over route 101 we get the unknot no matter which choice we make in the east. Thus the alternating sum corresponding to this knot and this choice of detours is $I(6_1) - I(8_3) - I(0) + I(0)$.

Goussarov's theorem says that the two notions of finite type invariants agree up to some renumbering:

Theorem 1 (Goussarov [3]) *Any Vassiliev type n invariant is a Goussarov type $2n$ invariant and any Goussarov type $2n$ or $2n+1$ invariant is a Vassiliev type n invariant.*



The key to the understanding of this theorem is the figure on the left, which indicates that a single Vassiliev style crossing change (left part of the figure) can be achieved using two Goussarov style detour moves (right part of the figure). Indeed, if none or just one of the detours is taken, the knot-part displayed remains unbraided, and only if both detours are taken do we get braiding. This too will be made precise later in this paper.

Our paper is only partially about proving Theorem 1. The theorem says that the two notions of finite type invariants are equivalent. Thus if we start from

the Goussarov notion and study it along the same lines as the standard study of the Vassiliev notion, we must meet the same objects: chord diagrams, $4T$ relations, etc., even if we pretend to know nothing about Vassiliev finite type invariants and about Theorem 1. Hence our plan is to carry out an independent study of the Goussarov theory with the hope that we encounter some familiar objects as we go. This we do in Section 3 which to our taste is the most elegant part of this paper. Before that, in Section 2, we quickly review the basics of the Vassiliev theory. This review is not a prerequisite for the study of the Goussarov theory (or else we would be defeating our own purpose), and we embark upon it merely for the purpose of comparison and to establish what we mean by the word "study". Finally, in Section 4 we use some of the results of Section 3 to give an easy proof of Theorem 1.

A different easy proof of Theorem 1 is in Conant's [2].

Acknowledgements This article is the written form of a lecture given by the author at the Goussarov Day at RIMS, Kyoto, September 25, 2001. It follows research done jointly with my student Haggai Scolnicov in 1998/99 and conversations with M. Hutchings in 1997. I wish to thank Shlomi Kotler for pointing out an error in a previous version of Figure 6 and A. Referee for further comments.

This paper is also available electronically at: `arXiv:math.GT/0111267`
and at: `http://www.math.toronto.edu/~drorbn/papers/Bracelets/`

2 A quick review of the Vassiliev finite type theory

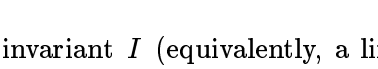
The purpose of this section is to recall how chord diagrams and the $4T$ relations arise in the Vassiliev theory of finite type invariants.

Let $\mathcal{K}_n^V$ denote the space of all formal linear combinations of n-singular knots, knots with n "double points" that locally look like $\times$, modulo the benign "differentiability relations" which will be described shortly. Let $\delta^V = \delta^V_{n+1} : \mathcal{K}^V_{n+1} \to \mathcal{K}^V_n$ be the linear map defined on a singular knot K by picking one of the double points $\times$ in K and then mapping K to the difference of the knots obtained by resolving $\times$ to and overcrossing $\times$ and to an undercrossing $\times$:

$$\delta^V : \times \mapsto \times - \times.$$

As it stands, δ^V is not well defined because it may depend on the choice of the double point to be resolved. We fix this by dividing $\mathcal{K}_n^V$ by differentiability

relations, which are exactly the minimal relations required in order to make δ^V well defined. In figures, the differentiability relations are the relations



(As usual in knot theory, this equation represents the whole family of relations obtained from the figures drawn by completing them to knots in all possible ways, but where all the "picturelets" (like and) are completed in the same manner).

We denote the adjoint of δ^V by ∂_V and call it "the derivative". It is a map $\partial_V : (\mathcal{K}_n^V)^\star \to (\mathcal{K}_{n+1}^V)^\star$. The name "derivative" is justified by the fact that $(\partial_V I)(K)$ for some $I \in (\mathcal{K}_n^V)^\star$ and some generator $K \in \mathcal{K}_{n+1}^V$ is by definition the difference of the values of I on two "neighboring" n-singular knots, in harmony with the usual definition of derivatives for functions on $\mathbb{R}^d$.

