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Loop spaces of configuration spaces and finite type invariants

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Abstract The total homology of the loop space of the configuration space of ordered distinct n points in $\mathbf{R}^m$ has a structure of a Hopf algebra defined by the 4-term relations if $m \geq 3$. We describe a relation of between the cohomology of this loop space and the set of finite type invariants for the pure braid group with n strands. Based on this we give expressions of certain link invariants as integrals over cycles of the above loop space.

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Dedicated to Professor Mitsuyoshi Kato on his sixtieth birthday

1 Introduction

The purpose of this article is to present a new approach for finite type invariants of braids based on the loop spaces of configuration spaces. For a smooth manifold M with a base point x_0 we denote by ΩM the loop space consisting of the piecewise smooth loops $\gamma : I \to M$ with $\gamma(0) = \gamma(1) = x_0$. The main object is the loop space $\Omega \mathrm{Conf}_n(\mathbf{R}^m)$ where $\mathrm{Conf}_n(\mathbf{R}^m)$ stands for the configuration space of ordered distinct n points in $\mathbf{R}^m$. It follows from the work of Cohen and Gitler [6] that, in the case $m \geq 3$, the total homology of the loop space $\Omega \mathrm{Conf}_n(\mathbf{R}^m)$ has a structure of the universal enveloping algebra of a Lie algebra defined by 4-term relations in the graded sense. Based on this one can show that the space of finite type invariants for pure braids is naturally isomorphic to the total cohomology of $\mathrm{Conf}_n(\mathbf{R}^m)$ if $m \geq 3$. The total homology of the loop space $\Omega \mathrm{Conf}_n(\mathbf{R}^m)$ has a uniform structure for $m \geq 3$ except a shift of grading. In this article, we mainly deal with the case $m = 3$. From our point of view the space of weight systems for Vassiliev invariants for knots in the sense of [13] lies in the direct sum $\bigoplus_{n \geq 2} H^*(\Omega \mathrm{Conf}_n(\mathbf{R}^m))$. The de Rham cohomology of $\Omega\mathrm{Conf}_n(\mathbf{R}^m)$ is described by Chen's iterated integrals of Green forms. We show that a certain link invariant can be expressed as an integral of a differential form on $\Omega\mathrm{Conf}_n(\mathbf{R}^m)$ over a cycle defined associated with a link. It is still work in progress to obtain a general formula for any finite type invariant. Let us recall that based on the work of Guadagnini, Martellini and Mintchev [8] for Chern-Simons perturbative theory, Bott and Taubes developed a method to express a knot invariant by means of iterated integrals of Green forms associated with Feynman diagrams. In this approach one needs to add integrals associated with graphs with trivalent vertices to obtain a knot invariant. Our method does not use trivalent vertices, but we add integrals of some non closed differential forms on the loop spaces of configuration spaces to obtain topological invariants.

The paper is organized in the following way. In Section 2, we give a description of the homology of the loop space $\Omega\mathrm{Conf}_n(\mathbf{R}^m)$ by means of the graded 4-term relations. In Section 3, we briefly review Chen's iterated integrals describing the de Rham cohomology of loop spaces. In Section 4, we discuss a relation between the above homology and finite type invariants for braids. In Section 5, we express certain link invariants in terms of an integral of a differential form on the loop space of the configuration spaces. Section 6 is devoted to a brief overview of a work in progress in [7] on the homology of the loop space of orbit configuration spaces associated to Fuchsian groups. A more detailed account of this subject will appear elsewhere.

2 Homology of the loop spaces

We denote by $\mathrm{Conf}_n(X)$ the configuration space of ordered distinct n points in a space X. Namely, we set

$$\mathrm{Conf}_n(X) = \{(x_1, \cdots, x_n) \in X^n \ ; \ x_i \neq x_j \text{ if } i \neq j\}.$$

Let Δ_{ij} be the diagonal set of X^n defined by $x_i = x_j$. We will deal with the loop space $\Omega\mathrm{Conf}_n(X)$ of the configuration space $\mathrm{Conf}_n(X)$ with a fixed base point x_0.

For the rest of this section we consider the case $X = \mathbf{R}^m$ with $m \geq 3$. The boundary of a tubular neighbourhood of the diagonal set Δ_{ij} is identified with the tangent sphere bundle of X and we have a map

$$\gamma_{ij} : S^{m-1} \to \mathrm{Conf}_n(\mathbf{R}^m), \ 1 \leq i < j \leq n,$$

so that $\gamma_{ij}(S^{m-1})$ has the linking number 1 with the diagonal set Δ_{ij}. Since S^{m-1} is considered to be the suspension of its equator S^{m-2}, we have a natural map

$$\alpha_{ij} : S^{m-2} \to \Omega\mathrm{Conf}_n(\mathbf{R}^m),\ 1 \leq i < j \leq n,$$

induced from γ_{ij}.

First we describe the cohomology ring of the configuration space $\mathrm{Conf}_n(\mathbf{R}^m)$. Let ω be a homogeneous $(m-1)$-form on $\mathbf{R}^m \setminus \{\mathbf{0}\}$ defining a standard volume form of the unit sphere S^{m-1}. Namely, ω satisfies

$$\int_{S^{m-1}} \omega = 1,$$
$$\omega(\lambda\mathbf{x}) = (\mathrm{sgn}\ \lambda)^m \omega(\mathbf{x})$$

for $\mathbf{x} \in \mathbf{R}^m \setminus \{\mathbf{0}\}$ and $\lambda \in \mathbf{R} \setminus \{\mathbf{0}\}$. Here sgn λ stands for $\lambda/|\lambda|$. We define the Green form ω_{ij} by

$$\omega_{ij}(\mathbf{x}_1, \cdots, \mathbf{x}_n) = \omega(\mathbf{x}_j - \mathbf{x}_i), \quad \mathbf{x}_1, \cdots, \mathbf{x}_n \in \mathbf{R}^m,$$

which is an $(m-1)$-form on $\mathrm{Conf}_n(\mathbf{R}^m)$ for $i \neq j$. The de Rham cohomology class $[\omega_{ij}]$ defines an integral cohomology class and is dual to the homology class $[\gamma_{ij}]$. We put $\xi_{ij} = [\omega_{ij}]$. It is known that the cohomology ring $H^*(\mathrm{Conf}_n(\mathbf{R}^m); \mathbf{Z})$ is generated by ξ_{ij}, $1 \leq i \neq j \leq n$, with relations

$$\begin{aligned}
&\xi_{ij}^2 = 0\\
&\xi_{ij} = (-1)^m \xi_{ji}\\
&\xi_{ij}\xi_{jk} + \xi_{jk}\xi_{ik} + \xi_{ik}\xi_{ij} = 0,\ i < j < k
\end{aligned}$$

where deg $\omega_{ij} = m - 1$ (see [5]).

In general, the total homology

$$H_*(\Omega M; \mathbf{Z}) = \bigoplus_{j \geq 0} H_j(\Omega M; \mathbf{Z})$$

of the loop space of M is equipped with a product

$$H_i(\Omega M; \mathbf{Z}) \otimes H_j(\Omega M; \mathbf{Z}) \to H_{i+j}(\Omega M; \mathbf{Z})$$

induced from the composition of loops. We also have a coproduct

$$H_k(\Omega M; \mathbf{Z}) \to \oplus_{i+j=k} H_i(\Omega M; \mathbf{Z}) \otimes H_j(\Omega M; \mathbf{Z})$$

induced from the cup product homomorphism on cochains. Let us now investigate the total homology of the loop space $\Omega\mathrm{Conf}_n(\mathbf{R}^m)$ in the case $m \geq 3$ as a Hopf algebra with the above product and coproduct.

Let us first consider the simplest example $\Omega\mathrm{Conf}_2(\mathbf{R}^m)$. Here the configuration space $\mathrm{Conf}_2(\mathbf{R}^m)$ is homotopy equivalent to S^{m-1}. The structure of the total homology of the loop space of a sphere was determined by Bott and Samelson [2] (see also [4]). We have isomorphisms of Hopf algebras

$$H_*(\Omega\mathrm{Conf}_2(\mathbf{R}^m);\mathbf{Z}) \cong H_*(\Omega S^{m-2};\mathbf{Z}) \cong \mathbf{Z}[X_{12}]$$

where $\mathbf{Z}[X_{12}]$ stands for the polynomial algebra with one indeterminate with $\deg X_{12} = m-2$ and X_{12} corresponds to the homology class represented by α_{12} defined above.

In general we have relations among the homology classes $X_{ij} = [\alpha_{ij}]$ analogous to the 4-term relations. We set $X_{ij} = (-1)^{m-2}X_{ji}$ for $i > j$.

Proposition 2.1 *In $H_*(\Omega\mathrm{Conf}_n(\mathbf{R}^m);\mathbf{Z})$, the homology classes X_{ij} satisfy the relation*

$$[X_{ij}, X_{ik} + X_{jk}] = 0$$

for $i < j < k$. Here $\deg X_{ij} = m-2$ and we consider the Lie bracket in the graded sense.

Proof We fix i, j and k with $1 \le i < j < k \le n$ and define

$$\varphi : S^{m-1} \times S^{m-1} \to \mathrm{Conf}_n(\mathbf{R}^m)$$

in the following way. We take $\mathbf{u} \in \mathbf{R}^m$ with $\|\mathbf{u}\| = 1$ and fix r_1 and r_2 such that $0 < r_1 < r_2 < 1$. For $\mathbf{x}_1, \mathbf{x}_2 \in S^{m-2}$ and $1 \le l \le n$ we set

$$\varphi_l(\mathbf{x}_1, \mathbf{x}_2) = \begin{cases} l\mathbf{u}, & l \ne j, k \\ i\mathbf{u} + r_1\mathbf{x}_1, & l = j \\ i\mathbf{u} + r_2\mathbf{x}_2, & l = k \end{cases}$$

and φ is defined to be $\varphi(\mathbf{x}_1, \mathbf{x}_2) = (\varphi_1(\mathbf{x}_1, \mathbf{x}_2), \cdots, \varphi_n(\mathbf{x}_1, \mathbf{x}_2))$. We denote by $\alpha \in H_{m-1}(S^{m-1};\mathbf{Z})$ the fundamental homology class. Let us notice that α determines a generator $\widehat{\alpha}$ of the total homology of the loop space $H_*(\Omega S^{m-1};\mathbf{Z})$. We consider the induced homomorphism

$$\varphi_* : H_{m-1}(S^{m-1} \times S^{m-1};\mathbf{Z}) \to H_{m-1}(\mathrm{Conf}_n(\mathbf{R}^m);\mathbf{Z}).$$

Then we have

$$\langle \xi_{ij}, \varphi_*(\alpha \times 1)\rangle = 1,\ \langle \xi_{ik}, \varphi_*(\alpha \times 1)\rangle = 0,\ \langle \xi_{jk}, \varphi_*(\alpha \times 1)\rangle = 0$$

and

$$\langle \xi_{ij}, \varphi_*(1 \times \alpha)\rangle = 0,\ \langle \xi_{ik}, \varphi_*(1 \times \alpha)\rangle = 1,\ \langle \xi_{jk}, \varphi_*(1 \times \alpha)\rangle = 1,$$

which implies

$$\varphi_*(\alpha \times 1) = \alpha_{ij}, \quad \varphi_*(1 \times \alpha) = \alpha_{ik} + \alpha_{jk}.$$

The map φ gives

$$\Omega\varphi : \Omega(S^{m-1} \times S^{m-1}) \to \Omega\mathrm{Conf}_n(\mathbf{R}^m).$$

Let us consider the induced homomorphism

$$\Omega\varphi_* : H_*(\Omega(S^{m-1} \times S^{m-1}); \mathbf{Z}) \to H_*(\Omega\mathrm{Conf}_n(\mathbf{R}^m); \mathbf{Z}).$$