Definition 2.1 A knot invariant I (equivalently, a linear functional on $\mathcal{K} = \mathcal{K}_0^V$) is of Vassiliev type n if its $(n+1)$-st (Vassiliev style) derivative vanishes, that is, if $(\partial_V)^{n+1} I \equiv 0$. (This definition is the analog of one of the standard definitions of polynomials on $\mathbb{R}^d$).

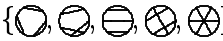
When thinking about finite type invariants, it is convenient to have in mind the following ladders of spaces and their duals, printed here with the names of some specific elements that we will use later:

$$\begin{array}{ccccccccccc}
\ldots \longrightarrow & \mathcal{K}_{n+1}^V & \xrightarrow{\delta^V} & \mathcal{K}_n^V & \xrightarrow{\delta^V} & \mathcal{K}_{n-1}^V & \longrightarrow \ldots & \xrightarrow{\delta^V} & \mathcal{K}_0^V = \mathcal{K} \\
\ldots \longleftarrow & (\mathcal{K}_{n+1}^V)^\star & \xleftarrow{\partial_V} & (\mathcal{K}_n^V)^\star & \xleftarrow{\partial_V} & (\mathcal{K}_{n-1}^V)^\star & \longleftarrow \ldots & \xleftarrow{\partial_V} & (\mathcal{K}_0^V)^\star = \mathcal{K}^\star \\
 & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & & & & \rotatebox{90}{$\in$} \\
 & \partial_V^{n+1} I \equiv 0 & & \partial_V^n I = W & & & & & I
\end{array} \tag{1}$$

We often study invariants of type n by studying their nth derivatives. Clearly, if I is of type n and $W = \partial_V^n I$, then $\partial_V W = 0$ ("W is a constant"). Glancing at (1), we see that W descends to a linear functional, also called W, on $\mathcal{K}_n^V / \delta^V \mathcal{K}_{n+1}^V$. The latter space is a familiar entity:

Proposition 2.2 *The space $\mathcal{K}_n^V / \delta^V \mathcal{K}_{n+1}^V$ is canonically isomorphic to the space $\mathcal{D}_n^V$ of n-chord diagrams, defined below.* □

Definition 2.3 An n-chord diagram is a choice of n pairs of distinct points on an oriented circle, considered up to orientation preserving homeomorphisms

of the circle. Usually an n-chord diagram is simply drawn as a circle with n chords (whose ends are the n pairs). The space $\mathcal{D}_n^V$ is the space of all formal linear combinations of n-chord diagrams. As an example, a basis for $\mathcal{D}_3^V$ is $\{$  $\}$.

Next, we wish to find conditions that a "potential top derivative" has to satisfy in order to actually be a top derivative. More precisely, we wish to find conditions that a functional $W \in (\mathcal{D}_n^V)^\star$ has to satisfy in order to be $\partial_V^n I$ for some invariant I. A first condition is that W must be "integrable once"; namely, there has to be some $W^1 \in (\mathcal{K}_{n-1}^V)^V$ with $W = \partial_V W^1$. Another quick glance at (1), and we see that W is integrable once iff it vanishes on $\ker \delta^V$, which is the same as requiring that W descends to $\mathcal{A}_n^V := \mathcal{D}_n^V / \pi(\ker \delta^V) = \mathcal{K}_n^V / (\operatorname{im} \delta^V + \ker \delta^V)$ (π is the projection $\mathcal{K}_n^V \to \mathcal{D}_n^V = \mathcal{K}_n^V / \delta^V \mathcal{K}_{n+1}^V$, and there should be no confusion regarding the identities of the δ^V's involved). Often elements of $(\mathcal{A}_n^V)^\star$ are referred to as "weight systems". A more accurate name would be "once-integrable weight systems".

We see that it is necessary to understand $\ker \delta^V$. In Figure 2 we show a family of members of $\ker \delta^V$, the "Topological 4-Term" ($T4T$) relations. Figure 3 explains how they arise from "lassoing a singular point". Figure 4 shows another family of members of $\ker \delta^V$. The following theorem says that this is all:

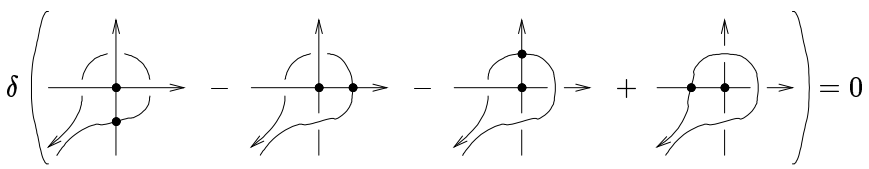


Figure 2: A Topological 4-Term ($T4T$) relation. Each of the four graphics in the picture represents a part of an n-singular knot (so there are $n-2$ additional singular points not shown), and, as usual in knot theory, the 4 singular knots in the equation are the same outside the region shown.

Theorem 2 (Stanford [5]) *The $T4T$ relations of Figure 2 and the TFI relations of Figure 4 span* $\ker \delta^V$. □

Pushing the $T4T$ and the TFI relations down to the level of chord diagrams, we get the well-known $4T$ and FI relations, which span $\pi(\ker \delta^V)$: (see e.g. [1])

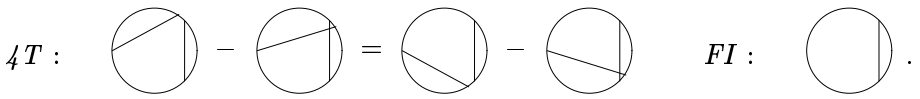


Figure 3: Lassoing a singular point: Each of the graphics represents an $(n-1)$-singular knot, but only one of the singularities is explicitly displayed. Start from the left-most graphic, pull the "lasso" under the displayed singular point, "lasso" the singular point by crossing each of the four arcs emanating from it one at a time, and pull the lasso back out, returning to the initial position. Each time an arc is crossed, the difference between "before" and "after" is the δ^V applied to an n-singular knot (up to signs). The four n-singular knot thus obtained are the ones making the Topological 4-Term relation, and δ^V applied to their signed sum is the difference between the first and the last $(n-1)$-singular knot shown in this figure; namely, it is 0.

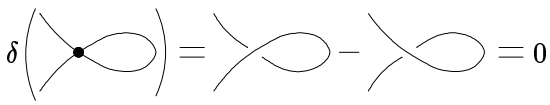


Figure 4: A Topological Framing Independence Relation (*TFI*)

We thus find that $\mathcal{A}_n^V = (\text{chord diagrams})/(4T \text{ and } FI \text{ relations})$, as usual in the theory of Vassiliev finite type invariants of knots.

The Fundamental Theorem of Finite Type Invariants, due to Kontsevich [4], asserts that (at least over $\mathbb{Q}$) this is indeed all: For every $W \in (\mathcal{A}_n^V)^\star$ there exists a type n invariant I with $W = \partial_V^n I$. In other words, every once-integrable weight system is fully integrable.

3 The Goussarov definition on its own

The purpose of this section is to tell the parallel story for the Goussarov theory of finite type invariants. Much of the mathematical content of this section is independent of that of the previous one. But we choose not to repeat the formal parts of the story, and to concentrate only on the "new stuff". Thus this section cannot be read independently.

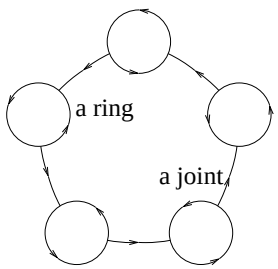
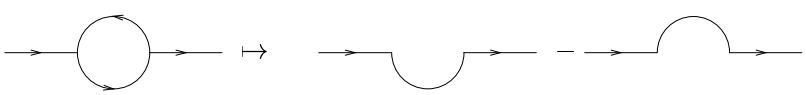


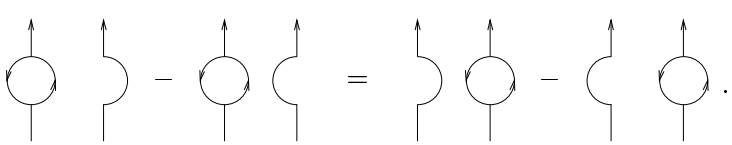
Figure 5: A 5-bracelet

In the Goussarov theory, what replaces the space $\mathcal{K}_n^V$ of formal linear combinations of n-singular knots (modulo differentiability) is the space $\mathcal{K}_n^G$ of formal linear combinations of knotted n-bracelets (modulo differentiability, defined later). A knotted n-bracelet is an embedding up to isotopy in $\mathbb{R}^3$ of an n-bracelet: a directed graph made of n rings and n joints. An example of a 5-bracelet appears in Figure 5. Figure 1 on page 2 can be made into an example of a knotted 2-bracelet by turning the dashed lines into solid lines and adding orientations in an appropriate manner.