The homology classes $\alpha \times 1$ and $1 \times \alpha$ determine the homology classes, say $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$, in $H_{m-2}(\Omega(S^{m-1} \times S^{m-1}))$. The total homology $H_*(\Omega(S^{m-1} \times S^{m-1}))$ is isomorphic to the graded commutative polynomial ring $\mathbf{Z}[x_1, x_2]$ with $\deg x_1 = \deg x_2 = m-2$. Here x_1 and x_2 correspond to $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$ respectively. We have $[x_1, x_2] = 0$ where the bracket is defined by $[x_1, x_2] = x_1x_2 - (-1)^m x_2 x_1$. This implies the relation $[X_{ij}, X_{ik} + X_{jk}] = 0$. □

The structure of $H_*(\Omega\mathrm{Conf}_n(\mathbf{R}^m); \mathbf{Z})$ as a Hopf algebra for $m \geq 3$ was determined by Cohen and Gitler [6]. We denote by $L_n(m)$ the graded free Lie algebra over $\mathbf{Z}$ generated by X_{ij}, $1 \leq i \neq j \leq n$, where the degree for X_{ij} is $m-2$. Let $\mathcal{T}$ denote the ideal of $L_n(m)$ generated by $X_{ij} - (-1)^m X_{ji}$ together with

$$\begin{aligned} &[X_{ij}, X_{ik} + X_{jk}], \quad i < j < k, \\ &[X_{ij}, X_{kl}], \quad i, j, k, l \text{ distinct} \end{aligned}$$

We define the Lie algebra $\mathcal{G}_n(m)$ by

$$\mathcal{G}_n(m) = L_n(m)/\mathcal{T}$$

and denote its universal enveloping algebra by $U\mathcal{G}_n(m)$. The following theorem is shown inductively by means of the fibration

$$\Omega\mathrm{Conf}_{n+1}(\mathbf{R}^m) \to \Omega\mathrm{Conf}_n(\mathbf{R}^m).$$

Theorem 2.2 [6] *If $m \geq 3$, then we have an isomorphism of graded Hopf algebras*

$$H_*(\Omega\mathrm{Conf}_n(\mathbf{R}^m); \mathbf{Z}) \cong U\mathcal{G}_n(m).$$

3 Chen's iterated integrals

Our next object is to describe the de Rham cohomology of the loop space $\Omega\mathrm{Conf}_n(\mathbf{R}^m)$. For this purpose we briefly review Chen's work on the de Rham cohomology of the loop space. Let $\omega_1, \cdots, \omega_q$ be differential forms on a smooth manifold M. Let Δ_q be the q-simplex defined by

$$\Delta_q = \{(t_1, \cdots, t_q) \ ; \ 0 \le t_1 \le \cdots \le t_q \le 1\}.$$

We have an evaluation map

$$\phi : \Delta_q \times \Omega M \to M^q$$

defined by

$$\phi(t_1, \cdots t_q; \gamma) = (\gamma(t_1), \cdots \gamma(t_q)).$$

Chen's iterated integral of the differential forms $\omega_1, \cdots, \omega_q$ along the path γ is by definition

$$\int_{\Delta_q} \phi^*(\omega_1 \times \cdots \times \omega_q)$$

Following Chen, we denote the above integral by

$$\int \omega_1 \cdots \omega_q.$$

Let p be the sum of the degrees of ω_j for $1 \le j \le q$. The iterated integral $\int \omega_1 \cdots \omega_q$ is considered to be a differential form of degree $p - q$ on the loop space ΩM. Let $B^{p,-q}(M)$ be the vector space over $\mathbf{R}$ spanned by the iterated integrals of the form $\int \omega_1 \cdots \omega_q$ where the sum of the degrees of ω_j, $1 \le j \le q$, is equal to p. As a differential form on the loop space ΩM

$$d \int \omega_1 \cdots \omega_q$$

is expressed up to sign the sum of

$$\int_{\Delta_q} \phi^* d(\omega_1 \times \cdots \times \omega_q)$$

and

$$\int_{\partial \Delta_q} \phi^*(\omega_1 \times \cdots \times \omega_q).$$

Thus we obtain the two differentials

$$d_1 : B^{p,-q}(M) \to B^{p+1,-q}(M), \quad d_2 : B^{p,-q}(M) \to B^{p,-q+1}(M).$$

The direct sum

$$\bigoplus_{p,q} B^{p,-q}(M)$$

has a structure of a double complex by the differentials d_1 and d_2. The associated total complex $B^\bullet(M)$ is a subcomplex of the de Rham complex of the loop space ΩM.

A basic result due to Chen [4] is formulated as follows. Let us suppose that the manifold M is simply connected. Then one has an isomorphism

$$H^j(\Omega M; \mathbf{R}) \cong H^j(B^\bullet(M))$$

for any j.

In the case when M is not simply connected, the fundamental group of M is related to the 0-dimensional cohomology of the bar complex $B^\bullet(M)$ in the following way. Each element of $H^0(B^\bullet(M))$ is represented by a linear combination of iterated integrals of 1-forms which is a function on the loop space ΩM depending only on the homotopy class of a loop. Thus we have a natural evaluation map

$$\pi_1(M, x_0) \times H^0(B^\bullet(M)) \to \mathbf{R},$$

which induces a bilinear pairing

$$\mathbf{R}\pi_1(M, x_0) \times H^0(B^\bullet(M)) \to \mathbf{R}.$$

Here $\mathbf{R}\pi_1(M, x_0)$ stands for the group ring of $\pi_1(M, x_0)$ over $\mathbf{R}$. Consider the increasing filtration

$$\mathbf{R} = \mathcal{F}^0 B^\bullet(M) \subset \mathcal{F}^1 B^\bullet(M) \subset \cdots \subset \mathcal{F}^k B^\bullet(M) \subset \mathcal{F}^{k+1} B^\bullet(M) \subset \cdots$$

defined by

$$\mathcal{F}^k B^\bullet(M) = \bigoplus_{q \leq k} B^{p,-q}(M).$$

This induces the increasing filtration

$$\mathcal{F}^k H^0(B^\bullet(M)) \subset \mathcal{F}^{k+1} H^0(B^\bullet(M))$$

on $H^0(B^\bullet(M))$. We denote by $\mathcal{I}$ the augmentation ideal of $\mathbf{R}\pi_1(M, x_0)$. It can be shown that for $\omega \in \mathcal{F}^k H^0(B^\bullet(M))$ the associated evaluation map

$$\mathbf{R}\pi_1(M, x_0) \to \mathbf{R}$$

factors through $\mathcal{I}^{k+1}$ and we obtain a bilinear map

$$\mathbf{R}\pi_1(M, x_0)/\mathcal{I}^{k+1} \times \mathcal{F}^k B^\bullet(M) \to \mathbf{R}.$$

It was shown by Chen (see [4]) that we have an isomorphism

$$\mathcal{F}^k H^0(B^\bullet(M)) \cong \mathrm{Hom}_{\mathbf{R}}(\mathbf{R}\pi_1(M, x_0)/\mathcal{I}^{k+1}, \mathbf{R}).$$

4 Finite type invariants for braids

The notion of finite type invariants for braids can be formulated by means of the group ring in the following way. We denote by P_n the pure braid group with n strands. As in the previous section we donote by $\mathcal{I}$ the augmentation ideal of the group ring $\mathbf{Z}P_n$. An invariant $v : P_n \to \mathbf{Z}$ is said to be of order k if the induced map $v : \mathbf{Z}P_n \to \mathbf{Z}$ factors through $\mathcal{I}^{k+1}$. The set of order k invariants for P_n with values in $\mathbf{Z}$ has a structure of a $\mathbf{Z}$-module and is identified with

$$\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}P_n/\mathcal{I}^{k+1}, \mathbf{Z})$$

and we denote it by $V_k(P_n)$. We have a natural inclusion $V_k(P_n) \subset V_{k+1}(P_n)$ and we set $V(P_n) = \bigcup_{k=0}^{\infty} V_k(P_n)$. We call $V(P_n)$ the space of finite type invariants for P_n with values in $\mathbf{Z}$. The above notion of finite type invariants for P_n is naturally generalized for B_n, where B_n stands for the braid group with n strands. We consider the ideal $\mathcal{J}$ in the group ring $\mathbf{Z}B_n$ generated by $\sigma_i - \sigma_i^{-1}$, for a standard system of generators σ_i, $1 \leq i \leq n-1$, for the braid group. Now an invariant $v : B_n \to \mathbf{Z}$ is defined to be of order k if v vanishies on $\mathcal{J}^{k+1}$. We denote by $V_k(B_n)$ the set of order k invariants for B_n with values in $\mathbf{Z}$ and we set $V(B_n) = \bigcup_{k=0}^{\infty} V_k(B_n)$.

Considering P_n as the fundamental group of the configuration space $\mathrm{Conf}_n(\mathbf{C})$ we readily obtain the isomorphism

$$\mathcal{F}^k H^0(B^\bullet) \cong V_k(P_n) \otimes \mathbf{R}$$

using Chen's theorem for fundamental groups (see [10]). The above isomorphism provides the expression of finite type invariants for braids in terms of iterated integrals of logarithmic forms. This is a prototype of the Kontsevich integral in [12].

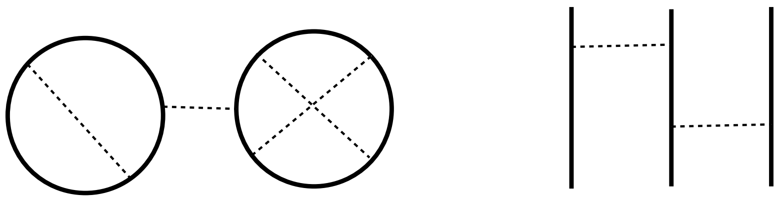

Figure 1: A chord diagrams and a horizontal chord diagram

Let us discuss a relation to the algebra of horizontal chord diagrams. First, we recall that a chord diagram is a collection of finitely many oriented circles with finitely many chords attached on them, regarded up to orientation preserving diffeomorphisms of the circles (see Figure 1). Here we assume that the endpoints of the chords are distinct and lie on the circles. Let $I_1 \sqcup \cdots \sqcup I_n$ be the disjoint union of n unit intervals. We fix a parameter by a map

$$p_j : [0,1] \to I_j$$

for each I_j, $1 \le j \le n$. A horizontal chord diagram on n strands with k chords is a trivalent graph constructed in the following way. We fix $t_1, \cdots, t_k \in [0,1]$ such that $0 < t_1 < \cdots < t_k < 1$. Let

$$(i_1, j_i), (i_2, j_2), \cdots, (i_k, j_k)$$

be pairs of distinct integers such that $1 \le i_p \le n$, $1 \le j_p \le n$, $p = 1, 2, \cdots, n$. We take k copies of unit intervals $C_1, C_2, \cdots, C_k$ and attach each C_ν to $I_1 \sqcup \cdots \sqcup I_n$ in such a way that it starts at $p_{i_\nu}(t_\nu)$ and ends at $p_{j_\nu}(t_\nu)$ for $1 \le \nu \le k$. In this way we obtain a graph with n strands $I_1 \sqcup \cdots \sqcup I_n$ and chords $C_1, \cdots, C_k$ attached to them. Such a graph is called a horizontal chord diagram on n strands with k chords. We consider chord diagrams up to orientation preserving homeomorphism. Let D_n^k denote the free $\mathbf{Z}$-module spanned by horizontal chord diagrams on n strands with k chords. We define Γ_{ij} as the horizontal chord diagram on n strands with one chord C_{ij} defined by the pair (i,j), $1 \le i < j \le n$. Then

$$D_n = \bigoplus_{k \ge 0} D_n^k$$

has a structure of an algebra generated by Γ_{ij}, where the product is defined by the concatenation of horizontal chord diagrams.