The replacement for δ^V is the map $\delta^G = \delta^G_{n+1} : \mathcal{K}^G_{n+1} \to \mathcal{K}^G_n$ defined by

$\delta^G :$  . (2)

The differentiability relation is the minimal relation which makes δ^G well defined:



We let the derivative ∂_G be the adjoint of δ^G, and just as in the Vassiliev theory, we can now define finite type invariants:

Definition 3.1 A knot invariant I (equivalently, a linear functional on $\mathcal{K} = \mathcal{K}_0^G$) is of Goussarov type n if its $(n+1)$-st (Goussarov style) derivative vanishes, that is, if $(\partial_G)^{n+1} I \equiv 0$.

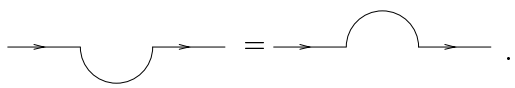
Just like in the Vassiliev case, we have ladders

$$\begin{array}{ccccccccc}
\dots \longrightarrow & \mathcal{K}^G_{n+1} & \xrightarrow{\delta^G} & \mathcal{K}^G_n & \xrightarrow{\delta^G} & \mathcal{K}^G_{n-1} & \longrightarrow \dots & \xrightarrow{\delta^G} & \mathcal{K}^G_0 = \mathcal{K} \\
\dots \longleftarrow & (\mathcal{K}^G_{n+1})^\star & \xleftarrow{\partial_G} & (\mathcal{K}^G_n)^\star & \xleftarrow{\partial_G} & (\mathcal{K}^G_{n-1})^\star & \longleftarrow \dots & \xleftarrow{\partial_G} & (\mathcal{K}^G_0)^\star = \mathcal{K}^\star. \\
 & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & & & & \rotatebox{90}{$\in$} \\
 & \partial^{n+1}_G I \equiv 0 & & \partial^n_G I = W & & & & & I
\end{array} \tag{3}$$

For the same reasons as in the Vassiliev case we are lead to be interested in the space $\mathcal{K}^G_n/\delta^G\mathcal{K}^G_{n+1}$. This is the space on which "(Goussarov style) weight systems" are defined, and it is the parallel of the space of chord diagrams in the Vassiliev case:

Proposition 3.2 *The space $\mathcal{K}^G_n/\delta^G\mathcal{K}^G_{n+1}$ is canonically isomorphic to the space $\mathcal{D}^G_n$ of formal linear combinations of "cyclically ordered n-component links", which are simply n-component links along with a cyclic order on their components.*

Proof Dividing by $\delta^G\mathcal{K}^G_{n+1}$ is the same as imposing the equality



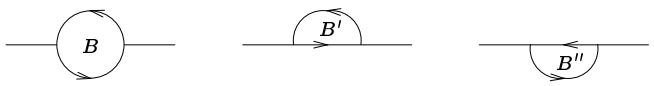
In English, this equality reads "it doesn't matter how joints are embedded, they can be moved modulo $\delta^G\mathcal{K}^G_{n+1}$". So what remains modulo $\delta^G\mathcal{K}^G_{n+1}$ is just the manner in which the rings are knotted. But this is precisely a cyclically ordered n-component link. □

In the case of the Vassiliev theory, we saw that $\mathcal{K}^V_n/(\ker\delta^V + \operatorname{im}\delta^V)$ is the the famed space $\mathcal{A}^V_n$ of chord diagrams modulo $4T$ and FI relations, whose dual is the space of weight systems. To see what we get in the Goussarov theory, we first have to understand $\ker\delta^G$.