As is the case of the chord diagrams for knots, we have a natural $\mathbf{Z}$ module homomorpshim

$$w : V_k(P_n) \to \mathrm{Hom}_{\mathbf{Z}}(D_n^k, \mathbf{Z})$$

where $w(v)$ is called the weight system for $v \in V_k(P_n)$. It can be shown that $w(v)$ vanishes on the ideal $\mathcal{T}$ in D_n generated by

$$\begin{aligned} &[\Gamma_{ij}, \Gamma_{ik} + \Gamma_{jk}], \quad [\Gamma_{ij} + \Gamma_{ik}, \Gamma_{jk}] \quad i < j < k, \\ &[\Gamma_{ij}, \Gamma_{kl}], \quad i, j, k, l \text{ distinct.} \end{aligned}$$

We set $A_n = D_n/\mathcal{T}$ and $A_n^k = D_n^k / D_n^k \cap \mathcal{T}$. We can show that w induces an isomorphism of $\mathbf{Z}$ modules

$$V_k(P_n)/V_{k-1}(P_n) \cong \mathrm{Hom}_{\mathbf{Z}}(A_n^k, \mathbf{Z})$$

(see [10]).

The total homology $H_*(\Omega \mathrm{Conf}_n(\mathbf{R}^m); \mathbf{Z})$ is interpreted as the algebra of horizontal chord diagrams on n strands modulo the 4-term relations in the graded sense. The homology group $H_{m(k-2)}(\Omega\mathrm{Conf}_n(\mathbf{R}^m);\mathbf{Z})$ corresponds to the subspace consisting of k horizontal chords. The cohomology

$$H^{m(k-2)}(\Omega\mathrm{Conf}_n(\mathbf{R}^m);\mathbf{Z})$$

is identified with the space of weight systems on such horizontal chord diagrams with values in $\mathbf{Z}$. We obtain the following theorem (see also [6]).

Theorem 4.1 *For $m \geq 3$ we have an isomorphism of $\mathbf{Z}$-modules*

$$H^{k(m-2)}(\Omega\mathrm{Conf}_n(\mathbf{R}^m);\mathbf{Z}) \cong V_k(P_n)/V_{k-1}(P_n).$$

Let us suppose that m is even. Then we have an isomorphism of Hopf algebras

$$H^*(\Omega\mathrm{Conf}_n(\mathbf{R}^m);\mathbf{Z}) \cong V(P_n).$$

The above theorem is generalized to finite type invariants for the full braid group in the following way. The symmetric group $\mathcal{S}_n$ acts naturally on the cohomology ring

$$H^*(\Omega\mathrm{Conf}_n(\mathbf{R}^m);\mathbf{Z})$$

where the action is induced by the permutations of the n components of $\mathrm{Conf}_n(\mathbf{R}^m)$. Then we have an isomorphism

$$H^*(\Omega\mathrm{Conf}_n(\mathbf{R}^m);\mathbf{Z}) \bullet \mathbf{Z}[\mathcal{S}_n] \cong V(B_n)$$

where the left hand side stands for the semidirect product with respect to the above action (see [11]).

Let $\mathcal{A}$ be the vector space over $\mathbf{R}$ spanned by chord diagrams on a circle modulo the 4-term relations. The dual space

$$\mathcal{A}^* = \mathrm{Hom}_{\mathbf{R}}(\mathcal{A}, \mathbf{R})$$

is the space of weight systems for Vassiliev invariants of knots. We have a surjective map

$$c : \bigoplus_{n\geq 2} A_n \to \mathcal{A}$$

obtained by taking the closure of a horizontal chord diagram as in Figure 2. By means of this construction we have the following theorem.

Theorem 4.2 *We have an injective homomorphism of vector spaces*

$$\mathcal{A}^* \to \bigoplus_{n\geq 2} H^*(\Omega\mathrm{Conf}_n(\mathbf{R}^m);\mathbf{R}).$$

for $m \geq 3$.

Certain cohomology classes in $\bigoplus_{n\geq 2} H^*(\Omega\mathrm{Conf}_n(\mathbf{R}^m);\mathbf{R})$ play a role of weight systems for Vassiliev invariants. It would be an interesting problem to characterize geometrically such cohomology classes.

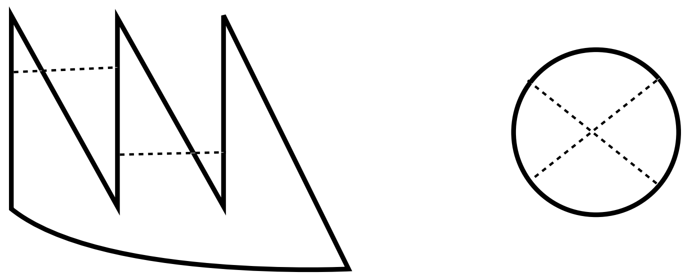

Figure 2: Closing a horizontal chord diagram

5 Integral representations

To explain the idea we interpret the Gauss linking number formula in terms of an integral on the loop space $\Omega\mathrm{Conf}_2(\mathbf{R}^3)$. Let

$$\phi : I \times \Omega\mathrm{Conf}_2(\mathbf{R}^3) \to \mathrm{Conf}_2(\mathbf{R}^3)$$

be the evaluation map defined by $\phi(t,\gamma) = \gamma(t)$. We pull back ω_{12} and integrate along the fibre of the projection map $p : I \times \Omega\mathrm{Conf}_2(\mathbf{R}^3) \to \Omega\mathrm{Conf}_2(\mathbf{R}^3)$ to obtain the 1-form

$$\int_I \phi^*\omega_{12}$$

defined on the loop space $\Omega\mathrm{Conf}_2(\mathbf{R}^3)$. This differential form on the loop space is denoted by $\int \omega_{12}$.

A two-component link $L = K_1 \cup K_2$ is given by a map

$$f : S^1 \times S^1 \to \mathrm{Conf}_2(\mathbf{R}^3).$$

Let us consider the induced map

$$\Omega f : \Omega(S^1 \times S^1) \to \Omega\mathrm{Conf}_2(\mathbf{R}^3).$$

A fundamental homology class of $S^1 \times S^1$ defined by a chain $\mu : I \times I \to S^1 \times S^1$ gives in a natural way a 1-chain $\widehat{\mu} : I \to \Omega(S^1 \times S^1)$ (see [4]). We set $\widehat{c}_f = \Omega f_*(\widehat{\mu})$ which is a 1-cycle of the loop space $\Omega\mathrm{Conf}_2(\mathbf{R}^3)$. Now we can express the linking number as

$$lk(K_1, K_2) = \int_{\widehat{c}_f} \int \omega_{12}.$$

Our next object is to describe the de Rham cohomology of the loop space

$$H^*_{DR}(\Omega\mathrm{Conf}_n(\mathbf{R}^m)).$$

We see that the configuration space $\mathrm{Conf}_n(\mathbf{R}^m)$ is simply connected when $m \geq 3$. Thus we can apply Chen's theorem to compute the de Rham cohomology of $\Omega\mathrm{Conf}_n(\mathbf{R}^m)$. Let us recall that the de Rham cohomology $H^*_{DR}(\Omega\mathrm{Conf}_n(\mathbf{R}^m))$ is generated by the Green forms ω_{ij}, $1 \leq i < j \leq n$. We have the following theorem.

Theorem 5.1 *If $m \geq 3$, then the de Rham cohomology*

$$H^{k(m-2)}_{DR}(\Omega\mathrm{Conf}_n(\mathbf{R}^m))$$

is represented by iterated integrals of the form

$$\sum a_{i_1 j_1 \cdots i_k j_k} \int \omega_{i_1 j_1} \cdots \omega_{i_k j_k} + (\textit{iterated integrals of length} < k)$$

where the coefficients $a_{i_1 j_1 \cdots i_k j_k} \in \mathbf{Z}$ satisfy the 4-term relations.

Here we say that $a_{i_1 j_1 \cdots i_k j_k}$ satisfy the 4-term relation if $X_{i_1 j_1 \cdots i_k j_k} \mapsto a_{i_1 j_1 \cdots i_k j_k}$ gives a well-defined $\mathbf{Z}$-module homomorphism $U\mathcal{G}(m) \to \mathbf{Z}$. In particular, in the case $n = 2$, the de Rham cohomology $H^*_{DR}(\Omega\mathrm{Conf}_2(\mathbf{R}^m))$ is spanned by the iterated integrals

$$1, \int \omega_{12}, \int \omega_{12}\omega_{12}, \cdots, \int \omega_{12} \cdots \omega_{12}, \cdots$$

As an application of the above description of the de Rham cohomology we give an integral representation of an order 2 invariant. Let L be a 3-component link, which is represented by $f : S^1 \times S^1 \times S^1 \to \mathrm{Conf}_3(\mathbf{R}^3)$. As in the case of the Gauss formula for the linking number we can construct a 2-cycle $\widehat{c}_f$ of the loop space $\Omega\mathrm{Conf}_3(\mathbf{R}^3)$ associated with f in the following way. We start from a cubic 3-chain

$$\mu : I^3 \to S^1 \times S^1 \times S^1$$

corresponding to the fundamental homology class of the torus $S^1 \times S^1 \times S^1$. Then μ gives a 2-chain

$$\widehat{\mu} : I^2 \to \Omega(S^1 \times S^1 \times S^1).$$

Composing with

$$\Omega f : \Omega(S^1 \times S^1 \times S^1) \to \Omega\mathrm{Conf}_3(\mathbf{R}^3)$$

we obtain a 2-cycle $\widehat{c}_f$ of the loops space $\Omega\mathrm{Conf}_3(\mathbf{R}^3)$.

There is a 3-form ϕ_{123} on $\mathrm{Conf}_3(\mathbf{R}^3)$ such that the relation

$$\omega_{12} \wedge \omega_{23} + \omega_{23} \wedge \omega_{13} + \omega_{13} \wedge \omega_{12} = d\phi_{123}$$

is satisfied. Now the iterated integral

$$\int(\omega_{12}\omega_{23}+\omega_{23}\omega_{13}+\omega_{13}\omega_{12})+\int\phi_{123}$$

is a closed 2-form on the loop space $\Omega\mathrm{Conf}_3(\mathbf{R}^3)$. The integral

$$\int_{\widehat{c}_f}\left(\int(\omega_{12}\omega_{23}+\omega_{23}\omega_{13}+\omega_{13}\omega_{12})+\int\phi_{123}\right)$$

of the above 2-form on the loop space over the 2-cycle $\widehat{c}_f$ defined above is an invariant of L. A relation of this invariant to known invariants will be discussed elsewhere. We see that the first term

$$I_1=\int_{\widehat{c}_f}\left(\int(\omega_{12}\omega_{23}+\omega_{23}\omega_{13}+\omega_{13}\omega_{12})\right)$$

is expressed by chord diagrams with 2 chords on 3 circles. It is important to notice that I_1 is not a link invariant. A prescription suggested by Chern-Simons perturbation theory is to compensate this integral by adding an integral defined by graphs with trivalent vertices (see [8] and [3]). From our point of view, instead of an integral associated with trivalent graphs, we add

$$I_2=\int_{\widehat{c}_f}\int\phi_{123}$$

to obtain an integration of a closed 2-form on the loop space over a 2-cycle. By suspension we obtain an $(m-1)$-cycle of $\Omega\mathrm{Conf}_3(\mathbf{R}^m)$ associated with $f:S^1\times S^1\times S^1\to\mathrm{Conf}_3(\mathbf{R}^m)$ for $m>3$ and we have an integral analogous to the above I_1+I_2 in the case $m>3$ as well.