Here are three families of elements in $\ker\delta^G$:

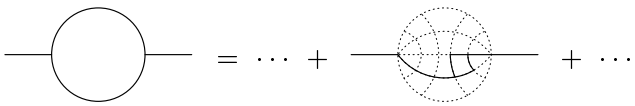
(1) Let B be a bracelet that has an 'empty ring' — a ring that bounds an embedded disk that does not intersect any other ring or joint. Then $B \in \ker\delta^G$. (Indeed, if a ring is empty then its two resolutions as in Equation (2) are isotopic).

(2) Let B be a bracelet and let B' be the bracelet obtained from B by reversing the orientation of one of the rings. Then $B + B' \in \ker \delta^G$. (No words needed).

(3) 

Let B, B' and B'' be bracelets related as above. (To be specific: All rings and joints may be knotted, including the parts drawn above. The parts not shown must be knotted in the same way for B, B' and B''. And finally, apart from orientations any two of B, B' and B'' share a "half-ring".) Then $B' + B'' - B \in \ker \delta^G$. (No words needed).

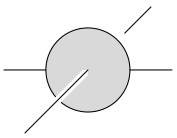
Let *lin* be the span of these three families within $\ker \delta^G$. The rationale for this name is that modulo *lin*, bracelets become "multi-linear" in "the span of their rings" (with the third type of elements, for example, becoming "additivity relations"). Anyway, in $\mathcal{K}_n^G/lin$ we can use this "linearity" repeatedly (and also some isotopies) to subdivide the span of rings to tiny pieces that contain very little:



(4)

Hence $\mathcal{K}_n^G/lin$ is spanned by a rather simple type of bracelets:

Definition 3.3 We say that a bracelet has simple rings if all of its rings bound embedded disks whose interior intersects the bracelet transversely and exactly once. (See an example of a simple ring on the right).



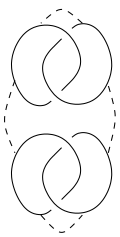
Proposition 3.4 *The space $\mathcal{K}_n^G/lin$ is spanned by bracelets with simple rings.*

Proof Let B be an n-bracelet. Find n immersed disks whose boundaries are the rings of B so that there are no triple intersections between them (this is easy; you can even arrange those n disks to have at most clasp intersections). Now subdivide all of those disks to pieces of uniform small size as in Equation (4) (make those subdivisions sufficiently generic so that the different mesh lines do not intersect each other and/or the joints). If the pieces are small enough, they must be empty (and hence zero mod *lin*) or at most one thing may cut through any given piece. □

It is time for the chord diagrams of the Vassiliev theory to make their appearance in the Goussarov theory:

Proposition 3.5 *For even n, the space $\mathcal{K}_n^G/(lin + \operatorname{im}\delta^G)$ (which is still bigger than the desired $\mathcal{K}_n^G/(\ker\delta^G + \operatorname{im}\delta^G)$) is isomorphic to the space of $\frac{n}{2}$-chord diagrams. For odd n the space $\mathcal{K}_n^G/(lin + \operatorname{im}\delta^G)$ is empty.*

Proof By Proposition 3.4 we can reduce to bracelets with simple rings and as in Proposition 3.2 we may forget their joints. What remains is cyclically ordered n component links, each of whose components is "simple", meaning that it forms a Hopf link with another component, and there's no further knotting or linking. For odd n, such pairing of the components is impossible. For even n we have a cyclically ordered set of size n (the components) whose elements are paired up. This is exactly a chord diagram with n vertices and $\frac{n}{2}$ chords.



As an example, the figure on the left shows a bracelet with simple rings whose corresponding chord diagram is . As appropriate when moding out by $\operatorname{im}\delta^G$, the joints appear "transparent".

One still needs to show that "Hopf pair bracelets" such as the one on the left, which directly correspond to chord diagrams, do not get killed or identified with each other by *lin*. This can be done by noting that appropriate products of linking numbers of rings detect Hopf pair bracelets and annihilate *lin*. We leave the details to the reader. □

There are two further families of elements in $\ker\delta^G$, the *G4T* elements and the *GFI* elements, shown in Figure 6. We leave it to our readers to verify that modulo $\operatorname{im}\delta^G$ these elements become the *4T* and the *FI* relations between chord diagrams:

Proposition 3.6 *For even n, the space $\mathcal{K}_n^G/(lin + G4T + GFI + \operatorname{im}\delta^G)$ is isomorphic to the space $\mathcal{A}_{n/2}^V$ of the Vassiliev theory. For odd n the space $\mathcal{K}_n^G/(lin + G4T + GFI + \operatorname{im}\delta^G)$ is empty.* □

Remark 3.7 In the light of the equivalence of the Goussarov theory and the Vassiliev theory (shown in the next section), it is clear that $lin + G4T + GFI + \operatorname{im}\delta^G = \ker\delta^G + \operatorname{im}\delta^G$, at least over $\mathbb{Q}$. I do not know if $lin + G4T + GFI = \ker\delta^G$.