Let $\widehat{c_1}$ and $\widehat{c_2}$ be p-chain and q-chain of the loop space ΩM. Then by the composition of loops we can define a $(p+q)$-chain denoted by $\widehat{c_1}\cdot\widehat{c_2}$ (see [4]). With this notation the above construction is generalized in the following way. Let L be an n-component link in $\mathbf{R}^3$. We consider an n-dimensional torus

$$T^n=\underbrace{S^1\times\cdots\times S^1}_{n}.$$

Then the link L gives a map $f:T^n\to\mathrm{Conf}_3(\mathbf{R}^3)$. For a q-dimensional subtorus $T^q\subset T^n$ we have a q-cycle $\alpha:I^q\to T^n$ corresponding to the fundamental homology class of T^q, which gives a $(q-1)$-chain

$$\widehat{\alpha}:I^{q-1}\to\Omega T^n.$$

Composing with

$$\Omega f:\Omega T^n\to\Omega\mathrm{Conf}_n(\mathbf{R}^3)$$

we obtain a $(q-1)$-cycle $\Omega f_*(\widehat{\alpha})$ of the loop space $\Omega\mathrm{Conf}_n(\mathbf{R}^3)$. Let $\widehat{c}_f$ be a k-cycle of $\Omega\mathrm{Conf}_n(\mathbf{R}^3)$ represented as the product of cycles of the above type for any subtorus of T^n. Then we have the following.

Theorem 5.2 *Let ω be a closed k-form on $\Omega\mathrm{Conf}_n(\mathbf{R}^3)$ given by iterated integrals as in Theorem 5.1. Then the integral*

$$\int_{\widehat{c}_f} \omega$$

over a cycle $\widehat{c}_f$ defined as above is a link invariant.

6 Orbit configuration spaces

The aim of this section is to give a brief review of the article [7] where we describe the relation between the homology of the loop spaces of the orbit configuration associated the action of Fuchsian groups acting on the upper half plane and finite type invariants for braids on surfaces. Let us consider the situation where a group Γ acts freely on a space X. We define the orbit configuration space by

$$\mathrm{Conf}_n^{\Gamma}(X) = \{(x_1, \cdots, x_n) \in X^n \ ; \ \Gamma x_i \cap \Gamma x_j = \emptyset \text{ if } i \neq j\}$$

where Γx stands for the orbit of $x \in X$ with respect to the action of Γ.

Let Γ be a Fuchsian group acting freely on the upper half plane $\mathbf{H}$. The quotient space $\mathbf{H}/\Gamma$ is an oriented surface denoted by Σ. For $d \geq 1$ we consider the action of Γ on $\mathbf{H} \times \mathbf{C}^d$ where Γ acts trivially on $\mathbf{C}^d$. We will describe the homology of the loop space of the orbit configuration space

$$H_*(\Omega\mathrm{Conf}_n^{\Gamma}(\mathbf{H} \times \mathbf{C}^d); \mathbf{Z}).$$

The factor $\mathbf{C}^d$ appears for the degree shifting. The reason why we consider the orbit configuration space rather than the configuration space itself is that the homology of the loop space of the former one has a more sensible structure in relation with finite type invariants for braids on surfaces.

We introduce the notion of horizontal chord diagrams on Σ. First, we recall chord diagrams on surfaces following [1]. Here we assume that the endpoints of the chords are distinct and lie on the circles. Let D be a chord diagram. We consider a continuous map $\gamma : D \to \Sigma$ and we denote by $[\gamma]$ its free homotopy class. We call such pair $(D, [\gamma])$ a chord diagram on Σ. We denote by $\mathcal{D}_\Sigma$ the vector space over $\mathbf{R}$ spanned by all chord diagrams on Σ and $\mathcal{D}_\Sigma^k$ its subspace spanned by chord diagrams with k chords. We define $\mathcal{A}(\Sigma)$ to be the quotient space of $\mathcal{D}_\Sigma$ modulo the 4-term relations. We refer the reader to [1] for a precise definition of the 4-term relations in this situation. The chord diagrams on a surface Σ are related to Vassiliev invariants for links in $\Sigma \times I$ in the following way. Let v be an order k invariant for links in $\Sigma \times I$. Then the associated weight system $w(v)$ defines a linear form $w(v) : \mathcal{D}^k_\Sigma \to \mathbf{R}$ satisfying the 4-term relations. As in shown in [1], $\mathcal{A}(\Sigma)$ has a structure of a Poisson algebra.

We fix a base point $\mathbf{x} = (x_1, \cdots, x_n) \in \mathrm{Conf}_n(\Sigma)$. The fundamental group $\pi_1(\mathrm{Conf}_n(\Sigma), \mathbf{x})$ is the pure braid group of Σ with n strands and is denoted by $P_n(\Sigma)$. We have a natural homomorphism

$$p : P_n(\Sigma) \to \bigoplus_{j=1}^{n} \pi_1(\Sigma, x_j)$$

and Ker p is denoted by $P_n(\Sigma)^0$. Notice that the direct sum $\bigoplus_{j=1}^n \pi_1(\Sigma, x_j)$ acts freely on the orbit configuration space $\mathrm{Conf}^\Gamma_n(H)$ and the quotient space is the configuration space $\mathrm{Conf}_n(\Sigma)$.

We denote by C^k_n the set of horizontal chord diagrams on n strands with k chords. For $\Gamma \in C^k_n$ consider a continuous map

$$f : \Gamma \to \Sigma$$

such that

$$f(p_i(0)) = f(p_i(1)) = x_i,\ 1 \le i \le n$$

and denote by $[f]$ its homotopy class. Here we consider a homotopy preserving the base point. We shall say that such horizontal chord diagram on Σ is based at $\mathbf{x} = (x_1, \cdots, x_n)$ We denote by $D^k_n(\Sigma)$ the free $\mathbf{Z}$ module spanned by pairs $(\Gamma, [f])$ for $\Gamma \in C^k_n$ and $f : \Gamma \to \Sigma$, based at $\mathbf{x}$. The subspace of $D^k_n(\Sigma)$ spanned by $f : \Gamma \to \Sigma$, such that each curve $f(p_i(t)), 0 \le t \le 1$, is homotopic to the point $\{x_i\}$ is denoted by $D^k_n(\Sigma)^0$.