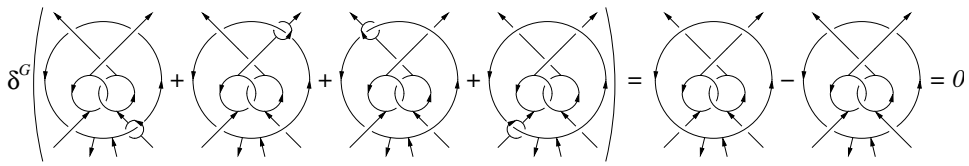
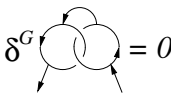
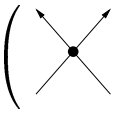


Figure 6: The $G4T$ family of elements of $\ker \delta^G$ (above) and the GFI family of elements of $\ker \delta^G$ (right).



4 The equivalence of the two definitions

As stated (in a slightly different form) in the introduction, the key to the proof of Theorem 1 is the (informal) equality

δ^V  $= (\delta^G)^2$

Let us turn this into a precise argument:

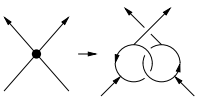
Proof of Theorem 1 An invariant I is of Vassiliev type n if it vanishes on $(\delta^V)^{n+1}(\mathcal{K}^V_{n+1})$ and is of Goussarov type $2n$ (respectively $2n+1$) if it vanishes on $(\delta^G)^{2n+1}(\mathcal{K}^G_{2n+1})$ (respectively $(\delta^G)^{2n+2}(\mathcal{K}^G_{2n+2})$). Thus we need to prove that

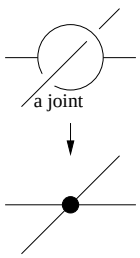
$$(\delta^G)^{2n+1}(\mathcal{K}^G_{2n+1}) \subset (\delta^V)^{n+1}(\mathcal{K}^V_{n+1}) \tag{5}$$

and that

$$(\delta^V)^{n+1}(\mathcal{K}^V_{n+1}) \subset (\delta^G)^{2n+2}(\mathcal{K}^G_{2n+2}). \tag{6}$$

The easier part is the proof of (6). Let $K \in \mathcal{K}^V_{n+1}$ be an $(n+1)$-singular knot, and let $B_K \in \mathcal{K}^G_{2n+2}$ be the $(2n+2)$-bracelet obtained from K by replacing every singular point with a pair of rings using the rule on the right. It is clear that $(\delta^V)^{n+1}(K) = (\delta^G)^{2n+2}(B_K)$, and as K was arbitrary, this proves (6).





Let us now prove (5). Let $B \in \mathcal{K}^G_{2n+1}$ be a $(2n+1)$-bracelet. We need to show that $(\delta^G)^{2n+1}(B)$ is in $(\delta^V)^{n+1}(\mathcal{K}^V_{n+1})$. Clearly it does not matter if we modify B by adding to it elements in $\ker \delta^G$, so using Proposition 3.4 we may assume that B has simple rings. A simple ring may loop around a joint or it may be Hopf-linked with another simple ring. In the former case, apply the rule on the left. In the latter case, apply the reverse of the rule in the first half of the proof. Doing so to all rings we get a singular knot K_B that has at least $n+1$ singularities (every ring in B contributes either 1 singularity or $\frac{1}{2}$ singularity, and B has $2n+1$ rings). If K_B has m singularities (with $m \geq n+1$), we have $(\delta^G)^{2n+1}(B) = (\delta^V)^m(K_B) \in (\delta^V)^m(\mathcal{K}^V_m) \subset (\delta^V)^{n+1}(\mathcal{K}^V_{n+1})$. □

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