We fix a base point $x_0 \in \Sigma$ and consider the fundamental group $\pi_1(\Sigma, x_0)$. Let us fix a path in Σ connecting x_0 to x_j and we identify the set of homotopy classes of paths from x_i to x_j with $\pi_1(\Sigma, x_0)$. For $\gamma \in \pi_1(\Sigma, x_0)$ we consider $(\Gamma_{ij}, [f]) \in D^k_n(\Sigma)^0$ such that $f(C_{ij})$ corresponds to $\gamma \in \pi_1(\Sigma, x_0)$ by the above identification. We denote this $(\Gamma_{ij}, [f])$ by $X_{ij,\gamma}$. We see that the direct sum

$$D_n(\Sigma)^0 = \bigoplus_{k \ge 0} D^k_n(\Sigma)^0$$

has a structure of an algebra over $\mathbf{Z}$ where the product is defined by the composition of chord diagrams. As an algebra $D^k_n(\Sigma)^0$ is generated by $X_{ij,\gamma}$, $1 \le i \neq j \le n$, $\gamma \in \pi_1(\Sigma, x_0)$. We have the following lemma.

Lemma 6.1 *The relation $X_{ij,\gamma} = X_{ji,\gamma^{-1}}$ holds for any $\gamma \in \pi_1(\Sigma, x_0)$.*

The direct sum

$$D_n(\Sigma) = \bigoplus_{k \geq 0} D_n^k(\Sigma)$$

has a structure of an algebra as well. For the subspace $D_n^0(\Sigma)$ spanned by the chord diagrams with empty chord, we have a natural injection

$$\iota_j : \pi_1(\Sigma, x_j) \to D_n^0(\Sigma), \ 1 \leq j \leq n$$

and we have an isomorphism of $\mathbf{Z}$ algebras

$$\bigoplus_{j=1}^{n} \mathbf{Z}\pi_1(\Sigma, x_j) \cong D_n^0(\Sigma).$$

We write μ_j for $\iota_j(\mu)$ where μ is an element of $\pi_1(\Sigma, x_j)$.

The direct sum

$$\Lambda_n = \bigoplus_{j=1}^{n} \mathbf{Z}\pi_1(\Sigma, x_j)$$

acts on $D_n(\Sigma)^0$ by the conjugation

$$\Gamma \mapsto \mu_j \Gamma \mu_j^{-1}, \ \Gamma \in D_n(\Sigma)^0, \mu \in \pi_1(\Sigma, x_j).$$

We have the following.

Lemma 6.2 *With respect to the above action we have*

$$\mu_l X_{ij,\gamma} \mu_l^{-1} = X_{ij,\gamma} \text{ for } l \neq i, j, \quad \mu_i X_{ij,\gamma} \mu_i^{-1} = X_{ij,\mu\gamma}.$$

Let $\mathcal{I}$ be the ideal of $D_n(\Sigma)^0$ generated by

$$[X_{ij,e}, X_{kl,e}], \quad i,j,k,l \text{ distinct}$$
$$[X_{ij,e}, X_{jk,e} + X_{ik,e}] \quad i,j,k \text{ distinct}$$

as a Λ_n module. We set

$$\mathcal{A}_n(\Sigma)^0 = D_n(\Sigma)^0/\mathcal{I}, \ \ \mathcal{A}_n(\Sigma) = D_n(\Sigma)/\mathcal{I}.$$

We have an action of Λ_n on $\mathcal{A}_n(\Sigma)^0$ by conjugation and the semidirect product $\mathcal{A}_n(\Sigma)^0 \bullet \Lambda_n$ with respect to this action is isomorphic to $\mathcal{A}_n(\Sigma)$.

Lemma 6.3 *We have the following relations in $\mathcal{A}_n(\Sigma)^0$.*

$$[X_{ij,\gamma}, X_{kl,\delta}] = 0, \quad i,j,k,l \text{ distinct}$$
$$[X_{ij,\gamma}, X_{jk,\delta} + X_{ik,\gamma\delta}] = 0 \quad i,j,k \text{ distinct}$$

The above 4-term relations appear naturally when we consider finite type invariants for $P_n(\Sigma)$. We denote by $V_k(P_n(\Sigma))$ the $\mathbf{Z}$ module of order k invariants of $P_n(\Sigma)$ with values in $\mathbf{Z}$. For $v \in V_k(P_n(\Sigma))$ the associated weight system $w(v)$ defines a $\mathbf{Z}$-module homomorphism

$$w(v) : D_n^k(\Sigma) \to \mathbf{Z}$$

satisfying the 4-term relations in Lemma 6.3. We have an injective homomorphism of $\mathbf{Z}$-modules

$$V_k(P_n(\Sigma))/V_{k-1}(P_n(\Sigma)) \to \mathrm{Hom}(\mathcal{A}_n^k(\Sigma), \mathbf{Z})$$

where $\mathcal{A}_n^k(\Sigma)$ denotes the submodule of $\mathcal{A}_n(\Sigma)$ spanned by chord diagrams with k chords.

The algebra $\mathcal{A}_n(\Sigma)^0$ and $\mathcal{A}_n(\Sigma)$ were introduced in [9] by the above generators and relations. In this article, they constructed an injective homomorphism

$$\mathbf{Z}P_n(\Sigma) \to \widehat{\mathcal{A}_n(\Sigma)} = \prod_{k \geq 0} \mathcal{A}_n^k(\Sigma)$$

as $\mathbf{Z}$ modules.

The homology of the loop space of our orbit configuration space is related to the algebra of horizontal chord diagrams on $\Sigma = \mathbf{H}/\Gamma$ in the following way.

Theorem 6.4 [7] *We have an isomorphism of Hopf algebras*

$$H_*(\Omega \mathrm{Conf}_n^\Gamma(\mathbf{H} \times \mathbf{C}^d); \mathbf{Z}) \cong \mathcal{A}_n(\Sigma)^0$$

for any $d \geq 1$.

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